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A Model of Near-Rational Exuberance *

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ABSTRACT

We study how the use of judgement or "add-factors" in forecasting may disturb the set of equilibrium outcomes when agents learn using recursive methods. We isolate conditions under which new phenomena, which we call exuberance equilibria, can exist in a standard self-referential environment. Local indeterminacy is not a requirement for existence. We construct a simple asset pricing example and find that exuberance equilibria, when they exist, can be extremely volatile relative to fundamental equilibria.

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1 Introduction

Judgement is a fact of life in macroeconomic forecasting. It is widely understood that even the most sophisticated econometric forecasts are adjusted before presentation. This adjustment is so pervasive that it is known as the use of "add-factors"—subjective changes to the forecast which depend on the forecaster's assessment of special circumstances that are not well summarized by the variables that are included in the econometric model.[1] Recently, some authors have argued that economic theory needs to take explicit account of the effects of judgement on the behavior of macroeconomic systems.[2]

We wish to think of the news or add-factor that modifies the forecast as a qualitative, unique, commonly understood economy-wide variable. An example of a judgemental adjustment is suggested by Reifschneider, *et al.* (1997), when they discuss the "financial headwinds" that were thought to be inhibiting U.S. economic growth in the early to mid-1990s. As they discuss, the headwinds add-factor was used to adjust forecasts over a period of many quarters. It was communicated to the public prominently in speeches by Federal Reserve Chairman Alan Greenspan. It was thus widely understood throughout the economy and was highly serially correlated. This is the type of variable we have in mind, although by no means would we wish to restrict attention to this particular example. Other examples might include the Y2K millennium bug, or the 9/11 terrorist attacks in the U.S., as well as a host of more minor events thought to influence economic performance.

Conventional wisdom among economists suggests that judgement is all to the good in macroeconomic forecasting. Models are, of course, crude approximations of reality and must be supplemented with other information not contained in the model. While we motivate our ideas in terms of macroeconomic forecasting, our framework applies more generally to economic environments where expectations and qualitative judgements about the effects

[1] See Reifschneider, Stockton, and Wilcox (1997) for a discussion of the extent of judgemental adjustment in macroeconomic forecasting at the Federal Reserve.

[2] See Svensson (2003, 2005). Related work includes Svensson and Tetlow (2005) and Jansson and Vredin (2001).

of unique events play an important role.

Our focus in this paper is on how the add-factor or judgemental adjustment of forecasts may create more problems than it solves. In particular, we study ways in which judgemental adjustment may become self-fulfilling. We study macroeconomic models in which expectations play an important role. Following Evans and Honkapohja (2001), we replace the rational expectations (RE) assumption with one of recursive learning. This involves the assignment of a well-chosen perceived law of motion, an econometric forecasting model, to the agents. We supplement that model with a qualitative, judgemental adjustment variable and study the resulting dynamics.

The main contribution of the paper is to define the concept of an *exuberance equilibrium*. We impose three requirements on the judgementally-adjusted system to define this concept. The first is that any equilibrium reached is a RE equilibrium with limited information. For this we use the consistent expectations equilibrium (CEE) concept.[3] The second is a Nash equilibrium in the inclusion of judgement—given that all agents are using judgementally adjusted forecasts, no agent wishes to discontinue using the judgemental adjustment. The last requirement is expectational stability or learnability of the equilibrium.

Our Nash equilibrium does not correspond exactly to a RE equilibrium. This is because the judgement variable is assumed to be unavailable in the statistical part of the forecasting. We think of this as reflecting the separation of the econometric forecasting unit from the actual decision makers. Decision makers treat the econometric forecast as an input to which they are free to add the judgement variable. The judgementally adjusted forecasts are the basis for the decisions and actions of the agents, but the judgement variable cannot be extracted by the econometric forecasting unit and converted into a statistical time series that can be utilized in an econometric forecasting model. In a similar vein the decision makers face a dichotomy in their use of judgement: they either incorporate the variable as an add-factor or they ignore it and directly use the econometric forecast. This inability of the

[3] See Sargent (1991), Marcet and Sargent (1995) and Hommes and Sorger (1998).

decision makers to transmit to the econometric forecasters in a quantitative way the judgemental aspects behind their final economic decisions is the source of the deviation from full RE and the reason for our use of the term "near-rationality."[4]

We isolate conditions under which exuberance equilibria exist in a standard dynamic framework in which the state of the system depends on expectations of future endogenous variables. We show that exuberance equilibria exhibit higher asymptotic variance of the endogenous variable than the standard RE equilibrium. (The standard RE equilibrium is also a solution in our framework.) As an application we study a simple univariate asset-pricing model. We interpret the exuberance equilibrium in the asset-pricing model as an example of "excess volatility."[5] In this paper we do not discuss policy applications. For a discussion of the implications for monetary policy the reader is referred to related work in Bullard, Evans, and Honkapohja (2008).

Our results may lead one to view the possibility of exuberance equilibria as particularly worrisome, as exuberance equilibria may exist even in otherwise benign circumstances. In particular, we show that exuberance is a clear possibility even in the case where the underlying RE equilibrium is unique (*a.k.a.* determinate). Thus an interesting and novel finding is the possibility of "sunspot-like" equilibria, but without requiring that the underlying RE equilibrium of the model is indeterminate.[6] In a sense, we find "sunspot-like"

[4] The term "near rationality" has been used elsewhere in the literature, often to mean less-than-full maximization of utility. See, for example, Akerlof and Yellen (1985) and Caballero (1995). Ball (2000) analyzes a model where the agents use a forecasting model that does not encompass the equilibrium law of motion—a "restricted perception." Our concept is based on full optimization but subject to the restriction that some information is not quantifiable—"judgement." Our concept of near rationality is discussed further in Section 5.

[5] "Exuberance" (which in our equilibria leads to both positive and negative deviations from the fundamentals solution) has a long informal tradition as a potential explanation of asset price "bubbles." For its possible role in "financial fragility" see Lagunoff and Schreft (1999).

[6] Indeterminacy and sunspot equilibria are distinct concepts, as discussed in Benhabib and Farmer (1999). We consider only linear models, for which the existence of stationary sunspot equilibria requires indeterminacy—see for example Propositions 2 and 3 of Chiappori and Guesnerie (1991).

equilibria without indeterminacy.

2 Economies with judgemental adjustment

We study systems in which agents use recursive algorithms to learn equilibria. These systems can converge to a RE equilibrium provided certain conditions are met. The conditions are outlined in some detail in Evans and Honkapohja (2001), and the key condition is known as expectational stability. In this paper, we alter the econometric model that the agents use to forecast in order to allow them to use judgemental adjustment.

We consider a scalar version of the model studied in Bullard, Evans and Honkapohja (2008)

$$y_t = \beta y_{t+1}^e + u_t + w_t \tag{1}$$

in which β is a scalar parameter and y_t is the state of the system. Here u_t and w_t are stochastic shocks, assumed to be white noise. The second shock w_t is included to allow for partial correlation of the fundamental shocks with the judgemental adjustment to expectations, as discussed below. This extension to correlated judgment was not considered in Bullard, Evans and Honkapohja (2008).

Equations of the type (1) often describe economies linearized at a steady state. We have normalized the steady state component to zero, so there is no constant term. We use the notation y_{t+1}^e to represent the expectations of the agents in the model, which may initially be non-rational. These expectations are formed through the use of an econometric model, and the expectations from this model are denoted by $E_t^\star y_{t+1}$. We then add a judgemental adjustment variable to this econometric forecast, denoted by ξ_t. Thus expectations are given by

$$y_{t+1}^e = E_t^\star y_{t+1} + \xi_t. \tag{2}$$

Without the addition of judgement, this system would be the same as the systems analyzed in Evans and Honkapohja (2001). For those systems, convergence properties are well-established. Our goal in this paper is to understand how the addition of judgemental adjustment may influence convergence properties and lead to the existence of new equilibria in the economy.

