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Partial Multidimensional Inequality Orderings

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Abstract:
The paper investigates how comparisons of multivariate inequality can be made robust to varying the intensity of focus on the share of the population that are more relatively deprived. It follows the dominance approach to making inequality comparisons, as developed for instance by Atkinson (1970), Foster and Shorrocks (1988) and Formby, Smith, and Zheng (1999) in the unidimensional context, and Atkinson and Bourguignon (1982) in the multidimensional context. By focusing on those below a multidimensional inequality "frontier", we are able to reconcile the literature on multivariate relative poverty and multivariate inequality. Some existing approaches to multivariate inequality actually reduce the distributional analysis to a univariate problem, either by using a utility function first to aggregate an individual's multiple dimensions of well-being, or by applying a univariate inequality analysis to each dimension independently. One of our innovations is that unlike previous approaches, the distribution of relative well-being in one dimension is allowed to affect how other dimensions influence overall inequality. We apply our approach to data from India and Mexico using monetary and non-monetary indicators of well-being.

Keywords: Inequality, multidimensional comparisons, stochastic dominance

JEL Classification: D3, I3

1 Introduction

In a recent review of the literature on multivariate inequality measurement, Weymark (2004) concludes that "(...) Although much has already been learned about multidimensional normative inequality indices, much more remains to be discovered. Compared to the theory of univariate inequality measurement, the analysis of multidimensional inequality is in its infancy." (p.29) This paper is a contribution to that young literature.

The method we develop and empirically apply is very much in the spirit of Amartya Sen's conceptual framework for thinking of inequality. In that framework, it is important to make clear at the outset what one is comparing across individuals (Sen 1982). The paper adopts the view that there can be several dimensions to well-being, and that comparisons of well-being across individuals should therefore be multidimensional. In addition, in making normative judgements on distributions of well-being, it is important to make explicit the ethical norms that are used. This has been clearly argued in the context of measuring both inequality and poverty (Sen 1973 and Sen 1976). The paper makes these judgements explicit by using classes of multidimensional inequality indices that are defined on the basis of explicit normative criteria.

In doing this, we build on the dominance approach to making inequality comparisons, as developed for instance by Atkinson (1970), Foster and Shorrocks (1988) and Formby, Smith, and Zheng (1999) in the unidimensional context, and Atkinson and Bourguignon (1982) in the multidimensional context. One advantage of this approach is that it is capable of generating inequality orderings that are robust over broad classes of inequality indices and over broad classes of aggregation rules across dimensions of well-being.

We start with the framework for multivariate *poverty* comparisons developed in Duclos, Sahn, and Younger (2006). Two modifications of that approach make it suitable to inequality analysis. First, rather than consider absolute values of multiple measures of well-being, we normalize them by a reference value, usually their mean. The robust poverty comparisons of Duclos, Sahn, and Younger (2006) thus become robust *relative* poverty comparisons. These can be of interest in their own right (see for instance Sen 1983), but they also permit sensible analysis of inequality if the poverty lines are allowed to span a suitably large range, extending beyond the least deprived people in the population.

Second, by taking a relative poverty approach we are able to focus on "downside" inequality aversion. Specifically, we consider inequality indices that can give greater weight to those positioned at the bottom of the well-being distribution. This is done by focusing our comparisons on individuals below a multidimensional poverty "frontier" that functions like a poverty line in a single dimension: people beyond this frontier do not contribute to relative poverty. In this approach, inequality is the limiting case of relative poverty, the case in which the poverty frontier is so far from the origin that everyone's relative position in the well-being distribution can have an impact on relative poverty. We show how our orderings can also be considered to be "frontier-robust".

It is of course possible to think of making multidimensional relative poverty comparisons by performing univariate comparisons independently for each dimension of well-being. But that does not allow the level of well-being in one dimension to influence our assessment of how other dimensions affect overall relative poverty, something that we argue any reasonable multivariate comparison should consider. Thus, an important feature of the inequality and relative poverty tests we develop is that they take into account the dependence between two measures of well-being when making multivariate comparisons. This will be important when that dependence is stronger for one population than it is for another. In such cases, univariate comparisons carried out in each dimension separately can yield results that differ from the genuine multivariate comparisons developed here. For example, population A may have lower univariate relative poverty than population B for two measures of well-being χ and ξ. But if A also has a greater dependence between χ and ξ, then it may also have *higher* bivariate relative poverty than B despite the univariate differences. And of course, the opposite is also true. In practice, we find that one-at-a-time univariate comparisons conclude that one population has lower inequality than another too easily, and that it is relatively rare to find greater multivariate inequality in A than in B when we do not find greater univariate inequality in A than in B.

It is ethically important to suppose that the dependence between dimensions of well-being matters. In particular, someone who is relatively worse-off in terms of ξ contributes more to the relative poverty and inequality measure if he is also relatively worse-off in terms of χ. Without this conviction, one could just as well study each dimension of well-being separately.

To highlight further the importance of the dependence between multiple measures of well-being, one of the key theoretical results of our paper is that if population B has a greater covariance between the two dimensions than population A does, it is impossible for B to dominate A at first or second order over the entire domain of the joint distribution of relative well-being. Thus, it is impossible to draw a robust conclusion that B has lower inequality than A at first and second orders, regardless of the dispersions of the marginal distributions, though conclusions about relative poverty may still be possible. This gives a decisive role to the covariance between dimensions of well-being; it is analogous to the role of the mean in univariate generalized Lorenz comparisons, where a distribution with a lower mean cannot stochastically dominate one with a greater mean regardless of the dispersions of the two distributions (Shorrocks 1983).

We also extend our approach to making robust inequality comparisons both in a relative and in an absolute sense. Relative inequality comparisons involve comparing the ratios of income to average income across individuals. Absolute inequality comparisons involve comparing distances between incomes and mean income across individuals. As mentioned above, relative inequality comparisons can be thought of as a limiting case of relative poverty comparisons, namely, when the relative poverty frontier extends over everyone in the population. Absolute inequality comparisons can be thought of as a limiting case of absolute poverty comparisons, when the poverty frontier extends beyond the least deprived individuals in the population and when deprivation is measured by the absolute income difference with the mean.

To gain a better understanding of how our proposed comparisons work in practice, we apply them to several simulated distributions and also to two diverse sources of data and different dimensions of well-being: the 1999 Demographic and Health Survey from India, where we rely on an important health indicator, the hemoglobin concentration (g/Dl) of women aged 15-49, and an index of the assets owned by those women's households; and from Mexico's 2008 *Evaluación Nacional del Logro Académico de Centros Escolares* (National Evaluation of Academic Attainment in High Schools), where the indicators of well-being involve achievement tests administered to all high school students in Mexico, in addition to an asset index constructed in a similar manner to the India data. These applications are of considerable interest since little is known empirically about multidimensional inequality rankings. In order to add further relevance to our empirical work, we provide the sampling distribution of the estimators that are needed to make inferences about the true population rankings.

The stochastic dominance tests that we use yield very strong results: if we can reject the null of non-dominance, we can conclude that one population has greater inequality for broad classes of inequality measures that include arbitrary aggregation across dimensions of well-being and across individuals. As a result, we should expect that it is relatively difficult to reject the null. In fact, for many of the distributions that we study, both simulated and real, this is the case. Compared to our previous work on multivariate poverty comparisons, we find it significantly more difficult to reject the null of non-dominance. One reason for this is that poverty can be lower in one population than another either because it has greater means or because it has lower dispersions in the marginal distributions, while relative poverty or inequality comparisons can differ only due to greater dispersions. But an equally important reason, highlighted in our simulations, is that the covariance between dimensions of well-being assumes an overwhelming importance as we extend the test domain to the least deprived observations in the sample. Poverty comparisons do not usually consider these wealthier people (by the focus axiom), but inequality comparisons must do so. In many cases, it will be possible to draw robust conclusions for a (relative) poverty comparison, but not for an inequality comparison, because of the increased role that the covariances play for such comparisons.

2 Multiple indicators of relative well-being

As with several other papers in the literature, we will for simplicity mostly focus on the 2-dimensional case. Let x and y then be two indicators of individual well-being *normalized* by some norm. These indicators could be, for instance, income, expenditures, calorie consumption, height, cognitive ability, *etc.*, normalized by what is deemed to be enjoyed by a representative individual in a society. As we will discuss below, these indicators of normalized well-being can be obtained by taking the distance between *non-normalized* indicators of well-being χ and ξ and some norm for each, yielding x and y, respectively.

One alternative to thinking of x and y as *two* indicators of normalized individual well-being is to define a function $U(\chi, \xi)$ that aggregates χ and ξ into an overall measure of individual well-being, and think of the distance between this and a norm defined in units of overall well-being. This approach is fairly common in the (limited) literature on multivariate inequality measures — see for instance Weymark (2004) for a discussion. In essence, it reduces the multivariate problem to a more familiar univariate one, but at a significant cost: it requires specifying a particular definition of $U(\chi, \xi)$, something that is necessarily arbitrary. By avoiding such "two-step" aggregations, our approach provides more general inequality comparisons.

Let the distribution of these two indicators χ, ξ in the population be given by an $n \times 2$ matrix denoted as A, where n is the number of individuals. Let the domain of admissible distributions be denoted as Ξ. We will represent inequality indices by P_A for inequality in A. For any $A, B \in \Xi$, we will therefore say that A is more unequal than B according to index P if and only if $P_A > P_B$. We also need two alternative definitions of inequality indices.

Definition 1 *P is a strongly relative inequality index if and only if $P_A = P_{A\Gamma}$ for all 2×2 diagonal matrices Γ with elements $\gamma_{ii} > 0$ ($i = 1, 2$) for which A and $A\Gamma$ are both members of Ξ.*

Definition 1 is analogous to the scale invariance axiom in univariate inequality analysis. Let **1** be a matrix whose entries are all equal to 1.

Definition 2 *P is a strongly translatable inequality index if and only if $P_A = P_{A+\mathbf{1}\Gamma}$ for all 2×2 diagonal matrices Γ for which A and $A+\mathbf{1}\Gamma$ are both members of Ξ.*

Definition 2 is also analogous to the translation invariance axiom for the analysis of univariate inequality. These definitions are in the spirit of those found in Tsui (1995) — see also Weymark (2004). But unlike the univariate case, they may not be uniformly acceptable. For instance, we may well feel that, if everyone's education level is doubled, the contribution of other indicators (such as health or income) to overall inequality should be affected. In that case, we might not want to use strongly relative inequality indices since these indices will remain invariant to such a change.

