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« Income Inequality Games »

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Income Inequality Games*

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Abstract

The paper explores different applications of the Shapley value for either inequality or poverty measures. We first investigate the problem of source decomposition of inequality measures, the so-called additive income sources inequality games, baed on the Shapley Value, introduced by Chantreuil and Trannoy (1999) and Shorrocks (1999). We show that multiplicative income sources inequality games provide dual results compared with Chantreuil and Trannoy's ones. We also investigate the case of multiplicative poverty games for which indices are non additively decomposable in order to capture contributions of sub-indices, which are multiplicatively connected with, as in the Sen-Shorrocks-Thon poverty index. We finally show in the case of additive poverty indices that the Shapley value may be equivalent to traditional methods of decomposition such as subgroup consistency and additive decompositions.

Key-words and phrases: Inequality, Poverty, Shapley, Source decomposition.

JEL Codes: D31, D63.

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1 Introduction

1.1 Inequality Measurement and Decompositions

In theses last few decades, the theory of inequality measurement have been connected with many properties of decomposition (or aggregation as can be seen in Bourguignon (1979)). This idea of decomposition is not really new in the sense that it may be viewed as an extension of the independence axiom we find in many fields such as in theory of choice under risk and uncertainty. This has lead to a widespread line of research, that of subgroup decomposition, see e.g. Battacharya and Mahalonobis (1967), Rao (1969), Cowell (1980), Shorrocks (1980, 1984) among others, showing that the subgroup decomposition for entropy, Atkinson-Kolm-Sen or Gini inequality measures enable two estimators (or three in particular cases) to be derived: within- and between-group indices of inequality.

A second line of research, for which the paper is concerned with, is another decomposition framework, that of income source decompositions: see e.g. Rao (1969), Shorrocks (1982, 1983, 1999), Lerman and Yitzhaki (1985), Morduch and Sicular (2002) among others. In other words, if incomes are linearly composed of L sources of income (factors), it is then possible to capture the contribution of each source to the overall amount of inequality. In 1992, Auvray and Trannoy proposed to adapt the Shapley (1953) value, a famous tool of cooperative game theory, in order to address the possibility to compute income sources contributions, showing for instance that the Shapley value applied to the variance of income in the case of L factors yields the same result compared with the well-known variance analysis.

1.2 Shapley Value and Factor Decompositions: Simple Examples

Let us expose simple examples to introduce the concepts of contribution and Shapley value. Consider an index $I(\cdot)$ of a variable $\mathbf{x}$. One basic way for splitting a variable is to assume that $\mathbf{x} = \mathbf{x}^1 + \mathbf{x}^2$, i.e. source decomposition. One standard problem is the decomposition of the index. Decomposition means that we wish to have coefficients $\omega = (\omega^1, \omega^2)$ such as, per source,

$$I(\mathbf{x}) = \omega^1 I(\mathbf{x}^1) + \omega^2 I(\mathbf{x}^2).$$

This decomposition cannot be unique. For instance, for the first one, a *natural* decomposition is simply

$$I(\mathbf{x}) = \frac{I(\mathbf{x})}{I(\mathbf{x}^1) + I(\mathbf{x}^2)} I(\mathbf{x}^1) + \frac{I(\mathbf{x})}{I(\mathbf{x}^1) + I(\mathbf{x}^2)} I(\mathbf{x}^2),$$

hence $\omega^1 = \omega^2$ (which might not be reasonable).

If $I(\cdot)$ denotes the variance operator, then for the sum

$$\begin{aligned} Var(\mathbf{x}) &= Var(\mathbf{x}^1 + \mathbf{x}^2) = Var(\mathbf{x}^1) + Var(\mathbf{x}^2) + 2cov(\mathbf{x}^1, \mathbf{x}^2) \\ &= \frac{Var(\mathbf{x})}{Var(\mathbf{x}^1) + Var(\mathbf{x}^2)} Var(\mathbf{x}^1) + \frac{Var(\mathbf{x})}{Var(\mathbf{x}^1) + Var(\mathbf{x}^2)} Var(\mathbf{x}^2) \\ &= \left(1 + \frac{2cov(\mathbf{x}^1, \mathbf{x}^2)}{Var(\mathbf{x}^1) + Var(\mathbf{x}^2)}\right) Var(\mathbf{x}^1) + \left(1 + \frac{2cov(\mathbf{x}^1, \mathbf{x}^2)}{Var(\mathbf{x}^1) + Var(\mathbf{x}^2)}\right) Var(\mathbf{x}^2). \end{aligned}$$

An alternative is to consider

$$\begin{aligned} Var(\mathbf{x}) &= Var(\mathbf{x}^1+\mathbf{x}^2) = Var(\mathbf{x}^1) + \left(\frac{Var(\mathbf{x})}{Var(\mathbf{x}^2)} - \frac{Var(\mathbf{x}^1)}{Var(\mathbf{x}^2)}\right) Var(\mathbf{x}^2) \\ &= Var(\mathbf{x}^1) + \left(\frac{cov(\mathbf{x}^2, 2\mathbf{x}^1+\mathbf{x}^2)}{Var(\mathbf{x}^2)}\right) Var(\mathbf{x}^2). \end{aligned}$$

A marginal or "*incremental sharing*" may be one of the two decompositions:

$$\begin{cases} I(\mathbf{x}) = I(\mathbf{x}^1) + [I(\mathbf{x}) - I(\mathbf{x}^1)] \\ I(\mathbf{x}) = [I(\mathbf{x}) - I(\mathbf{x}^2)] + I(\mathbf{x}^2) \end{cases}$$

This technique can be considered, e.g., in the context of priority rationing. Note that it is also possible to consider an average of those incremental sharing techniques, i.e.

$$I(\mathbf{x}) = \frac{1}{2}[I(\mathbf{x}^1) + I(\mathbf{x}) - I(\mathbf{x}^2)] + \frac{1}{2}[I(\mathbf{x}) - I(\mathbf{x}^1) + I(\mathbf{x}^2)]$$

also called Shapley value (a uniform average of all incremental methods). A more flexible approach is to seek for functions

$$I(\mathbf{x}) = \phi(I(\mathbf{x}^1), I(\mathbf{x}), I(\mathbf{x}-\mathbf{x}^1)) + \phi(I(\mathbf{x}^2), I(\mathbf{x}), I(\mathbf{x}-\mathbf{x}^2)).$$

In this framework, it is also very *natural* to seek for desirable properties of the decomposition. Based on a set of simple axioms, the Shapley value yields i.e. $\phi(\mathbf{x},\mathbf{y},\mathbf{z}) = [\mathbf{y}-\mathbf{z}+\mathbf{x}]/2$,

$$I(\mathbf{x}) = \frac{1}{2}\left[I(\mathbf{x}) - I(\mathbf{x}^2) + I(\mathbf{x}^1)\right] + \frac{1}{2}\left[I(\mathbf{x}) - I(\mathbf{x}^1) + I(\mathbf{x}^2)\right].$$

On this basis, Auvray and Trannoy's result (1992) is the following: the natural decomposition of the variance $\sum_\ell cov\left(\mathbf{x}^\ell, \mathbf{x}\right)$ for all $\ell \in \{1, 2, \ldots, L\}$ yields the same contribution of source ℓ to the overall inequality as the Shapley value, that is $cov\left(\mathbf{x}^\ell, \mathbf{x}\right)$. In 1999, Shorrocks verified this result related to the variance analysis and further introduced the notion of groups in order to make the Shapley value consistent with the first line of research (subgroup decomposition in which the population is partitioned in many groups) and also with the original perspective of the Shapley value whose interest lies in players (groups) coalitions.

1.3 Overview of the paper

Is the Shapley value a new direction in the measurement of inequality? Actually it is an old new direction. Old because many applications have been done in this last decade (see e.g. Sastre and Trannoy (2002), Israeli (2007) and Devicienti (2008) for diverse (econometric) applications). New because the technique reveals more, in particular with multiplicative games as shown in the paper) and because it may always offer new games for which the derived contributions respect the standard axiom. Also, it tells in many cases that values must not be applied to any social index respecting some precise properties. This is precisely the aim of the paper: to review the existing games (additive income sources ones), to make obvious extensions (multiplicative games), and to show that, with respect to some axioms, that the Shapley value may be welcome or not.

