

MULTI-SERVICE SERIAL COST SHARING:  
A CHARACTERIZATION OF THE MOULIN-SHENKER RULE

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**Abstract:** We focus on the Moulin-Shenker cost sharing rule as a natural extension of the serial rule to multi-service facilities where services are personalized. We show that it is the unique *regular* rule that is compatible with *scale invariance* and *self consistency*.

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# 1 Introduction

This paper focusses on the problem of allocating the cost of usage of a production facility jointly owned by a fixed group of agents. Instead of technologies generating a single (divisible) output, here we concentrate on those multi-commodity situations where each of the (divisible) goods is personalized. Each of the agents  $i = 1, 2, \dots, n$  has an interest in  $q_i$  units of good  $i$ , and we look for an equitable way of distributing the corresponding total cost  $c(q_1, q_2, \dots, q_n)$  among the  $n$  agents. Then, the more eligible devices will be dependent on the level of individual demands. In the literature, different solutions have been proposed, for instance Aumann-Shapley pricing (Aumann-Shapley (1974), Billera and Heath (1982)), the ordinally proportional rule (Sprumont (1997)), the Shapley-Shubik mechanism (Shubik (1962)), the Friedman-Moulin rule (Friedman and Moulin (1995)) and the Moulin-Shenker rule (Sprumont (1997)).

In this paper we focus on the latter, the Moulin-Shenker rule, that was analyzed by Sprumont (1997). Just like the Friedman-Moulin rule, it is an extension of the serial rule (Moulin and Shenker (1992a, 1992b), Moulin (1996)) in the sense that for each *homogeneous* cost sharing problem it proposes the serial cost shares for the naturally related one-dimensional problem. Sprumont (1997) argues that cost shares should not depend on the conventions used to measure an agent's demand. The principle that requires robustness of a cost sharing rule with respect to essentially any transformation of measuring scales, is called *ordinality*. Then, consequently, this plea for transformation robust mechanisms rules out the Friedman-Moulin rule since it is not even scale invariant. All other earlier mentioned mechanisms satisfy scale invariance, but ordinality is only consistent with Shapley-Shubik, Moulin-Shenker and

the ordinally proportional rule. Sprumont (1997) provides a very compact characterization of the Moulin-Shenker rule, but this does not rely on ordinality or scale invariance at all. The characterizing set of axioms consists of one rather technical axiom, *partial differentiability*, and the interesting *serial principle*. The serial principle is the natural extension of the property *independence of size of larger demands* (Moulin and Shenker (1992a)). It seems to be the most essential feature for serial cost sharing, by which the smaller agents are protected against possibly excessive behavior of the larger demanders. In this paper we show that the serial principle is implied by the combination of *self consistency*, *no exploitation* and *continuity* of the cost sharing rule as a function of demands. Now, in order to obtain a full characterization of the Moulin-Shenker rule we include in addition the property *scale invariance* and a very weak technical condition *null homogeneity*. So we do not need the full power of ordinality in order to characterize the Moulin-Shenker rule. It suffices to focus on the classic property of *scale invariance* that incorporates robustness of the rule against all linear transformations of scale.

## 2 The model and definitions

Throughout the paper we will concentrate on a fixed and finite group of agents  $N = \{1, 2, \dots, n\}$ . Its members jointly own some production facility for some set of goods. The output goods are personalized in the sense that there is at most one interested agent for each output. So we can speak of the set of goods  $N$ , where good  $i \in N$  is for agent  $i \in N$ . A particular level of demanded output can then be described by a vector  $q \in \mathbb{R}_+^N$ , where the  $i$ -th coordinate  $q_i$  is the demand of agent  $i$  for good  $i$ . Then the demand space is partially ordered by the natural ordering

$\leq$ . For all  $q, q' \in \mathbb{R}_+^N$ ,  $q < q'$  if and only if  $q_j \leq q'_j$  for all  $j \in N$  with strict inequality for at least one coordinate. Whenever  $q_j < q'_j$  for all  $j \in N$  then we write  $q \ll q'$ . The power set of  $N$  is denoted by  $\mathcal{P}(N)$ . For  $q \in \mathbb{R}_+^N$  and  $S \in \mathcal{P}(N)$ ,  $q_S$  is the demand profile obtained from  $q$ , where the demands of the players in  $N \setminus S$  are set to 0. The profile out of  $q \in \mathbb{R}_+^N$  where the demand of a player  $i$  is interchanged with  $t \in \mathbb{R}_+$  is denoted by  $(q^{-i}, t)$ . The Euclidean norm of a vector  $q \in \mathbb{R}_+^N$  is denoted by  $\|q\|$ .

We assume that all information about the costs involved with bringing production up to a certain level is given by a *cost function*  $c : \mathbb{R}_+^N \rightarrow \mathbb{R}_+$ . In this paper we will only be concerned with cost functions  $c$  that are continuously differentiable and increasing, i.e. if  $x < y$  then  $c(x) < c(y)$ . Moreover the partial derivatives of  $c$  are supposed to be bounded away from 0 and  $\infty$ , i.e. there are  $a(c), b(c) > 0$  such that

$$a(c) \leq D_i c \leq b(c) \text{ for all } i \in N.$$

Here  $D_i c$  denotes the  $i$ -th partial derivative, which is continuous by assumption. In addition there are no fixed costs, which amounts to the condition  $c(0) = 0$ . The class of all such cost functions is denoted by  $\mathcal{C}$ . Furthermore, a cost function is called *normalized* if  $D_i c(0) = 1$  for all  $i \in N$ .

A *cost sharing problem* is an ordered pair  $(q, c) \in \mathbb{R}_+^N \times \mathcal{C}$ . The class of all cost sharing problems is denoted by  $\mathcal{G}$ . A *cost sharing rule* is a mapping  $x : \mathcal{G} \rightarrow \mathbb{R}_+^N$  associating to each cost sharing problem  $(q, c)$  an efficient vector of cost shares  $x(q, c) \in \mathbb{R}_+^N$ , i.e.  $\sum_{i \in N} x_i(q, c) = c(q)$ .

Sprumont (1997) argues that cost sharing rules should be invariant under all *ordinal transformations* of the cost sharing problem. Modelling a cost sharing problem differently, by just redefining the measure scales

should not affect the outcome of the allocation process. The allocation should not depend on the conventions used to measure the demands. For our purposes, we need only the more familiar notion of *scale invariance*, which imposes independence of rescaling units of the measuring scales.

**Definition** A function  $f : \mathbb{R}_+^N \rightarrow \mathbb{R}_+^N$  is a positive linear transformation of scales if there is  $\lambda \in \mathbb{R}_{++}^N$  such that  $f(q) = (\lambda_1 q_1, \dots, \lambda_n q_n)$  for all  $q \in \mathbb{R}_+^N$ . A cost sharing rule  $x$  is *scale invariant* (SI) if for all such mappings  $f$  and all cost sharing problems  $(q, c) \in \mathcal{G}$  it holds that  $x(q, c) = x(f(q), c \circ f^{-1})$ .