The above nevertheless suggests that we can think of at least two types of normalizations to each indicator of well-being. The first normalization (Definition 1) is of a relative type, obtained by a scaling of the indicator by an arbitrary value, and the second type of normalization (Definition 2) is absolute and is obtained by a translation of the indicators by an arbitrary value. The mean is an obvious candidate for these arbitrary values in the context of inequality comparisons, but other distribution-dependent statistics (such as the median or the mode) could also be applied.

To implement these mean-normalization procedures, we can use distances between indicators of well-being and their population mean (for absolute inequality comparisons) or the same distances but divided by the mean (for relative inequality comparisons). For an indicator χ of non-normalized well-being with mean μ_χ, let then

$$x_\rho = \rho \left(\frac{\chi - \mu_\chi}{\mu_\chi} \right) + (1-\rho)\left(\chi - \mu_\chi\right). \qquad (1)$$

Absolute inequality in χ can be assessed by using x_0, and relative inequality by using x_1 (and similarly for another indicator of relative well-being, y_ρ, defined by replacing χ by ξ in (1)). The use of x_0 and x_1 in indices of inequality will make the indices strongly translatable and strongly relative in x, respectively (and similarly for y_0 and y_1 with respect to ξ). Intermediate inequality in χ and ξ can be assessed by using $0 < \rho < 1$. For expositional simplicity, we will however sometimes omit the indices ρ from x_ρ and y_ρ.

We then assume that we wish to compute an aggregate index of inequality based on the distribution of x and y. Denote by

$$\lambda(x,y) : \Re^2 \rightarrow \Re \left| \frac{\partial \lambda(x,y)}{\partial x} \leq 0, \frac{\partial \lambda(x,y)}{\partial y} \leq 0 \right. \tag{2}$$

a summary measure of the degree of relative deprivation of an individual. Note that the derivative conditions in (2) mean that different indicators can each contribute to decreasing overall deprivation. We make the differentiability assumptions for expositional simplicity, but they are not strictly necessary so long as $\lambda(x,y)$ is non-decreasing over x and y.

We may wish to focus on those with the greatest degree of deprivation.This can be done by drawing an inequality frontier separating those with lower and those with greater relative deprivation. We can think of this frontier as a series of points at which overall relative deprivation is kept constant at a critical value. This frontier is assumed to be defined implicitly by a locus of the form $\lambda(x,y) = 0$, and is analogous to the usual downward-sloping indifference curves in the (x,y) space. As in the poverty literature, (x,y) values that lie beyond this frontier do not contribute to aggregate relative deprivation. Thus, to obtain an inequality measure in the usual sense, the frontier would need to be set beyond the most extreme values of x and y. The set of those over whom we want to aggregate relative deprivation is then obtained as:

$$\Lambda(\lambda) = \{(x,y) \,|(\lambda(x,y) \geq 0\}\,. \tag{3}$$

Consider Figure 1 with thresholds z_x and z_y in dimensions of indicators x and y. The dotted $\lambda_1(x,y)$ line gives an "intersection" frontier: it considers someone to be relatively deprived only if he is deprived in *both* of the two dimensions of x and y, and therefore if he lies within the dashed rectangle of Figure 1. $\lambda_2(x,y)$ (the L-shaped, dashed line) gives a "union" frontier: it considers someone to be relatively deprived if he is deprived in *either* of the two dimensions, and therefore if he lies below or to the left of the dotted line. Finally, the continuous $\lambda_3(x,y)$

line provides an intermediate approach. Someone can be relatively deprived even if $y > z_y$, if his x value is sufficiently low to lie to the left of $\lambda_3(x, y) = 0$.

To define multidimensional inequality indices more precisely, let the joint distribution of x and y be denoted by $F(x, y)$. For analytical simplicity, we focus on classes of inequality indices that are additive across individuals. An additive inequality index that combines the two dimensions of well-being can be defined generally as $P(\lambda)$,

$$P(\lambda) \;=\; \int\int_{\Lambda(\lambda)} \pi(x, y; \lambda)\, dF(x, y), \tag{4}$$

where $\pi(x, y; \lambda)$ is the contribution to inequality of an individual with relative well-being indicators x and y. By the definition of the inequality frontier, we have that

$$\pi(x, y; \lambda) \begin{cases} \geq 0 \text{ if } \lambda(x, y) \geq 0 \\ = 0 \text{ otherwise.} \end{cases} \tag{5}$$

The π function in equation (5) is thus the weight that the inequality measure attaches to someone who is inside the inequality frontier. That weight could be 1 (for a count of how many are relatively deprived), but it could take on many other values as well, depending on the inequality measure of interest.

A *bi-dimensional dominance surface* can now be defined as:

$$D^{\alpha_x, \alpha_y}(z_x, z_y) = \int_0^{z_y} \int_0^{z_x} (z_x - x)^{\alpha_x} (z_y - y)^{\alpha_y}\, dF(x, y) \tag{6}$$

for integers $\alpha_x \geq 0$ and $\alpha_y \geq 0$. This dominance surface aggregates products of distances between indicators x and y and thresholds z_x and z_y — these distances are usually referred to as poverty gaps in the poverty literature. We can also rewrite (6) as

$$D^{\alpha_x, \alpha_y}(z_x, z_y) = \int_0^{z_x} (z_x - x)^{\alpha_x} \left[\int_0^{z_y} (z_y - y)^{\alpha_y}\, dF(y|x) \right] dF(x), \tag{7}$$

where $F(x)$ is the univariate (or marginal) distribution function of x and $F(y|x)$ is the distribution of y conditional on x. This says that the bivariate dominance curve can be thought of as the integral of the univariate dominance curves for y, conditional on x, weighted by the gaps in x, $\int_0^{z_x} (z_x - x)^{\alpha_x} dF(x)$.

We generate the dominance surface by varying the values of z_x and z_y over an appropriately chosen domain, with the height of the surface given by equation (6). In particular, if the domain of the integration is the entire (x, y) plane, then $D^{\alpha_x,\alpha_y}(z_x, z_y)$ qualifies as a usual measure of inequality. $D^{0,0}(z_x, z_y) = F(z_x, z_y)$ generates a bivariate cumulative density function of relative well-being. Note also that (6) is a two-dimensional generalization of the FGT index (Foster, Greer, and Thorbecke 1984) defined over gaps of relative well-being.

An important feature of the dominance surface is that it is influenced by the covariance between x and y, the two indicators of normalized well-being, because the integrand is multiplicative. Rewriting (6), we find indeed that

$$D^{\alpha_x,\alpha_y}(z_x, z_y) = \int_0^{z_y} (z_x - x)^{\alpha_x} \, dF(x) \int_0^{z_x} (z_y - y)^{\alpha_y} \, dF(y) + \text{cov}\left((z_x - x)^{\alpha_x}, (z_y - y)^{\alpha_y}\right). \tag{8}$$

The height of the dominance surface is therefore the product of the two unidimensional curves plus the covariance in the poverty gaps in the two dimensions. Thus, the higher the correlation between x and y, the higher the dominance surfaces, other things being equal.

Equation (8) highlights the critical importance of the covariance between dimensions of relative well-being in the two populations. For first- and second-order comparisons ($\alpha_x = 0, 1; \alpha_y == 0, 1$), the integrals in the first term on the right-hand side are equal for all distributions when the values of z_x and z_y are beyond the highest values of x and y in the populations. In this region, the dominance surfaces can differ only if the covariances between the poverty gaps in each dimension differ across the populations. This is true even if the univariate distributions are significantly more unequal in one population.

3 Dominance conditions

Our inequality comparisons make use of orders of dominance, s_x and s_y in the x and in the y dimensions, which will correspond respectively to $s_x = \alpha_x + 1$ and $s_y = \alpha_y + 1$. The parameters α_x and α_y also capture the aversion to inequality in the x and in the y dimensions, respectively.

To describe the class of inequality measures for which the dominance surfaces defined in equation (6) are sufficient to establish multidimensional inequality orderings, assume first that π in (4) is left differentiable[1] with respect to x and y over the set $\Lambda(\lambda)$. Denote by π^x the first derivative[2] of $\pi(x, y; \lambda)$ with respect to

[1] This differentiability assumption is made for expositional simplicity. It could be relaxed.

[2] The derivatives include the implicit effects of x and y on $\lambda(x, y)$.

x; by π^y the first derivative of $\pi(x,y;\lambda)$ with respect to y; by π^{xy} the derivative of $\pi(x,y;\lambda)$ with respect to both x and y; and treat similar expressions accordingly.

Let λ^+ be an uppermost inequality frontier, *i.e.*, a frontier that encompasses all of those individuals whose normalized well-being could eventually enter into $P(\lambda)$. We can then define the following two classes of bidimensional relative poverty indices:

$$\Pi^{1,1}(\lambda^+) = \left\{ P(\lambda) \;\middle|\; \begin{array}{l} \Lambda(\lambda) \subset \Lambda(\lambda^+); \\ \pi(x,y;\lambda) = 0, \text{ whenever } \lambda(x,y) \geq 0; \\ \pi^x(x,y;\lambda) \leq 0 \text{ and } \pi^y(x,y;\lambda) \leq 0 \; \forall x,y; \\ \pi^{xy}(x,y;\lambda) \geq 0, \; \forall x,y; \end{array} \right\} \tag{9}$$

and

$$\Pi^{2,2}(\lambda^+) = \left\{ P(\lambda) \;\middle|\; \begin{array}{l} P(\lambda) \in \Pi^{1,1}(\lambda^+); \\ \pi^x(x,y;\lambda) = 0, \pi^y(x,y;\lambda) = 0, \pi^{xx}(x,y;\lambda) = 0, \\ \pi^{yy}(x,y;\lambda) = 0, \text{ and } \pi^{xy}(x,y;\lambda) = 0 \text{ when } \lambda(x,y) = 0; \\ \pi^{xyy}(x,y;\lambda) \leq 0 \text{ and } \pi^{xxy}(x,y;\lambda) \leq 0, \; \forall x,y; \\ \text{and } \pi^{xxyy}(x,y;\lambda) \geq 0, \; \forall x,y. \end{array} \right\} \tag{10}$$

The conditions for membership in $\Pi^{1,1}(\lambda)$ require that the inequality indices be decreasing in both x and y. They also demand that this decrease be stronger the lower the level of the other dimension of relative well-being: $\pi^{xy}(x,y;\lambda) \geq 0$. This is equivalent to an assumption of "substitutability" between the dimensions of well-being. We return below to the interpretation and the role of this assumption.