Without addressing the role of group coalition together with source "coalitions" (see e.g. Charpentier and Mussard (2010)), we focus on income sources inequality/poverty games for which factors are multiplicative. This enables us to examine first the main results derived from inequality games with additive income sources in order to point out some dual results obtained within a multiplicative framework. Loosely speaking, in many cases, as can be seen in Chantreuil-Trannoy (1999) and Morduch and Sicular (2002), the contribution of a particular source to the overall amount of inequality can be negative. We point out the fact that Morduch and Sicular's (2002) uniform addition is nothing else than a sub-additive game implying systematically negative contributions. This problem of income source decomposition of inequality measures, the so-called (additive) multiplicative income sources inequality games in which the factors are multiplicative (additive) is reviewed and discussed in Section 2 and 3. We subsequently investigate the relevance of the use of the Shapley value in the case of poverty measures and population sub-groups (Section 4). Indeed, we show that the Shapley value may be equivalent to traditional techniques of decomposition such as subgroup consistency and additive decomposition. Accordingly, the Shapley value may be seen as a non desirable method since it reveals nothing more. On the contrary, for non-additive poverty indices, the Shapley Value helps to capture contributions.

2 Additive Income Sources Inequality Games

In this Section, we briefly review some Chantreuil and Trannoy's (1999) results about decomposition by income sources, namely additive income sources inequality games, in order to make further distinctions with multiplicative income sources inequality games.

2.1 Notations, Definitions, Axioms

Let $I(\mathbf{x})$ be a generic term to denote inequality measures where the vector of incomes $\mathbf{x} \in \mathbb{D} \equiv \bigcup_{n=1}^{\infty} \mathbb{R}^n$ is composed of L incomes sources, $L \in \mathbb{N}^*$ where $\mathbb{N}^*$ denotes the positive part of the set of integers and where $I : \mathbb{D}_+ \to \mathbb{R}_+$, $\mathbb{R}_+$ being the set of nonnegative real numbers and $\mathbb{R}_{++}$ that of positive real numbers. Imagine a population with n income units, where x_i represents individual i's income. Suppose that each income is desegregated into L sources, $\ell \in \{1, \ldots, L\}$ such as:

$$\mathbf{x} = (x_1, x_2, \ldots, x_i, \ldots, x_n) = \left(\sum_{\ell=1}^{L} x_1^{\ell}, \sum_{\ell=1}^{L} x_2^{\ell}, \ldots, \sum_{\ell=1}^{L} x_i^{\ell}, \ldots, \sum_{\ell=1}^{L} x_n^{\ell} \right),$$

where $x_i = \sum_{\ell=1}^{L} x_i^{\ell}$ is the income of individual i decomposed into its L sources. Let $\mathbf{x}^{\ell}$ be the $n-$dimensional vector of the ℓ-th . The challenge in inequality decomposition by factor components is to decompose an index of inequality in order to get:

$$I : \mathbb{D}_+ \to \mathbb{R}_+ \text{ such as } I(\mathbf{x}) = \sum_{\ell=1}^{L} \omega^{\ell} I\left(\mathbf{x}^{\ell}\right),$$

where ω^ℓ is a weight attached to inequality of source ℓ. In order to obtain a linear combination of inequality by sources $I\left(\mathbf{x}^\ell\right)$ for any given weights, it is then possible to compute the percentage contribution of the ℓ-th source to the overall amount of inequality that is, $\omega^\ell I\left(\mathbf{x}^\ell\right)/I(\mathbf{x})$. This allows the main sources generating inequality to be identified in order to plan policies of redistribution thanks to these indicators, see e.g. Fei, Ranis and Kuo (1978) which is one of the first attempt to. These decompositions have exercised the mind of many researchers because few indices are decomposable by income sources and because decompositions often yield non linear structure, or because one tries to compute contributions close to $\omega^\ell I\left(\mathbf{x}^\ell\right)$ without verifying the minimal requirements introduced e.g. by Shorrocks (1982). Following these requirements, Shorrocks (1982) found that the variance and the coefficient of variation squared are the only natural decomposable inequality measures. This result is compatible with the Shapley value decomposition, which yields exactly the same contributions than the natural ones derived from the variance and the coefficient of variation squared (Shorrocks (1999) and Chantreuil and Trannoy (1999)).

Shapley's *algorithm* enables marginal impacts as well as contributions of each ℓ factor to the overall index $I(\cdot)$ to be captured. This is because, as it is usual in cooperative games theory, players have the possibility to form coalitions in order to improve their payoff (reduce their cost). Instead of looking for coalitions of players, it is possible to use the Shapley values to derive the contribution of each factor to the overall index. For this purpose, we must imagine the sequence of eliminations of the L factors in $\mathcal{L} = \{1, \ldots, \ell, \ldots, L\}$ that provides $L!$ permutations. We must also imagine the sequence of eliminations of all variable coalitions, namely the set $\mathcal{S}$. Then, the Shapley value enables many functional forms to be expressed for the index $I\left(\mathbf{x}(S)\right)$.

Definition 2.1 *An additive income sources inequality game is a couple $(\mathcal{L}, F_I)^+$ such as:*

$$\mathbf{x} : 2^{\mathcal{L}} \to \mathbb{D}\ ,\ \textit{for all } \mathcal{S} \in 2^{\mathcal{L}},\ \mathcal{S} \neq \emptyset$$

$$\Longrightarrow\ \mathbf{x}(\mathcal{S}) := \left(\sum_{\ell=1}^{s} x_1^\ell, \sum_{\ell=1}^{s} x_2^\ell, \ldots, \sum_{\ell=1}^{s} x_i^\ell, \ldots, \sum_{\ell=1}^{s} x_n^\ell \right),\ s := |\mathcal{S}|.$$

The Value V_I or equivalently the characteristic Function F_I for any given index (inequality or poverty discussed later) is a function of function, that is,

$$V_I \equiv F_I : 2^{\mathcal{L}} \to \mathbb{R}^s \to \mathbb{R}_+ \textit{ such as } F_I = I \circ \mathbf{x},$$

where $\mathbf{x}(\emptyset) = 0$ by convention. Consequently:

$$F_I(\mathcal{L}) = I(\mathbf{x}(\mathcal{L})) = I(\mathbf{x}) \textit{ and } F_I(\mathcal{S}) = I(\mathbf{x}(\mathcal{S})).$$

In $(\mathcal{L}, F_I)^+$ the factor ℓ is dropped from the set $\mathcal{L}$, then the number of remaining variables is $s = L - 1$ (the cardinality of $\mathcal{S}$). In measuring the difference between the inequality index over all variables $I(L)$ and the index after eliminating factor ℓ, we get one possible marginal impact associated with factor ℓ. Repeating the discarding of ℓ over all possible sets of variables 'coalitions' $\mathcal{S}$ provides all marginal impacts. Averaging these marginal impacts allows the contribution of the factor ℓ to be determined. We then obtain $L!$ marginal impacts for all ℓ factors. To gauge the contribution of the factor ℓ to $I(\cdot)$, namely $\mathcal{C}^\ell$ for all $\ell \in \mathcal{L}$, the Shapley value yields:

$$\mathcal{C}^\ell(\mathcal{S}; F_I) = \sum_{s=0}^{L-1} \sum_{\mathcal{S} \subseteq \mathcal{L} \setminus \{\ell\}} \frac{(L-1-s)!s!}{L!} \Delta_\ell F_I(\mathcal{S}), \tag{1}$$

where

$$\Delta_\ell F_I(\mathcal{S}) := F_I(\mathcal{S} \cup \{\ell\}) - F_I(\mathcal{S}) \tag{2}$$

represents the marginal impacts of variable ℓ and where

$$F_I(\{\emptyset\}) = 0, \tag{3}$$

by convention.[1] Chantreuil et Trannoy (1999) show that the Shapley value is a consistent rule of decomposition since the sum of all contributions provides the overall inequality index. They however demonstrate that the marginality rule, that is, the use of Eq. (2) is not coherent for many inequality measures (the regular ones).