2.1 The nature of judgemental adjustment

The judgemental add-factor allows the agent to adjust the forecast in response to qualitative, unique events that may have an impact on the economy. Let us denote the qualitative "news" as η_t.

$$\eta_t = f w_t + \hat{\eta}_t. \tag{3}$$

We assume $f \geq 0$. In other words, the news consists of both information on the shock w_t as well as exogenous white noise $\hat{\eta}_t$. (We assume that w_t and $\hat{\eta}_t$ are independent.) The latter can be interpreted either as extraneous randomness or as measurement error of w_t. The news η_t is thought to have a persistent impact on y_{t+1} which is not captured by the econometric forecast $E_t^{\star} y_{t+1}$.

Following Bullard, Evans and Honkapohja (2008) we assume that the amount of judgemental adjustment contained in y_{t+1}^e satisfies

$$(1 - \rho L)\, \xi_t = \eta_t, \tag{4}$$

where L is a lag operator, or $\xi_t = \rho \xi_{t-1} + \eta_t$. Thus the expected effect of past qualitative events, $\rho \xi_{t-1}$, plus the effect of today's news η_t, gives the complete judgemental adjustment at time t. A key property is persistence in the judgement, which we think is natural, given the examples described above, and because the endogenous variable has corresponding persistence in the equilibrium.

We stress that the assumption that ξ_t follows an $AR\,(1)$ process has been adopted here for analytical convenience only. We caution the reader not to think of the judgemental adjustment as a quantifiable variable even though it is written in $AR\,(1)$ form. Past judgements certainly have an impact on past forecasts, and in that sense they could be quantified. But an adjustment for the Korean War would not be comparable to adjustments for wage and price controls or for the Y2K millennium bug, or for a host of other, more minor events, and for this reason the past judgements do not provide useful time series information.

2.2 Econometric forecasts

In the recursive learning literature, agents are assigned a *perceived law of motion* (PLM) which summarizes how they use available economic data to form expectations about the future. The PLM is chosen to be consistent with the time-series properties of the equilibrium law of motion under RE. The actual evolution of the system (1) will depend in part on how agents are forming expectations, including any judgemental adjustments they are making.

For the system with judgement, we show below that an ARMA(1,1) process

$$y_t = by_{t-1} + v_t - av_{t-1}, \tag{5}$$

is appropriate since in equilibrium y_t will follow an ARMA(1,1) process. In this equation v_t is stochastic and $|a| < 1$ and $|b| < 1$ are parameters. This can be written as

$$y_t = \theta(L)\, v_t, \tag{6}$$

where

$$\theta(L) = \frac{1 - aL}{1 - bL}.$$

Then

$$E_t^{\star} y_{t+1} = by_t - av_t = [b\theta(L) - a]\, v_t \tag{7}$$

is the best forecast in a mean square error sense for the given PLM. We refer to (7) as the *econometric forecast.* This part of the agents' forecast depends on the econometric model alone.

2.3 Exuberance equilibrium

We can now deduce the *actual law of motion* (ALM) for this system by combining the expectations (2) and the effect of these expectations on the state of the economy (1). This yields

$$\begin{aligned} y_t &= \beta y^e_{t+1} + u_t + w_t \\ &= \beta \left(\frac{b-a}{1-bL} \right) v_t + \frac{\beta}{1-\rho L} \eta_t + u_t + w_t \\ &= \beta \left(\frac{b-a}{1-bL} \right) \left(\frac{1-bL}{1-aL} \right) y_t + \frac{\beta}{1-\rho L} \eta_t + u_t + w_t. \end{aligned}$$

Solving for y_t and using (3), the ALM can be written as

$$y_t = \frac{1-aL}{\beta\,(a-b) + 1 - aL} \left[(1-\rho L)^{-1} \beta (f w_t + \hat{\eta}_t) + (u_t + w_t) \right]. \qquad (8)$$

The judgement term $\hat{\eta}_t$ enters the ALM, since judgement is affecting the expectations of agents and through that channel is affecting actual macroeconomic outcomes y_t. Of course, the agents could decide not to use a judgemental adjustment, in which case the ALM would not involve this term, and this too would be an equilibrium. A critical aspect of the analysis will be to develop the conditions under which economic actors will use a judgemental adjustment, given that all other agents are making such an adjustment and causing the actual law motion to be (8).

We proceed by defining the concept of an *exuberance equilibrium*. The general concept is that of an equilibrium in which the judgemental adjustment influences the actual evolution of the economy. The exuberance concept has three parts, and we discuss each in detail below. We first require that the PLM of the agents, their econometric forecast, is consistent with the actual data being generated by the economy. This is a type of non-falsifiability assumption for the econometric model. We use the consistent expectations equilibrium concept to impose this condition.[7] A second requirement is that there is a Nash equilibrium in the use of judgement, so that given that all agents are using a judgementally-adjusted forecast, no agent wishes to discontinue using that forecast. Finally, since our agents are learning using recursive algorithms, we impose learnability as a requirement for an exuberance equilibrium.

[7] CEE and rational expectations equilibrium with limited information are equivalent in our linear settings.

3 Analysis

3.1 Consistent expectations

The main idea behind consistent expectations equilibrium is that econometricians in the model should be unable to reject their model of the economy. The econometric model should provide a good fit to the data produced by the economy itself. We impose this idea through a requirement that the autocovariance generating function of the econometric model (the PLM) is equivalent to the autocovariance generating function generated by the economy (the ALM).[8] We can analytically verify the existence of a solution to the equation implied by this statement for the univariate case.

The autocovariance generating function for the PLM is given by

$$G_{PLM}(z) = \sigma_v^2 \frac{(1-az)\left(1-az^{-1}\right)}{(1-bz)\left(1-bz^{-1}\right)} \tag{9}$$

where σ_v^2 is the variance of v, and z is a complex scalar.[9] For the ALM, the autocovariance generating function is the sum of three such functions

$$G_{ALM}(z) = G_{\hat{\eta}}(z) + G_u(z) + G_w(z)$$

by the independence of $\hat{\eta}$, u, and w. The autocovariance generating function for the ALM (8) is

$$\begin{aligned} G_{ALM}(z) \quad = \quad & \frac{(1-az)\left(1-az^{-1}\right)}{(1-\rho z)\left(1-\rho z^{-1}\right)} \times \\ & \left\{ \frac{\beta^2\sigma_{\hat{\eta}}^2 + (1-\rho z)\left(1-\rho z^{-1}\right)\sigma_u^2}{\left[\beta(a-b)+1-az\right]\left[\beta(a-b)+1-az^{-1}\right]} \right. \\ & \left. + \frac{\sigma_w^2(f\beta+1-\rho z)(f\beta+1-\rho z^{-1})}{(\beta(1-a)+1-az)(\beta(1-a)+1-az^{-1})} \right\}. \end{aligned}$$

We use these functions to demonstrate the following result in Appendix A.

Lemma 1 *There exists a CEE with* $b=\rho$ *and* $a \in [0,\rho]$.

[8] See e.g. Hommes and Sorger (1998) and Branch and McGough (2005).

[9] See Brockwell and Davis (1991, pp. 417-420), or Hamilton (1994, pp. 266-268).

Intuitively, $b = \rho$ in order to match the induced autoregressive process arising from judgement. Any other value of b would lead to a higher-order process for y_t that would not be consistent with the PLM.