Note that $\pi^{xx}(x,y;\lambda) \geq 0 \; \forall x,y$ and $\pi^{yy}(x,y;\lambda) \geq 0 \; \forall x,y$ are conditions that are implied by the continuity conditions $\pi^{xx} = 0$ and $\pi^{yy} = 0$ at the frontier and by the conditions $\pi^{xyy}(x,y;\lambda) \leq 0$ and $\pi^{xyy}(x,y;\lambda) \leq 0$. The conditions for membership in $\Pi^{2,2}(\lambda)$ thus require that the inequality indices be convex in both x and y, and that they therefore obey the principle of transfers in both of these dimensions. This assumption seems more natural in an inequality context than in a welfare/absolute poverty context. It would indeed seem to make sense that overall multidimensional equality be monotonically increasing in the equality of either dimension, everything else being the same. The conditions for $\Pi^{2,2}(\lambda)$ also require that the transfer principle be more important in one dimension of relative well-being the lower the level of the other dimension of relative well-being. Finally, they also impose that the second-order derivative in one dimension of well-being be convex in the level of the other indicator of well-being. This is equivalent to saying that the concern for inequality in one dimension is convex in the level of the other indicator. This is a regularity condition that demands that equalizing transfers in the x dimension become progressively less important as the value of y is increased.[3] We also return to this below.

To see how this differs from another popular definition of convexity, we can introduce the following definition.

Definition 3 *(Bistochastic majorization as a multi-attribute version of the Pigou-Dalton transfer, Weymark 2004) For all $A, B \in \Xi$ for which $A \neq B$, A is more unequal than B if $B = \Sigma A$ for some $n \times n$ bistochastic matrix Σ that is not a permutation matrix.*

As Savaglio (2006) writes, this definition "(...) is a sort of *decomposability* property, which allows [orderings] to be coherent with an inequality measurement via an additive evaluation function" (p.90) — see also Dardanoni (1995). Figure 2 illustrates, however, how a bi-stochastic transformation can increase inequality in well-being as measured by a utility function U. Assume that an initial distribution A of well-being is made of points a and d. Assume also that a bi-stochastic transformation moves point a to point b and point d to point c in order to generate a new distribution B of well-being made of points b and c. The bistochastic transformation moves point a and d closer to the center (given by $\overline{x}$ and $\overline{y}$) in both dimensions at the same time and at the same rate.

Overall well-being (or utility $U(x, y)$) was the same at U^0 for each point a and d in A initially; now it is lower for individual b (at U^1) than for individual c (larger than U^1). Reducing inequality simultaneously and equi-proportionately in each dimension at the same time thus increases inequality in the U dimension. This therefore suggests that a bi-stochastic transformation of A into B might lead to *greater* inequality in B.

The conditions for membership in $\Pi^{2,2}(\lambda)$ fortunately do not impose the bistochastic majorization condition. They only imply that inequality should fall if, as in Figure 3, points $\{a, c\}$ were moved (simultaneously and at the same speed) towards point b, or if points $\{a, g\}$ were moved towards point d, or if points $\{c, i\}$ were moved towards point f. These properties follow from the signs of the

[3] The classes $\Pi^{1,1}$ and $\Pi^{2,2}$ are reminiscent of the classes of welfare functions used by Atkinson and Bourguignon (1982). Atkinson and Bourguignon (1982) nevertheless allow for possibly different signs for $\pi^{xy}(x, y)$ and $\pi^{xxyy}(x, y)$ since they also consider the case of functions that show "complementarity" in indicators. They do not, however, allow for $\Lambda(\lambda^+)$ to exclude anyone.

second-order derivatives, $\pi^{xx}(x, y; \lambda) \geq 0$ and $\pi^{yy}(x, y; \lambda) \geq 0\ \forall x, y$. The conditions for membership in $\Pi^{2,2}(\lambda)$ do not, however, require that inequality should fall if points $\{a, i\}$ were moved closer together towards point e — this would be implied, however, by the bistochastic majorization principle.

The sign of the third-order derivatives, $\pi^{xyy}(x, y; \lambda) \leq 0\ \forall x, y$, also imply that the fall in inequality (as measured by members of the $\Pi^{2,2}(\lambda)$) will be larger if points $\{a, g\}$ are moved towards point d than if points $\{c, i\}$ are moved towards point f. Similarly, the conditions for membership in $\Pi^{2,2}(\lambda)$ also require that the fall in inequality will be larger if points $\{g, i\}$ are moved towards point h than if points $\{a, c\}$ are moved towards point b. Furthermore, the condition that $\pi^{xxyy}(x, y; \lambda) \geq 0,\ \forall x, y$ implies that replacing in Figure 3 points $\{g, i\}$ and $\{e, e\}$ by points $\{h, h\}$ and $\{d, f\}$, respectively, will reduce inequality by more than if points $\{d, f\}$ and $\{b, b\}$ are replaced by points $\{e, e\}$ and $\{a, c\}$.

The inequality impact of the correlation between attributes, as captured by the $\pi^{xy}(x, y; \lambda) \geq 0$ condition, is also important. We can illustrate this in three different ways:

1. First, if we were to replace points $\{c, g\}$ by points $\{a, i\}$ on Figure 3, then bivariate inequality would need to fall, though univariate inequality would remain unchanged.

2. Second, a movement from points $\{a, i\}$ to points $\{d, f\}$ would decrease univariate inequality in the y dimension and would leave univariate inequality in the x dimension unchanged. A movement from points $\{a, i\}$ to points $\{d, f\}$ would, however, not necessarily decrease bivariate inequality since such a movement would increase the correlation between the attributes.

3. Third, moving point i beyond f towards c, and moving point a beyond d towards g, will eventually *increase* bivariate inequality, since $\{c, g\}$ is less equal than $\{a, i\}$. This is despite the fact that univariate inequality in both of the x and y dimensions never increases (and sometimes *falls*) in that movement.

Note that this substitutability assumption is probably more defendable in a multidimensional inequality context than in a multidimensional poverty context. It would seem indeed that replacing points $\{c, g\}$ by points $\{a, i\}$ on Figure 3 should almost certainly reduce relative welfare disparities between the individuals, although there might be situations in which that change might increase absolute poverty — see for instance the discussion in Duclos, Sahn, and Younger (2006).

Let then $\Delta P(\lambda) = P_A(\lambda) - P_B(\lambda)$ and $\Delta D^{s_x-1,s_y-1}(z_x, z_y) = D_A^{s_x-1,s_y-1}(z_x, z_y) - D_B^{s_x-1,s_y-1}(z_x, z_y)$. This leads to the following dominance relationships for $s_x = s_y = 1, 2$:

Theorem 1 **(Π^{s_x,s_y} relative poverty dominance)**

$$\Delta P(\lambda) > 0, \ \forall P(\lambda) \in \Pi^{s_x,s_y}(\lambda^+)$$
$$\textit{iff}\ \Delta D^{s_x-1,s_y-1}(z_x, z_y) > 0, \ \forall (z_x, z_y) \in \Lambda(\lambda^+). \tag{11}$$

Proof: See appendix. ■

Theorem 1 says that it is possible to order relative poverty across distributions A and B by checking whether condition (11) holds. If condition (11) holds, then relative poverty is larger in A than in B for all of the relative poverty indices that belong to the class $\Pi^{s_x,s_y}(\lambda^+)$, $s_x = s_y = 1, 2$.

Several remarks follow from Theorem 1.

Remark 1 *Inequality dominance (that is, dominance over the entire ranges of possible values for x and y) is obtained by letting λ^+ lie beyond the largest values of x and y. Then (11) implies that inequality is larger in A than in B.*

Remark 2 Π^{s_x,x_y} *dominance does not imply univariate dominance in either of the two indicators.*

Remark 3 *For bivariate $\Pi^{2,2}$ inequality dominance, we need $\Delta \operatorname{cov}(x, y) > 0$, that is, that the covariance of the indicators be greater in A than in B.*

Remark 4 *An array of tests of absolute and relative inequality dominance in each dimension can be made with Theorem 1 by using x_ρ and y_ρ with different values for ρ. For instance, using x_0 and y_0 leads to a test of absolute inequality dominance in each dimension; using x_0 and y_1 leads to a test of absolute inequality dominance in the x and of relative inequality dominance in the y dimension; and so on.*

Remark 5 *Because of the normalizations used to obtain x_ρ and y_ρ (see (1)),* $\Pi^{1,1}$ *relative poverty dominance is only feasible over a range of x_ρ and y_ρ that does not extend to the largest value of these variables. Since $\int z dF_{x_\rho}(z) = \int z dF_{y_\rho}(z) = 0$ for all values of ρ, it is indeed not possible to find $\Delta D^{0,0}(z_x, z_y) > 0$ for all values of (z_x, z_y). Hence,* $\Pi^{1,1}$ *inequality dominance is not possible.*

Remark 6 *If the $\pi(x, y; \lambda)$ relative deprivation function in (4) were separable in x and y, the cross-derivatives involved in the definition of the $\Pi^{s_x,s_y}(\lambda^+)$ classes would all be zero. That would inter alia imply that the impact on π of changing x would need to be independent of the value of y and would need to depend (potentially) only on the value of x. Such separability is implicit when one is checking for multivariate inequality dominance by performing univariate comparisons independently for each dimension of well-being.*

Such a separability assumption has undesirable consequences in the context of multivariate inequality measurement. Consider an example involving a transfer of an indicator of cognitive ability (i.e., achievement tests) (the variable x) between Bill Gates and John School, and assume as above that $\pi^{xx}(x, y; \lambda) \geq 0$. Also assume that despite his vastly superior income (variable y), Bill Gates happens to score lower on achievement tests than John School. Separability of the $\pi(x, y; \lambda)$ in x and y would imply that a transfer of ability from John School to Bill Gates would necessarily reduce overall inequality. This would seem undesirable since that transfer would also increase the welfare distance between the two individuals.

4 Estimation and inference

We now consider the estimation of the surfaces derived above as well as statistical inference on them. This can be seen as a generalization of the procedures followed in Duclos, Sahn, and Younger (2006) to the case of surfaces, curves and indices whose thresholds and individual functions of contributions to total poverty are subject to sampling variability because they depend on unknown characteristics and moments of the distribution (*e.g.,* the means μ_χ and μ_ξ of the variables).