Definition 2.2 *An inequality index is Shapley decomposable if,*

$$I(\mathbf{x}(\mathcal{L})) = \sum_{\ell=1}^{L} \mathcal{C}^\ell(\mathcal{S}, F_I). \tag{SHAP}$$

In order to prove that the Shapley value is a good candidate to decompose different inequality measures, the authors show that the method enables usual axioms to be satisfied.[2]

2.2 Axioms

Axiom 2.1 Relevance: *For any $\mathbf{x} \in \mathbb{D}_+$, an index of inequality $I(\cdot)$ respects the relevance principle if, for all $\ell \in \mathcal{L}$,*

$$I(\mathbf{x}(\mathcal{L})) \geq \mathcal{C}^\ell(\mathcal{S}, F_I) \geq 0. \tag{REL}$$

Axiom 2.2 Normalization: *An index of inequality is normalized if, for any $\mathbf{x} = (\varepsilon, \varepsilon, \dots, \varepsilon) \in \mathbb{D}_+$, then*

$$I(\mathbf{x}) = 0. \tag{NM}$$

Axiom 2.3 Linear Homogeneity: *An index of inequality is absolute if, for any $\mathbf{x}, \mathbf{y} \in \mathbb{D}_+$ such as $\mathbf{y} = (\varepsilon, \varepsilon, \dots, \varepsilon)$, then*

$$I(\mathbf{x} + \mathbf{y}) = I(\mathbf{x}). \tag{LH}$$

Axiom 2.4 (Positive) Homogeneity of Order 0: *An index of inequality is relative if, for any $\mathbf{x} \in \mathbb{D}_+$ and $\lambda > 0$,*

$$I(\lambda \mathbf{x}) = I(\mathbf{x}). \tag{HOM^0}$$

[1] This convention may be used for relative inequality and poverty indices only.

[2] We have ourselves added the (REL) axiom below in order to investigate the problem of uniform addition.

Axiom 2.5 (Positive) Homogeneity of Order 1: *An index of inequality is homogeneous of degree one if, for any* $\mathbf{x} \in \mathbb{D}_+$ *and* $\lambda \geq 1$,

$$I(\lambda \mathbf{x}) = \lambda I(\mathbf{x}). \qquad (\mathrm{HOM}^1)$$

Axiom 2.6 Symmetry: *For all* $\mathbf{x}$, $\mathbf{y} \in \mathbb{D}_+$ *such as* $\mathbf{y} = \Pi \mathbf{x}$, Π *being a* $n \times n$ *permutation matrix, then*

$$I(\mathbf{x}) = I(\mathbf{y}). \qquad (\mathrm{SM})$$

Axiom 2.7 Population Principle: *Let* $\mathbf{x} \in \mathbb{D}_+$ *and* $\mathbf{x}^{(t)}$ *being obtained after concatenating* $\mathbf{x}$ t *times. For all* $t \in \mathbb{N}^* \setminus \{1\}$,

$$I(\mathbf{x}^{(t)}) = I(\mathbf{x}). \qquad (\mathrm{PP})$$

Axiom 2.8 Transfer Principle (Schur-convexity): *For all* $\mathbf{x}$, $\mathbf{y} \in \mathbb{D}_+$ *such as* $\mathbf{y} = \mathbf{B}\mathbf{x}$, $\mathbf{B}$ *being a* $n \times n$ *bistochastic matrix, then*

$$I(\mathbf{x}) \geq I(\mathbf{y}). \qquad (\mathrm{TP})$$

The authors find that the contributions issued from the Shapley value, Shapley contributions from now on (denoted by $\mathcal{C}^\ell(\mathcal{S}, F_I)$) are linear combinations of inequality in source ℓ, so that, Shapley contributions respect almost all axioms an inequality index has to satisfy: (HOM^0) or (LH), (SM), (TR) and (PP). They also remark, for any given index, that additive income sources inequality games do not provide systematically the natural contributions of the variance or the coefficient of variation squared and do not always lead to the respect of (REL) for indices being either (HOM^0) or (LH), this point being discussed in the next Section. On the other hand, the authors point out another problem linked with the hierarchy of the factor components. The Shapley value does not ascribe a same contribution for any given source splitting in two or more sources. This provides a problem of hierarchy in the sources when each factor may be broken down in sub-sources. We do not focus on this problem, where solutions may be found in Chantreuil-Trannoy (1999) with the Nested-Value approach, in Shorrocks (1999) with the Owen value.

In sum, the Shapley Value is appealing for economists since it brings out the contribution of each variable ℓ without recourse to econometric models for which the quality of fit and all tests have to be checked. This is also very appealing for policy purposes since gauging contributions yields the ability for analysts and decision makers to plan many policy orientations and consequently to apprehend their impacts in being aware of the axioms satisfied by the Shapley contributions.

3 Multiplicative Income Sources Inequality Games

3.1 Examples and motivations

The problem of multiplicative income sources is appealing when measuring wage inequalities, because it enables, for example, the impact of worked hours per individuals x_i^h and the impact of the nominal wage per worked hours and per individuals x_i^w to be computed. This problem of computing contributions may be appealing for policy purpose, e.g. for a Sarkozy's (french) policy such as "work more to earn more". This implies income inequality variations. But in which item? Inequality in hours or inequality in wages per worked hours ? This is precisely why it is important to find a methodology to compute contributions. Another suitable example is that of prices and quantities when measuring inequalities of consumption. Indeed, it is possible that agents face different prices for the same commodity and/or consume different quantities. Thus, for policy purposes, it would be interesting to gauge the contribution of quantities and prices when computing inequalities in consumption expenditures. For ease of exposition, the first case (nominal wage and worked hours) will be occasionally considered.

In this section we study the possibility to have multiplicative factor components by supposing that each income may be multiplicatively desegregated into L sources, $\ell \in \{1, \ldots, L\}$ such as:

$$\mathbf{x} = (x_1, x_2, \ldots, x_i, \ldots, x_n) = \left(\prod_{\ell=1}^{L} x_1^\ell, \prod_{\ell=1}^{L} x_2^\ell, \ldots, \prod_{\ell=1}^{L} x_i^\ell, \ldots, \prod_{\ell=1}^{L} x_n^\ell \right), \tag{4}$$

where $x_i = \prod_{\ell=1}^{L} x_i^\ell$ is the income of individual i decomposed into its L multiplicative sources. Let μ be the mean income of the population and μ^ℓ that of source ℓ.

Remark 3.1 *From a mathematical point of view, Eq. (4) is equivalent to an* additive *decomposition of log-incomes,*

$$\log \mathbf{x} = (\log x_1, \log x_2, \ldots, \log x_i, \ldots, \log x_n) = \left(\sum_{\ell=1}^{L} \log x_1^\ell, \sum_{\ell=1}^{L} \log x_2^\ell, \ldots, \sum_{\ell=1}^{L} \log x_i^\ell, \ldots, \sum_{\ell=1}^{L} \log x_n^\ell \right).$$

Nevertheless, relevant axioms on inequality measures for incomes are obviously not the same as inequality measures for log-incomes. Thus, based on the set of natural axioms defined in Section 2.2, it might be natural to study multiplicative decomposition.

