VERSION SPACES AND GENERALIZED MONOTONE BOOLEAN FUNCTIONS

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Abstract

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Subsequently, we consider the monoid of all conjunctive operators on a set and show that this monoid is algebraically isomorphic to the monoid of all binary relations on this set. In this development, two operators, positive content and positive closure, play an important role.

The results are then applied to the version space of all monotone hypotheses of a set of binary examples also called the class of all monotone extensions of a partially defined Boolean function, to clarify its lattice theoretic properties.

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Version Spaces and Generalized Monotone Boolean Functions *

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Abstract. We consider generalized monotone functions $f : X \mapsto \{0,1\}$ defined for an arbitrary binary relation $\preceq$ on X by the property $x \preceq y \Rightarrow f(x) \leq f(y)$. These include the standard monotone (or positive) Boolean functions, regular Boolean functions and other interesting functions as special cases. It is shown that a class of functions is closed under conjunction and disjunction (i.e., a distributive lattice) if and only if it is the class of monotone functions with respect to some quasi-order $\preceq$.
We consider the monoid of all conjunctive operators on a set and show that this monoid is algebraically isomorphic to the monoid of all binary relations on this set. In this development, two operators, positive content and positive closure, play an important role.
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Keywords: machine learning, version spaces, lattices, ordinal classification, Boolean functions, monotone functions, generalized monotone functions, regular functions, Horn functions, positive content, positive closure, partially defined Boolean functions.

1 Introduction

It is well known that the class of all Boolean functions is closed under conjunction and disjunction (hence forms a distributive lattice (e.g., [11])). The same holds for the class of all monotone (also called positive) Boolean functions. In this paper, we point out that the monotonicity can be defined in quite a general setting, still maintaining the property that the class of generalized monotone functions forms a distributive lattice. In addition to the standard monotone Boolean functions, the generalized

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monotonicity includes such Boolean functions as regular [1, 9, 21, 23], aligned [5], Q-transitive [6] and g-transitive [6] functions. Although Horn functions [17, 19] are not monotone in our sense, some part of the theory can also be applied to them.

More precisely, given a ground set X, we consider functions $f : X \mapsto \{0,1\}$. Note, that these functions are just the characteristic functions of the subsets of X. For *any* binary relation $\preceq$ on X, we say that f is *monotone* with respect to $\preceq$ if $x \preceq y$ implies $f(x) \leq f(y)$ for all $x, y \in X$, where $\leq$ is the usual inequality on $\{0,1\}$. The case of a quasi-order $\preceq$ is particularly interesting, since it is shown that a class of functions is closed under conjunction and disjunction if and only if it is the class of monotone functions with respect to some quasi-order $\preceq$.

We then consider the operators defined on the class of all functions. After introducing the notion of conjunctive operators, we show that the set of all binary relations and the class of all conjunctive operators are isomorphic, if viewed as monoids under composition of relations and composition of operators, respectively. In this proof, special operators, called the positive content and the positive closure, are introduced and utilized.

Monotone functions (in the generalized sense) have been studied in logical analysis of data ([6, 10]), where extensions (i.e., Boolean functions) which are consistent with a given data sets (i.e., partially defined Boolean functions) are sought. This is because the generalized monotonicity often embodies the structure inherent in the data set under consideration. Given a binary relation $\preceq$, an interesting problem in this area is to investigate mathematical properties of the class of all monotone extensions of a given data set. We show that this class , also called a *version space* in machine learning is also closed under conjunction and disjunction. In order to clarify the lattice structure of this version space, it becomes clear that the above operators, positive content and positive closure, play an important role. In particular, the map π from the class of all monotone functions to the class of all monotone extensions can be described by using such operators, and it provides an algorithm to determine minimal representations of a given monotone extension.

2 Preliminaries

2.1 Functions and lattices

Given a finite set X, we consider the class of characteristic functions,

$$\mathcal{B}(X) = \{f \mid f : X \mapsto \{0,1\}\}.$$

The order $\leq$ is defined on $\{0,1\}$ by $0 \leq 0, 1 \leq 1$ and $0 \leq 1$. In particular, if $X = \{0,1\}^n$, then $\mathcal{B}(X)$ denotes the class of Boolean functions of n variables. In this paper, it is assumed that the reader is familiar with a basic knowledge of Boolean functions [16, 21, 22]. For $f \in \mathcal{B}(X)$, define

$$T(f) = \{x \in X \mid f(x) = 1\},$$
$$F(f) = \{x \in X \mid f(x) = 0\}.$$

We denote $f \leq g$ if $T(f) \subseteq T(g)$ holds. The relation $\leq$ on $\mathcal{B}(X)$ is a partial order. Two functions *top* $\top$ and *bottom* $\perp$ in $\mathcal{B}(X)$ are defined by $\top(x) = 1$ and $\perp(x) = 0$ for all $x \in X$ (i.e., $T(\top) = X$ and $T(\perp) = \emptyset$), respectively. Obviously $f \leq \top$ and $\perp \leq f$ hold for all $f \in \mathcal{B}(X)$.

Consider a subset $\mathcal{L}$ of $\mathcal{B}(X)$. If $f, g \in \mathcal{L}$, then the smallest element larger than both f and g in the sense of $\leq$ is called the least upper bound (lub) of f and g, and this element is denoted by $f \sqcup g$. Similarly, the greatest lower bound (glb) of f and g is denoted $f \sqcap g$. A subset $\mathcal{L} \subseteq \mathcal{B}(X)$ is called a *sublattice* of $\mathcal{B}(X)$ if $\mathcal{L}$ is closed under $\sqcap$ and $\sqcup$. The smallest element $f_{\min}$ and the largest element $f_{\max}$ of a lattice $\mathcal{L}$, if they exist, are called the *universal bounds* of $\mathcal{L}$: $f_{\min} \leq g \leq f_{\max}$ for all $g \in \mathcal{L}$.