To start with, note that the $D^{s_x-1,s_y-1}(z_x, z_y)$ functions defined in (6) above can be seen as a special case of the more general class of bidimensional surfaces defined as

$$D = \int_0^{g_\chi(\mu_\chi)} \int_0^{g_\xi(\mu_\xi)} h(\chi, \xi; \mu_\chi, \mu_\xi) dF(\chi, \xi), \tag{12}$$

where g_χ, g_ξ and h are continuous and differentiable functions of μ_χ and μ_ξ. A natural estimator of D is obtained by replacing F by its empirical counterpart, $\hat{F}$, and the μ_χ and μ_ξ by their sampling values. To see this better, suppose that we have a random sample of N independently and identically distributed (IID) observations drawn from the joint distribution of χ and ξ. We can write these observations of χ^L and ξ^L, drawn from a population $L = A, B$, as (χ_i^L, ξ_i^L),

$i = 1, \ldots, N$. Let $I(\cdot)$ be the indicator function, which is equal to 1 if its argument is true and 0 otherwise. This gives:

$$\begin{aligned} \hat{D}^L &= \int_0^{g_\chi(\hat{\mu}_\chi^L)} \int_0^{g_\xi(\hat{\mu}_\xi^L)} h(\chi, \xi; \hat{\mu}_\chi^L, \hat{\mu}_\xi^L) d\hat{F}^L(\chi, \xi) \\ &= N^{-1} \sum_{i=1}^N I(\chi_i^L \leq g_\chi(\hat{\mu}_\chi^L) I(\xi_i^L \leq g_\xi(\hat{\mu}_\xi^L) h(\chi_i^L, y_i^L; \hat{\mu}_\chi^L, \hat{\mu}_\xi^L). \end{aligned} \tag{13}$$

Denoting $f_+ = \max(f, 0)$, the dominance surfaces defined in (6) are obtained from (13) by setting

$$\begin{aligned} &g_\chi(\mu_\chi) = z_x \mu_\chi \\ &g_\xi(\mu_\xi) = z_y \mu_\xi \\ &h(\chi, \xi; \mu_\chi, \mu_\xi) = \left(z_x - \frac{\chi}{\mu_\chi}\right)_+^{\alpha_x} \left(z_y - \frac{\xi}{\mu_\xi}\right)_+^{\alpha_y} \end{aligned} \tag{14}$$

for relative inequality both in χ and in ξ; by setting

$$\begin{aligned} &g_\chi(\mu_\chi) = z_x + \mu_\chi \\ &g_\xi(\mu_\xi) = z_y \mu_\xi \\ &h(\chi, \xi; \mu_\chi, \mu_\xi) = (z_x - (\xi - \mu_\chi))_+^{\alpha_x} \left(z_y - \frac{\xi}{\mu_\xi}\right)_+^{\alpha_y} \end{aligned} \tag{15}$$

for absolute inequality in χ and relative inequality in ξ; and by setting

$$\begin{aligned} &g_\chi(\mu_\chi) = z_x + \mu_\chi \\ &g_\xi(\mu_\chi) = z_y + \mu_\xi \\ &h(\chi, \xi; \mu_\chi, \mu_\xi) = (z_x - (\chi - \mu_\chi))_+^{\alpha_x} (z_y - (\xi - \mu_\xi))_+^{\alpha_y} \end{aligned} \tag{16}$$

for absolute inequality both in χ and in ξ. Substituting the above into (13) gives estimators of the various combinations of absolute and relative dominance surfaces discussed above. For arbitrary α_x and α_y, (13) then has the convenient property of being a simple sum of IID variables, even if we allow for the fact that χ and ξ will generally be correlated across observations.

The following theorem provides the asymptotic sampling distribution of the general case given by (13) under relatively minor conditions and in the case in which we have a sample from each of two populations, A and B, that may or may not have been drawn independently from each other.

Theorem 2 *Let the joint population moments of order 2 of* $\chi^A + \xi^A + h(\chi^A, \xi^A; \mu_\chi^A, \mu_\xi^A)$ *and* $\chi^B + \xi^B + h(\chi^B, \xi^B; \mu_\chi^B, \mu_\xi^B)$ *be finite. Then*

$N^{1/2}\left(\hat{D}^A - D^A\right)$ and $N^{1/2}\left(\hat{D}^B - D^B\right)$ are asymptotically normal with mean zero, with asymptotic covariance structure given by ($L, M = A, B$)

$$\begin{array}{l}\lim_{N\to\infty} N\textit{cov}\left(\hat{D}^L, \hat{D}^M\right) \\ = E\left[\left(m_\chi^L \chi^L + m_\xi^L \xi^L + I(\chi^L \le g_\chi(\mu_\chi^L))I(\xi^L \le g_\xi(\mu_\xi^L))h(\chi^L, \xi^L; \mu_\chi^L, \mu_\xi^L)\right)\right. \\ \cdot\left.\left(m_\chi^M \chi^M + m_\xi^M \xi^M + I(\chi^M \le g_\chi(\mu_\chi^M))I(\xi^M \le g_\xi(\mu_\xi^M))h(\chi^M, \xi^M; \mu_\chi^M, \mu_\xi^M)\right)\right] \\ \quad -E\left[m_\chi^L \chi^L + m_\xi^L \xi^L + I(\chi^L \le g_\chi(\mu_\chi^L))I(\xi^L \le g_\xi(\mu_\xi^L))h(\chi^L, \xi^L; \mu_\chi^L, \mu_\xi^L)\right] \\ \cdot E\left[m_\chi^L \chi^L + m_\xi^L \xi^L + I(\chi^L \le g_\chi(\mu_\chi^L))I(\xi^L \le g_\xi(\mu_\xi^L))h(\chi^L, \xi^L; \mu_\chi^L, \mu_\xi^L)\right]\end{array} \tag{17}$$

where

$$\begin{array}{l} m_\chi^L = g'_\chi(\mu_\chi^L)\int_0^{g_\xi(\mu_\xi^L)} h(g_\chi(\mu_\chi^L), \xi; \mu_\chi^L, \mu_\xi^L) f(g_\chi(\mu_\chi^L), \xi) d\xi \\ + \int_0^{g_\chi(\mu_\chi^L)} \int_0^{g_\xi(\mu_\xi^L)} h^{\mu_\chi}(\chi, \xi; \mu_\chi^L, \mu_\xi^L) dF^L(\chi, \xi) \end{array} \tag{18}$$

and

$$\begin{array}{l} m_\xi^L = g'_\xi(\mu_\xi^L)\int_0^{g_\chi(\mu_\chi^L)} h(\chi, g_\xi(\mu_\xi^L); \mu_\chi^L, \mu_\xi^L) f(\chi, g_\xi(\mu_\xi^L)) d\chi \\ + \int_0^{g_\chi(\mu_\chi^L)} \int_0^{g_\xi(\mu_\xi^L)} h^{\mu_\xi}(\chi, \xi; \mu_\chi^L, \mu_\xi^L) dF^L(\chi, \xi), \end{array} \tag{19}$$

and where $f(\chi, \xi)$ is the joint density of χ and ξ.
Proof: See the Appendix (Section 8).

When the samples from the populations A and B are independent, the variance of each of $\hat{D}^A$ and $\hat{D}^B$ can be found by using (17) and by replacing N by N_A and N_B respectively. The covariance between the two estimators is then zero. The elements of the asymptotic covariance matrix in (17) can be estimated consistently using their sample equivalents. Further details are provided in the Appendix of Section 8.

5 Simulation exercises

To provide a better understanding of how our proposed multivariate inequality comparisons are likely to work in practice, we undertake simulations that compare a variety of dist ributions. In all cases, we create two populations of 200,000 individuals and joint distributions of x and y that are bivariate normal. We always shift these distributions so that they include no negative values, so the means of the marginal distributions are usually around 5 or 6. We then sample from those populations and compare the samples. In most cases, we use a sample size of 1000, which is roughly the size of large sub-samples (by region, say) in common household surveys like the Living Standards Measurement Surveys or the Demographic and Health Surveys. All the results reported here are for 100 such samples.

The phrase "non-statistical results" refers to comparisons made from simply comparing the surface estimates of $D^{s_x-1,s_y-1}(z_x, z_y)$ without carrying out any formal statistical testing. The phrase"statistical results" refers to statistical inference results. In testing for inequality dominance, we follow the intersection-union approach proposed by Kaur, Prakasa Rao, and Singh (1994) and recently extended by Davidson and Duclos (2006). We posit a null hypothesis of non-dominance of A by B and an alternative hypothesis of dominance of A by B. We reject the null and accept the alternative that B has less inequality than A if and only if the t statistics at all of the test points exceed the usual 5% critical value of the normal distribution.

Since the theory we present above stresses the importance of the covariance between dimensions of well-being, our first simulations vary the correlation between x and y in the first population, while keeping it at zero for the second. In all cases, the variance of the marginal distributions is one. Table 1 gives the results. For a sample size of 1000 and a difference in correlation of 0.2, which is plausible for several of the actual distributions that we will examine in Section 6, there are no statistically significant comparisons between the two samples, even though about half of the non-statistical comparisons find that the dominance surface estimates for sample 1 are above those for sample 2 everywhere. Even for very large differences in correlations of 0.6, there are relatively few statistically significant differences between the dominance surfaces, even though virtually all of the non-statistical comparisons would appear conclusive. The results are similar across absolute and relative comparisons.

Table 2 shows the number of rejections at each point in the (x, y) domain where we test for differences in the surfaces. The origin is at the lower left, and the first column and last row give the coordinates for x and y. This particular result is for relative inequality comparisons when population 1 has a correlation of 0.6 and population 2 has no correlation between x and y. It is clear that the reason that we cannot reject the null often is that there are relatively few rejections in the areas of the surface where x is relatively large and y is relatively small, or vice-versa. This is clearly a problem of statistical power: there are too few observations in these regions of the surfaces to infer dominance with sufficient confidence. Unfortunately, even when we increase the sample sizes to 10,000, we get relatively few significant differences. (See the last two columns of Table 1.) Thus, for the typical samples from national household surveys that are conducted in developing countries, it may be difficult to reject the null for samples that differ only in their correlation between dimensions of well-being. Nevertheless, it will often be possible to find differences in *relative poverty* for poverty measures that do not extend to the upper left and lower right corners of the dominance surfaces. This would include a wide range of intersection relative poverty measures, but no union measures.