3.2 Aims and Scope

The problem of decomposing an inequality index is a non trivial one, as mentioned in Section 1.2. Indeed, for the generalized entropy inequality measures (see Shorrocks (1980) or Cowell (1980)), we get:

$$S_c(\mathbf{x}) = \begin{cases} \sum_{i=1}^{n} \frac{(x_i/\mu)^c - 1}{nc(c-1)} & c \in \mathbb{R} - 0 - 1 \\ \frac{1}{n} \sum_{i=1}^{n} \log \frac{\mu}{x_i} & c = 0 \\ \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{\mu} \log \frac{x_i}{\mu} & c = 1. \end{cases}$$

One possible decomposition by income sources (being positive) is (see e.g. Cheng and Li (2006)):

$$S_c(\mathbf{x}) = \begin{cases} \sum_{i=1}^{n} \frac{\left(\frac{\prod_{\ell=1}^{L} x_i^\ell}{\sum_{\ell=1}^{L} \mu^\ell}\right)^c - 1}{nc(c-1)} & c \in \mathbb{R} - 0 - 1 \\ \frac{1}{n} \sum_{i=1}^{n} \log \frac{\sum_{\ell=1}^{L} \mu^\ell}{\prod_{\ell=1}^{L} x_i^\ell} & c = 0 \\ \frac{1}{n} \sum_{i=1}^{n} \frac{\prod_{\ell=1}^{L} x_i^\ell}{\sum_{\ell=1}^{L} \mu^\ell} \log \frac{\prod_{\ell=1}^{L} x_i^\ell}{\sum_{\ell=1}^{L} \mu^\ell} & c = 1. \end{cases}$$

This example shows that the generalized entropy inequality measures are not trivially decomposable by sources (when they are multiplicative, and additive too). Indeed, without taking recourse to Taylor expansions or econometric models where the error term may be "non normal", it is difficult to get $S_c(\mathbf{x}) = \sum_{\ell=1}^{L} \omega^\ell S_c(\mathbf{x}^\ell)$ or at least $S_c(\mathbf{x}) = f\left(S_c(\mathbf{x}^1), \ldots, S_c(\mathbf{x}^L)\right)$, where contributions satisfy the usual axioms of the literature.

Taking the Gini index seems also to provide an impossibility:

$$G = \sum_{i=1}^{n}\sum_{r=1}^{n} \frac{|x_i - x_r|}{2n^2\mu} = \sum_{i=1}^{n}\sum_{r=1}^{n} \frac{\left|\prod_{\ell=1}^{L} x_i^\ell - \prod_{\ell=1}^{L} x_r^\ell\right|}{2n^2\left(\sum_{\ell=1}^{L}\mu^\ell\right)}.$$

3.3 Multiplicative Income Sources Inequality Games

Let us first define multiplicative income sources games before connecting the results with the fundamental properties of inequality measurement.

Definition 3.1 *A multiplicative income sources game is a couple* $(\mathcal{L}, F_I)^\times$ *such as:*

$$\mathbf{x} : 2^{\mathcal{L}} \to \mathbb{D}\ \forall \mathcal{S} \in 2^{\mathcal{L}},\ \mathcal{S} \neq \emptyset \implies \mathbf{x}(\mathcal{S}) := \left(\prod_{\ell=1}^{s} x_1^\ell, \prod_{\ell=1}^{s} x_2^\ell, \ldots, \prod_{\ell=1}^{s} x_i^\ell, \ldots, \prod_{\ell=1}^{s} x_n^\ell\right),\ s := |\mathcal{S}|,$$

$$F_I : 2^{\mathcal{L}} \to \mathbb{D} \to \mathbb{R}_+ \text{ such as } F_I = I \circ \mathbf{x},\ \mathbf{x}(\emptyset) := 0,\ F_I(\{\emptyset\}) := 0.$$

Remark 3.2 *The Shapley value associated with a multiplicative income sources game* $(\mathcal{L}, F_I)^\times$ *is:*

$$\mathcal{C}^\ell(\mathcal{S}; F_I) = \sum_{s=0}^{L-1}\sum_{\mathcal{S}\subseteq\mathcal{L}\setminus\{\ell\}} \frac{(L-1-s)!s!}{L!}\left(F_I\left(\mathcal{S}\cup\{\ell\}\right) - F_I(\mathcal{S})\right) \text{ such as } \sum_{\ell=1}^{L}\mathcal{C}^\ell\left(\mathcal{S}; F_I\right) = I\left(\mathbf{x}\right).$$

Proof.
Straightforward. Just remark as in Chantreuil and Trannoy (1999) for additive income sources games that $\mathcal{C}^\ell(\mathcal{S}; F_I)$ may be negative, that is, a source of income may contribute to decrease overall inequalities, e.g. transfers to poor, child support benefits, etc. may contribute to decrease the inequalities *via* redistributive policies (this point being discussed *infra*). ■

Proposition 3.1 *Let* $\mathbf{x}^\ell = (\varepsilon, \varepsilon, \ldots, \varepsilon)$*, then for any income source inequality games being* $(\mathcal{L}, F_I)^\times$ *or* $(\mathcal{L}, F_I)^+$

$$\mathcal{C}^\ell(\mathcal{S}; F_I) = \sum_{s=0}^{L-1}\sum_{\mathcal{S}\subseteq\mathcal{L}\setminus\{\ell\}} \frac{(L-1-s)!s!}{L!}\left(F_I\left(\mathcal{S}\cup\{\ell\}\right) - F_I(\mathcal{S})\right)$$

such as
$(\imath)$ $(\mathcal{L}, F_I)^\times \implies \mathcal{C}^\ell(\mathcal{S}; F_I) = 0$ *if* $I(\mathbf{x})$ *satisfies* (HOM^0)

(ii) $(\mathcal{L}, F_I)^+ \Longrightarrow \mathcal{C}^\ell(\mathcal{S}; F_I) = 0$ *if* $I(\mathbf{x})$ *satisfies* (LH) [Chantreuil-Trannoy (1999)]
(iii) $(\mathcal{L}, F_I)^\times \Longrightarrow \mathcal{C}^\ell(\mathcal{S}; F_I) \geq 0$ *if* $I(\mathbf{x})$ *satisfies* (HOM^1)
(iv) $(\mathcal{L}, F_I)^+ \Longrightarrow \mathcal{C}^\ell(\mathcal{S}; F_I) \leq 0$ *if* $I(\mathbf{x})$ *satisfies* (HOM^0) [Chantreuil-Trannoy (1999)]

Proof.
Straightforward. Just remark the close interrelation we obtain between multiplicative and additive income sources inequality games à la Chantreuil and Trannoy (1999). ■

Corollary 3.1 *Let* $\mathbf{x}^h$ *and* $\mathbf{x}^w$ *be the vectors of wages per worked hours and the number of worked hours, respectively. If the gross salary per individual is defined to be* $x_i = x_i^h \cdot x_i^w$ *such as* $\mathcal{L} = \{h, w\}$*, then the contribution of worked hours and wage per worked hours issued from a two-sources multiplicative income inequality game* $(\mathcal{L}, F_I)^\times$ *(resp. additive* $(\mathcal{L}, F_I)^+$ *) are expressed as:*

$$\mathcal{C}^h(\mathcal{S}; F_I) = \frac{1}{2}\left[F_I\left(\{h\} \cup \{w\}\right) - F_I(\{w\})\right] + \frac{1}{2}\left[F_I\left(\{h\}\right) - F_I(\{\emptyset\})\right]$$

$$\mathcal{C}^w(\mathcal{S}; F_I) = \frac{1}{2}\left[F_I\left(\{w\} \cup \{h\}\right) - F_I(\{h\})\right] + \frac{1}{2}\left[F_I\left(\{w\}\right) - F_I(\{\emptyset\})\right],$$

where for any $\mathbf{x}^h = (\varepsilon, \ldots, \varepsilon)$ $\forall \varepsilon \in \mathbb{R}_{++}$*,*
(i) $(\mathcal{L}, F_I)^\times \Longrightarrow I(\mathbf{x})$ *respects* (REL) *with* $\mathcal{C}^h(\mathcal{S}; F_I) = 0$; $\mathcal{C}^w(\mathcal{S}; F_I) = I(\mathbf{x})$ *if* $I(\mathbf{x})$ *satisfies* (HOM^0)
(ii) $(\mathcal{L}, F_I)^+ \Longrightarrow I(\mathbf{x})$ *respects* (REL) *with* $\mathcal{C}^h(\mathcal{S}; F_I) = 0$; $\mathcal{C}^w(\mathcal{S}; F_I) = I(\mathbf{x})$ *if* $I(\mathbf{x})$ *satisfies* (LH)
(iii) $(\mathcal{L}, F_I)^\times \Longrightarrow I(\mathbf{x})$ *respects* (REL) *if* $I(\mathbf{x})$ *satisfies* (HOM^1)
(iv) $(\mathcal{L}, F_I)^+ \Longrightarrow I(\mathbf{x})$ *does not respects* (REL) *if* $I(\mathbf{x})$ *satisfies* (HOM^0) *since* $\mathcal{C}^h(\mathcal{S}; F_I) \leq 0$ *and* $\mathcal{C}^w(\mathcal{S}; F_I) \geq I(\mathbf{x})$.