A lattice $\mathcal{L}$ is *distributive* if $f \sqcup (g \sqcap h) = (f \sqcup g) \sqcap (f \sqcup g)$ holds for all $f, g, h \in \mathcal{L}$. The lattice $\mathcal{B}(X)$ is obviously a distributive lattice with $f_{\max} = \top$ and bottom $f_{\min} = \perp$, such that $f \sqcup g = f \vee g$ and $f \sqcap g = f \wedge g$, where the binary operations $\vee$ and $\wedge$ are the usual operators respectively called *disjunction* and *conjunction*. By convention, the operator is sometimes omitted from an expression; e.g., $f \wedge g$ may be written as fg. It is clear that $\mathcal{L}$ is a distributive sublattice of $\mathcal{B}$ if it is closed under conjunction and disjunction, since $f \sqcup g = f \vee g$ and $f \sqcap g = f \wedge g$ hold in such $\mathcal{L}$, and the distributive law $f \vee (g \wedge h) = (f \vee g) \wedge (f \vee g)$ always holds.

2.2 Generalized monotone functions

Let $\preceq$ be an *arbitrary* binary relation on X. A function $f \in \mathcal{B}(X)$ is called *monotone* with respect to $\preceq$ if $x \preceq y$ implies $f(x) \leq f(y)$ for any $x, y \in X$. The class of monotone functions with respect to $\preceq$ is denoted by $\mathcal{M}(X_{\preceq})$. As we shall see later, a binary relation $\preceq$ is particularly interesting if it is a *quasi-order* (i.e., *reflexive*: $x \preceq x$ for all $x \in X$, and *transitive*: $x \preceq y$ and $y \preceq z$ imply $x \preceq z$ for all $x, y, z \in X$).

Note that the class $\mathcal{B}(X)$ itself is monotone with respect to the equality relation $=$, i.e., $\mathcal{B}(X) = \mathcal{M}(X_{=})$. Now consider the case $X = \{0, 1\}^n$. For the ordinary inequality $\leq$ between vectors (i.e., $x \leq y \Leftrightarrow x_j \leq y_j$ for all j), a function $f \in \mathcal{M}(X_{\leq})$ has been traditionally called *monotone* (or *positive*). If it is necessary to distinguish, we say standard monotone functions and generalized monotone functions, respectively.

A function $f \in \mathcal{M}(X_{\leq})$ that satisfies the following additional condition is called *regular* [1, 9, 21, 23]: $f_{x_i=0, x_j=1} \leq f_{x_i=1, x_j=0}$ for any $i < j$, where $f_{x_i=a, x_j=b}$ is the restriction of f to the space with $x_i = a$ and $x_j = b$. It is known that a regular function is monotone in the above sense if $\preceq$ is defined by $x \preceq y \Leftrightarrow \sum_{j\leq k} x_j \leq \sum_{j\leq k} y_j$ for all $k \in \{1, 2, \ldots, n\}$.

There are still other types of (generalized) monotone functions. A function $f \in \mathcal{B}(X)$ is *aligned* [5] if it is monotone with respect to the relation $\preceq$ defined by $x \preceq y \Leftrightarrow x_i < y_i$ for $i \in \{1, 2, \ldots, n\}$ implies $\sum_{j\leq i} x_j \geq \sum_{j\leq n} y_j$. A monotone function f is *Q-transitive* [6], if $\preceq$ is defined as follows: Given an $m \times n$ real matrix Q, $x \preceq y \Leftrightarrow Qx \leq Qy$. It is interesting to see that a standard monotone function and a regular function are special cases of a Q-transitive function when Q is the identity matrix and when $Q_{ij} = 1$ if and only if $i \geq j$, respectively. Finally, a monotone function is *g-transitive* if, given a function $g : \{0, 1\}^n \mapsto \mathbb{R}$, $x \preceq y$ holds if and only if $g(x) \leq g(y)$, where $\mathbb{R}$ denotes the set of real numbers. For example, if $g(x) = \sum_{j=1}^{n} x_j$, then a function f is monotone with respect to $\preceq$ if and only if it is a positive symmetric function, where a function f is called *symmetric* if $f(x) = f(y)$ holds for all $x, y \in \{0, 1\}^n$ with $\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j$.

For our discussion, the following theorem is essential; it says that $\mathcal{M}(X_\preceq)$ is a distributive sublattice of $\mathcal{B}(X)$.

Theorem 1. *For any binary relation $\preceq$ on X, $\mathcal{M}(X_\preceq)$ is closed under conjunction and disjunction, and contains $\top$ and $\bot$.*

Proof. Let $f, g \in \mathcal{M}(X_\preceq)$. By definition, for any $x \preceq y$, $x \in T(f)$ implies $y \in T(f)$ and $x \in T(g)$ implies $y \in T(g)$. Thus $x \in T(f) \cap T(g) (= T(f \wedge g))$ implies $y \in T(f) \cap T(g)$. Hence $f \wedge g$ is monotone with respect to $\preceq$, and $\mathcal{M}(X_\preceq)$ is closed under conjunction. Similarly for disjunction, since $x \in T(f) \cup T(g)$ and $x \preceq y$ obviously imply $y \in T(f) \cup T(g)$. The second statement $\top, \bot \in \mathcal{M}(X_\preceq)$ is also obvious since $\top$ and $\bot$ are monotone with respect to any relation $\preceq$.

Definition 1. *A subset $\mathcal{L} \subseteq \mathcal{B}(X)$ is called an* $\wedge$-semilattice *if $f \wedge g \in \mathcal{L}$ holds for all $f, g \in \mathcal{L}$. Such an $\mathcal{L}$ is called* topped *if $\top \in \mathcal{L}$.*

Example 1. There are $\wedge$-semilattices $\mathcal{L} \subseteq \mathcal{B}(X)$, which are not closed under disjunction. A Boolean function f is called *Horn* if it has a CNF (conjunctive normal form) such that each clause in it has at most one positive literal. It is well known that the class of all Horn functions $\mathcal{C}_{\text{Horn}}$ is closed under conjunction but not under disjunction.