For instance, if the output of some good is measured by weight, scale invariance tells us that the cost shares should not depend on the fact that we expressed the amounts in kilos instead of tons. We like to stress the gap between ordinality and scale invariance: ordinality requires also invariance with respect to all non-linear increasing transformations of the measuring scales.

Also we like robustness of a rule with respect to small changes in the data that is used to model the cost sharing problem. For instance, small perturbations of the demand profile should not result in large changes in the cost shares.

**Definition** A cost sharing rule  $x$  is *continuous* (CONT) if for all  $c \in \mathcal{C}$ , the mapping  $q \mapsto x(q, c)$  is continuous.

Less familiar will be the following property that prescribes the limiting behavior of a rule on a very specific class of cost sharing problems ,

**Definition** A cost sharing rule  $x$  is *null homogeneous* (NHOM) if for

all cost sharing problems  $(q, c) \in \mathcal{G}$  with  $D_i c(0) = 1$  for all  $i \in N$ , it holds that, for all  $i \in N$ ,

$$\lim_{t \downarrow 0} \frac{x_i(te_N, c)}{c(te_N)} = \frac{1}{|N|}$$

where  $e_N \in \mathbb{R}_+^N$  is the vector with all coordinates 1.

Null homogeneity expresses the feeling that when a problem resembles a homogeneous problem very much, then the individual cost shares should be almost the same (compared to the size of the problem) in case all demands are equal (in absolute terms). All earlier mentioned rules satisfy NHOM, and it is in this sense that it can be considered as very weak. It can not be used to distinguish between other well-known cost sharing rules.

### 3 The Moulin-Shenker rule

Just like the Aumann-Shapley pricing mechanism, the Friedman-Moulin serial extension determines cost shares through measuring the marginal cost along some curve in the demand space towards the aggregate demand. If  $q$  is an ordered demand profile, i.e.  $q_i \leq q_j$  whenever  $i \leq j$ , then the latter mechanism uses the unique increasing curve that meets at 0 and the intermediate levels  $(q_1, q_1, \dots, q_1), (q_1, q_2, q_2, \dots, q_2), \dots, (q_1, q_2, q_3, \dots, q_{n-1}, q_{n-1}), q$  and is linear inbetween. If this curve is denoted by  $\gamma$  then agent  $i \in N$  is charged  $\int_\gamma D_i c$ . When the argument of  $\gamma$  is seen as a time parameter, it can be considered to describe a production process. Typically, by taking the same type of production curve for all cost structures, this yields an additive method. As is pointed out in Friedman and Moulin (1995) and

Kolpin (1996) additivity is too demanding in combination with the serial idea and scale invariance.

Consider the following system of differential equations for  $(q, c) \in \mathcal{G}$ . Focus on a mapping  $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+^N$  such that for all  $i \in N$

$$D_i \gamma(t) = \begin{cases} \frac{1}{D_i c(\gamma(t))} & \text{for all } t \geq 0 \text{ s.t. } \gamma_i(t) < q_i \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

It can be shown that there is a unique mapping  $\gamma^{q,c}$  that satisfies the above equalities. Essentially, this is due to the fact that the mapping  $q \mapsto \frac{1}{D_i c(q)}$  satisfies a Lipschitz condition (see Coddington and Levinson (1955)). Note that this curve heavily depends on the demand profile  $q$  and the cost function  $c \in \mathcal{C}$ . The solution  $\gamma^{q,c}$  can also be interpreted as a production device. Suppose that the intensity at which an agent  $i$  is served by means of the plan  $\gamma^{q,c}$  at moment  $t$  is measured by the corresponding marginal cost  $D_i c(\gamma^{q,c}(t)) D_i \gamma^{q,c}(t)$ . Then  $\gamma^{q,c}$  can be intuitively interpreted as a device by which goods are distributed with equal intensity for those agents that are not fully served at  $t$ , since for those agents  $i$ , it holds  $D_i c(\gamma^{q,c}(t)) D_i \gamma^{q,c}(t) = 1$ .

The Moulin-Shenker rule now charges agent  $i$  for the marginal costs  $D_i c$  along the production device  $\gamma^{q,c}$  in the cost sharing problem  $(q, c)$ .

**Definition** Let  $(q, c) \in \mathcal{G}$ . The *Moulin-Shenker rule*  $x^{\text{MS}}$  determines the individual cost shares by taking the integral of all marginal cost along the curve  $\gamma^{q,c}$ , which solves the above system of differential equations (1). Then for all  $i \in N$ ,

$$x_i^{\text{MS}}(q, c) = \int_0^\infty D_i c(\gamma^{q,c}(s)) D_i \gamma^{q,c}(s) ds. \quad (2)$$

The Moulin-Shenker rule is a serial extension that captures the ideas of scale invariance and demand monotonicity, at the cost of additivity. As one might expect, this can only be achieved when such a production device varies not only with the profile of demands but also with the cost structure at hand.

The rest of the paper is devoted to studying the Moulin-Shenker rule as a special member of the general class of all *path generated* cost sharing methods, which propagates the *serial principle* as the common characteristic. The serial principle prevents the smaller agent to get overexposed to the consequences of contingent excessive behavior of other agents. This principle was formulated for the one good case by Moulin and Shenker (1992a) as *independence of size of larger demands*. The corresponding formulation makes use of intercomparison of individual demands. But, typically, the asymmetric multi-good case lacks a natural ordering by which demands can be directly compared. Still, if a mechanism is singled out for some fairness properties, then there is just one consistent way of comparing the demands, and that is by comparing the size of the corresponding cost shares for the problem at hand. The serial principle then urges that once the mechanism values the demand of an agent  $i$  lower than that of agent  $j$ , any further increase of agent  $j$ 's demand should have no effect at all on agent  $i$ 's cost share.

**Definition** A cost sharing rule satisfies the *serial principle* if for all cost sharing problems  $(q, c) \in \mathcal{G}$  it holds that for all  $i \in N$  and  $j \in N \setminus \{i\}$  with  $x_j(q, c) \geq x_i(q, c)$  it holds that for  $t \geq q_j$ ,

$$x_i((q^{-j}, t), c) = x_i(q, c).$$

In general, though the names are quite suggestive, being a serial extension is not sufficient for a method to satisfy the serial principle. The



Friedman-Moulin rule illustrates this distinction; it is a serial extension only.

Next we will define the class of *path generated cost sharing methods*. The idea is in fact adopted from Sprumont (1997), but notations are different.

For  $S \subseteq N$  a *path* in  $\mathbb{R}_+^S$  is a continuous mapping  $\pi : \mathbb{R}_+ \rightarrow \mathbb{R}_+^S$  with  $\pi(0) = 0$ . The path  $\pi$  is increasing if  $\pi_i(t) < \pi_i(t')$  for all  $i \in S$  if only  $t < t'$ . In our setting, with the argument of  $\pi$  thought of as being time, an increasing path may be considered as a program for production. At time  $t$  an amount of good  $i$  equal to  $\pi_i(t)$  units is produced for agent  $i$ .