Proof.
Obvious from Proposition 3.1. ■

As explained before, one possible issue in the factor component measurement of inequality is to find a technique for which contributions are, after axiomatic validations, indices of inequality, that is, respecting (PP), (SM), (TR), (NM), (HOM^0) for relative indices, (LH) for absolute ones, (HOM^1) for some inequality, deprivation or distance indices. As can be seen *supra* and in Chantreuil and Trannoy (1999), Shapley contributions are inequality indices except in the case where the latter respect (HOM^0) since in this case (REL) is violated.

However, many authors defend the idea that contributions have not to be inequality indices (e.g. Shorrocks (1982)) and defend, for instance, the negativity of a given contribution especially when the source is equally distributed. Accordingly, Morduch and Sicular (2002) proposed the principle of uniform addition in the case of natural decompositions. For instance the natural decomposition of the variance (see the Introduction) yields, for source ℓ, an absolute *natural contribution* of $cov(\mathbf{x}^\ell, \mathbf{x}) =: S^\ell$.

Axiom 3.1 Uniform Addition: *For any* $\mathbf{x} \in \mathbb{D}_+$ *and* $\mathbf{y} := (\varepsilon, \ldots, \varepsilon) \in \mathbb{D}_+$, *then*

$$I(\mathbf{x} + \mathbf{y}) \leq I(\mathbf{x}). \tag{UA}$$

Morduch and Sicular (2002) noticed, on the basis of many examples that, the uniform addition does not imply systematically the negativity of a natural contribution:

Proposition 3.2 : Morduch and Secular (2002):

$$I(\cdot) \text{ satisfies } (UA) \nRightarrow S^{\mathbf{y}} \leq 0.$$

Precisely, the authors give examples, showing that the Gini index yields nil contributions whereas the Theil index yields negative contributions. Moreover, the coefficient of variation squared provides either nil or negative contributions with respect to two different analytic forms of contribution one can derive. The authors deplored, when one factor is equally distributed, that natural contributions may be nonnegative. From the Shapley contribution view, the axiom (UA) is just a matter of sub-additivity (super-additivity) games, implying negative (positive) contributions.

Axiom 3.2 Sub-additivity: *For any* $\mathbf{x}, \mathbf{y} \in \mathbb{D}_+$, *then*

$$I(\mathbf{x} + \mathbf{y}) \leq I(\mathbf{x}) + I(\mathbf{y}). \tag{SUB}$$

Indeed, remark in the case where $\mathbf{y}$ is an egalitarian income distribution, that (SUB) implies (UA) since $I(\mathbf{y}) = 0$. It appears that (UA) is a sub-additive income sources inequality game for which the Shapley contributions are always negative:

Lemma 3.1 *For any sub-additive income sources inequality game, that is* $(\mathcal{L}, F_I)^+$ *where* $I(\mathbf{x})$ *satisfies* (SUB) *and where* $\mathbf{x}^\ell = (\varepsilon, \varepsilon, \ldots, \varepsilon)$, *then* $\mathcal{C}^\ell \leq 0$.

Proof.
Obvious. ■

On the contrary, let us expose the super-additivity:

Axiom 3.3 Super-additivity: *For any* $\mathbf{x}, \mathbf{y} \in \mathbb{D}_+$, *then*

$$I(\mathbf{x} + \mathbf{y}) \geq I(\mathbf{x}) + I(\mathbf{y}). \tag{SUP}$$

Accordingly, we get the reverse result:

Lemma 3.2 *For any super-additive income sources inequality game, that is* $(\mathcal{L}, F_I)^+$ *where* $I(\mathbf{x})$ *satisfies* (SUP) *and for any given* $\mathbf{x}^\ell$ *(being an egalitarian distribution or not), then* $I(\mathbf{x})$ *respects* (REL).

Proof.

One has to show by contraposition that $\mathcal{C}^{\ell}(\cdot) < 0$ for all $\ell \in \mathcal{L}$ implies strict sub-additivity. If $\mathcal{C}^{\ell}(\cdot) < 0$, then $F(\mathcal{S} \cup \{\ell\}) - F(\mathcal{S}) < 0$. Note, by definition of an inequality measure that $F(\{\ell\}) \geq 0$ for any given ℓ, thus $F(\mathcal{S} \cup \{\ell\}) < F(\mathcal{S}) + F(\{\ell\})$. ■

We might also examine what happens in the case of a super(sub)-additivity game within a multiplicative framework. We would see, on the contrary, that only super-additivity makes sense. But finally, one sees that the Shapley value is flexible because:

• one can argue with (SUP) that the Shapley contributions has to react exactly as income inequality measures (by the respect of (REL));

• or one can argue with (SUB) that Shapley contributions may be dependent to another axiomatic shape (see the most popular, Shorrocks (1982)) and has to be negative when (UA) is respected. This is furthermore relevant with the literature on the Gini decomposition by factor (e.g. Lerman and Yitzhaki (1985)) where (negative) correlations between sources may decrease the amount of overall inequality. Without making a difference between (SUB) and (SUP), for which future researches may be done to introduce a link with source correlations, one has to see that if an index satisfies (SUB) as the absolute Gini index, then the Shapley contribution for an egalitarian income source distribution will be always nonpositive. This is not true with natural decompositions.

This result is also linked with the nature of the index of inequality being either relative or absolute (and also with the nature of the game being either multiplicative or additive). But, another example showing that the Shapley value is quite flexible is the possibility to reconciliate both relative and absolute measures in a two-source game:

Proposition 3.3 *Let $(\mathcal{L}, F_I)_2^{\times}$ be a non super-additive income inequality game, which is a two sources multiplicative (resp. additive $(\mathcal{L}, F_I)_2^{+}$) income inequality game. If a max-min Value for inequality indices is computed $\forall \ell \in \{h, w\}$ and for any $\mathbf{x}^h = \{\varepsilon, \ldots, \varepsilon\}$ $\forall \varepsilon \in \mathbb{R}_+$*

$$\mathcal{C}^{\ell}(\mathcal{S}; F_I) = \max\left\{\min\left\{\sum_{s=0}^{L-1} \sum_{\mathcal{S} \subseteq \mathcal{L} \setminus \{\ell\}} \frac{(L-1-s)!s!}{L!}\left(F_I\left(\mathcal{S} \cup \{\ell\}\right) - F_I(\mathcal{S})\right); I(\mathbf{x})\right\}; 0\right\}$$

then,

($\imath$) $\mathcal{C}^{\ell}(\mathcal{S}; F_I)$ respects REL *for all $(\mathcal{L}, F_I)_2^{\times}$ and $(\mathcal{L}, F_I)_2^{+}$ and for all $\mathbf{x}^h, \mathbf{x}^w \in \mathbb{D}_+$*

($\imath\imath$) $(\mathcal{L}, F_I)_2^{+} \Longrightarrow \mathcal{C}^h(\mathcal{S}; F_I) = 0$; $\mathcal{C}^w(\mathcal{S}; F_I) = I(\mathbf{x})$ if $I(\mathbf{x})$ satisfies HOM^1 *or* HOM^0.

Proof.

Straightforward. ■

This yields the ability to reconciliate the contribution one may obtain either with absolute or relative inequality measures in the case of a two sources inequality game that is either additive or multiplicative.

In sum, income sources inequality games have to be applied in many cases:

• when the natural decomposition is impossible;

• when the natural decomposition exists but when natural contributions diverge in their values and in their signs;

• and when the uniform addition does not produce a natural negative contribution (since the Shapley value is always negative in the case of additive games).

4 Multiplicative Sub-indices Poverty Games

In this section, focus is put on poverty indices being either additive or not in order to prove, as in the previous Section, that the Shapley approach may help to compute contributions or not. We begin by the additive case for which the application of the Shapley value is not welcome since the contributions are monotonically equivalent to the natural ones. Subsequently, we explore the non-additive poverty indices, with a multiplicative structure, in order to bring out additive Shapley contributions.