As shown in Appendix A, there are interesting limiting cases: when $\sigma_\eta^2 = \sigma_{\hat{\eta}}^2 + f^2\sigma_w^2 \to 0$, so that the relative variance of the judgement process is small, $a \to \rho$, while for $\sigma_u^2 + \sigma_w^2 \to 0$, so that the relative variance of the fundamental process is small, $a \to 0$. Since a solution $a \in [0, \rho]$ always exists,[10] the conditions for a CEE can always be met.

We now ask whether individual rationality holds with respect to inclusion of the judgement variable in making forecasts.

3.2 Incentives to include judgement

The agents can choose not to use the judgementally adjusted forecast. But when all other agents are using the judgemental adjustment, they cause the ALM of the system to be given by equation (8). An individual agent faces the question of whether the outcomes generated by this ALM can be better forecast with a model that incorporates the judgemental adjustment, or by one that ignores the judgemental adjustment. If each agent chooses the former course, then all agents will use the judgementally adjusted forecast and the ALM will remain as given in equation (8). Of course, if each agent chooses the latter course, then no agent will use the judgementally adjusted forecast, and the judgement variable will not affect equilibrium outcomes. The latter case leads to the standard rational expectations equilibrium viewed here as a "no exuberance equilibrium".

This individual rationality condition can be assessed by comparing the forecast error variance of the two models, (2) and (7), when all other agents are including judgement in their forecasts and hence causing the ALM to be given by (8). Using the condition from the consistent expectations calculation

[10] Appendix A also makes it clear that there is a second, negative value of a that equates the two autocovariance generating functions. We found that the other conditions for exuberance equilibrium are not met at this value of a, and we refer to it only in passing in the remainder of the paper.

that $b = \rho$, we note that $v_t = \left(\frac{1-\rho L}{1-aL}\right) y_t$. The econometric forecast is therefore given by

$$E_t^{\star} y_{t+1} = \frac{\rho - a}{1 - \rho L} v_t = \frac{\rho - a}{1 - aL} y_t \tag{10}$$

whereas the judgementally adjusted forecast is given by

$$y_{t+1}^{e} = \frac{\rho - a}{1 - aL} y_t + \frac{1}{1 - \rho L} \eta_t. \tag{11}$$

Is it possible for the variance of the judgementally adjusted forecast to be lower than the variance of the econometric forecast? It is. Consider the special case when $\sigma_{\hat{\eta}}^2, \sigma_w^2 \to 0$ so that the positive root $a \to \rho$. Then it is shown in Appendix B that, apart from additive terms in u_t that are identical for the two forecasts, the forecast errors are

$$FE|_{u=0} = \frac{1 + \beta f - (\rho + kf)L}{1 - \rho L} w_{t+1} + \frac{\beta - kL}{1 - \rho L} \hat{\eta}_{t+1},$$

where $k = 1$ if judgement is included and $k = 0$ for the case of no judgement.

For the term involving $\hat{\eta}_{t+1}$ we have, with no judgement,

$$FE_{NJ}^{\hat{\eta}} = \frac{\beta}{1 - \rho L} \hat{\eta}_{t+1},$$

whereas the forecast error with judgement is

$$FE_{J}^{\hat{\eta}} = \frac{\beta \left(1 - \beta^{-1} L\right)}{1 - \rho L} \hat{\eta}_{t+1}.$$

Thus, as $\sigma_{\hat{\eta}}^2 \to 0$ the ratio between the variances of these terms is[11]

$$\frac{Var[FE_{J}^{\hat{\eta}}]}{Var[FE_{NJ}^{\hat{\eta}}]} = 1 + \beta^{-2} - 2\beta^{-1}\rho.$$

This is less than one if and only if

$$\rho\beta > \frac{1}{2}. \tag{12}$$

[11] See, for instance, Harvey (1981, p. 40). The variance of $x_t = [(1 + mL) / (1 - \ell L)] \epsilon_t$ is $\left[\left(1 + m^2 + 2\ell m\right) / \left(1 - \ell^2\right)\right] \sigma_\epsilon^2$.

For the term involving w_{t+1} we get for the relevant variances

$$Var[FE^w_{NJ}] - Var[FE^w_J] \asymp \frac{f}{1+\beta f}\left(2\rho - \frac{2\rho + f}{1+\beta f}\right),$$

where $\asymp$ means "is positively proportional to." It is seen that the term in the brackets is positive for all $f \geq 0$ when $\beta\rho > 1/2$.

By continuity, it follows that if $\beta > 1/2$ there are non-trivial judgement processes (with $\rho > 1/2\beta$ and $\sigma^2_{\hat{\eta}}, \sigma^2_w > 0$ sufficiently small) for which the agents have incentives to include the process as an add factor in their forecasts. The preceding argument considered the limiting case $a \to \rho$, but as we will numerically show below, it is not necessary for a to be close to ρ for our results to hold.

It is useful to consider further aspects of this comparison. For simplicity we now assume that $\sigma^2_w = 0$ so that there is no correlation between fundamental shocks and judgement. Polar opposite to the case $\sigma^2_{\hat{\eta}} \to 0$ is the special case $\sigma^2_u \to 0$, in which case the positive root $a \to 0$. It can be shown that in this case

$$Var[FE_J] - Var[FE_{NJ}] = \left(\frac{(\beta^{-1} - \rho)^2}{1-\rho^2}\right)\sigma^2_\eta.$$

This can never be less than zero, so that agents will not make the judgemental adjustment in this case.

By continuity, for the case of uncorrelated judgement, we deduce that for $R = \sigma^2_\eta/\sigma^2_u$ large it will not be individually rational to include judgement, but if $\rho\beta > \frac{1}{2}$ there are values of $R = \sigma^2_\eta/\sigma^2_u \in (0, \infty)$ such that agents rationally choose to use a judgementally adjusted forecast, given that all other agents are doing so.[12] The conclusion that it can be optimal to judgementally adjust the econometric forecast is striking since this forecast already reflects the effects of judgement on the time series properties of the observable variables.

Figure 1 illustrates our points numerically for the case of uncorrelated judgement. For $a < \rho$, the forecast error variances involve the variance of an

[12] The case with $a \approx \rho$ is a near-common factor representation of the time series, but the required variances remain continuous in the parameters, as can be seen from the formulae in Appendix B.

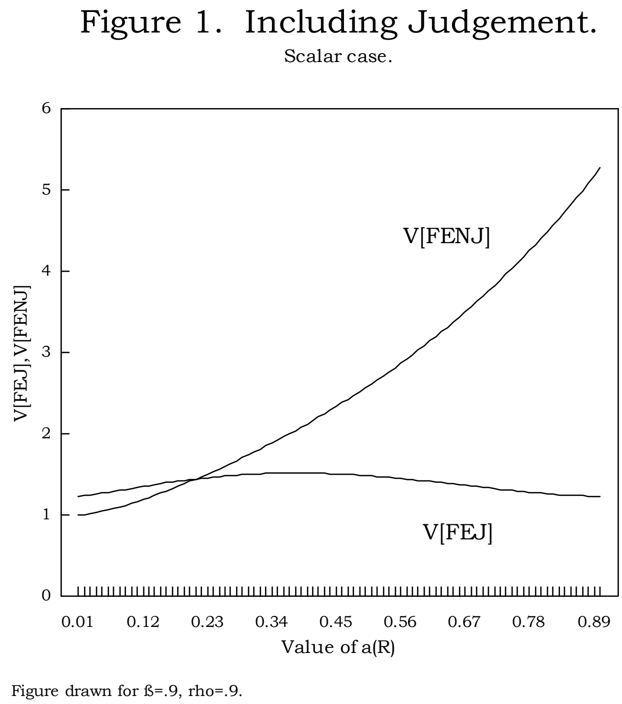

Figure 1: The variance of the forecast error, with (FEJ) and without (FENJ) judgement. The variance can be lower with judgement included, even for values of a far from ρ.