Suppose that for each pair  $(d, c) \in \mathcal{G}$  we have an increasing path  $\pi^{d,c,S}$  for  $S$  such that for each  $q \in \mathbb{R}_+^S$  there is  $t \in \mathbb{R}_+$  with  $\pi^{d,c,S}(t) > q$ . Such a path will be considered to describe a fictitious production plan for coalition  $S$  from level  $d \in \mathbb{R}_+^N$ . Possibly such a plan will depend on the exogeneous information of costs that is summarized by  $c \in \mathcal{C}$ .

$\Pi$  is defined as the collection of all those paths, one for each triple  $(d, c, S) \in \mathbb{R}_+^N \times \mathcal{C} \times \mathcal{P}(N)$ . We will refer to  $\Pi$  as a *path collection*. A path collection  $\Pi$  defines for each cost sharing problem  $(q, c) \in \mathcal{G}$  a production plan, *casu quo* a path  $\pi$  for  $N$  in the following way.

We start at production level 0. Initially, we take the path for  $N$ ,  $\pi^{0,c,N}$  as a production device, telling us for each moment in time what is produced for the individual agents. So follow  $\pi^{0,c,N}$  up to the earliest moment  $t_1$  that some agents  $M_1 \subseteq N$  are satisfied, i.e.

$$\pi_i^{0,c,N}(t_1) = q_i \text{ for all } i \in M_1.$$

Define  $\pi$  on  $[0, t_1]$  by  $\pi(t) = \pi^{0,c,N}(t)$ . Let  $d^1$  denote the total demand that is processed so far,  $d^1 = \pi(t_1)$ . Still, an agent  $i \in N \setminus M_1$  needs  $q_i - d_i^1$  units of good  $i$  in order to be satisfied. Next, we take  $\pi^{d^1,c,N \setminus M_1}$  as the additional production plan for  $N \setminus M_1$  until the first moment  $t_2$  that some agents  $M_2 \subseteq N \setminus M_1$  are satisfied, i.e.

$$\pi^{d^1,c,N \setminus M_1}(t_2) = q_i - d_i^1 \text{ for all } i \in M_2.$$

The definition of  $\pi$  is now completed up to moment  $t_1 + t_2$  by

$$\pi(t + t_1) := d^1 + (0_{M_1}, \pi^{d^1,c,N \setminus M_1}(t)) \text{ for all } t \in (0, t_2].$$

Let  $d^2 = \pi(t_1 + t_2)$ . Follow the production device  $\pi^{d^2,c,N \setminus (M_1 \cup M_2)}$  until moment  $t_3$  where the first agents  $M_3 \subseteq N \setminus (M_1 \cup M_2)$  are fulfilled with their remaining needs  $q_{M_3} - d_{M_3}^2$ . Then define, for all  $t \in (0, t_3]$ ,

$$\pi(t + t_1 + t_2) = d^2 + (0_{M_1 \cup M_2}, \pi^{d^2,c,N \setminus (M_1 \cup M_2)}(t)).$$

In this way we can go on and complete the definition of  $\pi$ . We just proceed by determining time levels  $t_4, t_5, \dots$  and corresponding groups of agents  $M_4, M_5, \dots$  until the first moment  $t_1 + \dots + t_k$  such that there are no remaining demanders, i.e.  $N \setminus (M_1 \cup \dots \cup M_k) = \emptyset$ . Note that  $\pi(t) = q$  when  $t > t_1 + t_2 + \dots + t_k$ . We will say that  $\pi$  is *the path for*  $(q, c)$  *generated by*  $\Pi$ .

**Definition** The solution for the cost sharing problem  $(q, c) \in \mathcal{G}$  generated by a path collection  $\Pi$  is the vector  $x^\Pi(q, c) \in \mathbb{R}_+^N$  defined as follows. Let  $\pi$  be the path for  $(q, c)$  generated by  $\Pi$ . Suppose that according to  $\pi$  agent  $i$  is satisfied at moment  $t_i$ . Without loss of generality, assume that  $t_i \leq t_j$  whenever  $i \leq j$  for all  $i, j \in N$ . We split the successive cost increments  $c(\pi(t_{i+1})) - c(\pi(t_i))$  equally among the agents requiring service on the interval  $(t_i, t_{i+1}]$ . By assumption

this is the set of agents  $\{i + 1, i + 2, \dots, n\}$ . Then this boils down to  $x_1^\Pi(q, c) = \frac{c(\pi(t_1))}{N}$  as the cost share for agent 1, while the cost shares for the other agents  $i \in N$  are inductively defined through

$$x_i^\Pi(q, c) = x_{i-1}^\Pi(q, c) + \frac{c(\pi(t_i)) - c(\pi(t_{i-1}))}{n - i + 1}.$$

By varying over all cost sharing problems in  $\mathcal{G}$  this yields a cost sharing rule  $x^\Pi$ , generated by the path collection  $\Pi$ .

We will also say that in the above definition the cost shares for the problem  $(q, c)$  are generated by  $\Pi$ . Note that in essence for a path generated method only the images of the paths are of importance for determining the cost allocation; any other parametrization of the paths determines the same rule. Keeping this in mind, one should have no problem with the following.

**Lemma 3.1** *Let  $f : \mathbb{R}_+^N \rightarrow \mathbb{R}_+^N$  be a linear transformation of measuring scales. Suppose  $x$  is a scale invariant cost sharing rule that is generated by a path collection  $\Pi = \{\pi^{d,c,S} \mid (d, c, S) \in \mathbb{R}_+^N \times \mathcal{C} \times \mathcal{P}(N)\}$ . If the cost shares for  $(q, c) \in \mathcal{G}$  are generated by  $\pi$ , then the cost shares for  $(f^{-1}(q), c \circ f)$  are generated by  $f^{-1} \circ \pi$ .*

A simple but important observation is that each path generated method indeed satisfies the serial principle. Sprumont (1997) proves the converse of this statement for all continuous mechanisms.

**Lemma 3.2 (Sprumont (1997))** *A continuous cost sharing mechanism satisfies the serial principle if and only if it is generated by a path collection.*

Fix a cost function  $c \in \mathcal{C}$ . For each  $d \in \mathbb{R}_+^N$ , let  $c^d \in \mathcal{C}$  be the cost function that relates each increase of demand  $q$  after  $d$  to the corresponding incremental cost, i.e.  $c^d(q) = c(d + q) - c(q)$  for all  $q \in \mathbb{R}_+^N$ . An ordered pair  $(d, S) \in \mathbb{R}_+^N \times \mathcal{P}(N)$  gives rise to a system of differential equations in the following way. Let  $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+^N$  be such that for all  $t \in \mathbb{R}_+$  and all  $i \in S$

$$D_i \gamma(t) = \frac{1}{D_i c^d(\gamma(t))}. \quad (3)$$

By the regularity assumptions on  $c$  this system has a unique solution, which we will denote by  $\gamma^{d,c,S}$ . Then by varying over all triples  $(d, c, S) \in \mathbb{R}_+^N \times \mathcal{C} \times \mathcal{P}(N)$  this gives rise to a path collection  $\Gamma$ , which in turn generates the Moulin-Shenker rule. Note that  $\gamma^{d,c,S} = \gamma^{0,c^d,S}$  for all  $(d, c, S) \in \mathbb{R}_+^N \times \mathcal{C} \times \mathcal{P}(N)$ .