4.1 Equivalence Between Additive Decomposition and Shapley Value

In 1991, Foster and Shorrocks demonstrated that a wide class of decomposable indices of poverty may be obtained by defining a property of subgroup consistency and a property of additive decomposition, both properties being closely related. We show that the Shapley rule allowing for capturing contributions, namely the "subgroup poverty game", is not different from these properties.

Notations.

Let $n \in \mathbb{N}^*$ and $\mathbb{D} = \bigcup_{n=1}^{\infty} \mathbb{R}^n$. A poverty index is a function defined with a vector of incomes $\mathbf{x} \in \mathbb{R}_+^n$ and a poverty line $z \in \mathbb{R}_{++}$ such as $P : \mathbb{D}_+ \times \mathbb{R}_{++} \to \mathbb{R}_+$. In the sequel the equivalence relation $\overset{\mathbf{c}}{\Longleftrightarrow}$ will be used to indicate an equivalence between contributions issued from two methods of decomposition.

Compared with the previous Section, the Shapley value enables poverty indices to be expressed as $P(\mathbf{x}(S))$, but in this case we do not search for subsets of factors in $\mathbf{x}$ but subsets of partitions of the overall population in many groups. Decomposable poverty indices by population subgroups enable one to compute the contribution of each group of the population of size $n(\mathbf{x})$ to the overall amount of poverty. Let an economy composed of L groups (being gender, races, etc.) such as $\mathcal{L} = \{1, \ldots, \ell, \ldots, L\}$. The first requirement in poverty decomposition analysis is that poverty indices are weighted average of poverty within each group ℓ, $P(\mathbf{x}_\ell; z)$, of the population:

Definition 4.1 *For any $\ell \in \{1, 2, \ldots, L\}$ and $L \geq 2$ and for any $\mathbf{x}_\ell \in \mathbb{D}_+$, a poverty index is decomposable if,*

$$P(\mathbf{x}_1, \ldots, \mathbf{x}_\ell, \ldots, \mathbf{x}_L; z) = \sum_{\ell=1}^{L} \omega_\ell \; P(\mathbf{x}_\ell; z). \tag{DEC}$$

We next require that poverty indices are not trivial.

Definition 4.2 *For all* $\mathbf{x}, \mathbf{x}' \in \mathbb{D}_+$*, a poverty index is trivial if,*

$$P(\mathbf{x}; z) = P(\mathbf{x}'; z).$$

Example.

The widely used Foster-Greer-Thorbecke (1984, FGT) measures of poverty are:

$$FGT^{\alpha} = \frac{1}{n(\mathbf{x})} \sum_{i=1}^{q} \left(\frac{z - x_i}{z} \right)^{\alpha} \; \forall \alpha > 0, \text{ where } \frac{z - x_i}{z} = 0 \; \forall x_i > z,$$

where q is the number of poor in the population.

In order to show that Shapley contributions are not welcome in the (DEC) case, let us define poverty games as follows:

Definition 4.3 *A subgroup poverty game is a couple* $(\mathcal{L}, F_P)$ *such as:*

$$\mathbf{x} : 2^{\mathcal{L}} \to \mathbb{D}_+ \text{ for all } \mathcal{S} \in 2^{\mathcal{L}}, \; \mathcal{S} \neq \emptyset \implies \mathbf{x}(\mathcal{S}) := (\mathbf{x}_1, \ldots, \mathbf{x}_s), \; s := |\mathcal{S}|.$$

The characteristic Function F_P *for poverty indices is a function of function:*

$$F_P : 2^{\mathcal{L}} \to \mathbb{D}_+ \times \mathbb{R}_{++} \to \mathbb{R}_+ \text{ such as } F_P = P \circ \mathbf{x},$$

where $\mathbf{x}(\emptyset) = 0$ *by convention and for all poverty measures defined such as:*

$$P(\mathbf{x}(\mathcal{S}); z) = \sum_{\ell=1}^{s} \omega_{\ell} \, P(\mathbf{x}_{\ell}; z).$$

Remark 4.1 *It is obvious that both methods provide equivalent contributions:* $(\mathcal{L}, F_P) \overset{c}{\Longleftrightarrow}$ (DEC)*.*

In poverty decomposition analysis a stronger requirement than (DEC) is welcome. Foster and Shorrocks (1991) introduced the subgroup consistency principle, of the following general form:

$$P(\mathbf{x}; z) = f\left(\frac{1}{n(\mathbf{x})} \sum_{i=1}^{n(\mathbf{x})} \phi(x_i) \right) \text{ where } f' \geq 0. \tag{SC}$$

This property is widely used to understand poverty variations. Again, the use of the Shapley value seems to be inappropriate since equivalent contributions are found. For this purpose, let us define subgroup consistent poverty games:

Definition 4.4 *A subgroup consistent poverty game is a couple* $(\mathcal{L}, F_P)^C$ *such as:*

$$\mathbf{x} : 2^{\mathcal{L}} \to \mathbb{D}_+ \text{ for all } \mathcal{S} \in 2^{\mathcal{L}}, \; \mathcal{S} \neq \emptyset \implies \mathbf{x}(\mathcal{S}) := (\mathbf{x}_1, \ldots, \mathbf{x}_s), \; s := |\mathcal{S}|,$$

$$F_P : 2^{\mathcal{L}} \to \mathbb{D}_+ \times \mathbb{R}_{++} \to \mathbb{R}_+ \text{ such as } F_P = P \circ \mathbf{x}, \text{ where } \mathbf{x}(\emptyset) = 0,$$

$$P(\mathbf{x}(\mathcal{S}); z) = f\left[\frac{1}{n(\mathbf{x}(\mathcal{L}))} \sum_{i=1}^{n(\mathbf{x}(\mathcal{S}))} \phi(x_i) \right].$$

Theorem 4.1 *Let f be (positively) homogeneous of degree one, then the Shapley value of the subgroup poverty game* $(\mathcal{L}, F_P)^C \overset{\mathbf{c}}{\Longleftrightarrow}$ (DEC) *if, and only if,* $f(x) = cx$.

Proof.
(*Necessity*). Let $L = 2$ and assume one individual per group. From (DEC), the contribution of group 1 is $\frac{1}{2}P(\mathbf{x}_1; z)$ which is by definition equal to $\frac{1}{2}f[\phi(x_1)]$. From the Shapley value the contribution of group 1 is:

$$\mathcal{C}^1(\mathcal{S}; P_I) = \frac{1}{2}f\left[\frac{1}{2}\phi(x_1) + \frac{1}{2}\phi(x_2)\right] - \frac{1}{2}f\left[\frac{1}{2}\phi(x_2)\right] + \frac{1}{2}f\left[\frac{1}{2}\phi(x_1)\right].$$

If the Shapley value of the subgroup poverty game $(\mathcal{L}, F_P) \overset{\mathbf{c}}{\Longleftrightarrow}$ (DEC) we can equalize the contribution issued from (DEC) and from Shapley, that is:

$$\frac{1}{2}f\left[\frac{1}{2}\phi(x_1) + \frac{1}{2}\phi(x_2)\right] - \frac{1}{2}f\left[\frac{1}{2}\phi(x_2)\right] + \frac{1}{2}f\left[\frac{1}{2}\phi(x_1)\right] = \frac{1}{2}f\left[\phi(x_1)\right].$$

Let $\phi(x_\ell) =: u_\ell$, by homogeneity of degree one, we get:

$$\frac{1}{2}f\left[\frac{1}{2}(u_1 + u_2)\right] = \frac{1}{4}f\left[u_2\right] + \frac{1}{4}f\left[u_1\right].$$

Hence,

$$f\left[\frac{1}{2}(u_1 + u_2)\right] = \frac{1}{2}f\left[u_2\right] + \frac{1}{2}f\left[u_1\right],$$

which is Jensen's equation of solution $f(x) = cx + r$ for some constant r. We then obtain exactly Foster and Shorrock's (1991) result about subgroup consistent poverty indices, that is, f is linear and increasing (by definition). By convention we have $F_P(\{\emptyset\}) = 0$. Let $F_P(\{\emptyset\}) = f(0)$, this implies that $r = 0$ and therefore $f(x) = cx$, which is consistent with positive homogeneity of degree one.
(*Sufficiency*). It follows directly from contraposition. ■