As another example, let $X = \{1, 2, \ldots, n\}$ and let $f \in \mathcal{B}(\{0, 1\}^n)$ be a Horn function. Then each $x \in T(f)$ is considered as a map $x : X \mapsto \{0, 1\}$ (i.e., $T(x) = \{j \mid x_j = 1\}$). It is known that the class $\mathcal{H}_f = \{x \mid x \in T(f)\}$ is closed under conjunction but not under disjunction [19]. $\mathcal{H}_f$ is topped if $f(11 \cdots 1) = 1$ holds.

If we consider the complement of Horn functions in the above description, we can define classes which are closed under disjunction but not under conjunction.

Let $\mathcal{L} \subseteq \mathcal{B}$ be a topped $\wedge$-semilattice. Then $\mathcal{L}$ is not a sublatttice of $\mathcal{B}$, unless it is closed under disjunction. Nevertheless, even if it is not closed under disjunction, then we can define the operator $\sqcup$, where $\sqcup$ is not equal to $\vee$, such that $\mathcal{L}$ becomes a lattice, see ([11])

Lemma 1. *A topped $\wedge$-semilattice $\mathcal{L}$ is a lattice, where $f \sqcap g = f \wedge g$ and $f \sqcup g = \bigwedge\{h \mid f \vee g \leq h\}$ hold.*

3 The Quasi-Order Induced by $\mathcal{L}$

In this section, we show that a topped $\wedge$-semilattice $\mathcal{L}$ on a finite set X induces a quasi-order $\sqsubseteq_{\mathcal{L}}$ on X. We then discuss relationships between $\mathcal{L}$ and $\mathcal{M}(X_{\sqsubseteq_{\mathcal{L}}})$, and between $\preceq$ and $\sqsubseteq_{\mathcal{L}}$ for $\mathcal{L} = \mathcal{M}(X_{\preceq})$.

3.1 Relationship between $\mathcal{L}$ and $\mathcal{M}(X_{\sqsubseteq})$

Definition 2. *Let X be a finite set, and let $\mathcal{L} \subseteq \mathcal{B}(X)$ be a topped $\wedge$-semilattice. Let*

$$m_x = \bigwedge\{g \in \mathcal{L} \mid x \in T(g)\},\ x \in X. \tag{1}$$

Then the relation $\sqsubseteq_{\mathcal{L}}$ induced by $\mathcal{L}$ is defined by

$$x \sqsubseteq_{\mathcal{L}} y \Leftrightarrow y \in T(m_x).$$

The subscript $\mathcal{L}$ of $\sqsubseteq$ is usually omitted unless confusion arises.

Note that $m_x \in \mathcal{L}$ holds since $\mathcal{L}$ is closed under conjunction. By definition (1), it is also obvious that

$$x \in T(m_x) \tag{2}$$

always hold.

Example 2. Let $\mathcal{L} = \mathcal{M}(X_{\leq})$, i.e., the class of monotone functions in the traditional sense. Then m_x is represented by the term obtained from the minterm of x by deleting all negative literals. For example, $x = (10011)$ has the minterm $x_1\bar{x}_2\bar{x}_3x_4x_5$ and the function m_x is represented by $x_1x_4x_5$. Thus $x \sqsubseteq y$ holds if and only if y satisfies the term $x_1x_4x_5$. In this case, it is not difficult to see that $x \sqsubseteq y \Leftrightarrow x \leq y$ holds.

For $\mathcal{L} = \mathcal{C}_{\text{Horn}}$, m_x is represented by the conjunction of all Horn clauses that contain x. As a single literal is a Horn clause, m_x for $x = (10011)$ for example is represented by its minterm $x_1\bar{x}_2\bar{x}_3x_4x_5$. Thus, $x \sqsubseteq y \Leftrightarrow x = y$.

Finally, let $\mathcal{L} = \mathcal{H}_f$ for a Horn function f (see Example 1 for its definition), where $X = \{1, 2, \ldots, n\}$. Denote $i \vdash_f j$ if all $x \in T(f)$ with $x_i = 1$ satisfy $x_j = 1$. Then $T(m_i) = \{j \mid i \vdash_f j\}$ holds, and we obtain $x \sqsubseteq y \Leftrightarrow i \vdash_f j$. By Boolean algebra, it can be shown that $i \vdash_f j$ holds if and only if clause $(\bar{x}_i \vee x_j)$ is an implicate of f.

Lemma 2. *Let $\mathcal{L} \subseteq \mathcal{B}(X)$ be a topped $\wedge$-semilattice, and let $\sqsubseteq$ be the relation induced by $\mathcal{L}$. Then $x \sqsubseteq y$ holds if and only if $m_y \leq m_x$ holds.*

Proof. If $x \sqsubseteq y$, then $y \in T(m_x)$. Since $m_x \in \mathcal{L}$, this implies $m_y = \bigwedge\{g \in \mathcal{L} \mid y \in T(g)\} \leq m_x$. Conversely, assume $m_y \leq m_x$. By (2), we have $y \in T(m_y) \subseteq T(m_x)$, implying $x \sqsubseteq y$.

Lemma 3. *Let $\mathcal{L} \subseteq \mathcal{B}(X)$ be a topped $\wedge$-semilattice, and let $\sqsubseteq$ be the relation induced by $\mathcal{L}$. Then $\sqsubseteq$ is a quasi-order on X.*

Proof. By (2), $x \in T(m_x)$ holds, i.e., $\sqsubseteq$ is reflexive. Now let $x \sqsubseteq y$ and $y \sqsubseteq z$. By Lemma 2, we have $m_y \leq m_x$ and $m_z \leq m_y$, and hence $m_z \leq m_x \Leftrightarrow x \sqsubseteq z$; i.e., $\sqsubseteq$ is transitive. Thus $\sqsubseteq$ is a quasi-order.