Note that in our setting the serial principle implies *positivity*, i.e.  $q_i > 0$  implies  $x_i(q, c) > 0$  for all  $i \in N$  and all problems  $(q, c) \in \mathcal{G}$ . For every non-positive mechanism possibly free-riders enter the picture. Any increase of any agent's demand causes a rise of total cost, so the impact on total cost of any non-zero demander is considered to be positive. Consequently, positivity can be considered as compelling for our purposes. Another resulting principle encompasses the fairness concept, that agents cannot profit from others just by their presence. This criterium is better known as *no exploitation* (NOEXP); if an agent has demand 0 then his share of total costs should not exceed 0.

It is easily seen that that for every positive cost sharing rule the content of null homogeneity is exactly rendered, for all *normalized* cost functions  $c \in \mathcal{C}$ , by

$$\lim_{t \downarrow 0} \frac{x_i(te_N, c)}{x_j(te_N, c)} = 1 \quad \text{for all } i, j \in N. \quad (4)$$

Sprumont (1997) shows that among the class of all path generated methods there is only one for which all partial derivatives w.r.t. the demand input exist, and that is  $x^{\text{MS}}$ . Actually, it can be shown that  $x^{\text{MS}}$  is continuously differentiable. Before we are ready for another characterization, we need to focus on another feature of the Moulin-Shenker rule first.

In Moulin and Shenker (1992) we find the property *free lunch* which combines a mild form of justice with a weak form of *consistency* (see also Kolpin (1994) and Thomson (1990, 1995)). To generalize this idea we develop the notion of *self-consistency*. This notion makes it possible to link outcomes for problems of different size.

Essentially, a cost sharing mechanism is used as an instrument of evaluation; the agent with the larger cost share can be considered to have a larger demand. In this way, for a problem  $(q, c)$ , all the demands are *equally valued by a cost sharing rule*  $x$  if and only if  $x_i(q, c) = x_j(q, c)$  for all  $i, j \in N$ .

Fix a cost sharing problem  $(q, c)$  and a cost sharing rule  $x$ . Suppose that we provide all agents with equally valued parts of their demands; agent  $i$  gets  $d_i \leq q_i$  such that  $x_i(d, c) = x_j(d, c)$  for all  $i, j \in N$ . Then the *reduced* cost sharing problem is defined by the profile of unfulfilled demands  $q - d$ , and the cost data for any level of production beyond  $d$  as is summarized by  $c^d$ . Now *self-consistency* allows for determining the final cost shares by independently solving the problems  $(d, c)$  and  $(q - d, c^d)$  and taking the sum over the corresponding outcomes.

In the same spirit we deal with those situations where there are some zero demanders. It is reasonable to require that just their presence should have no effect on the allocation of costs for the other agents. Suppose again that  $d$  is a demand profile smaller than  $q$ , such that the non-zero demanders are equally evaluated by the mechanism  $x$ . Then

*self-consistency* proposes  $x(q, c)_S = x(d, c)_S + x(q - d, c^d)_S$ , where  $S$  is the set of the non-zero demanders for  $q$ . So, if cost shares differ, then this is not due to the part of the problem that the agents are equally charged for, but due to asymmetries in the related reduced problem.

**Definition** A cost sharing rule  $x$  satisfies *self-consistency* (SCONS) if for all cost sharing problems  $(q, c) \in \mathcal{G}$  such that  $q_{N \setminus S} = 0_{N \setminus S}$  for some  $S \in \mathcal{P}(N)$  and  $d \leq q$  such that  $x_i(d, c) = x_j(d, c)$  for all  $i, j \in S$ ,

$$x(q, c)_S = x(d, c)_S + x(q - d, c^d)_S.$$

**Lemma 3.3** *A continuous cost sharing mechanism satisfies no exploitation and self consistency only if it is generated by a collection of paths.*

**Proof** Let  $x$  be a continuous cost sharing mechanism that satisfies no exploitation and self consistency. We will define a collection of paths by which  $x$  is generated. Let  $p^{0,c,N}$  be the set  $\{q \in \mathbb{R}_+^N \mid x_i(q, c) = \frac{c(q)}{n} \text{ for all } i \in N\}$ . We claim the following:

(i) For each  $t \geq 0$  there is exactly one  $q \in p^{0,c,N}$  with  $c(q) = t$ .

(ii) If  $q, q' \in p^{0,c,N}$ ,  $q \neq q'$  either  $q \ll q'$  or  $q' \ll q$ .

First we will prove (i). The first step consists of showing that there is at least one such  $q$  for all  $t \in \mathbb{R}_+$ . For  $t = 0$ , obviously there is only such  $q$  and that is  $q = 0$ . Let  $t > 0$  and define  $A(t)$  to be the isocost surface for  $c$  at level  $t$ . Denote the unit simplex in  $\mathbb{R}_+^N$  by  $\Delta^N$ . Then  $h : A(t) \rightarrow \Delta^N$  with  $h(q) = \frac{q}{\sum_{i \in N} q_i}$  for all  $q \in A(t)$  defines a homeomorphism. Denote its continuous inverse by  $h^{-1}$ . Next define the mapping  $g : \Delta^N \rightarrow \Delta^N$  by  $g(q) = \frac{1}{t}x(h^{-1}(q), c)$ . Then by continuity of both  $q \mapsto x(q, c)$  and  $h^{-1}$ , it follows that  $g$  is continuous. Note that for all  $q \in \Delta^N$  it holds that  $\{q_i = 0 \Rightarrow g_i(q) = 0\}$  by no exploitation. We are ready if we prove that there is a  $z \in \Delta^N$  such that  $g_i(z) = \frac{1}{n}$  for all  $i \in N$ . Define  $G : \Delta^N \rightarrow \Delta^N$  by

$$G_i(q) = \frac{q_i + \max\{\frac{1}{n} - g_i(q), 0\}}{1 + \sum_{j \in N} \max\{\frac{1}{n} - g_j(q), 0\}} \quad \text{for all } i \in N.$$

We claim that  $G(z) = z$  implies  $g_i(z) = \frac{1}{n}$  for all  $i \in N$ . Suppose the opposite,  $G(z) = z$  while not  $g_i(z) = \frac{1}{n}$  for all  $i \in N$ . Then there are  $k, l \in N$  such that  $g_k(z) < \frac{1}{n} < g_l(z)$ . Hence,  $\sum_{j \in N} \max\{\frac{1}{n} - g_j(z), 0\} > 0$

and therefore  $G_i(z) < z_i + \max\{\frac{1}{n} - g_i(z), 0\}$ . For  $i = k$  this converts to  $G_k(z) < z_k$ , which leads to contradiction.

Observe that  $G(\Delta^N) \subset \Delta^N$ , since by no exploitation  $G_i(q) > 0$ , also for the case  $q_i = 0$ . But now we are there, since by invoking Brouwer's Theorem we guarantee existence of such a fixed point  $z$  for  $G$ .