Note that such an approach relies on a poverty index where the average is taken over $n(\mathbf{x}(\mathcal{L}))$. A strict application of the Shapley value (averaging over $n(\mathbf{x}(\mathcal{S}))$ would have been possible but yields nonsensical results, indeed:

Theorem 4.2 *For any* $(\mathbf{x}, z) \in \mathbb{D}_+ \times \mathbb{R}_{++}$ *and* $P(\mathbf{x}(\mathcal{S}); z) = f\left(\frac{1}{n(\mathbf{x}(\mathcal{S}))}\sum_{i=1}^{n(\mathbf{x}(\mathcal{S}))}\phi(x_i)\right)$, $(\mathcal{L}, F_P)^C \overset{\mathbf{c}}{\Longleftrightarrow}$ (DEC) *if, and only if,* $P(\mathbf{x}; z)$ *is trivial.*

Proof.
(*Necessity*). Suppose the equivalence between $(\mathcal{L}, F_P)^C$ and (DEC). Let $L = 2$. Equalizing the contribution of group 1 by both approaches and letting $\phi(x_\ell) =: u_\ell$ yields:

$$\frac{1}{2}f\left[\frac{1}{2}u_1 + \frac{1}{2}u_2\right] = \frac{1}{2}f\left[u_2\right]. \tag{5}$$

Let $u_2 = 0$ then;

$$f\left[\frac{1}{2}u_1\right] = f(0).$$

Let $x := \frac{u_1}{2}$ and $f(0) = c$, thus for all x:

$$f(x) = c.$$

Therefore,

$$f(x) = P(\mathbf{x}; z) = c = f\left(x'\right) = P\left(\mathbf{x}'; z\right).$$

(*Sufficiency*). Use contraposition. ∎

On the basis of the previous results, it is difficult to imagine that the Shapley value constitutes a new direction in the measurement of contributions in the case of additive poverty indices. We could have examined (SUB) or (SUP). In this case, as in practise (DEC) is associated with weights such as $\omega_\ell \leq 1$ for all $\ell \in \mathcal{L}$. It follows that (DEC) $\Longrightarrow$ (SUB), that is, group coalitions enable poverty to be fought (united we stand, divided we fall). This may be convenient to explain Shapley contributions differently, that is, in line with its original perspective. But from a computational purpose, natural contributions and Shapley contributions are identical (see Remark 4.1 and Theorem 4.1).

Let us now examine the non-additive decomposable poverty measures (either in additive or multiplicative games).

4.2 On Measuring the Impact of Sub-Indices

Additive and Multiplicative Games.

We turn to a different problem, for which the Shapley value may provide new directions. Imagine that an index is a function of sub-indices. Taking permutations over a set of sub-indices would help to derive the contribution of each sub-index to the overall index. This is slightly different from our previous problems in which we took permutations over sets of sources (or subgroups throughout the $\mathbf{x}$ vector) to compute the corresponding contributions. Indeed, for an additive (multiplicative) income source inequality game, we took permutations over the source set $\mathcal{L}$ (or the group one) such as:

$$\mathbf{x} : 2^{\mathcal{L}} \to \mathbb{D}_+ \text{ for all } \mathcal{S} \in 2^{\mathcal{L}},\ \mathcal{S} \neq \emptyset \implies \mathbf{x}(\mathcal{S}) := (\mathbf{x}_1, \ldots, \mathbf{x}_s),\ s := |\mathcal{S}|.$$

Now imagine the index is a function such as:

$$I(\mathbf{x}) = f\left(H^1(\mathbf{x}), \ldots, H^\ell(\mathbf{x}), \ldots, H^L(\mathbf{x})\right) =: f\left(\mathbf{x}^H(\mathcal{L})\right),$$

where H^ℓ for all $\ell \in \{1, \ldots, L\}$ are L different indices (for any given functional forms), for some function f and where $\mathcal{L} = \{H^1(\mathbf{x}), \ldots, H^\ell(\mathbf{x}), \ldots, H^L(\mathbf{x})\}$. The problem of taking permutations over a set of sub-indices to determine the impact of the different sub-indices on the overall index $I(\mathbf{x})$ may be viewed as follows:

$$\mathbf{x}^H : 2^{\mathcal{L}} \to \mathbb{D}_+ \text{ for all } \mathcal{S} \in 2^{\mathcal{L}},\ \mathcal{S} \neq \emptyset \implies \mathbf{x}^H(\mathcal{S}) := \left(H^1(\mathbf{x}), \ldots, H^s(\mathbf{x})\right),\ s := |\mathcal{S}|$$

$$F_I : 2^{\mathcal{L}} \to \mathbb{D}_+ \to \mathbb{R}_+ \text{ such as } F_I = f \circ \mathbf{x}^H,\ \mathbf{x}^H(\emptyset) := 0,\ F_I(\{\emptyset\}) := 0$$

where

$$F_I(\mathcal{L}) = f\left(\mathbf{x}^H(\mathcal{L})\right) = I(\mathbf{x}).$$

Intuitively, it follows that the nature of f will establish the structure of the contributions of all sub-indices. For instance, if we define an additive sub-index game as $\left(\mathcal{L}, f_+\left(\mathbf{x}^H\right)\right)$ and a multiplicative sub-index game as $\left(\mathcal{L}, f_\times\left(\mathbf{x}^H\right)\right)$, the contributions issued from both methods do not coincide (except in non desirable cases in which $I(\mathbf{x})$ and $H^s(\mathbf{x})$ take the same values in both games) e.g.:

$$I_A(\mathbf{x}) = H^1(\mathbf{x}) \times H^2(\mathbf{x}) \times H^3(\mathbf{x})\ ;\ I_B(\mathbf{x}) = H^1(\mathbf{x}) + H^2(\mathbf{x}) + H^3(\mathbf{x}),$$

where $H^1(\mathbf{x}), H^2(\mathbf{x}), H^3(\mathbf{x})$ represent three sub-indices. It is clear that $I_A(\mathbf{x})$ and $I_B(\mathbf{x})$ take different values in almost all cases. As mentioned above, there is an obvious equivalence between additivity property such as (DEC) and Shapley contributions. Again, there is an obvious equivalence between the natural contribution computed on $I(\mathbf{x}_B)$ and Shapley contributions. The natural contributions computed on $I(\mathbf{x}_A)$ induces some problem since they do not sum up to $I(\mathbf{x}_A)$. However, the Shapley value gives:

Lemma 4.1 $\left(\mathcal{L}, f_\times\left(\mathbf{x}^H\right)\right)$ *implies the respect of* (SHAP) *and* (REL).

Proof.
Obvious. ■

The result means that the Shapley contributions for a multiplicative sub-index game always lead to contributions that sum up to the overall index. Moreover, Shapley contributions are always positive (respect (REL)) whereas the logarithmic transformation may provide negative contributions.[3]

This result points out two things:

• Shapley contributions are consistent since it is possible to compute the contribution of sub-indices;[4]

• it may be more relevant compared with the log transformation.[5]

Mixture: the Sen-Shorrocks-Thon Index.

Contrary to the previous case, we aim at showing that the use of the Shapley value enables contributions to be computed when sub-indices are linked with a mixture of additive and multiplicative sub-indices. Let n be the number of income units, q the number of poor individuals,

[3] Both methods may produce quite identical contributions. For instance if $I(\mathbf{x}) = 4 \times 5$, the log yields 46.3% for the former and 53.7% for the latter. The (relative) Shapley contributions are 47.5% and 42.5%.

[4] If sub-indices are themselves related to other sub-indices, one has to use a Nested procedure as in Chantreuil-Trannoy (1999) to avoid the problem of hierarchical structure.