Table 3 gives results for comparisons when the standard deviations of the marginal distributions in population 2 are a multiple of those in distribution 1 for both x and y. As equation (8) shows, these surfaces cannot differ over the entire (x, y) domain because they have the same covariance of zero. Thus, we report statistical differences for all test points except the extreme one at the upper right of the test domain that is just beyond the maximum value of x and y. These are, then, relative poverty comparisons, valid for a very wide range of relative poverty lines and both union and intersection poverty measures. Once the ratio of standard deviations reaches 1.4, well within the range that we find in real samples in the following section, we begin to have a significant number of cases in which the dominance surfaces differ statistically. Thus, for samples with strong univariate differences in inequality, we should also find bivariate differences, except at the extreme of the distribution where only the correlation matters.

Table 4, however, shows that a relatively modest correlation in one population can offset even rather large differences in univariate dispersions, and vice-versa. In the first two columns we compare population 2 with univariate standard deviations that are 1.6 times as large as those for population 1. In the first column, there is no correlation in either population and, excluding the extreme test point of the test domain, there are many significant differences between the dominance surfaces. In column 2, population 1 now has correlation between x and y of 0.2. This greatly reduces the number of significant differences between dominance surfaces drawn from these two populations. In such cases, one-at-time univariate comparisons will reject the null of equality between the two samples too easily because they do not consider the correlation between dimensions.

Column 3 shows that there are some (though relatively few) significant differences between samples if the correlation between x and y in population 1 is 0.6 higher than that in population 2. However, all of these significant differences vanish if we increase the standard deviations in the second sample by even a modest

20 percent. Overall, in cases where one population has greater correlation while the other has greater univariate variances, it could be uncommon to conclude that there is a statistically significant difference in multidimensional inequality, because of the conflicting effects of univariate inequality and joint inequality.

Thus far, our multivariate inequality comparisons are more likely not to reject the null of non-dominance than one-at-a-time univariate comparisons, but it is also possible that a correlation between x and y helps to resolve contradictory univariate results. In particular, it is possible that one univariate test rejects the null while another does not, but the bivariate test does reject the null of non-dominance. It is also possible that one univariate test shows significantly less inequality in distribution A while the other shows less in distribution B, and the bivariate test still shows greater inequality in one or the other distribution, if it has a large covariance between x and y.

Table 5 gives results for some such cases. In the first column, population 2 has a somewhat smaller standard deviation of x and a larger standard deviation of y than population 1. In this case, even a large correlation between x and y in population 1 is insufficient to produce statistically significant differences in the bivariate comparisons. However, if the variances of y are the same in both populations, the larger correlation in the first population is now sufficient to reject the null about half of the time.

6 Examples

We turn now to examples using real data. We focus on the interesting cases in which the bivariate tests that we propose produce different results than one-at-a-time univariate comparisons for the same dimensions of well-being. Table 6 gives results from the 1999 Demographic and Health Survey (DHS) for India. It considers relative poverty and inequality in two dimensions of well-being: the hemoglobin concentration (g/Dl) of women aged 15-49, and an index of the assets owned by those women's households. Hemoglobin is an important health indicator. Low hemoglobin concentrations cause a variety of health problems and have been shown to reduce physical productivity — see for instance Haas and Brownlie (2001) and Horton and Ross (2003). Household assets are a good proxy for a household's material well-being (Sahn and Stifel 2000 and Sahn and Stifel 2003). The index is constructed as the first factor from a factor analysis of the household's water source, type of toilet facility, the household head's years of schooling, and indicators of whether or not the household has electricity, a radio, a television, a refrigerator, and a bicycle. Since this distribution is centered around zero, it is shifted rightward so that it has no negative values.

The comparisons in Table 6 are between the Indian states of Goa and Rajasthan. Note that Goa has lower Gini coefficients for both assets and hemoglobin concentrations, but also has a significantly higher correlation between those two variables. The bottom section of the table shows that Goa dominates Rajasthan in both dimensions of well-being independently: across the entire distribution of assets or hemoglobin, Goa's dominance curve is below Rajasthan's, and these differences are statistically significant. Based on this information, one would conclude that Goa has less inequality than Rajasthan.

However, the bivariate comparison (11) cannot reject the null that Goa has more inequality than Rajasthan. Table 7 shows the t-statistics for the null hypothesis that the two states' dominance surfaces are equal at each of 100 equally spaced points across the entire domain of the joint distribution of assets and hemoglobin concentrations. These differences change sign, indicating that the dominance surfaces cross, and the differences are rarely statistically significant; when they are statistically significant, they also sometimes indicate that Goa has more relative poverty. Thus, the higher correlation between assets and hemoglobin in Goa is sufficient to nullify the conclusion drawn from the univariate comparisons.

On the other hand, it is also possible that the univariate comparisons are inconclusive or contradictory (showing that A dominates B in one dimension while the opposite is true in the other), yet the bivariate comparison rejects the null of non-dominance. Tables 8 and 9 show similar comparisons for Goa and the state of Himachal Pradesh. In this case, Himachal Pradesh has lower asset inequality, but we cannot reject the null that Goa has less inequality of hemoglobin concentrations. The bivariate comparison, however, clearly rejects the null of non-dominance across the entire domain of the dominance surfaces, so we reject the null in favor of greater bivariate inequality in Goa.[4]

To have a sense of how common these results are, Table 10 summarizes univariate and bivariate comparisons across all possible combinations of states in the 1999 India DHS. The first frame is for relative inequality comparisons, for which the data are divided by their mean. The second frame is for absolute inequality comparisons (the mean is subtracted from all data). The third frame is for non-normalized data; these comparisons are the same as the poverty compar-

[4] This example is also a caution that simple recourse to the Ginis and correlation coefficients can be misleading. What matters is the dispersion for each variable and the dependence between them over the entire distribution.

isons developed earlier in Duclos, Sahn, and Younger (2006), but with the poverty frontier extended beyond all of observations in the sample. Thus, they are welfare comparisons.

A striking result in Table 10 is how few cases of bivariate inequality dominance there are: only 33 for relative inequality and 16 for absolute inequality (out of 325 possible comparisons across Indian states) at dominance order (2,2), and somewhat more for order (3,3), especially for the absolute inequality case (70 out of 325). This compares to 125 cases of bivariate dominance for the welfare comparisons at order (2,2).

Closer examination of the table shows that there are two reasons for this. First, the last two columns show non-statistical comparisons. That is, if one sample surface is everywhere below another, we conclude that it dominates, regardless of the statistical significance of that difference. This yields welfare dominance in 197 cases, relative inequality dominance in 129, and absolute inequality dominance in 90. This is to be expected insofar as the non-normalized welfare distributions can differ either because the means differ or the dispersions differ. Because the inequality comparisons normalize the data, they can differ only if the dispersions around the mean differ. Previous work on incomes (Datt and Ravallion (1992)) and anthropometry (Sahn and Younger (2005)) has shown that distributional differences are often dominated by different means rather than dispersions. So it is not surprising to find fewer differences when examining normalized distributions.

Table 10 also shows that differences in dominance surfaces are less likely to be statistically significant for the inequality comparisons. For relative inequality, only 33 of the 129 cases where the surfaces do not cross are statistically significant, compared to 125 of 197 for the welfare comparisons. This is despite the fact that our samples are relatively large – averaging about 1100 women per state (but with some as low as 280).

A second observation about Table 10 is that the interesting cases — those for which the bivariate and the "one-at-a-time" univariate comparisons come to different conclusions — are relatively rare. For the inequality comparisons, there are *no* statistically significant cases of bivariate dominance when both of the univariate dominance tests are insignificant.[5] (See the first row of each block.) When the univariate comparisons both reject the null and are in agreement — the second row in each block — the bivariate case is statistically insignificant a little more

[5]The fact that there is only one such case for welfare suggests that these results may depend on the variables that we have chosen. In our prior work on poverty (Duclos, Sahn, and Younger (2006a)), we found many more such cases when studying household expenditures per capita and children's height-for-age.

than half the time, and the bivariate surfaces actually cross fairly often. (See the last two columns.) This is far more common than with the welfare comparisons. Finally, when the univariate results are inconsistent, either because population A dominates population B in one dimension while the reverse is true in the other, or because one difference is statistically significant while the other is not, it is relatively rare that the bivariate comparison can resolve this conflict. (See the third row of each block.) The welfare comparisons are able to do this significantly more often.

Our second example is more encouraging. The data come from the 2008 *Evaluación Nacional del Logro Académico de Centros Escolares* (National Evaluation of Academic Attainment in High Schools), a test given to all high school students in Mexico. In addition, a sample of students' households included information on asset holdings, with which we have constructed an asset index. The index is based on a factor analysis of the assets, and we have exponentiated the result to ensure that all values are positive. Because the test scores were standardized, we use instead each student's national percentile rank in the distribution of scores. Results of comparisons across Mexican states are summarized in Table 11, in a manner analogous to Table 10.

In these comparisons, we reject the null of non-dominance for the relative inequality comparisons in about one-third of the cases, though rejections for the absolute case remain relatively rare, as in the India DHS data. The larger than normal sample sizes (about 6700 observations per state, on average) appear to help since statistical and non-statistical results are more closely in line, especially for relative inequality and for welfare comparisons. The share (about one third) of rejections is now substantially closer to the share for welfare comparisons (about half) in this case than it was in any of the India DHS comparisons. It is also notable that the bivariate comparisons are able to resolve inconsistent univariate comparisons in a non-trivial number of cases for relative (28/255), absolute (26/253), and welfare (37/244) comparisons, unlike the India data, where such cases were rare for the inequality comparisons. Furthermore, it is quite rare for the second-order bivariate comparisons for relative inequality (17/144) and welfare (4/200) to reverse the univariate comparisons when they are consistent and reject the null of non-dominance. The bivariate comparisons are more demanding, so we would expect them to reject less often than the one-at-a-time univariate comparisons. In this Mexican illustration, therefore, making use of the entire bivariate distributions does not seem significantly to prevent making robust comparisons of inequality across distributions.

7 Conclusion

The paper has considered multivariate relative poverty and inequality comparisons. We do so in the spirit of Sen's "partial orderings" and follow the stochastic dominance approach to distributional comparisons. The approach also draws on our previous work addressing multidimensional *absolute poverty* comparisons. Here, we use similar methods, but first normalize the data, either relatively (dividing by the mean) or absolutely (shifting the mean to zero). We call poverty comparisons on such normalized variables "relative poverty" comparisons; extending the relevant poverty frontier to the limits of the joint enables making inequality comparisons. As in the stochastic dominance literature, the comparisons are made robust to the choice of any particular poverty or inequality index that is a member of some class. An important feature of our approach is that it is also robust to aggregation procedures across dimensions of well-being. We also derive the sampling distribution of our estimators, thus allowing our distributional comparisons to be robust to sampling variability.