In the next lemma we consider the class of all functions which are monotone with respect to the quasi-order $\sqsubseteq$ induced by a topped $\wedge$-semilattice $\mathcal{L}$. This class is denoted by $\mathcal{M}(X_\sqsubseteq)$. Define the *disjunctive closure* of $\mathcal{L}$ by

$$Cl_\vee(\mathcal{L}) = \{g \mid g = \bigvee_{f \in S} f,\ S \subseteq \mathcal{L}\}$$

Lemma 4. *Let $\mathcal{L} \subseteq \mathcal{B}(X)$ be a topped $\wedge$-semilattice, and let $\sqsubseteq$ be the quasi-order induced by $\mathcal{L}$. Then $\mathcal{L} \subseteq \mathcal{M}(X_\sqsubseteq)$ and $Cl_\vee(\mathcal{L}) = \mathcal{M}(X_\sqsubseteq)$.*

Proof. Take any $f \in \mathcal{L}$. Then $x \in T(f) \Rightarrow m_x \leq f$ (by (1)), and therefore, $x \in T(f)$ and $x \sqsubseteq y$ imply $y \in T(m_x) \subseteq T(f)$. This shows that f is monotone with respect to $\sqsubseteq$, i.e., $f \in \mathcal{M}(X_\sqsubseteq)$, and hence $\mathcal{L} \subseteq \mathcal{M}(X_\sqsubseteq)$. To prove the second statement, it suffices to show that any $f \in \mathcal{M}(X_\sqsubseteq)$ belongs to $Cl_\vee(\mathcal{L})$, since $Cl_\vee(\mathcal{L}) \subseteq \mathcal{M}(X_\sqsubseteq)$ is clear from Theorem 1 (i.e., $\mathcal{M}(X_\sqsubseteq)$ is closed under disjunction). For such f, take an arbitrary $x \in T(f)$. Then any $y \in T(m_x)$ (i.e., $x \sqsubseteq y$) satisfies $y \in T(f)$. Hence $m_x \leq f$ by (1). Therefore,

$$f = \bigvee_{x \in T(f)} m_x, \tag{3}$$

since $\bigvee m_x \leq f$ is implied by $m_x \leq f$ and $f \leq \bigvee m_x$ is implied by $m_x(x) = 1$ for all $x \in T(f)$.

Corollary 1. *If $\mathcal{L} \subseteq \mathcal{B}(X)$ is closed under conjunction and disjunction, and contains $\top$ and $\bot$, then $\mathcal{L} = \mathcal{M}(X_{\sqsubseteq})$ holds for the quasi-order $\sqsubseteq$ induced by $\mathcal{L}$.*

Putting Theorem 1 and this corollary together, we have the next theorem.

Theorem 2. *A class of functions $\mathcal{L} \subseteq \mathcal{B}(X)$ is closed under conjunction and disjunction, and contains $\top$ and $\bot$, if and only if it is the class of monotone functions with respect to a quasi-order $\preceq$ on X.*

Note that this theorem does not exclude the possibility that $\mathcal{M}(X_{\preceq_1}) = \mathcal{M}(X_{\preceq_2})$ for some $\preceq_1$ and $\preceq_2$, where $\preceq_1$ is a quasi-order but $\preceq_2$ is not.

Example 3. As a case in which X is not equal to $\{0,1\}^n$, consider the class $\mathcal{L} = \mathcal{H}_f$ defined for a Horn function $f : \{0,1\}^n \mapsto \{0,1\}$ (see Examples 1 and 2). Then the set $\mathcal{M}(X_{\sqsubseteq})$ is the collection of functions u (cf. the notion of *upset* in [11]) such that $i \in T(u)$ and $i \vdash_f j$ imply $j \in T(u)$. It can be further shown, by the definition of $\vdash_f$ given in Example 2, that $\mathcal{M}(X_{\sqsubseteq}) = \mathcal{H}_{f'}$ holds, where f' is the Horn function given by $f' = \bigwedge_{i \vdash_f j}(\bar{x}_i \vee x_j)$. Now recall that, if $\mathcal{H}_f$ is closed under not only conjunction but also disjunction, then $\bar{f}$ is called *submodular* [14]. It is known that such an f has a CNF of the form $\bigwedge_{i \vdash_f j}(\bar{x}_i \vee x_j)$, which is the same as the above f'. This is not surprising because $\mathcal{M}(X_{\sqsubseteq})$ is closed under conjunction and disjunction (Theorem 1).

3.2 Relationship between $\preceq$ and $\sqsubseteq$

Let $\preceq$ be a binary relation on X. Recall that $\mathcal{M}(X_{\preceq})$ is a topped lattice closed under conjunction and disjunction (Theorem 1). Therefore, $\mathcal{L} = \mathcal{M}(X_{\preceq})$ induces a quasi-order $\sqsubseteq$ on X (Lemma 3). In this section, we discuss the relationship between the relations $\preceq$ and $\sqsubseteq$.

First of all, it is easy to see that

$$x \preceq y \Rightarrow x \sqsubseteq y \tag{4}$$

holds. For this, assume $x \preceq y$. Then any $g \in \mathcal{M}(X_{\preceq})$ satisfies $g(x) \leq g(y)$; i.e., $x \in T(g)$ implies $y \in T(g)$. Therefore $y \in T(m_x)$ holds, where $m_x = \bigwedge\{g \in \mathcal{M}(X_{\preceq}) \mid x \in T(g)\}$, concluding $x \sqsubseteq y$.

Theorem 3. *Let $\sqsubseteq$ be the quasi-order induced by $\mathcal{M}(X_{\preceq})$, where $\preceq$ is a binary relation on X. Then $\mathcal{M}(X_{\preceq}) = \mathcal{M}(X_{\sqsubseteq})$.*

Proof. Note first that (4) implies $\mathcal{M}(X_{\sqsubseteq}) \subseteq \mathcal{M}(X_{\preceq})$. To prove the converse, i.e., $\mathcal{M}(X_{\preceq}) \subseteq \mathcal{M}(X_{\sqsubseteq})$, take an $f \in \mathcal{M}(X_{\preceq})$. Then for any $x \in T(f)$, we have $m_x \leq f$ by definition (1). Therefore, $x \in T(f)$ and $x \sqsubseteq y$ imply $y \in T(m_x) \subseteq T(f)$, i.e., f is monotone with respect to $\sqsubseteq$. This proves $f \in \mathcal{M}(X_{\sqsubseteq})$.