Now, we turn to proving uniqueness. Suppose that for  $q^1, q^2 \in p^{0,c,N}$  such that  $q^1 \neq q^2$ , it holds that  $c(q^1) = c(q^2)$ . Define  $q^* := q^1 \vee q^2$ . Then, in particular it holds that  $q^* > q^1$  and thus  $c(q^*) > c(q^1)$ . By self-consistency we have  $x(q^*, c) = x(q^1, c) + x(q^* - q^1, c^{q^1})$  and  $x(q^*, c) = x(q^2, c) + x(q^* - q^2, c^{q^2})$ . Since  $x(q^1, c) = x(q^2, c)$ , it holds that

$$x(q^* - q^1, c^{q^1}) = x(q^* - q^2, c^{q^2}). \quad (5)$$

For each  $i \in N$  it holds either  $q_i^* = q_i^1$  or  $q_i^* = q_i^2$ , so by no exploitation the cost share of agent  $i$  is either 0 in the cost sharing problem  $(q^* - q^1, c^{q^1})$  or in the cost sharing problem  $(q^* - q^2, c^{q^2})$ . But then by equality (5) we get  $x(q^* - q^1, c^{q^1}) = 0$ , and consequently  $x(q^1, c) = x(q^*, c)$ . This gives the desired contradiction, since

$$c(q^1) = \sum_{i \in N} x_i(q^1, c) = \sum_{i \in N} x_i(q^*, c) = c(q^*).$$

So, this proves our first claim.

Then, a direct consequence of (i) is that each  $t \geq 0$  defines a unique bundle  $y(t) \in p^{0,c,N}$  such that  $c(y(t)) = t$ . We will prove that the mapping  $y : t \mapsto y(t)$  is continuous. Continuity at  $t = 0$  is obvious. Suppose on the contrary that there is  $t^* > 0$  and a sequence  $t_1, t_2, \dots$  in  $\mathbb{R}_{++}$  such that  $\lim_{k \rightarrow \infty} t_k = t^*$ , while the sequence  $y(t_1), y(t_2), \dots$  does not converge to  $y(t^*)$ . Take  $\epsilon > 0$  such that  $B_\epsilon := \{z \in \mathbb{R}_+^N \mid \|y(t^*) - z\| < \epsilon\}$  is contained in  $\mathbb{R}_+^N$ , while there is a subsequence  $t'_1, t'_2, \dots$  of  $t_1, t_2, \dots$  such that for each  $k \in N$ ,  $y(t'_k) \notin B_\epsilon$ . Define  $r := \max_{k \in N} t'_k$ .



Then the sequence  $y(t'_1), y(t'_2), \dots$  is contained in the compact set  $\{z \in \mathbb{R}_+^N \mid c(z) \leq r\} \setminus B_\epsilon$ . Hence, there exists a subsequence  $t''_1, t''_2, \dots$  of  $t'_1, t'_2, \dots$  such that  $y(t''_1), y(t''_2), \dots$  converges, say to  $q$ . Observe that  $q \neq y(t^*)$ . By continuity of  $x$ , it holds for all  $i \in N$ ,

$$x_i(q, c) = \lim_{k \rightarrow \infty} x_i(y(t''_k), c) = \lim_{k \rightarrow \infty} \frac{c(y(t''_k))}{n} = \lim_{k \rightarrow \infty} \frac{t''_k}{n} = \frac{t^*}{n}.$$

Consequently, also  $q \in p^{0,c,N}$  and  $c(q) = t^*$ , but with  $y(t^*)$  as the unique vector satisfying these conditions, we reached a contradiction. So,  $y$  is continuous.

We will now prove (ii). Take  $q, q' \in p^{0,c,N}$  such that  $q \neq q'$ . Then (i) implies  $c(q) \neq c(q')$ . Without loss of generality we will assume that  $c(q') < c(q)$ . Suppose that not  $q' \ll q$ . By the continuity of  $y$ , there is a maximal  $t' < c(q')$  such that  $y(t') \in [0, q]$ . Hence, by self consistency,

$$x(q, c) = x(y(t'), c) + x(q - y(t'), c^{y(t')}).$$

But for all  $i \in N$ , we have  $x_i(q, c) - x_i(y(t'), c) = \frac{1}{n}(c(q) - c(y(t')))$ , and therefore corresponding to  $x$  all shares in the problem  $(q - y(t'), c^{y(t')})$  are equal. However,  $t'$  is taken such that  $(q - y(t'))_i = 0$  for at least one coordinate  $i \in N$ . Then, by no exploitation, the corresponding cost share of agent  $i$  is 0, hence the corresponding cost shares for the others are also 0. On the other hand, cost shares sum up to the total cost  $c^{y(t')}(q - y(t'))$ , which equals  $c(q) - c(y(t')) = c(q) - t'$ . But recall that  $t' < c(q') < c(q)$ , which yields

$$0 = \sum_{i \in N} x_i(q - y(t'), c^{y(t')}) = c(q) - t' > 0,$$

a contradiction. Therefore  $q' \ll q$ , which ascertains the validity of our second claim.

Now (i) together with (ii) show that  $p^{0,c,N}$  is the image of a path, which we will denote by  $\pi^{0,c,N}$ .

We proceed as follows. Define for all  $d \in p^{0,c,N}$  and nonempty sets  $S \subset N$  the set  $p^{d,c,S}$  by

$$\{q \in \mathbb{R}_+^S \mid x_i((0_{N \setminus S}, q), c^d) = \frac{1}{|S|} c^d((0_{N \setminus S}, q)) \text{ for all } i \in S\}.$$

Then, essentially by the same reasoning as before, it follows that  $p^{d,c,S}$  is the image of a path  $\pi^{d,c,S}$ . Take again  $d'$  as element of one of the previously defined sets  $p^{d,c,S}$ , and let  $d^1 := d + (0_{N \setminus S}, d')$ . Define for all  $S^1 \subset S, S^1 \neq \emptyset$ , the set  $p^{d^1,c,S^1}$  by

$$\{q \in \mathbb{R}_+^{S^1} \mid x_i((0_{N \setminus S^1}, q), c^{d^1}) = \frac{1}{|S^1|} c^{d^1}((0_{N \setminus S^1}, q)) \text{ for all } i \in S^1\}.$$

Again, essentially the same techniques as before show that this is the image of a path for  $S^1$ . In exactly the same way we proceed inductively by defining paths for coalitions of decreasing size. At the end of this procedure there still may be combinations  $(d, c, S)$  left for which  $\pi^{d,c,S}$  is not defined; for any of those triples we take  $\pi^{d,c,S}$  to be an arbitrary path. Then this completes the definition of a path collection  $\Pi$ .