ARMA(2, 2) process, which is given in Appendix B. The Figure is drawn for $\beta = .9$ and $\rho = .9$, which is a possible case. To draw Figure 1, we consider changes in R resulting from changes in σ_u^2 with σ_η^2 fixed. The variances of the forecast errors with and without judgement are plotted on the vertical axis, while the implied a is plotted on the horizontal axis. The special cases (i) $R \to \infty$, $a \to 0$ and (ii) $R \to 0$, $a \to \rho$, which were discussed above are borne out in the Figure. The Figure also shows intermediate cases, and indicates that a does not have to be particularly close to ρ for the individual rationality condition to be met. In fact, the two forecast error variances are equal at $a \approx .21$, which is far from the value of ρ in this example, which is .9.

We conclude that the conditions for exuberance equilibria to exist are quite likely to be met for a wide range of relative variances R provided both β and ρ are relatively close to one.[13]

3.3 Learnability

Because we have agents using recursive algorithms to learn, a plausibility condition for exuberance equilibrium is that any candidate equilibrium is learnable. We follow the literature on recursive learning to impose this requirement.[14]

In the current context, the CEE formulated above takes the form of an ARMA(1,1) process. Estimation of ARMA(1,1) processes is usually done using maximum likelihood techniques, taking us beyond standard least squares estimation. Recursive maximum likelihood (RML) algorithms are available and they have formal similarities to recursive least squares estimation.[15] Because this technical analysis is relatively unfamiliar, we confine the formal details to Appendix C. However, the results are easily summarized. Let a_t and b_t denote estimates at time t of the coefficients of the ARMA forecast function (7). Numerical computations using RML indicate convergence of (a_t, b_t) to (a, ρ), where $a > 0$ is the CEE value given in Lemma 1. Thus this CEE is indeed stable under learning. Moreover, in Section 3.4 we state a formal convergence result as part of our existence theorem.

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[13] One suspects that the individual rationality constraint is binding at values $\rho\beta < 1/2$. For example, if $\rho = .7$ and $\beta = .7$, an analogous numerical exercise shows that the judgementally adjusted forecast is never preferable to the econometric forecast.

[14] Evans and Honkapohja (2001) gives a systematic treatment of adaptive learning and its implications in macroeconomics. Evans and Honkapohja (1999), Marimon (1997) and Sargent (1993, 1999) provide surveys of the field.

[15] They are also called Recursive Prediction Error (RPE) algorithms—see Evans and Honkapohja (1994) and Marcet and Sargent (1995) for other uses of RPE methods in learning.

3.4 Existence and properties of equilibrium

We now collect the various results above. The following theorem gives the key results about existence of an exuberance equilibrium and characterizes its asymptotic variance:

Theorem 2 *Consider the model with judgement and suppose that* $\beta > 1/2$. *Then*
(i) for appropriate AR(1) judgement processes there exists an exuberance equilibrium and
(ii) the exuberance equilibrium has a higher asymptotic variance than the RE equilibrium.

Proof. (i) The preceding analysis has verified that the first two conditions for an exuberance equilibrium defined in Section 2.3, consistent expectations and incentives to include judgement, are met for all $\sigma_\eta^2 > 0$ sufficiently small. In Appendix C it is proved that the third condition also holds, that is, the CEE is stable under RML learning, when $\sigma_\eta^2 > 0$ is sufficiently small.

(ii) The RE equilibrium for the model is $y_t = u_t + w_t$ since $0 < \beta < 1$ and u_t and w_t are *iid* with mean zero. The exuberance equilibrium with $a > 0$ can be represented as the ARMA(1,1) process $y_t = \rho y_{t-1} + v_t - a v_{t-1}$ where a solves equation (16) given in Appendix A. From (15) of Appendix A it can be seen that $\sigma_v^2 a \left[\beta\left(a-\rho\right)+1\right] = \sigma_u^2\rho + \sigma_w^2(1+f\beta)\rho$. Thus

$$\sigma_v^2 = \frac{\rho}{a(\beta(a-\rho)+1)}\sigma_u^2 + \frac{(f\beta+1)\rho}{a(\beta(a-\rho)+1)}\sigma_w^2 > \sigma_u^2 + \sigma_w^2,$$

since $a < \rho$, $f \geq 0$ and $0 < \beta, \rho < 1$. Next, using the formula for the variance of an ARMA(1,1) process we have

$$\sigma_y^2 = \frac{1+a^2-2\rho a}{1-\rho^2}\sigma_v^2$$

and since $\frac{1+a^2-2\rho a}{1-\rho^2} > 1$, the result follows. ■

The theorem states that in an exuberance equilibrium, the variance of the state variable y_t is larger than it would be in a fundamental RE equilibrium.

This is because the REE has $y_t = u_t + w_t$, so that $\sigma_y^2 = \sigma_u^2 + \sigma_w^2$, but in an exuberance equilibrium $\sigma_y^2 > \sigma_u^2 + \sigma_w^2$.[16]

4 An Asset Pricing Example

A simple univariate example of the framework (1) is given by the standard present value model of asset pricing. One convenient way of obtaining this equation is the model of Brock and Hommes (1998). In their framework agents are myopic mean-variance maximizers who choose the quantity of riskless and risky assets in their portfolio to maximize expected value of a quadratic utility function of end of period wealth.[17]

We modify their framework to allow for shocks to the supply of the risky asset.[18] For convenience we assume homogeneous expectations and constant known dividends. The temporary equilibrium is given by

$$p_{t+1}^e + d - R_f p_t = s_t,$$

where d is the dividend, p_t is the price of the asset and $R_f > 1$ is the rate of return factor on the riskless asset. Here s_t is a linear function of the random supply of the risky asset per investor, assumed *i.i.d.* for simplicity.[19] Defining $y_t = p_t - \bar{p}$, where $\bar{s} = Es_t$ and $\bar{p} = (d - \bar{s})/(R_f - 1)$, we obtain (1) with $\beta = R_f^{-1}$, $u_t = -R_f^{-1}(s_t - \bar{s})$ and $w_t \equiv 0$. We assume that $0 < \beta < 1$, so that the model is determinate, that is, under RE there is a unique nonexplosive solution, given by the "fundamentals" solution $y_t = u_t$. In particular, under RE, sunspot solutions do not exist.

[16] To examine the amount of correlation between the fundamental $u_t + w_t$ and the judgement innovation η_t, we also considered the limit $\sigma_{\tilde{\eta}}^2 \to 0$ and computed the correlation for different values of σ_u^2, σ_w^2 and f under the constraints of learning convergence and inclusion of judgement is a CEE. For example, if $\beta = \rho = 0.95$, $f = 1$ and $\sigma_{\tilde{\eta}}^2$ very small, the correlation can be pushed beyond 0.9 before the conditions start to fail.

[17] There are, of course, alternative ways to derive the same equation.

[18] The role of share supply has recently been stressed in the finance literature, e.g. see Ofek and Richardson (2003) and Hong, Scheinkman and Xiong (2006).

[19] Using the notation of Brock and Hommes (1998) $s_t = a\sigma^2 z_{st}$, where σ^2 is the conditional variance of excess returns (assumed constant), a is a parameter of the utility function and z_{st} is the (random) asset supply.