Let us define the *reflexive transitive closure* of a binary relation $\preceq$ as the smallest quasi-order that contains $\preceq$. We now show, via a few lemmas, that the relation $\sqsubseteq$ is the reflexive transitive closure of $\preceq$. Define a function $\uparrow x$ by

$$T(\uparrow x) = \{y \in X \mid x \preceq y\}. \tag{5}$$

Note that $T(\uparrow x)$ can be empty, since $\preceq$ may even not be reflexive.

Lemma 5. *Let $\preceq$ be a binary relation on X. Then:*

a) $\preceq$ *reflexive* $\Leftrightarrow x \in T(\uparrow x)$ *for all* $x \in X$,
b) $\preceq$ *transitive* $\Leftrightarrow \uparrow x \in \mathcal{M}(X_{\preceq})$ *for all* $x \in X$.

Proof. a) Immediate from the definition of $\uparrow x$ in (5).

b) Suppose $\preceq$ is transitive. To prove that $\uparrow x$ is monotone, let $y \in T(\uparrow x)$ and $y \preceq z$. Then we have $x \preceq y$ and $y \preceq z$, and $x \preceq z$ by transitivity. Therefore, $z \in T(\uparrow x)$. This implies $\uparrow x \in \mathcal{M}(X_{\preceq})$. To prove the converse, assume that $x \preceq y$ and $y \preceq z$, but $x \not\preceq z$. Then $y \in T(\uparrow x)$ and $z \notin T(\uparrow x)$ for $y \preceq z$. This shows that $\uparrow x$ is not monotone with respect to $\preceq$.

Lemma 6. *Let $\preceq$ be a binary relation on X, and let $m_x = \bigwedge\{g \in \mathcal{M}(X_{\preceq}) \mid x \in T(g)\}$. Then $\preceq$ is a quasi-order if and only if $\uparrow x = m_x$ holds for all $x \in X$.*

Proof. First assume that $\preceq$ is a quasi-order. By Lemma 5, we have $x \in T(\uparrow x)$ and $\uparrow x \in \mathcal{M}(X_{\preceq})$. Then $x \in T(\uparrow x)$ implies $m_x \leq \uparrow x$. We now show $\uparrow x \leq m_x$, i.e., $\uparrow x \leq g$ for all $g \in \mathcal{M}(X_{\preceq})$ with $g(x) = 1$. For this, assume $g(x) = 1$ and $y \in T(\uparrow x)$. Then, since $x \preceq y$ and $x \in T(g)$, we have $y \in T(g)$. This proves $\uparrow x \leq g$.

To prove the converse, assume $\uparrow x = m_x$. Then $\uparrow x(x) = m_x(x) = 1$ holds. Furthermore $m_x \in \mathcal{M}(X_{\preceq})$ is clear because $\mathcal{M}(X_{\preceq})$ is closed under conjunction. Thus $\preceq$ is a quasi-order by Lemma 5.

Theorem 4. *Let $\preceq$ be a binary relation on X. Then the quasi-order $\sqsubseteq$ induced by $\mathcal{M}(X_{\preceq})$ is the reflexive transitive closure of $\preceq$.*

Proof. Considering (4), it is sufficient to prove that $\preceq$ equals $\sqsubseteq$ whenever $\preceq$ is a quasi-order on X. Let $x \sqsubseteq y$. Since $x \sqsubseteq y \Leftrightarrow y \in T(m_x)$ and $\preceq$ is a quasi-order by assumption, Lemma 6 says that $x \sqsubseteq y \Leftrightarrow y \in T(\uparrow x)$. Thus $x \preceq y$.

Corollary 2. *If $\preceq$ is a quasi-order on a finite set X, then $\preceq = \sqsubseteq$ holds, where $\sqsubseteq$ is the quasi-order induced by $\mathcal{M}(X_{\preceq})$.*

3.3 Disjunctive representation of generalized monotone functions

In this subsection, suppose that $\preceq$ is a quasi-order on X. Then we define an equivalence relation μ on X by $x\mu y \Leftrightarrow m_x = m_y$. According to lemma 6, we have $x\mu y \Leftrightarrow (\uparrow x = \uparrow y) \Leftrightarrow (x \preceq y$ and $y \preceq x)$. The equivalence classes $[x]_\mu$ form a partially ordered set denoted by X/μ. Now, given an $f \in \mathcal{M}(X_\preceq)$, it is easy to see that every equivalence class $[x]_\mu$ satisfies either $[x]_\mu \subseteq T(f)$ or $[x]_\mu \subseteq F(f)$. Let

$$\min T(f) = \{x \in T(f) \mid \text{ no } y \in T(f) \text{ satisfies } y \preceq x \text{ and } x \not\preceq y\}.$$

As $\min T(f)$ is also a disjoint union of some equivalence classes $[x]_\mu$, we select one representative from each equivalence class and denote the resulting set of representatives by $R(\min T(f))$. The next lemma describes a method to represent a monotone function.

Theorem 5. *Let $\preceq$ be a quasi-order on X, and let $f \in \mathcal{M}(X_\preceq)$. Then f has the disjunctive representation:*

$$f = \bigvee_{x \in R(\min T(f))} m_x. \tag{6}$$

This representation is irredundant (in the sense that no m_x can be removed without changing the function f) and is unique.

Proof. First note that $x \preceq y \Leftrightarrow \uparrow y \leq \uparrow x \Leftrightarrow m_y \leq m_x$, by Lemma 6. Thus it is clear that representation (3) leads to the above representation (6). The representation (6) is irredundant and unique, since, by definition, two $x, y \in R(\min T(f))$ satisfy neither $x \preceq y$ nor $y \preceq x$, and any x, y in the same equivalence class $[z]_\mu$ satisfy $m_x = m_y$.