It is now an easy exercise to show that it constitutes  $x$ , or  $x = x^\Pi$ . Let  $q \in \mathbb{R}_+^N$ . Suppose  $q \in \pi^{0,c,N}(\mathbb{R}_+)$ . Then according to  $x^\Pi$ , costs  $c(q)$  are split equally. But recall the definition of  $p^{0,c,N}$  which contains  $q$ , in order to see that the same division is made in case of  $x$ . If  $q \notin \pi^{0,c,N}(\mathbb{R}_+)$ , then let  $t_1$  be the first moment that  $\pi^{0,c,N}$  meets the demands of the agents  $N_1$ . Let  $d^1 := \pi^{0,c,N}(t_1)$  and suppose that  $(q - d^1)_{N \setminus N_1} \in \pi^{d^1,c,N \setminus N_1}(\mathbb{R}_+)$ . First notice that  $x_i(d^1, c) = \frac{1}{n} c(d^1)$  for all  $i \in N$ . Suppose that the vector of remaining demands  $(q - d^1)_{N \setminus N_1}$  is on the path for  $N \setminus N_1, \pi^{d^1,c,N \setminus N_1}$ . Then as a consequence

$$x(q - d^1, c^{d^1}) = \frac{1}{|N \setminus N_1|} c^{d^1}(q - d^1).$$

Thus, by self consistency for all  $i \in N \setminus N_1$ ,

$$\begin{aligned} x(q, c) &= x(d^1, c) + x(q - d^1, c^{d^1}) \\ &= \frac{1}{|N|} c(d^1) + \frac{1}{|N \setminus N_1|} c^{d^1}(q - d^1) = x^\Pi(q, c). \end{aligned}$$

If not  $(q - d^1)_{N \setminus N_1} \in \pi^{d^1, c, N \setminus N_1}(\mathbb{R}_+)$ , then proceed by following  $\pi^{d^1, c, N \setminus N_1}$  up to the first moment  $t_2$  that some agents  $N_2 \subset N \setminus N_1$  are satisfied with the present production level. Then the previous reasoning can just be replicated until, finally, a point is reached at which the remaining demand bundle is on the corresponding path for the remaining demanders.  $\square$

Especially, Lemma 3.3 shows that a continuous cost sharing rule with the properties self consistency and no exploitation satisfies the serial principle. However, self consistency is fundamentally different from the serial principle. For instance, it is easy to define path generated cost sharing rules, that satisfy no exploitation and continuity and fail to obey self consistency. Furthermore, splitting cost equally for all cost sharing problems defines a self consistent rule that neither satisfies no exploitation nor the serial principle.

We are now ready for the main result.

**Theorem 3.4** *There is only one continuous, null homogeneous, scale invariant cost sharing rule that satisfies the self consistency and no exploitation, and that is the Moulin-Shenker rule.*

**Proof** It is clear that the Moulin-Shenker rule obeys all the enlisted principles.

Now suppose that  $x$  is a cost sharing rule satisfying CONT, NHOM, SI, NOEXP and SCONS. We will show that  $x = x^{\text{MS}}$  in the following way.

By Lemma 3.3 it follows that  $x$  is generated by a path collection  $\Pi$ . Thus  $x$  satisfies the serial principle according to Lemma 3.2. There is no unique way to describe  $\Pi$ ; all other path collections resulting from choosing other parametrizations for the paths in  $\Pi$  generate  $x$  as well. Then it suffices to prove that a path collection by which  $x$  is generated can be chosen such that it equals  $\Gamma$ , one of the path collections corresponding to the Moulin-Shenker rule. Therefore the theorem will be proved if we show that, starting with an arbitrary path collection  $\Pi$  generating  $x$ , for all  $(d, c, S) \in \mathbb{R}_+^N \times \mathcal{C} \times \mathcal{P}(N)$  the path  $\pi^{d,c,S} \in \Pi$  is equal to  $\gamma^{d,c,S}$  up to parametrization.

We claim that there is a parametrization  $\bar{\pi}$  of  $\pi^{0,c,N}$ , which is a solution to the above system of differential equations (3). Then  $\bar{\pi}$  must coincide with  $\gamma^{0,c,N}$  by uniqueness of the solution.

Then by simple variations the same reasoning shows that all paths of type  $\pi^{0,c^d,S}$  are equal to  $\gamma^{0,c^d,S}$  up to parametrization, for all  $d \in \mathbb{R}_+^N$  and  $S \in \mathcal{P}(N)$ .

First, we will show that  $\pi^{d,c,S}(\mathbb{R}_+) = \pi^{0,c^d,S}(\mathbb{R}_+)$ . We need only to consider those profiles  $d$ , which can actually be produced using the path collection  $\Pi$  and the above construction. Suppose we have an inequality

instead and that a cost sharing problem  $(q, c)$  the path constructed from  $\Pi$  reaches the profile  $d$  after a specific period.

Then the path  $\pi^{d,c,S}$  is used in the above construction from the very moment where all agents in  $N \setminus S$  are satisfied with production. Still, their individual completion times for production may differ. At least the agents in  $S$  will have made equal contributions to the procedure of equally splitting incremental costs for raising production levels, since they have not completed yet. So we have  $x_i(d, c) = x_j(d, c)$  for all  $i, j \in S$ .

Let  $q \in d + (\pi^{d,c,S}(\mathbb{R}_+), 0_{N \setminus S})$  and assume that  $q \notin d + (\pi^{0,c^d,S}(\mathbb{R}_+), 0_{N \setminus S})$ . Then also the payments for the cost sharing problem  $(q, c)$  according to the rule  $x$  are the same for the individual agents in  $S$ , or  $x_i(q, c) = x_j(q, c)$  for all  $i, j \in S$ . Applying SCONS gives

$$x_i(q, c) = x_i(d, c) + x_i(q - d, c^d) \text{ for } i \in S.$$

So, actually the cost shares for the reduced cost sharing problem  $(q - d, c^d)$  must be equal for the agents in  $S$ ,  $x_i(q - d, c^d) = x_j(q - d, c^d)$  for all  $i, j \in S$ . Recall the construction of sharing the cost in the cost sharing problem  $(q - d, c^d)$ . First the production plan  $\pi^{0,c^d,S}$  is used in order to define the first production level  $y$  at which a set  $S'$  of agents in  $S$  are satisfied with the production so far. By assumption, however, this cannot be the profile  $q - d$ . So  $y < q - d$ . Now the incremental cost for bringing production from level 0 up to  $y$  are split equally among the members of  $S$ . Then the procedure continues in order to divide the remaining costs  $c^d(q - d) - c^d(y)$  among the agents  $S \setminus S'$ , which is a nonempty set. Because  $c^d(q - d) - c^d(y) > 0$  this means that there is at least one agent in  $S \setminus S'$  that pays more than any of the agents in  $S'$ . So there are differences in cost shares of agents in  $S$  which gives the desired contradiction.

In the proof we roughly distinguish between four steps.