Theorem 2 shows that the basic asset pricing model is consistent with excess volatility. If investors incorporate judgemental factors that are strongly serially correlated, they will find that this improves their forecasts, but in an exuberance equilibrium this will also generate significant stationary asset price movements in excess of those associated with fundamental factors. The stationarity of our exuberance movements is in marked contrast to the literature on rational asset price bubbles. Because the latter are explosive, the literature on rational bubbles has been punctuated by controversy and complicated by the need to construct valid tests for non-stationary bubbles. Exuberance equilibria offer an alternative approach to modeling bubbles within a stationary time series framework.

A natural question is whether the excess volatility associated with an exuberance equilibrium is economically meaningful, or if the exuberance conditions outlined in Theorem 2 are only met for situations in which the variance σ_y^2 is just trivially larger than the fundamental variance. In particular, it is of interest to know if the excess volatility effect isolated in the theorem is large enough to be comparable to empirical estimates of the degree of excess volatility in financial data. One famous calculation due to Shiller (1981) put the ratio of the standard deviation of U.S. stock prices to the standard deviation of prices based on fundamental alone at between 5 and 13.[20]

Table 1 provides illustrative calculations of exuberance equilibria for representative parameter values. In the REE, $y_t = u_t$, the standard deviation of prices is σ_u The Table gives relative volatility results in exuberance equilibria for different ratios σ_ξ/σ_u, ranging from 0.5 to 2.0. Since $\sigma_\xi^2 = \sigma_\eta^2/\left(1-\rho^2\right)$ the innovation variance associated with judgement is quite modest for high values of ρ. We examine an empirically plausible case where the discount factor $\beta = 0.95$, and where the degree of serial correlation ρ is relatively high. A dash in the table indicates that an exuberance equilibrium does not exist for the indicated parameter values. The entries in the table are a measure

[20]Shiller (1981) actually compared the variance of equity prices to the variance of their ex post price (the present value of actual future dividends), but the latter must exceed the variance of the fundamentals price under rational expectations.

Table 1. Excess Volatility

	σ_ξ/σ_u			
	0.50	1.00	1.50	2.00
$\rho = 0.70$	1.54	2.74	–	–
$\rho = 0.80$	1.85	3.62	5.58	–
$\rho = 0.90$	2.70	5.82	9.11	12.43
$\rho = 0.95$	3.99	8.75	13.64	18.56

Table 1: Exuberance equilibria in the asset pricing model. A dash indicates that exuberance equilibrium does not exist. Entries give the ratio of the standard deviation of y to the standard deviation of u.

of excess volatility corresponding to Shiller's (1981) concept, namely, σ_y/σ_u. The results indicate that these measures are often in the range of 5 to 13 estimated by Shiller.

We conclude based on this illustrative calculation that the model can generate substantial excess volatility without difficulty. We remark that if we push the discount factor β closer to unity, the degree of excess volatility can rise to very high levels for high degrees of serial correlation, with σ_y many hundreds of times larger than σ_u. In this sense, the model can generate arbitrarily large amounts of excess volatility. We think these results are striking since they indicate that the possibility of near-rational exuberance in model that are usually regarded as well-behaved. In addition to excess volatility, there are, of course, other stylized facts and puzzles concerning asset prices, involving the related issues of Sharpe ratios, predictability of excess returns, and the equity premium puzzle.[21] Because our exuberance equilibria generate both excess volatility and positively serially correlated prices, we think our enlarged class of equilibria may be able to shed light on these phenomena. However, investigation of these issues would require a serious calibration of the model, and we reserve this topic for future research.

[21] See, for example, Lettau and Uhlig (2002).

5 Rationality tests of judgement

Our exuberance equilibrium is near-rational but not fully rational. There are two ways in which we have imposed assumptions that deviate from full rationality. First, the judgement process ξ_t is assumed not directly available to (or usable by) econometric forecasters, who rely purely on the observables y_t. This seems realistic because ξ_t is the adjustment the judgemental forecasters believe is appropriate due to "unique" qualitative events. This procedure thus reflects a natural division of labor in which the econometricians produce the best statistical forecast based on the observable variables of interest, and the judgemental forecasters modify these forecasts as they think is appropriate. Although $\xi_t = y^e_{t+1} - E_t y^{\star}_{t+1}$ may possibly be obtainable by the econometricians (at least with a lag), we would expect the judgemental forecasters to resist the incorporation of ξ_t into the econometric model.

Furthermore, older ξ_{t-j} represent different unique events, unrelated to the current judgemental variable. Econometric models sometimes incorporate dummy variables (or other proxies) to capture the quantitative effects of qualitative events, but as the events become more distant such variables tend to get dropped and rolled into the unobserved random shocks in order to preserve degrees of freedom. The impact of recent qualitative events could be estimated by incorporating dummy variables into the econometric model, but for forecasting purposes this would be unhelpful, and would still leave the problem of forecasting the future impact of qualitative factors to the judgemental forecasters.

The second way in which our exuberance equilibrium is not fully rational is that the incentive condition is assumed dichotomous. This also seems realistic, since its inclusion is determined by the judgemental forecaster. Furthermore, econometric tests of whether "all" of ξ_t should have been included would often have low power. Suppose we allowed for just a proportion $k \in [0, 1]$ of the judgement to be included in the forecast. It can be shown that the minimum MSE in the univariate case occurs at $k = \beta\rho$.

Table 2. Test rejection rates

	ρ					
	0.7	0.8	0.85	0.9	0.95	0.99
$n = 120$	47.6	5.1	0.1	0	0.4	2.0
$n = 240$	86.8	15.9	1.4	0	0.4	1.7
$n = 480$	99.7	48.0	3.5	0	0.4	1.0

Table 2: Exuberance equilibria in the asset pricing model. Percent of test rejections at 5 percent level of the null hypothesis that including judgement is fully rational, that is, Ho: k=1. Results given are based on 1000 replications.

For $\beta\rho$ near one, rationality tests using

$$y_{t+1} - y^e_{t+1} = (1-k)\,\xi_t + \zeta_{t+1} \tag{13}$$

of the null hypothesis $H_0 : k = 1$ have low power, and considerable data would be required to detect that not all of ξ_t should be optimally included.

We illustrate this point in Table 2, which takes into account both aspects of bounded rationality discussed above. Suppose that econometricians do have access ex post to the judgementally adjusted forecasts y^e_{t+1}, and therefore to $\xi_t = y^e_{t+1} - E^\star_t y_{t+1}$, and that they estimate (13) and test the null hypothesis $H_0 : k = 1$ that the inclusion of judgement is fully rational. For the purposes of this test we set the discount factor at $\beta = 1 - 0.05/12 = 0.9958$ in line with a real monthly risk-free rate of return of 0.05/12.[22] We also set $\sigma^2_\xi/\sigma^2_u = 1.0$. The three sample sizes shown correspond to 10, 20 and 40 years of monthly data and the nominal significance level of the test is set at 5%. When ρ is below 0.8 one would expect to eventually detect a deviation from full rationality. However, it can be seen that for ρ at or above 0.85, rejection of the null is unlikely even with 40 years of data. In particular, for $\rho = 0.9$ or $\rho = 0.95$ any deviation of the judgmental forecasts from full rationality would be virtually undetectable except with enormous sample sizes. Furthermore, these cases correspond to large, empirically plausible values of excess volatil-

[22] In the Brock and Hommes (1998) set-up, β is the inverse of the risk-free real rate-of-return factor. The value chosen here corresponds to an annual discount rate of 5% p.a., but the results of Table 2 are quite similar if 3% p.a. (or 7% p.a.) is used.

ity: for the parameter settings of Table 2 we have excess volatility measures of $\sigma_y/\sigma_u = 8.40$ for $\rho = 0.9$ and $\sigma_y/\sigma_u = 16.39$ for $\rho = 0.95$.