Based on simulated distributions as well as asset and health variables from the 1999 DHS in India and mathematics and Spanish test scores and assets from the 2008 ENLACE in Mexico, we gain some practical experience and insights into the methods we propose. Our empirical applications suggests that finding bivariate inequality differences may be difficult with some variables and typical survey data sample sizes. In particular, bivariate inequality comparisons using assets and health variables in India do not reject the null of non-dominance nearly as often as welfare comparisons do for these variables. Further, it is rare for the bivariate comparisons to reject the null of non-dominance when the univariate comparisons do not. Comparisons using mathematics test scores and assets in Mexico are more revealing, as there are more rejections of the null, and more cases in which these bivariate comparisons "resolve" inconsistent univariate comparisons in each dimension alone. It is likely that results with other variables and distributions will differ, especially if we consider incomes or expenditures as one of the dimensions of well-being. That is certainly an interesting avenue for future research.

Table 1: Results for simulating different correlations

Correlation in sample 1	0.2		0.6		0.2	
Sample size	1000		1000		10,000	
	Statistical	Non-Statistical	Statistical	Non-Statistical	Statistical	Non-Statistical
Share of relative dominance results	0.00	0.48	0.24	0.94	0.17	0.93
Share of absolute dominance results	0.00	0.45	0.17	0.94	0.20	0.96

Notes: Statistical tests at 95% confidence level

Shares are out of 100 comparisons
Both distributions are normal with mean=5 and variance=1

Table 2: Number of rejections (out of 100) per test point, relative inequality comparisons

1.58	48	71	83	95	99	100	100	100	100	100
1.26	91	99	100	100	100	100	100	100	100	100
1.17	96	100	100	100	100	100	100	100	100	100
1.10	97	100	100	100	100	100	100	100	100	100
1.05	98	100	100	100	100	100	100	100	100	99
1.00	98	100	100	100	100	100	100	100	100	99
0.95	98	100	100	100	100	100	100	100	100	97
0.90	99	100	100	100	100	100	100	100	100	89
0.84	99	99	100	100	100	100	100	100	98	69
0.73	93	97	98	98	98	98	98	96	92	40
0.00	0.75	0.84	0.90	0.96	1.01	1.05	1.10	1.16	1.25	1.59

Notes: See notes to previous table. Correlation between x and y in sample 1 is 0.6. The first column and last row are the coordinates in the (x, y) plane where comparisons are made. x and y values are normalized by their respective means.

Table 3: Results for simulating different variances

Ratio of standard deviations, sample 2/sample 1	1.2		1.4		1.6	
Sample size	1000		1000		1000	
	Statistical	Non-Statistical	Statistical	Non-Statistical	Statistical	Non-Statistical
Share of relative dominance results	0.01	0.92	0.47	1.00	0.87	1.00
Share of absolute dominance results	0.01	0.91	0.44	1.00	0.87	1.00

Notes: Statistical tests at 95% confidence level. Tests exclude the last test point at the extreme of the distribution. Shares are out of 100 comparisons. Both distributions are normal with mean=5 and no covariance. Variance in sample 1 is 1.

Table 4: Results for simulating different variances

Ratio of standard deviations, sample 2/sample 1	1.6	1.6	1.0	1.2
Correlation in sample 1	0.0	0.2	0.6	0.6
Share of relative dominance results	0.87	0.28	0.24	0.00
Share of absolute dominance results	0.87	0.30	0.17	0.00

Notes: Statistical tests at 95% confidence level. Tests exclude the last test point at the extreme of the distribution. Shares are out of 100 comparisons. Sample size is 1000. Both distributions are normal with mean=5. Variance in distribution 1 is 1.

Table 5: Results for simulating different variances for x and y

Ratio of standard deviation for (x, y) distribution 2 / distribution 1	(0.8, 1.2)	(0.8, 1.0)
Correlation in distribution 1	0.6	0.6
Share of relative dominance results	0.00	0.47
Share of absolute dominance results	0.00	0.45

Notes: Statistical tests at 95% confidence level. Tests exclude the last test point at the extreme of the distribution. Shares are out of 100 comparisons. Sample size is 1000. Both distributions are normal with mean=5. Variance in distribution 1 is 1.

Table 6: Descriptive Statistics and Univariate Relative Inequality Dominance Comparisons for Household Assets and Women's Hemoglobin Concentrations, Goa and Rajasthan

	Goa	Rajasthan
Gini coefficients		
Assets	0.292	0.452
Hemoglobin	0.076	0.089
Correlation	0.176	0.047

Difference Between Dominance Curves
t-statistics for difference

Sample decile	Assets	Hemoglobin
0.1	-20.51	-1.88
0.2	-25.36	-3.36
0.3	-25.10	-4.50
0.4	-26.05	-5.43
0.5	-26.53	-6.46
0.6	-27.03	-7.10
0.7	-30.49	-7.39
0.8	-42.83	-7.43
0.9	-15.31	-7.04

Source: 1999 DHS for India.
Notes:
The t-statistic tests the difference of Goa - Rajasthan, so a negative value indicates that Goa has less relative poverty than Rajasthan.
Poverty lines are set and differences are tested at normalized values of assets and hemoglobin found at each decile of the combined samples.
Normalization is relative, *i.e.*, data are divided by their means. Dominance order is 2.

Table 7: Bivariate Relative Inequality Dominance Comparisons for Household Assets and Women's Hemoglobin Concentrations, Goa and Rajasthan

		t-statistics for difference in the surfaces									
	3.47	-0.62	-1.30	-1.71	-2.06	-2.50	-2.55	-1.85	-0.76	-0.32	1.48
	2.20	-0.37	-0.90	-1.15	-1.37	-1.61	-1.55	-1.09	-0.40	0.30	0.47
	1.69	-0.10	-0.46	-0.56	-0.67	-0.75	-0.60	-0.16	0.34	0.77	0.40
	1.28	0.28	0.12	0.19	0.24	0.40	0.66	1.14	1.67	2.12	1.66
Household Assets	1.01	0.46	0.41	0.54	0.63	0.82	1.04	1.44	1.87	2.27	1.89
	0.83	0.40	0.46	0.62	0.67	0.78	0.90	1.15	1.44	1.73	1.43
	0.67	0.26	0.35	0.44	0.39	0.36	0.31	0.35	0.44	0.57	0.21
	0.52	0.00	0.15	0.28	0.28	0.27	0.15	0.09	0.10	0.17	-0.23
	0.32	-0.93	-0.95	-0.86	-0.84	-0.83	-0.94	-0.98	-0.91	-0.97	-1.66
	0.18	-0.68	-0.98	-0.97	-0.91	-0.83	-0.92	-0.91	-0.80	-0.93	-2.22
		0.79	0.87	0.93	0.98	1.02	1.05	1.09	1.13	1.18	1.63
		Hemoglobin Concentration									

Source: 1999 DHS for India.

Notes:

The t-statistic tests the difference of Goa - Rajasthan, so a negative value indicates that Goa has less relative poverty than Rajasthan.

Poverty lines are set and differences are tested at normalized values of assets and hemoglobin found at each decile of the combined samples.

Normalization is relative, *i.e.* data are divided by their means.

Dominance orders are (2,2).

Table 8: Descriptive Statistics and Univariate Absolute Inequality Dominance Comparisons for Household Assets and Women's Hemoglobin Concentrations, Goa and Himachal Pradesh

	Goa	Himachal Pradesh
Absolute Gini coefficients		
Assets	0.506	0.443
Hemoglobin	0.922	0.896
Correlation	0.176	-0.008
Difference Between Dominance Curves		
	t-statistics for difference	
Sample decile	Assets	Hemoglobin
0.1	8.06	2.09
0.2	9.22	2.03
0.3	10.40	2.02
0.4	11.15	1.99
0.5	11.24	1.92
0.6	11.04	1.80
0.7	10.01	1.66
0.8	7.76	1.50
0.9	5.97	1.38

Source: 1999 DHS for India.

Notes:

The t-statistic tests the difference of Goa - Himachal Pradesh, so a negative value indicates that Goa has less absolute inequality than Himachal Pradesh.

Poverty lines are set and differences are tested at normalized values of assets and hemoglobin found at each decile of the combined samples.

Normalization is absolute, *i.e.*, means are subtracted from the data.

Dominance order is 3.

Table 9: Bivariate Absolute Inequality Dominance Comparisons for Household Assets and Women's Hemoglobin Concentrations, Goa and Himachal Pradesh

		t-statistics for difference in the surfaces									
	2.35	2.57	2.88	3.05	3.14	3.26	3.26	3.12	2.79	2.41	6.41
	1.38	2.82	3.18	3.46	3.70	3.94	4.13	4.34	4.65	5.04	4.86
	0.77	2.89	3.37	3.74	4.07	4.41	4.69	4.99	5.38	5.85	6.56
	0.30	2.92	3.56	4.04	4.47	4.93	5.31	5.70	6.15	6.66	8.94
Household Assets	0.01	2.89	3.63	4.19	4.69	5.21	5.64	6.03	6.46	6.91	10.46
	-0.19	2.82	3.62	4.21	4.73	5.28	5.72	6.12	6.53	6.96	11.06
	-0.36	2.74	3.56	4.18	4.72	5.29	5.73	6.13	6.56	7.03	11.19
	-0.56	2.53	3.36	3.98	4.50	5.05	5.47	5.83	6.19	6.58	10.88
	-0.75	2.25	3.04	3.62	4.12	4.64	5.03	5.35	5.67	6.02	10.03
	-0.90	1.99	2.68	3.24	3.74	4.24	4.63	5.00	5.42	5.91	8.96
		-2.38	-1.39	-0.78	-0.28	0.22	0.62	1.02	1.52	2.12	6.81
		Hemoglobin Concentration									

Source: 1999 DHS for India.
Notes: The t-statistic tests the difference of Goa - Himachal Pradesh, so a negative value indicates that Goa has less absolute inequality than Himachal Pradesh.
Poverty lines are set and differences are tested at normalized values of assets and hemoglobin found at each decile of the combined samples.
Normalization is absolute, *i.e.*, means are subtracted from the data.
Dominance orders are (3,3).