Theorem 5 is an extension of the result known as the unique DNF form of prime implicants for the standard monotone functions [16, 21]. However, in the general case m_x is not necessarily a conjunction of literals.

Example 4. Let $X = \{0,1\}^4$, and define a quasi-order $\preceq$ by $x \preceq y \Leftrightarrow x_1+x_2 \leq y_1+y_2$, and $x_3 + x_4 \leq y_3 + y_4$. Now consider the function f defined by $f = x_1x_2 \vee (x_1x_3x_4 \vee x_2x_3x_4)$. Then f is monotone with respect to $\preceq$. Furthermore, $R(\min T(f))$ is for example given by the set of equivalence classes $\{(1100), (0111)\}$, and the unique representation (6) of f becomes

$$f = m_{(1100)} \vee m_{(0111)} = x_1x_2 \vee (x_1x_3x_4 \vee x_2x_3x_4).$$

3.4 Dual theory of generalized monotone functions

The results obtained so far can be dualized in a way that is similar to the 'principle of duality' known in Boolean algebra and lattice theory (cf., [11, 16, 21, 22]). We present a summary of such results in this subsection, without detailed proofs, since most of them can be done in the symmetric manner.

Call a subset $\mathcal{L} \subseteq \mathcal{B}(X)$ an $\vee$-*semilattice* if $f \vee g \in \mathcal{L}$ holds for all $f, g \in \mathcal{L}$. Such an $\mathcal{L}$ is *bottomed* if $\perp \in \mathcal{L}$. Even if it is not closed under conjunction, Theorem 1 can be modified to show that a bottomed $\vee$-semilattice is a lattice, in which $\sqcap$ is not equal to $\wedge$.

Definition 2 is modified as follows. Given a bottomed $\vee$-semilattice $\mathcal{L} \subseteq \mathcal{B}(X)$, let

$$M_x = \bigvee \{g \in \mathcal{L} \mid x \in F(g)\}, \ x \in X. \tag{7}$$

Then the relation $\sqsubseteq_{\mathcal{L}}$ induced by $\mathcal{L}$ is defined by: $y \sqsubseteq_{\mathcal{L}} x \Leftrightarrow y \in F(M_x)$. It is not difficult to see that this definition of $\sqsubseteq_{\mathcal{L}}$ is the same as that in Definition 2, if $\mathcal{L}$ is closed both under conjunction and disjunction.

In the standard case of $X = \{0, 1\}^n$ and $\preceq = \leq$, M_w for $w \in X$ is given by the disjunction of literals x_j such that $w_j = 0$. For exmple, $w = (010110)$ gives $M_w = x_1 \vee x_3 \vee x_6$.

Define the *conjunctive closure* of $\mathcal{L}$ by

$$Cl_{\wedge}(\mathcal{L}) = \{g \mid g = \bigwedge_{f \in S} f, \ S \subseteq \mathcal{L}\}$$

Also define the function $\downarrow x$ by

$$T(\downarrow x) = \{y \in X \mid y \preceq x\}. \tag{8}$$

Then the whole discussion in Section 3 can be dualized just by considering the following correspondences: $\vee \leftrightarrow \wedge$, $m_x \leftrightarrow M_x$ and $\uparrow x \leftrightarrow \downarrow x$. Note that the statement $\uparrow x = m_x$ in Lemma 6 should read $\downarrow x = \bar{M}_x$ (i.e., complemented), and the representation (3) in Theorem 4 becomes

$$f = \bigwedge_{x \in F(f)} M_x.$$

Define $\max F(f)$ by

$$\max F(f) = \{x \in F(f) \mid \text{ no } y \in F(f) \text{ satisfies } x \preceq y \text{ and } y \not\preceq x\}.$$

Using the equivalence relation μ defined by $x \mu y \Leftrightarrow M_x = M_y$, we can define $R(\max F(f))$ by selecting one representative from each equivalence class of μ. Then we have the following dual version of Lemma 5.

Theorem 6. *Let $\preceq$ be a quasi-order on X, and let $f \in \mathcal{M}(X_{\preceq})$. Then f has the conjunctive representation:*

$$f = \bigwedge_{x \in R(\max F(f))} M_x. \tag{9}$$

This representation is irredundant (in the sense that no M_x is removed without changing the function f) and is unique.

Example 5. Consider the $X, \preceq$ and f in Example 4. Then $\max F(f)$ has two equivalence classes $\{(0011\}$ and $\{0101, 0110, 1001, 1010).\}$ Thus $R(\max F(f))$ is for example given by $\{(0011), (0101)\}$. By applying (7) to this case, we have $M_{(0011)} = x_1 x_2$ and $M_{(0101)} = (x_1 \vee x_3)(x_1 \vee x_4)(x_2 \vee x_3)(x_2 \vee x_4)$. Therefore the conjunctive representation (9) of f becomes

$$f = (x_1 \vee x_2)((x_1 \vee x_3)(x_1 \vee x_4)(x_2 \vee x_3)(x_2 \vee x_4)),$$

which is of course equal to $x_1 x_2 \vee x_1 x_3 x_4 \vee x_2 x_3 x_4$ obtained in Example 4.