Step 1: The properties NHOM, SP and SI allow us to specify  $D\pi^{0,c,N}$  up to multiplication with a scalar  $y$  under the assumptions of existence of  $D\pi^{0,c,N}$  and  $D\pi^{0,c,N} \gg 0$ . We claim that for all  $i \in N$

$$D_i\pi^{0,c,N}(0) = \frac{y}{D_i c(0)}.$$

This is proved as follows. Suppose all partial derivatives of  $\pi^{0,c,N}$  are strictly positive. Define the scale transformation  $f : \mathbb{R}_+^N \rightarrow \mathbb{R}_+^N$  by

$$f_i(u) = \frac{u_i}{D_i c(0)} \text{ for all } i \in N, u \in \mathbb{R}_+^N.$$

Then by scale invariance for any  $q \in \mathbb{R}_+^N$  the problem  $(q, c)$  is equivalent with  $(f^{-1}(q), c \circ f)$ . But the latter one is normalized in the sense that for all  $i$ ,  $D_i(c \circ f)(0) = D_i c(0) D_i f(0) = 1$ . Then by the serial principle and null homogeneity we have for all  $i, j \in N$ ,

$$\lim_{r \downarrow 0} \frac{x_i(re_N, c \circ f)}{x_j(re_N, c \circ f)} = 1.$$

But this will only be the case if for all  $i, j \in N$

$$\lim_{t \downarrow 0} \frac{\pi_i^{0,c \circ f, N}(t)}{\pi_j^{0,c \circ f, N}(t)} = 1.$$

Then by Lemma 3.1 for all  $i, j \in N$

$$\lim_{t \downarrow 0} \frac{f_i(\pi^{0,c,N}(t))}{f_j(\pi^{0,c,N}(t))} = 1.$$

Thus as a result

$$\lim_{t \downarrow 0} \frac{(\pi^{0,c,N})_i(t)}{(\pi^{0,c,N})_j(t)} = \frac{D_j c(0)}{D_i c(0)}$$

It is not difficult to prove the following. Let  $h, g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be continuous mappings for which  $h'(0)$  and  $g'(0)$  exist at 0, with  $g > 0$  on

$(0, \infty)$ ,  $g'(0) > 0$ , we have for  $\alpha \in \mathbb{R}_+$ ,

$$\lim_{t \downarrow 0} \frac{h(t)}{g(t)} = \alpha \implies h'(0) = \alpha g'(0).$$

Then this makes clear that by our regularity assumptions on  $D\pi^{0,c,N}(0)$  for all  $i, j \in N$ ,

$$\frac{D_i \pi^{0,c,N}(0)}{D_j \pi^{0,c,N}(0)} = \frac{D_j c(0)}{D_i c(0)}.$$

By taking  $y$  such that  $D_1 \pi^{0,c,N}(0) = \frac{y}{D_1 c(0)}$  we prove our claim.

For  $q \in \mathbb{R}_+^N$  we define  $t_q := \operatorname{argmin}\{\pi^{0,c,N}(t) \geq q\}$ . Then  $t_q$  stands for the first moment that  $\pi^{0,c,N}$  reaches the boundary of the cube  $\{u \in \mathbb{R}_+^N \mid u \leq q\}$ .

Step 2: Take  $d \in \pi^{0,c,N}(\mathbb{R}_+)$ ,  $d \neq 0$ . Note that  $d$  is a demand profile for which  $x$  determines equal cost shares. Assume now that  $D\pi^{0,c^d,N}(0)$  exists and  $D\pi^{0,c^d,N}(0) \gg 0$ . We claim that there is a  $y \in \mathbb{R}_+$  such that for all  $i \in N$

$$D_i \pi^{0,c,N}(t_d) = \frac{y}{D_i c(\pi^{0,c,N}(t_d))}. \quad (6)$$

Essentially this is proved with the techniques from Step 1 together with the property SCONS. Applying Step 1 for  $c^d$  instead of  $c$  immediately provides us with a  $y \in \mathbb{R}_+$  such that for all  $i \in N$

$$D_i \pi^{0,c^d,N}(0) = \frac{y}{D_i c^d(0)} = \frac{y}{D_i c(d)} = \frac{y}{D_i c(\pi^{0,c,N}(t_d))}. \quad (7)$$

On the other hand we find another expression for  $D_i \pi^{0,c^d,N}(0)$  by the relation between  $\pi^{0,c^d,N}$  and  $\pi^{0,c,N}$ . By SCONS and the fact that  $x_i(d, c) = \frac{c(d)}{|N|}$ , we have for all  $d' \geq d$ ,  $d' \in \pi^{0,c,N}(\mathbb{R}_+)$ ,  $i \in N$

$$x_i(d', c) = \frac{c(d)}{|N|} + x_i(d' - d, c^d).$$

Since  $d'$  is also a demand profile for which  $x$  determines equal cost shares, it holds for all  $i \in N$

$$x_i(d' - d, c^d) = \frac{c(d') - c(d)}{|N|}.$$

But  $x_i(d' - d, c^d) = x_j(d' - d, c^d)$  for all  $i, j \in N$  if and only if the first splitting point for the problem  $(d' - d, c^d)$  is  $d' - d$ , or equivalently  $d' - d \in \pi^{0, c^d, N}(\mathbb{R}_+)$ . So  $d' \in \pi^{0, c, N}(\mathbb{R}_+)$  if and only if  $d' \in \pi^{0, c^d, N}(\mathbb{R}_+) + d$ . This in turn implies  $\pi^{0, c^d, N}(\mathbb{R}_+) = \pi^{0, c, N}([t_d, \infty))$ . Since only the images of the paths matter we may assume that  $\pi^{0, c^d, N}(t) = \pi^{0, c, N}(t_d + t) - d$  for all  $t \in \mathbb{R}_+$ . But this gives for  $i \in N$ ,  $D_i \pi^{0, c^d, N}(0) = D_i \pi^{0, c, N}(t_d)$  and together with equality (6),

$$D_i \pi^{0, c, N}(t_d) = \frac{y}{D_i c(\pi^{0, c, N}(t_d))}. \quad (8)$$

Step 3: For almost every  $t \in \mathbb{R}_+$ , there is a  $y \in \mathbb{R}_+$  with

$$D_i \pi^{0, c, N}(t) = \frac{y}{D_i c(\pi^{0, c, N}(t))}. \quad (9)$$

The mapping  $\pi^{0, c, N}$  is monotonically increasing and therefore differentiable almost everywhere. If only  $D\pi^{0, c, N} \gg 0$  almost everywhere, then we are done: the result from Step 2 applies for almost every  $d \in \pi^{0, c, N}(\mathbb{R}_+)$ , which in turn implies (9).

Let

$$\tilde{\pi} := \pi^{0, c, N} \circ (c \circ \pi^{0, c, N})^{-1}. \quad (10)$$

Then  $\tilde{\pi}$  is a parametrization of  $\pi^{0, c, N}$  by the costs; for each  $t \in \mathbb{R}_+$  it holds that  $c(\tilde{\pi}(t)) = t$ . Take  $t \in \mathbb{R}_+$  and  $h > 0$ . Then,

$$\begin{aligned} \left\| \frac{\tilde{\pi}(t+h) - \tilde{\pi}(t)}{h} \right\| &\geq |N| \frac{c(\tilde{\pi}(t+h)) - c(\tilde{\pi}(t))}{hb(c)} \\ &= |N| \frac{h}{hb(c)} = |N| b(c)^{-1} > 0. \end{aligned}$$



This implies that whenever  $\tilde{\pi}$  is differentiable at  $t$ , then  $\tilde{\pi}'(t) \gg 0$ . But consequently  $\tilde{\pi}' \gg 0$  almost everywhere, since it is a monotonically increasing function. There is only one possibility, and that is  $D\pi^{0,c,N} \gg 0$  almost everywhere. This proves our claim.