Table 10: Summary of Cross-State Inequality and Welfare Dominance Comparisons for Household Assets and Women's Hemoglobin Concentration, India (325 comparisons)

Univariate Dominance	Bivariate dominance					
	Second order		Third order		2nd order, non-statistical	
	yes	no	yes	no	yes	no
Relative Inequality						
No univariate dominance in either dimension	0	74	0	36	0	3
Univariate dominance in both dimensions, consistent	26	31	37	57	123	37
Univariate results inconsistent	7	187	4	191	6	156
sub-total	33	292	41	284	129	196
Absolute Inequality						
No univariate dominance in either dimension	0	176	0	91	0	16
Univariate dominance in both dimensions, consistent	10	20	64	31	87	66
Univariate results inconsistent	6	113	6	133	3	153
sub-total	16	309	70	255	90	235
Welfare						
No univariate dominance in either dimension	1	44	1	37	0	9
Univariate dominance in both dimensions, consistent	90	24	94	27	167	18
Univariate results inconsistent	34	132	38	128	30	101
sub-total	125	200	133	192	197	128

Source: 1999 DHS for India

Table 11: Summary of Cross-State Inequality and Welfare Dominance Comparisons for Household Assets and Student Math Test Percentile, Mexico (465 comparisons)

Univariate Dominance	Bivariate dominance					
	Second order		Third order		2nd order, non-statistical	
	yes	no	yes	no	yes	no
	Relative Inequality					
No univariate dominance in either dimension	1	65	0	56	1	9
Univariate dominance in both dimensions, consistent	127	17	131	19	218	19
Univariate results inconsistent	28	227	25	234	14	204
sub-total	156	309	156	309	233	232
	Absolute Inequality					
No univariate dominance in either dimension	2	201	0	149	0	61
Univariate dominance in both dimensions, consistent	5	4	30	35	80	65
Univariate results inconsistent	26	227	8	243	3	256
sub-total	33	432	38	427	83	382
	Welfare					
No univariate dominance in either dimension	0	21	0	20	0	3
Univariate dominance in both dimensions, consistent	196	4	202	4	256	3
Univariate results inconsistent	37	207	44	195	30	173
sub-total	233	232	246	219	286	179

Source: ENLACE, Mexico.

8 Appendix

Proof of Theorem 1.

For $s_x = 1$ and $s_y = 1$ and for $s_x = 2$ and $s_y = 1$, the proof follows from Theorems 1 and 2 in Duclos, Sahn, and Younger (2006). For $s_x = 2$ and $s_y = 2$, start with

$$
\begin{aligned}
P(z_x(y), z_y) &= -\int_0^{z_x(z_y)} \pi^x(x, z_y; \lambda^+)\, D^{0,0}(x, z_y)\, dx && (20)\\
&+ \int_0^{z_y} z_\chi^{(1)}(y)\, \pi^x(z_x(y), y; \lambda^+)\, D^{0,0}(z_x(y), y)\, dy && (21)\\
&+ \int_0^{z_y} \int_0^{z_x(y)} \pi^{xy}(x, y; \lambda^+)\, D^{0,0}(x, y)\, dx\, dy. && (22)
\end{aligned}
$$

Integrating (22) by parts with respect to x and y, and imposing the continuity conditions characterizing the indices in $\Pi^{2,2}(\lambda^+)$ in (10), we find:

$$
\begin{aligned}
P(z_x(y), z_y) &= -\int_0^{z_x(z_y)} \pi^{xxy}(x, z_y; \lambda^+)\, D^{1,1}(x, z_y)\, dx && (23)\\
&+ \int_0^{z_y} z_\chi^{(1)}(y)\, \pi^{xxy}(z_x(y), y; \lambda^+)\, D^{1,1}(z_x(y), y)\, dy && (24)\\
&+ \int_0^{z_y} \int_0^{z_x(y)} \pi^{xxyy}(x, y; \lambda^+)\, D^{1,1}(x, y)\, dx\, dy. && (25)
\end{aligned}
$$

The rest of the proof follows from Theorem 1 in Duclos, Sahn, and Younger (2006).

Proof of Theorem 2.

Leaving out the superscript L for simplicity, we find

$$\hat{D} - D = \int_0^{g_\chi(\hat{\mu}_\chi)} \int_0^{g_\xi(\hat{\mu}_\xi)} h(\chi, \xi; \hat{\mu}_\chi, \hat{\mu}_\xi) d\hat{F}(\chi, \xi) - D \tag{26}$$

$$= \int_0^{g_\chi(\mu_\chi)} \int_0^{g_\xi(\mu_\xi)} h(\chi, \xi; \hat{\mu}_\chi, \hat{\mu}_\xi) d\hat{F}(\chi, \xi) \tag{27}$$

$$+ \int_0^{g_\chi(\mu_\chi)} \int_{g_\xi(\mu_\xi)}^{g_\xi(\hat{\mu}_\xi)} h(\chi, \xi; \hat{\mu}_\chi, \hat{\mu}_\xi) d\hat{F}(\chi, \xi) \tag{28}$$

$$+ \int_{g_\chi(\mu_\chi)}^{g_\chi(\hat{\mu}_\chi)} \int_0^{g_\xi(\mu_\xi)} h(\chi, \xi; \hat{\mu}_\chi, \hat{\mu}_\xi) d\hat{F}(\chi, \xi) \tag{29}$$

$$+ \int_{g_\chi(\mu_\chi)}^{g_\chi(\hat{\mu}_\chi)} \int_{g_\xi(\mu_\xi)}^{g_\xi(\hat{\mu}_\xi)} h(\chi, \xi; \hat{\mu}_\chi, \hat{\mu}_\xi) d\hat{F}(\chi, \xi) \tag{30}$$

$$-D. \tag{31}$$

Expressions (28), (29) and (30) are of asymptotically lower order than (27) and can thus be neglected. Expanding (27) and leaving out the terms of asymptotically lower order, we find

$$\hat{D} - D \cong \tag{32}$$

$$[\hat{\mu}_\chi - \mu_\chi]\, g'_\chi(\mu_\chi) \int_0^{g_\xi(\mu_\xi)} h(g_\chi(\mu_\chi), y; \mu_\chi, \mu_\xi) dF(y\,|g_\chi(\mu_\chi)\,) f_x(g_\chi(\mu_\chi)) \tag{33}$$

$$+ [\hat{\mu}_\chi - \mu_\chi] \int_0^{g_\chi(\mu_\chi)} \int_0^{g_\xi(\mu_\xi)} h^{\mu_\chi}(\chi, \xi; \mu_\chi, \mu_\xi) dF(\chi, \xi) \tag{34}$$

$$+ [\hat{\mu}_\xi - \mu_\xi]\, g'_y(\mu_\xi) \int_0^{g_\chi(\mu_\chi)} h(x, g_\xi(\mu_\xi); \mu_\chi, \mu_\xi) dF(x\,|g_\xi(\mu_\xi)\,) f_\xi(g_\xi(\mu_\xi)) \tag{35}$$

$$+ [\hat{\mu}_\xi - \mu_\xi] \int_0^{g_\chi(\mu_\chi)} \int_0^{g_\xi(\mu_\xi)} h^{\mu_\xi}(\chi, \xi; \mu_\chi, \mu_\xi) dF(\chi, \xi) \tag{36}$$

$$+ \int_0^{g_\chi(\mu_\chi)} \int_0^{g_\xi(\mu_\xi)} h(\chi, \xi; \mu_\chi, \mu_\xi) d(\hat{F} - F)(\chi, \xi), \tag{37}$$

where f_χ and f_ξ are the univariate density of χ and ξ, respectively. Rearranging, and defining m_χ and m_ξ as in (18), we find:

$$\hat{D} - D \cong m_\chi \left(N^{-1} \sum \chi_i - \mu_\chi \right) + m_\xi \left(N^{-1} \sum \xi_i - \mu_\xi \right) \tag{38}$$

$$+ \left(N^{-1} \sum I(\chi_i \leq g_\chi(\mu_\chi^L)) I(\xi_i \leq g_\xi(\mu_\xi^L)) h(\chi_i, \xi_i; \mu_\chi, \mu_\xi) \right) - D. \tag{39}$$

The existence of the appropriate population moments of order 1 lets us apply the law of large numbers to (13) for each distribution A and B. This implies that $\hat{D}$ is a consistent estimator of D. Given the existence of the population moments of order 2, the estimator in (13) is root-N consistent by the central limit theorem, and it is also asymptotically normal with asymptotic covariance matrix given by Theorem 2. If the samples are dependent, then the covariance between the estimator for A and for B is provided by Theorem 2 by setting $L = A$ and $M = B$.

In the case that $\alpha_x > 0$ and $\alpha_y > 0$ for (7), the computation of m_x simplifies since $h(g_\chi(\mu_\chi^L), y; \mu_\chi^L, \mu_\xi^L) = 0$ (and similarly for m_y). In this case, for relative inequality both in χ and in ξ, we have

$$\begin{aligned} &g'_\chi(\mu_\chi) = z_x \\ &g'_\xi(\mu_\xi) = z_y \\ &h^{\mu_\chi}(\chi, \xi; \mu_\chi, \mu_\xi) = \tfrac{\alpha_x \chi}{\mu_\chi^2} \left(z_x - \tfrac{\chi}{\mu_\chi}\right)_+^{\alpha_x - 1} \left(z_y - \tfrac{\xi}{\mu_\xi}\right)_+^{\alpha_y} \\ &h^{\mu_\xi}(\chi, \xi; \mu_\chi, \mu_\xi) = \tfrac{\alpha_y y}{\mu_\xi^2} \left(z_x - \tfrac{\chi}{\mu_\chi}\right)_+^{\alpha_x} \left(z_y - \tfrac{\xi}{\mu_\xi}\right)_+^{\alpha_y - 1}. \end{aligned} \tag{40}$$

For absolute inequality in χ and in ξ, we have

$$\begin{aligned} &g'_\chi(\mu_\chi) = 1 \\ &g'_\xi(\mu_\xi) = 1 \\ &h^{\mu_\chi}(\chi, \xi; \mu_\chi, \mu_\xi) = \alpha_x \left(1 - \tfrac{\chi - \mu_\chi}{z_x}\right)_+^{\alpha_x - 1} \left(1 - \tfrac{\xi - \mu_\xi}{z_y}\right)_+^{\alpha_y} \\ &h^{\mu_\xi}(\chi, \xi; \mu_\chi, \mu_\xi) = \alpha_y \left(1 - \tfrac{\chi - \mu_\chi}{z_x}\right)_+^{\alpha_x} \left(1 - \tfrac{\xi - \mu_\xi}{z_y}\right)_+^{\alpha_y - 1} \end{aligned} \tag{41}$$

and similarly for a combination of absolute and relative inequality.