From Table 2 we see that, for an exuberance equilibrium with ρ values above 0.85, decision makers are likely to conclude that the functional division of labor between econometricians, who supply forecasts based on the observable variable y_t, and judgemental forecasters, who adjust these forecasts to take account of perceived qualitative events omitted from the econometric model, is entirely appropriate. Because an exuberance equilibrium is a CEE, the econometricians are fully taking into account the predictable serial correlation properties of the variable being forecast. At the same time, the mean square forecast error is smaller for the judgemental forecasts than for the pure econometric forecast, and thus there is a clear gain to forecast performance in making use of the judgemental adjustment.

The uniqueness of qualitative events is also relevant to the issue at hand. Suppose, for example, that $\rho = 0.8$ and that rationality tests eventually indicate a statistically significant deviation from full rationality, with an estimated value near $k = 0.8$. It does not really seem plausible that forecasters would decide to downweight current judgemental adjustments, based on the finding that such adjustments over the last 20 years or so have been about 20% too high, since past judgemental adjustments mainly concerned different qualitative events, and since the adjustments may have been made by different judgemental forecasters. Furthermore, even if on this basis current judgement is downweighted, and even if this eventually results in the role of judgement being gradually extinguished over time, a new qualitative event will at some point suggest the need once again for judgement, with the judgement process again becoming persistent. In this sense, an economy in which exuberance equilibria exist always remains "subject to judgement."

6 Conclusions and possible extensions

We have studied how a new phenomenon, *exuberance equilibria*, may arise in standard macroeconomic environments. We assume that agents are learning in the sense that they are employing and updating econometric models used to forecast the future values of variables they care about. Unhindered, this learning process would converge to a RE equilibrium in the economies we study. We investigated the idea that decision-makers may be tempted to include judgemental adjustments to their forecasts if all others in the economy are similarly judgementally adjusting their forecasts. The judgemental adjustment, or add factor, is a pervasive and widely-acknowledged feature of actual macroeconometric forecasting in industrialized economies. We obtain conditions under which such add-factoring can become self-fulfilling, altering the actual dynamics of the economy significantly, but in a way that remains consistent with the econometric model of the agents.

In order to develop our central points we have made some strong simplifying assumptions. We have assumed that the exuberance or judgement variables take a simple autoregressive form, but this assumption is mainly made for convenience. While we do believe that judgemental adjustments exhibit strong positive serial correlation, a more complicated stationary stochastic process could instead be used. The assumption of identical judgements of different (representative) agents is correspondingly restrictive. Allowing for differences in judgements by individual agents would probably make the conditions for exuberance equilibrium more difficult to achieve. On the other hand, this could create new phenomena, such as momentum effects arising when a large fraction of agents begin to agree in their judgements.

The incorporation of judgment into decisions, in the form of adjustments to econometric forecasts, can have a self-fulfilling feature in the sense that decisions makers would believe *ex post* that their judgement had improved their forecasts. This result is similar in spirit to the self-fulfilling nature of sunspot equilibria, but with the novel feature that it can arise in determinate models in which there is a unique RE equilibrium that depends only on fundamentals. This widens the set of models in which self-fulfilling fluctuations might plausibly emerge. In particular, we have shown that exuberance equilibria can arise in the standard asset-pricing model, generating substantial excess volatility.

Appendices

A Conditions for CEE

It can be seen from the form of $G_{ALM}(z)$ that, for arbitrary a and b, the ALM is an ARMA(2,2) process. As we will now show, there are choices of a and b that yield $G_{PLM}(z) = G_{ALM}(z)$. These choices of a and b also have the property that the corresponding ALM takes an ARMA(1,1) form that matches the PLM. This is possible if a and b are chosen so that there is a common factor in the numerator and denominator of the expression on the right-hand side of $G_{ALM}(z)$.

We now set $G_{PLM}(z) = G_{ALM}(z)$, under the condition $b = \rho$, so that the poles of the autocovariance generating functions agree. This yields

$$\begin{aligned}\sigma_v^2 &= \left\{ \frac{\beta^2 \sigma_{\hat{\eta}}^2 + (1-\rho z)(1-\rho z^{-1})\sigma_u^2}{(\beta(a-b)+1-az)(\beta(a-b)+1-az^{-1})} \right. \\ &\quad \left. + \frac{\sigma_w^2 (f\beta + 1 - \rho z)(f\beta+1-\rho z^{-1})}{(\beta(1-a)+1-az)(\beta(1-a)+1-az^{-1})} \right\}\end{aligned}$$

or

$$\begin{aligned}&\sigma_v^2 \left[\beta(a-\rho)+1-az\right]\left[\beta(a-\rho)+1-az^{-1}\right] = \\ &\qquad \beta^2\sigma_{\hat{\eta}}^2 + (1-\rho z)\left(1-\rho z^{-1}\right)\sigma_u^2 + \sigma_w^2(f\beta+1-\rho z)(f\beta+1-\rho z^{-1})\end{aligned}$$

This equation can be written as

$$\begin{aligned}&\sigma_v^2 \left\{ \left[1+\beta(a-\rho)\right]^2 + a^2 \right\} - \sigma_v^2 a \left[\beta(a-\rho)+1\right]\left(z+z^{-1}\right) = \\ &\beta^2\sigma_{\hat{\eta}}^2 + \sigma_u^2\left(1+\rho^2\right) - \sigma_u^2\rho\left(z+z^{-1}\right) + \sigma_w^2[(f\beta+1)^2+\rho^2-(f\beta+1)\rho(z+z^{-1})].\end{aligned}$$

For the autocovariances of the PLM and ALM to be equal, the coefficients on the powers of z in this equation must be equal. Equating these we obtain the two equations

$$\sigma_v^2 \left\{ \left[1+\beta(a-\rho)\right]^2 + a^2 \right\} = \beta^2\sigma_{\hat{\eta}}^2 + \sigma_u^2\left(1+\rho^2\right) + \sigma_w^2[(f\beta+1)^2+\rho^2] \quad (14)$$

and

$$\sigma_v^2 a\left[\beta\left(a-\rho\right)+1\right]=\sigma_u^2\rho+\sigma_w^2(f\beta+1)\rho. \tag{15}$$

We wish to solve for a value of a such that $|a|<1$. Solving equation (15) for σ_v^2 and substituting the result into equation (14), at a CEE a solves

$$\hat{F}(a)\equiv\hat{c}_2a^2+\hat{c}_1a+\hat{c}_0=0 \tag{16}$$

with

$$\begin{aligned}\hat{c}_2 &= -\hat{t}(1+\beta^2)+\hat{s}\beta,\\ \hat{c}_1 &= -2\hat{t}\beta(1-\rho\beta)+\hat{s}(1-\rho\beta),\\ \hat{c}_0 &= -\hat{t}(1-\rho\beta)^2\end{aligned}$$

where $\hat{s}=\beta^2\sigma_{\hat{\eta}}^2+(1+\rho^2)\sigma_u^2+\sigma_w^2((1+f\beta)^2+\rho^2)$ and $\hat{t}=\sigma_u^2\rho+\sigma_w^2\rho(1+f\beta)$. Clearly, $\hat{F}(0)<0$ when $f\geq 0$. Next we consider $\hat{F}(\rho)$. Using Mathematica it is verified that

$$\hat{F}(\rho)=\beta\rho[\beta\sigma_{\hat{\eta}}^2+\sigma_w^2f(\beta f+1-\rho^2)].$$

Clearly, $\hat{F}(\rho)>0$ for $f\geq 0$. It follows that there exists a CEE with $0<a<\rho$. Furthermore, by inspecting $\hat{F}(\rho)$ it is seen that $a\to\rho$ when $\sigma_{\hat{\eta}}^2\to 0$. Also $a\to 0$ when $\sigma_u^2+\sigma_w^2\to 0$ since $\hat{t}\to 0$.