4 $\mathcal{M}$-operators on $\mathcal{B}(X)$

In this section we discuss operators on $\mathcal{B}(X)$ that are conjunctive, i.e mappings of the form $\psi : \mathcal{B}(X) \mapsto \mathcal{B}(X)$, that satisfy the condition: $\forall f, g \in \mathcal{B}(X) : \psi(f \wedge g) = \psi(f) \wedge \psi(g)$; or that are disjunctive $\forall f, g \in \mathcal{B}(X) : \psi(f \vee g) = \psi(f) \vee \psi(g)$. Both conjunctive and disjunctive operators are order preserving (monotone): $\forall f, g \in \mathcal{B}(X) : f \leq g \Rightarrow \psi(f) \leq \psi(g)$. In Boolean function theory these mappings arise in the context of approximation operators. As a typical example we mention the mapping $f \mapsto \nabla(f)$, where $\nabla(f)$ denotes the largest positive function contained in f. Some early observations on approximation operators can already be found in [16, 21, 22]. These operators have been used by Bioch and Ibaraki [3, 4] in the framework of decompositions. The idea of approximation operators has been generalized for Boolean functions by considering other orderings on $\{0, 1\}^n$ than the standard partial ordering $\leq$. This has been done by Bshouty [7] in the context of computational learning theory, and by Khardon and Roth [18] in the context of reasoning. In this section we generalize these ideas further to operators on $\mathcal{B}(X)$ that are conjunctive or disjunctive, where X is a (finite) set. (Note, that the condition $|X| \leq \infty$ is not always necessary). It appears that in this general setting approximation operators are highly related to the modal operators of necessity and possibility: $\Box$ and $\Diamond$ used in modal logic [8]. Therefore, many properties of the operators that depend on the properties of the relation $\preceq$ on X discussed here, and more, can be found in the literature on modal logic. The main difference with our discussion and the abstract framework in modal logic is that in modal logic the modal operators are applied to logical expressions rather than to (generalized) Boolean functions. In the next section we prove that the monoid of all conjunctive/disjunctive operators on $\mathcal{B}(X)$ is algebraically isomorphic to the monoid of all binary relations on the set X. Furthermore, it appears that there is a one-to-one correspondence between approximation operators and quasi-orders. In the last section of this paper we use the theory on (generalized) approximation operators to investigate the lattice structure of the version space of all monotone hypotheses on a binary data set.

4.1 Approximation operators

For a function $f \in \mathcal{B}(X)$, we call a function $g \in \mathcal{B}(X)$ a *major* (*minor*) of f if $f \leq g$ ($g \leq f$). It is *positive* (*negative*) if $g \in \mathcal{M}(X_{\preceq})$ ($g \in \mathcal{M}(X_{\preceq^I})$, where the inverse order $\preceq^I$ of $\preceq$ is defined by $x \preceq^I y \Leftrightarrow y \preceq x$. Then the largest positive minor and the smallest positive major of f are respectively defined as follows:

$$\triangledown_{\preceq}(f) = \bigvee\{g \mid g \text{ is a positive minor of } f\},$$
$$\blacktriangle_{\preceq}(f) = \bigwedge\{g \mid g \text{ is a positive major of } f\}.$$

The largest negative minor and the smallest negative major of are similarly defined. These operators are respectively denoted by $\blacktriangledown_{\preceq}$ and $\triangle_{\preceq}$. We will refer to the operators defined here as: *approximation operators.*

Obviously, if $f \in \mathcal{M}(X_{\preceq})$, then $f = \triangledown(f) = \blacktriangle(f)$ holds, and if $f \in \mathcal{M}(X_{\preceq^I})$, then $f =\triangle (f) = \blacktriangledown(f)$ holds. It easily follows from the definition that the approximation operators are are all order preserving(monotone); e.g, $f \leq g \Rightarrow \triangledown(f) \leq \triangledown(g)$. In the next fundamental lemma we show that the operator $\triangledown_{\preceq}$ is conjunctive. The negation operator $\neg$ used in this lemma is defined as follows: $\forall f \in \mathcal{B}(X) : \neg(f)(x) = \bar{f}(x)$, where $\bar{f}$ denotes the complement of f: $\bar{f}(x) = 1 - f(x)$. In the following we also use the obvious but important observation: $f \in \mathcal{M}(X_{\preceq}) \Leftrightarrow \bar{f} \in \mathcal{M}(X_{\preceq^I})$.

Lemma 7. *Let $\preceq$ be a relation on X. Then:*

a) $\forall f, g \in \mathcal{B}(X) : \triangledown_{\preceq}(f \wedge g) = \triangledown_{\preceq}(f) \wedge \triangledown(g)$.
b) $\triangledown_{\preceq} = \neg \triangle_{\preceq} \neg$.

Proof. a) Since the operator $\triangledown_{\preceq}$ is monotone, it follows that $\triangledown_{\preceq}(f \wedge g) \leq \triangledown_{\preceq}(g) \wedge \triangledown(g)$. Conversely, since $\triangledown_{\preceq}(f), \triangledown_{\preceq}(g) \in \mathcal{M}_{\preceq}$ and $\mathcal{M}(X_{\preceq})$ is closed under intersection we

have in addition: $\nabla_{\preceq}(g) \wedge \nabla(g) \leq \nabla_{\preceq}(f \wedge g)$.
b) This is immediate from the definition.

Now, in order to examine how to compute these functions, we restrict ourselves to the case of Boolean functions: $X = \{0,1\}^n$. Furthermore, we will restrict ourselves to binary relations that are *self-dual*.

Definition 3. *Let $\preceq$ be a relation on $X = \{0,1\}^n$. Then $\preceq$ is called* self-dual *if $x \preceq y \Leftrightarrow \bar{x} \preceq^I \bar{y} \Leftrightarrow \bar{y} \preceq \bar{x}$.*

Finally, let $\preceq$ be a binary relation on X. Then the reflexive transitive closure of $\preceq$ is denoted by $[\preceq]$. Obviously, $[\preceq]$ is the smallest quasi-order that contains $\preceq$.

Lemma 8. *Let $\preceq$ be a binary relation on $X = \{0,1\}^n$. If $\preceq$ is self-dual, then $[\preceq]$ is also self-dual.*

Proof. This is immediate from the definitions.