The expressions for $g'_\chi(\mu_\chi)$ and $g'_\xi(\mu_\xi)$ are the same if α_x and/or α_y equals 0, but then h^{μ_χ} and/or h^{μ_ξ} reduces to 0. In that case, there is however the complication of estimating $\int_0^{g_\xi(\mu_\xi)} h(g_\chi(\mu_\chi), \xi; \mu_\chi, \mu_\xi) f(g_\chi(\mu_\chi), \xi) d\xi$ in m_χ (and similarly for m_ξ). When $\alpha_x = 0$, this reduces to

$$\int_0^{g_\xi(\mu_\xi)} h(g_\chi(\mu_\chi), \xi; \mu_\chi, \mu_\xi) f(g_\chi(\mu_\chi), \xi) d\xi \tag{42}$$

$$= F(\xi = g_\xi(\mu_\xi) | \chi = g_\chi(\mu_\chi)) f_\chi(g_\chi(\mu_\chi)), \tag{43}$$

which is the distribution function of ξ at $g_\xi(\mu_\xi)$ conditional on χ being equal to $g_\chi(\mu_\chi)$ times the density of χ at $g_\chi(\mu_\chi)$. We can estimate this non-parametrically by kernel weighting the values of the conditional distribution function of ξ at

$g_\xi(\mu_\xi)$ across values of χ, using weights that depend on the distance between χ and $g_\chi(\mu_\chi)$. To see this more clearly, let a bivariate kernel $K_i(\chi, \xi)$ be defined as the product of two univariate kernels, $h_\chi^{-1}\phi\left(\frac{\chi-\xi_i}{h_\chi}\right)$ and $h_\xi^{-1}\phi\left(\frac{\xi-\xi_i}{h_\xi}\right)$ (both of them integrating to 1 over χ and ξ), for each observation $i = 1, ..., N$. We can then estimate (43) as

$$(Nh_\chi h_\xi)^{-1}\sum_i \int_{-\infty}^{g_\xi(\mu_\xi)} \phi\left(\frac{g_\chi(\mu_\chi)-\chi_i}{h_\chi}\right)\phi\left(\frac{z-\xi_i}{h_\xi}\right)dz \tag{44}$$

$$= (Nh_\chi)^{-1}\sum_i \phi\left(\frac{g_\chi(\mu_\chi)-\chi_i}{h_\chi}\right)\Phi\left(\frac{g_\xi(\mu_\xi)-\xi_i}{h_\xi}\right), \tag{45}$$

where $\Phi(z) = \int_{-\infty}^{z}\phi(u)du$. In a bivariate setting, the approximately optimal window for a Gaussian kernel and a multivariate normal density normalized to unit variance is given by $(1.25N)^{-1/6}$ — see Silverman (1986) for instance.

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Figure 1: Inequality frontiers

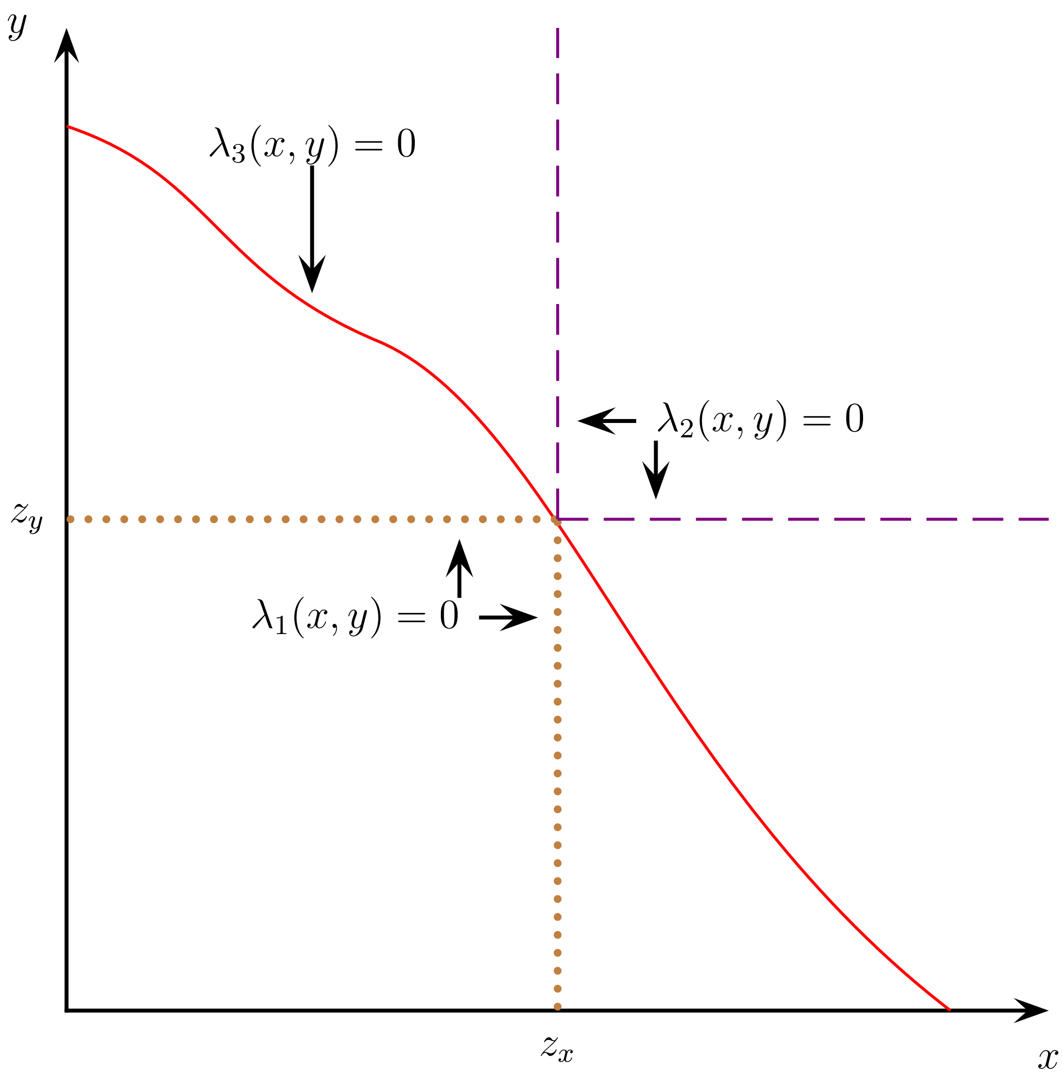

Figure 2: Bistochastic transformations may increase inequality in well-being

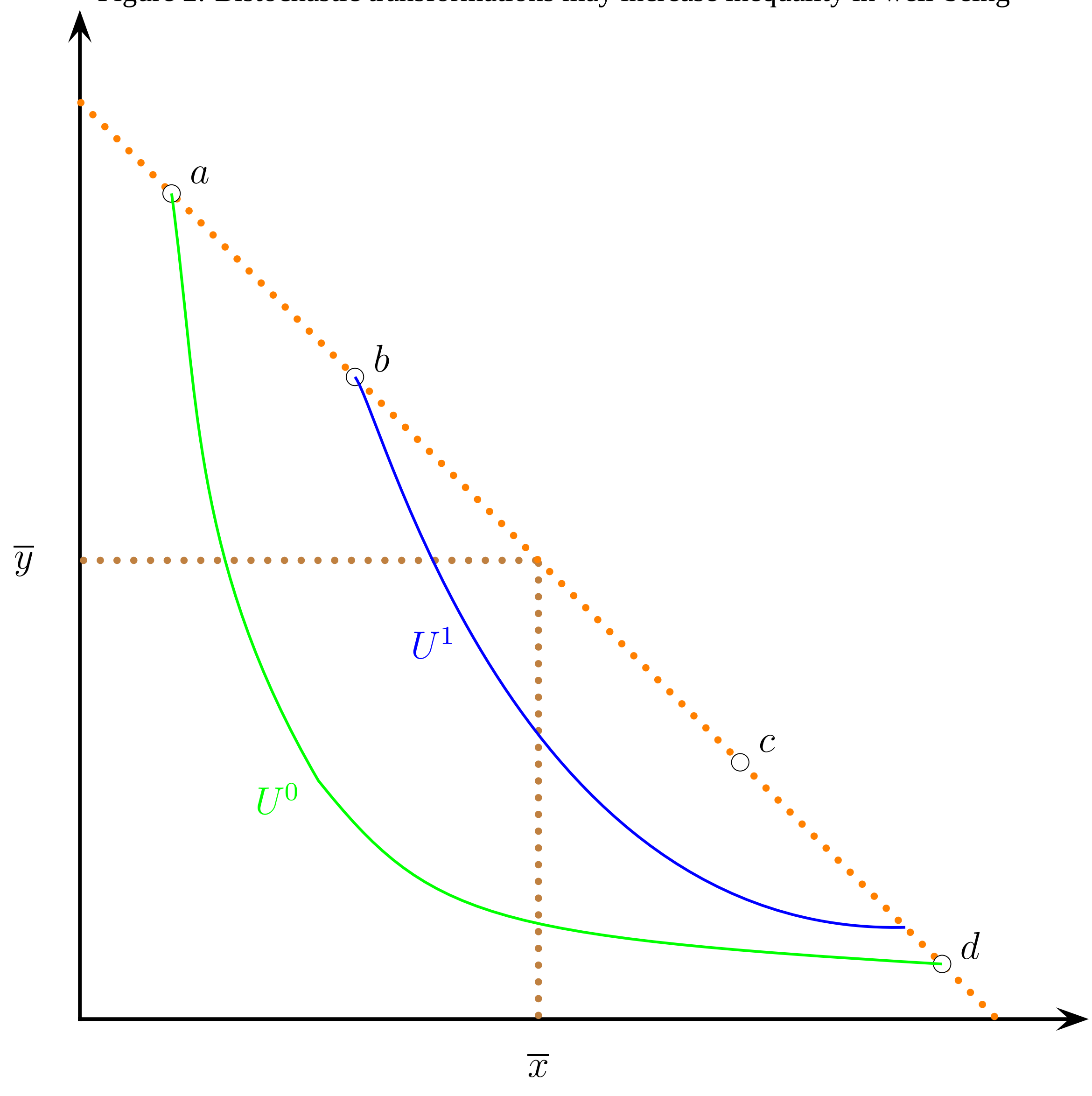

Figure 3: Inequality changes