Step 4: The last part of the proof is of rather technical nature. We will show now that the above  $\tilde{\pi}$  can be used to define the proper parametrization of  $\pi^{0,c,N}$  that we are looking for.

Note, that given the fact that  $\pi^{0,c,N}$  is monotonically increasing we have for almost all  $t \in \mathbb{R}_+$ :

(i):  $\pi^{0,c,N}$  is differentiable at  $(c \circ \pi^{0,c,N})^{-1}(t)$  and

$$D\pi^{0,c,N}((c \circ \pi^{0,c,N})^{-1}(t)) > 0.$$

(ii):  $c \circ \pi^{0,c,N}$  is differentiable at  $(c \circ \pi^{0,c,N})^{-1}(t)$  and

$$(c \circ \pi^{0,c,N})'((c \circ \pi^{0,c,N})^{-1}(t)) > 0.$$

So for the parametrization  $\tilde{\pi}$  of  $\pi^{0,c,N}$ , defined above by (10), the following equality holds almost everywhere, for all  $i \in N$

$$D_i \tilde{\pi}(t) = \frac{D_i \pi^{0,c,N}((c \circ \pi^{0,c,N})^{-1}(t))}{(c \circ \pi^{0,c,N})'((c \circ \pi^{0,c,N})^{-1}(t))} = \frac{1}{|N|} \frac{1}{D_i c(\tilde{\pi}(t))}.$$

Consider the curve  $\bar{\pi} := \tilde{\pi} \circ \phi$ , where  $\phi(t) = |N|t$  for all  $t \in \mathbb{R}_+$ . Then  $\bar{\pi}$  is a parametrization of  $\pi^{0,c,N}$  for which for almost all  $t \in \mathbb{R}_+$  it holds that for all  $i \in N$

$$D_i \bar{\pi}(t) = \frac{1}{D_i c(\bar{\pi}(t))}.$$

If we can show that this equality holds for all  $t \in \mathbb{R}_+$ , then we are done. Since then we showed that  $\bar{\pi}$  is actually the parametrization of  $\pi^{0,c,N}$  that

we were looking for, because  $\bar{\pi} = \gamma^{0,c,N}$ . The mapping  $\bar{\pi}$  is Lipschitz continuous: for all  $t_1, t_2 \in \mathbb{R}_+$ ,

$$\begin{aligned} \|\bar{\pi}(t_1) - \bar{\pi}(t_2)\| &= \|\tilde{\pi}(|N| t_1) - \tilde{\pi}(|N| t_2)\| \\ &\leq a(c)^{-1} |c(\tilde{\pi}(|N| t_1)) - c(\tilde{\pi}(|N| t_2))| \\ &= |N| a(c)^{-1} |t_1 - t_2|. \end{aligned}$$

So  $\bar{\pi}$  is absolutely continuous and therefore, for all  $i \in N$  and  $t \in \mathbb{R}_+$ ,

$$\bar{\pi}(t) = \int_0^t D_i \bar{\pi}(s) ds = \int_0^t D_i c(\bar{\pi}(s))^{-1} ds.$$

By the continuity of the mapping  $s \mapsto D_i c(\bar{\pi}(s))^{-1}$ , it follows that  $\bar{\pi}$  is differentiable and for all  $i \in N$

$$D_i \bar{\pi}(t) = \frac{1}{D_i c(\bar{\pi}(t))} \text{ for all } t \in \mathbb{R}_+.$$

But then  $\bar{\pi}$  is a solution of the system of differential equations that determines the Moulin-Shenker path  $\gamma^{0,c,N}$ . By uniqueness of the solution  $\bar{\pi}$  must coincide with  $\gamma^{0,c,N}$ . This proves our claim that  $\pi^{0,c,N}$  has the same image as  $\gamma^{0,c,N}$ .

□

In the above Theorem we could have replaced the characterizing property NHOM by the combination of some continuity requirement with respect to the cost function combined with *independence of irrelevant costs* (IIC). IIC states that for a problem  $(q, c) \in \mathcal{G}$  a cost sharing mechanism may only use cost information that is considered to be relevant for the profile  $q$ ; facing two problems  $(q, c_1)$  and  $(q, c_2)$  with  $c_1 = c_2$  on  $[0, q]$ , a cost sharing mechanism  $x$  satisfying IIC should determine the same cost shares for the two cost sharing problems. By its weakness, NHOM seems to be preferable as a characterizing property. A detailed

proof is omitted.

## 4 Concluding remark

In this paper we focussed on cost sharing problems associated with a fixed agent set  $N$ . In an easy way, it is possible to enlarge the setting to which our result applies. We will now describe the way we could have proceeded. We can define for  $S \subseteq N$  a cost sharing problem to be a pair  $(d, c)$ , where  $d$  is a demand bundle in  $\mathbb{R}_+^S$  and  $c$  is the corresponding cost function that relates each desired output  $y \in \mathbb{R}_+^S$  to its cost. We will take only those cost functions for consideration that satisfy similar regularity assumptions, like continuously differentiability and bounds on partial derivatives. Call  $\mathcal{G}^S$  the set of all cost sharing problems with agent set  $S$ . A cost sharing rule for  $S$  relates each problem in  $\mathcal{G}^S$  to an efficient vector of cost shares in  $\mathbb{R}_+^S$ . For the general class of cost sharing problems, consisting of all cost sharing problems with agent sets smaller or equal to  $N$ , a (generalized) cost sharing rule is defined as a mapping whose restriction to each set  $\mathcal{G}^S$  defines a cost sharing rule for  $S$ . The Moulin-Shenker rule trivially extends to a generalized cost sharing rule. Also the contents of the properties that are used for the above characterization, like CONT, NHOM and SI, are easily converted. Now focus on the following version of *self consistency*.

Let  $x$  be a generalized cost sharing rule. In this setting it is called a *self consistent* rule if only for all  $S \subseteq N$ ,  $(q, c) \in \mathcal{G}^S$  and  $0 \leq d \leq q$  such that  $x_i(d, c) = x_j(d, c)$  for all  $i, j \in S$  it holds

$$x(q, c) = x(d, c) + x(q - d, c^d).$$

Suppose we strengthen the no exploitation property, in the sense that not only a zero demander should pay nothing, but that he even can be totally removed from the cost sharing problem without altering the outcomes for the remaining agents. More formally, a generalized cost sharing rule  $x$  has the *null agent* property, if for all  $S \subseteq N$ , and problems  $(q, c) \in \mathcal{G}^S$  it holds for all  $i \in S$ ,

$$q_i = 0 \implies x_{S \setminus \{i\}}(q, c) = x(q_{S \setminus \{i\}}, c_i),$$

where  $c_i : \mathbb{R}^{S \setminus \{i\}} \rightarrow \mathbb{R}_+$  is the admissible cost function such that  $c_i(d) = c((0_{\{i\}}, d))$  for all  $d \in \mathbb{R}^{S \setminus \{i\}}$ . Then, by using essentially the same techniques as before, we obtain the following result.

**Theorem 4.1** *The generalized Moulin-Shenker rule is the unique rule that satisfies the properties continuity, null homogeneity, null agents, self consistency and scale invariance.*

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