There can be a second, negative root. However, our numerical results indicate that the CEE corresponding to the negative root is not learnable.

B Impact of Judgement

The induced ALM, as depicted in equation (8), is

$$y_t=\frac{1-aL}{\beta\left(a-b\right)+1-aL}\left[(1-\rho L)^{-1}\beta(fw_t+\hat{\eta}_t)+(u_t+w_t)\right]. \tag{17}$$

By substituting equation (17) into both (10) and (11), we can write the two types of forecasts in terms of the shocks u_t and η_t. These expressions become

$$E_t^{\star}y_{t+1}=\frac{\rho-a}{\beta\left(a-\rho\right)+1-aL}\left(\frac{\beta}{1-\rho L}\eta_t+u_t+w_t\right)$$

in the case of no judgement, and

$$y_{t+1}^{e} = \frac{\rho - a}{\beta\left(a - \rho\right) + 1 - aL}\left(\frac{\beta}{1 - \rho L}\eta_t + u_t + w_t\right) + \frac{1}{1 - \rho L}\eta_t$$

in the case of the judgementally adjusted forecast. The actual state of the economy at time $t+1$ is, from equation (17),

$$y_{t+1} = \frac{1 - aL}{\beta\left(a - \rho\right) + 1 - aL}\left(\frac{\beta}{1 - \rho L}\eta_{t+1} + u_{t+1} + w_{t+1}\right). \tag{18}$$

We can therefore compute forecast errors in each of the two cases. When computing these forecast errors, we save on clutter by ignoring the terms involving u, as these will be the same whether or not the agent judgementally adjusts the forecast. The forecast error in the case of no judgement can be written as

$$FE_{NJ} \equiv \left[y_{t+1} - E_t^{\star} y_{t+1}\right]|_{u=w=0} = \frac{\beta}{1 + \beta\left(a - \rho\right)} \frac{1}{\left[1 - \left(\frac{a}{1+\beta(a-\rho)}\right) L\right]}\eta_{t+1} \tag{19}$$

whereas in the case of a judgementally adjusted forecast it is

$$\begin{aligned} y_{t+1} - y_{t+1}^{e} \quad = \quad & \frac{1}{1 + \beta(a - \rho) - aL} \times \\ & \left\{\left[\frac{\beta f(1 - aL) - (\rho - a)\beta f L}{1 - \rho L} + (1 - aL)\right] w_{t+1} + \right. \\ & \left. \frac{\beta(1 - aL) - (\rho - a)\beta L}{1 - \rho L}\hat{\eta}_{t+1}\right\} \\ & - \frac{kfL}{1 - \rho L} w_{t+1} - \frac{kL}{1 - \rho L}\hat{\eta}_{t+1} \\ & + \text{term in } u_{t+1}, \end{aligned}$$

where $k = 1$ if judgement is included, and zero otherwise. These equations simplify to those given in the text for the case $a \to \rho$.

When $\sigma_{\hat{\eta}}^2 \to 0$, $\sigma_w^2 \to 0$, the relevant terms in the forecast error for assessing judgement are:

$$\frac{1 + \beta f - (\rho + kf)L}{1 - \rho L} w_{t+1} + \frac{\beta - kL}{1 - \rho L}\hat{\eta}_{t+1}.$$

For the second term the comparison is in the main text. For the term involving w_{t+1} we get for the relevant variances

$$Var\,|_{k=0} - Var\,|_{k=1} \asymp \frac{f}{1+\beta f}\left(2\rho - \frac{2\rho+f}{1+\beta f}\right),$$

where $\asymp$ means "is positively proportional to." It is seen that the term in the brackets is positive for all f when $\beta\rho > 1/2$. This implies that adding a small correlation between judgement and unobserved fundamentals does not alter the incentive condition for inclusion of judgement. In other words, if $\beta\rho > 1/2$ an individual agent will make the judgemental adjustment to the forecast for sufficiently small values of $\sigma^2_{\hat{\eta}}$ and σ^2_w.

Next, we give the formula to compute the variance of the ARMA(2,2) process needed for the numerics behind Figure 1. Apart from the lead coefficient $\beta/\left(1+\beta\left(a-\rho\right)\right)$, each forecast error process is in the generic class

$$x_t = \frac{1+m_1L+m_2L^2}{1-\ell_1L-\ell_2L^2}\epsilon_t.$$

Using the procedure described on pp. 26-27 of Granger and Newbold (1986), one can solve simultaneously for $Var(x_t)$, $Cov(x_t, x_{t-1})$ and $Cov(x_t, x_{t-2})$ to obtain

$$Var\left(x_t\right) = \frac{x_{num}}{x_{den}}\sigma^2_\epsilon, \tag{20}$$

where

$$x_{num} = \frac{\left(1+\ell_2\right)\ell_1\left(m_1+m_2\ell_1+m_2m_1\right)}{1-\ell_2} + \left(m_1+m_2\ell_1\right)\left(\ell_1+m_1\right) + \left(1+2m_2\ell_2+m_2^2\right)$$

and

$$x_{den} = 1 - \frac{\ell_1^2}{1-\ell_2} - \frac{\ell_2\ell_1^2}{1-\ell_2} - \ell_2^2.$$

Considering the forecast error in the case without judgement included, equation (19), we set $m_1 = m_2 = \ell_2 = 0$ and $\ell_1 = a/\left[1+\beta\left(a-\rho\right)\right]$ in equation (20). For the case with judgement, we set

$$\begin{aligned} m_1 &= -(1+a\beta)\,\beta^{-1}, \\ m_2 &= a\beta^{-1}, \\ \ell_1 &= \frac{a+\rho\,[1+\beta\,(a-\rho)]}{1+\beta\,(a-\rho)}, \\ \ell_2 &= \frac{-a\rho}{1+\beta\,(a-\rho)}, \end{aligned}$$

and a is determined by β, ρ and σ_η^2/σ_u^2.

C Recursive maximum likelihood

We now consider recursive estimation when the PLM is an ARMA(1,1) process, that is,

$$y_t = by_{t-1} + v_t + cv_{t-1},$$

where y_t is observed but the white noise process v_t is not observed. Let b_t and c_t denote the estimates of b and c using data through time $t-1$. The econometricians are assumed to use a recursive maximum likelihood (RML) algorithm, which we now describe.[23]

Let $\phi_t' = (b_t, c_t)$. To implement the algorithm an estimate ε_t of v_t is required. Let $\varepsilon_t = y_t - x_{t-1}'\phi_{t-1}$, where $x_{t-1}' = (y_{t-1}, \varepsilon_{t-1})$. y_t is given by $y_t = \beta\,[E_t^\star y_{t+1} + \xi_t] + u_t + w_t$, where $E_t^\star y_{t+1} = b_{t-1}y_t + c_{t-1}\varepsilon_t$. The RML algorithm is as follows

$$\begin{aligned} \psi_t &= -c_{t-1}\psi_{t-1} + x_t \\ \phi_t &= \phi_{t-1} + t^{-1}R_{t-1}^{-1}\psi_{t-1}\varepsilon_t \\ R_t &= R_{t-1} + t^{-1}(\psi_{t-1}\psi_{t-1}' - R_{t-1}). \end{aligned}$$