[5] A logarithmic transformation may be welcome, but it may violate some axioms, e.g. (NM). For instance for $\mathbf{x} = (0, 0, \ldots, 0)$ it is impossible to respect (NM). See also Xu and Osberg (2001) for the natural decomposition of the SST index and the use of log.

and z the poverty line. The overall poverty rate is $h = \frac{q}{n}$. Let x_i be the income of the ith poor person. The poverty gap ratio (sometime called relative poverty gap or poverty gap) is:

$$y_i = \begin{cases} \frac{z-x_i}{z}, & \forall z > x_i \\ 0, \end{cases} \tag{6}$$

for all q poor individuals. The arithmetic mean of the poverty gap ratio is:

$$\overline{y}_p = \sum_{i=1}^{q} \frac{1}{q} y_i.$$

Let $\mathbf{y}$ the vector of all poverty gap ratio for all q poor people. The Sen-Shorrocks-Thon (SST)[6] index is then:

$$SST = h\overline{y}_p \left(1 + G(\mathbf{y})\right) = f\left(h, \overline{y}_p\ , G(\mathbf{y})\right), \tag{7}$$

where $G(\mathbf{y})$ is the Gini index computed on poverty gap ratios, measuring the degree of inequalities of poverty. How can we capture the contribution of the sub-indices that account for SST as a whole ? Instead of looking for permutations over $\mathcal{L} = \{h,\ \overline{y}_p,\ (1 + G(\mathbf{y}))\}$ we seek to get permutations over the set $\mathcal{L}_1 = \{h,\ \overline{y}_p\}$ and over $\mathcal{L}_2 = \{h,\ \overline{y}_p\ ,\ G(\mathbf{y})\}$ (of cardinality L_1 and L_2) since $SST = h.\overline{y}_p + h.\overline{y}_p.G(\mathbf{y})$. Let us define two multiplicative sub-index games $\left(\mathcal{L}_i, f^i_\times\left(\mathbf{x}^H_i\right)\right)$ such as $\forall i \in \{1, 2\}$:

$$\mathbf{x}^H_i : 2^{\mathcal{L}_1} \to \mathbb{D}_+ \text{ for all } \mathcal{S}_i \in 2^{\mathcal{L}_i},\ \mathcal{S}_i \neq \emptyset \implies \mathbf{x}^H_i(\mathcal{S}_i) := \left(H^1_i(\mathbf{x}), \ldots, H^{s_i}_i(\mathbf{x})\right),\ s_i := |\mathcal{S}_i|$$

with $H^1_1(\mathbf{x}) = h$, $H^2_1(\mathbf{x}) = \overline{y}_p$ and $H^1_2(\mathbf{x}) = h$, $H^2_2(\mathbf{x}) = \overline{y}_p$, $H^3_2(\mathbf{x}) = G(\mathbf{y})$ such as:

$$F_{I_i} : 2^{\mathcal{L}}_i \to \mathbb{D} \to \mathbb{R}_+ \text{ such as } F_{I_i} = f^i \circ \mathbf{x}^H_i,\ \mathbf{x}^H_i(\emptyset) := 0,\ F_{I_i}(\{\emptyset\}) := 0\ \forall i \in \{1, 2\}.$$

In particular:

$$F_{I_1}(\mathcal{L}_1) = f^1\left(\mathbf{x}^H_1(\mathcal{L}_1)\right) = I^1(\mathbf{x}) = h.\overline{y}_p$$

and

$$F_{I_2}(\mathcal{L}_2) = f^2\left(\mathbf{x}^H_2(\mathcal{L}_2)\right) = I^2(\mathbf{x}) = h.\overline{y}_p.G(\mathbf{y}).$$

Proposition 4.1 *The SST index is decomposable according to two multiplicative sub-index games* $\left(\mathcal{L}_1, f^1\left(\mathbf{x}^H_1\right)\right)$ *and* $\left(\mathcal{L}_2, f^2\left(\mathbf{x}^H_2\right)\right)$.

Proof.
Applying the multiplicative sub-index game on each index $I^i(\mathbf{x})\ \forall i \in \{1, 2\}$ provides the Shapley contribution of each sub-index to the indices $I^i(\mathbf{x})$. Following the additivity property of the Shapley value it is possible to apply the Shapley value on each additive part of the SST index (instead of the overall index). It then follows that the contribution of the headcount ratio h to $I^1(\mathbf{x})$ and $I^2(\mathbf{x})$ is, respectively:

$$\mathcal{C}^h_{I_1} = \sum_{s_1=0}^{L_1-1} \sum_{\mathcal{S}_1 \subseteq \mathcal{L}_1 \setminus \{h\}} \frac{(L_1 - 1 - s_1)!s_1!}{L_1!} \left[F_{I_1}\left(\mathcal{S}_1 \cup \{h\}\right) - F_{I_1}\left(\mathcal{S}_1\right)\right]$$

[6] See Sen (1976), then the modification provided by Shorrocks (1995) that coincides with Thon's index (1979).

$$\mathcal{C}_{I_2}^h = \sum_{s_2=0}^{L_2-1} \sum_{\mathcal{S}_2 \subseteq \mathcal{L}_2 \setminus \{h\}} \frac{(L_2 - 1 - s_2)!s_2!}{L_2!} \left[F_{I_2}\left(\mathcal{S}_2 \cup \{h\}\right) - F_{I_2}\left(\mathcal{S}_2\right)\right].$$

This entails that the aggregate contribution of h to the overall amount of the SST index is:

$$\mathcal{C}_{SST}^h = \mathcal{C}_{I_1}^h + \mathcal{C}_{I_2}^h.$$

The same demonstration holds to find the contribution of the average poverty gap ratio $\overline{y}_p$ and the contribution of the Gini index of poverty gaps $G(\mathbf{y})$, respectively:

$$\mathcal{C}_{SST}^{\overline{y}_p} = \mathcal{C}_{I_1}^{\overline{y}_p} + \mathcal{C}_{I_2}^{\overline{y}_p}\ ,\ \mathcal{C}_{SST}^{G(\mathbf{y})} = \mathcal{C}_{I_1}^{G(\mathbf{y})} + \mathcal{C}_{I_2}^{G(\mathbf{y})}.$$

The SST index is then additively decomposed into three contributions representing the three sub-indices:

$$SST = \mathcal{C}_{SST}^h + \mathcal{C}_{SST}^{\overline{y}_p} + \mathcal{C}_{SST}^{G(\mathbf{y})}.$$

■

5 Conclusion

New directions to be taken in income inequality measurement, in particular *via* the use of Shapley contributions, is synonymous of either computing Shapley values in many circumstances or not in others.

For instance, the paper highlights, in the case of subgroup consistency poverty indices, that the use of the Shapley value is not welcome for the computation of contributions. Although group coalitions yields an alleviation of poverty in accordance with the cooperative game theory maxim "united we stand, divided we fall", Shapley contributions are identical to those derived from subgroup consistency or decomposable poverty indices. On the contrary, for any non additive poverty indices such as SST, the technique may help to compute the contribution of sub-indices.

In the case of inequality measures, the potential of the Shapely is rather immediate. As can be seen with the max min-Value, there exists some direction to reconciliate absolute and relative inequality measures in order to make their contributions identical since the sign of Shapley contributions may diverge with respect to the homogeneity type of the index. In particular, we have shown the close interrelation between homogeneity of degree zero, uniform additions and sub-additive income sources inequality games. Sub-additivity implies negative Shapley contributions of egalitarian income source distributions (the respect of uniform addition) whereas super-additivity may produce Shapley contributions, which are inequality indices respecting almost all axioms. An income source game is either additive or multiplicative since sources are additively or multiplicatively linked. Once again, the Shapley contributions may be seen as inequality indices because they are never negative if the game is multiplicative and the index homogeneous of degree one. On the other hand, Shapley contributions may be seen as indices that rely on a different axiomatic shape for which contributions may be negative when the index is homogeneous of degree zero (Chantreuil and Trannoy (1999)).

The fact that contributions may be inequality indices (Chantreuil and Trannoy (1999)) or built on a different axiomatic shape (Shorrocks (1982)) is relevant with the Shapley value. Moreover, income sources inequality games being either additive and multiplicative may be applied in many cases:

• when the natural decomposition is impossible;

• when the natural decomposition exists but when natural contributions diverge in their values and in their signs;

• and when the uniform addition does not produce a natural negative contribution.

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