The question of interest is whether ϕ_t converges to a CEE. Convergence

[23] For further details on the algorithm see Section 2.2.3 of Ljung and Soderstrom (1983). The algorithm is often called a recursive prediction error algorithm.

can be studied using the associated ordinary differential equation

$$\frac{d\phi}{d\tau} = R^{-1}E\psi_t(\phi)\varepsilon_t(\phi) \tag{21}$$

$$\frac{dR}{d\tau} = E\psi_t(\phi)\psi_t(\phi)' - R. \tag{22}$$

Here $y_t(\phi)$, $\psi_t(\phi)$ and $\varepsilon_t(\phi)$ denote the stationary processes for y_t, ψ_t and ε_t with ϕ_t set at a constant value ϕ. Using the stochastic approximation tools discussed in Marcet and Sargent (1989), Evans and Honkapohja (1998) and Chapter 6 of Evans and Honkapohja (2001), it can be shown that the RML algorithm locally converges provided the associated ordinary differential equation is locally asymptotically stable (analogous instability results are also available). Numerically, convergence of (21)-(22) can be verified using a discrete time version of the differential equation. A first-order state space form is convenient for computing the expectations $E\psi_t(\phi)\varepsilon_t(\phi)$ and $E\psi_t(\phi)\psi_t(\phi)'$ and this procedure was used for the numerical illustrations given in the main text.

We now prove convergence analytically for all $0 < \beta, \rho < 1$ with $\sigma^2_{\hat{\eta}}$ and σ^2_w sufficiently small. This completes the proof of part (i) in Theorem 1. We rewrite the system (21)-(22) in the form

$$\frac{d\phi}{d\tau} = (R)^{-1}g(\phi)$$

$$\frac{dR}{d\tau} = M_\psi(\phi) - R$$

where we have introduced the simplifying notation $g(\phi) = E\psi_t(\phi)\varepsilon_t(\phi)$ and $M_\psi(\phi) = E\psi_t(\phi)\psi_t(\phi)'$. An equilibrium $\bar{\phi}, \bar{R}$ of the system is defined by $g(\bar{\phi}) = 0$ and $\bar{R} = M_\psi(\bar{\phi})$. As mentioned in Appendix A, there can be two equilibrium values $\bar{\phi}' = (\rho, -a)$ determined by the solutions to the quadratic (16), but we here focus on the solution with $0 < a < 1$. Recall that for this solution $a \to \rho$ as $\sigma^2_\eta \to 0$.

Linearizing the system at the equilibrium point, it can be seen that the linearized system has a block diagonal structure, in which one block has the eigenvalues equal to -1 (with multiplicity four) and the eigenvalues of the other block are equal to those of the "small" differential equation

$$\frac{d\phi}{d\tau} = (\bar{R})^{-1} J(\bar{\phi})(\phi - \bar{\phi}), \tag{23}$$

where $J(\phi)$ is the Jacobian matrix of $g(\phi)$. The system (21)-(22) is therefore locally asymptotically stable if the coefficient matrix $(\bar{R})^{-1}J(\bar{\phi})$ of the two-dimensional linear system (23) has a negative trace and a positive determinant. Since $(\bar{R})^{-1} = (\det(\bar{R}))^{-1} adj(\bar{R})$ we have

$$\begin{aligned} Tr[(\bar{R})^{-1}J(\bar{\phi})] &= (\det(\bar{R}))^{-1} Tr[adj(\bar{R})J(\bar{\phi})] \text{ and} \\ \det[(\bar{R})^{-1}J(\bar{\phi})] &= \det[(\bar{R})^{-1}] \det[J(\bar{\phi})]. \end{aligned}$$

Now $\det(\bar{R}) > 0$ as $\bar{R}$ is a matrix of second moments and thus positive definite for $\sigma_\eta^2 > 0$. It thus remains to prove that $Tr[adj(\bar{R})J(\bar{\phi})] < 0$ and $\det[J(\bar{\phi})] > 0$ when $\sigma_\eta^2 > 0$ is sufficiently small.

We consider the values of $Tr[adj(\bar{R})J(\bar{\phi})]$ and $\det[J(\bar{\phi})]$ when $\sigma_\eta^2 \to 0$. Using the definition of ε_t, the explicit form of $g(\phi)$ is

$$\begin{aligned} g(\phi) &= E\psi_{t-1}(\phi)x'_{t-1}\left[(1-\beta b - \beta c)^{-1}\beta \begin{pmatrix} -bc \\ -c^2 \end{pmatrix} - \begin{pmatrix} b \\ c \end{pmatrix}\right] \\ &\quad + (1-\beta b-\beta c)^{-1}\beta\rho E\psi_{t-1}(\phi)\xi_{t-1}, \end{aligned}$$

where the moment matrices $E\psi_{t-1}(\phi)x'_{t-1}$ and $E\psi_{t-1}(\phi)\xi_{t-1}$ can be computed from the state space form

$$AX_t = CX_{t-1} + H\begin{bmatrix} u_t \\ \hat{\eta}_t \\ w_t \end{bmatrix}, \text{ with } X_t = \begin{pmatrix} y_t \\ \varepsilon_t \\ \xi_t \\ \psi_t \\ \psi_{t-1} \end{pmatrix},$$

$$A = \begin{pmatrix} 1 & -(1-\beta b)^{-1}\beta c & -(1-\beta b)^{-1}\beta & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -b & -c & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & -c & 0 \\ 0 & 0 & 0 & 0 & -c \end{pmatrix}, H = \begin{pmatrix} (1-\beta b)^{-1} & 0 & (1-\beta b)^{-1} \\ 0 & 0 & 0 \\ 0 & 1 & f \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For the limit $\sigma^2_{\hat{\eta}} \to 0$ and $\sigma^2_w \to 0$ we first set $\sigma^2_w = \lambda\sigma^2_{\hat{\eta}}$, where $\lambda > 0$ is arbitrary. It can be computed using Mathematica 4 (routine available on request) that $Tr[adj(\bar{R})J(\bar{\phi})]$ and $\det[J(\bar{\phi})]$ have the following properties as functions of (using temporary notation) $\omega \equiv \sigma^2_{\hat{\eta}}$:

$$\begin{aligned} \lim_{\omega\to 0} Tr[adj(\bar{R})J(\bar{\phi})] &= \lim_{\omega\to 0} \frac{d}{d\omega} Tr[adj(\bar{R})J(\bar{\phi})] = 0, \\ \lim_{\omega\to 0} \det[J(\bar{\phi})] &= \lim_{\omega\to 0} \frac{d}{d\omega} \det[J(\bar{\phi})] = 0, \end{aligned}$$

$$\lim_{\omega\to 0} \frac{d^2}{d\omega^2} Tr[adj(\bar{R})J(\bar{\phi})] = -\frac{4\beta^2\rho^2[\beta + f\lambda(1+f\beta-\rho^2)]^2}{(1-\beta\rho)(\rho^2-1)^6} < 0 \text{ and}$$

$$\lim_{\omega\to 0} \frac{d^2}{d\omega^2} \det[J(\bar{\phi})] = \frac{2\beta^2\rho^2[\beta + f\lambda(1+f\beta-\rho^2)]^2}{(\rho^2-1)^6} > 0.$$

The two latter derivatives are evidently locally smooth functions in view of the form of $g(\phi)$. Expressing $Tr[adj(\bar{R})J(\bar{\phi})]$ and $\det[J(\bar{\phi})]$ in terms of Taylor series these results show that

$$Tr[adj(\bar{R})J(\bar{\phi})] < 0 \text{ and } \det[J(\bar{\phi})] > 0$$

for $\sigma^2_\eta > 0$ sufficiently small. *Q.E.D.*

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