It is easy to see that the standard partial order on X and the order used in the definition of regular functions are self-dual. For a Boolean function $f \in \mathcal{B}(X)$, f^* and f^d are defined by

$$T(f^*) = \{\bar{x} \mid x \in T(f)\} \text{ and } T(f^d) = \{x \mid \bar{x} \in F(f)\},$$

where $\bar{x}$ is the binary vector obtained from x by complementing all elements. This may be alternatively denoted by $f^*(x) = f(\bar{x})$ and $f^d(x) = \bar{f}(\bar{x})$. The function f^d is known as the *dual function* of f. However, if $f \in \mathcal{M}(X_{\preceq}))$ then *not* necessarily $f^d \in \mathcal{M}(X_{\preceq})$.

Lemma 9. *Let $\preceq$ be a self-dual relation on X, and let $f \in \mathcal{M}(X_{\preceq})$ then $f^d \in \mathcal{M}(X_{\preceq})$.*

Proof. Since f^d is the negation of f^*, it is sufficient to prove that $f \in \mathcal{M}(X_{\preceq}) \Rightarrow f^* \in \mathcal{M}(X_{\preceq^I})$. So, let $x \preceq^I y \Leftrightarrow y \preceq x$. Then, by the self-duality of $\preceq$ we have $\bar{x} \preceq \bar{y}$. Since f is monotone this implies $f(\bar{x}) \leq f(\bar{y})$. Hence we have proved that $f^* \in \mathcal{M}(X_{\preceq^I})$.

The following relations between the approximation operators are already known in the theory of Boolean functions [16, 22] in the case that $\preceq$ equals the standard partial order $\leq$. However, here we generalize this result to the the case that $\preceq$ is self-dual.

Lemma 10. *Let $\preceq$ be a self-dual binary relation on $X = \{0,1\}^n$. Then the approximation operators $\nabla, \blacktriangle, \blacktriangledown, \triangle$, are related as follows.*

a) $\triangledown = *\blacktriangledown* = d\blacktriangle d = \neg \vartriangle \neg$,
b) $\blacktriangle = * \vartriangle * = d\triangledown d = \neg\blacktriangledown\neg$,
c) $\blacktriangledown = *\triangledown* = d \vartriangle d = \neg\blacktriangle\neg$,
d) $\vartriangle = *\blacktriangle* = d\blacktriangledown d = \neg\triangledown\neg$.

Proof. By definition $\triangledown_{\preceq}(f) = \bigvee\{g \mid g$ is a positive minor of $f\}$. Therefore, $\triangledown(f) = d\bigwedge\{g^d \mid g$ is a positive minor of $f\}$. Since by Lemma 9 g^d is a *positive* major of f^d, we have $\triangledown(f) = d\bigwedge\{h \mid h$ is a positive major of $f^d\} = d\blacktriangle d(f)$. The other results are proved similar.

Furthermore, in the standard case of Boolean functions we have [3, 4, 16, 21]:

Lemma 11. *Let f be a Boolean function, and assume that $\preceq = \leq$. Then:*

a) $\triangledown(f)$: *Remove negative literals in a CNF of f.*
b) $\blacktriangle(f)$: *Remove negative literals in a DNF of f.*
c) $\blacktriangledown(f)$: *Remove positive literals in a CNF of f.*
d) $\vartriangle(f)$: *Remove positive literals in a DNF of f.*

Note that, if in the above process all literals in a term of a DNF are removed, then the DNF becomes $\top$. Similarly, if all literals in a clause of a CNF are removed, then the CNF becomes $\bot$.

Example 6. i) Consider the Boolean f function defined by:

$$f = x_1\bar{x}_2 \vee x_2x_3.$$

Then $f^d = (x_1 \vee \bar{x}_2)(x_2 \vee x_3) = x_1x_2 \vee x_1x_3 \vee \bar{x}_2x_3$ and hence f has the following CNF:

$$f = (x_1 \vee x_2)(x_1 \vee x_3)(\bar{x}_2 \vee x_3).$$

Therefore, by Lemma 11, we obtain $\triangledown(f) = (x_1 \vee x_2)(x_1 \vee x_3)x_3 = x_1x_3 \vee x_2x_3$, $\blacktriangle(f) = x_1 \vee x_2x_3$, $\blacktriangledown(f) = \bot$ and $\vartriangle(f) = \top$.

ii) $f = x_1\bar{x}_2 \vee \bar{x}_1x_2 = (x_1 \vee x_2)(\bar{x}_1 \vee \bar{x}_2)$. Then we obtain: $\triangledown(f) = \bot$, $\blacktriangle(f) = x_1 \vee x_2$, $\blacktriangledown(f) = \bot$ and $\vartriangle(f) = \bar{x}_1 \vee \bar{x}_2$.

4.2 Isomorphism between operators and relations

In this subsection we show that the collection of all conjunctive operators is isomorphic to the set of all binary relations on X, when viewed as monoids. In this development, we introduce the box-operator $\Box_{\preceq}$ defined for a binary relation $\preceq$ and show some of its properties and its relationship to the largest positive minor of a function with respect to this relation.

Definition 4. *An operator* $\psi : \mathcal{B}(X) \mapsto \mathcal{B}(X)$ *is* conjunctive *if the following two properties hold.*

a) $\psi(f \wedge g) = \psi(f) \wedge \psi(g)$, *for* $f, g \in \mathcal{B}(X)$,
b) $\psi(\top) = \top$.

The collection of all conjunctive operators is denoted by $\mathcal{O}_\wedge(\mathcal{B}(X))$.

The conjunctive operators are just the homomorphisms of $\mathcal{B}(X)$ viewed as a topped semi-lattice with respect to conjunction. Note that $\mathcal{O}_\wedge(\mathcal{B}(X))$ is a monoid under composition of operators, since the composition $\psi_1\psi_2$ of $\psi_1, \psi_2 \in \mathcal{O}_\wedge(\mathcal{B}(X))$ is also a conjunctive operator (because $\psi_1\psi_2(f \wedge g) = \psi_1(\psi_2(f) \wedge \psi_2(g)) = \psi_1\psi_2(f) \wedge \psi_1\psi_2(g)$). In this monoid, the identity operator $\psi_=$ defined by $\psi_=(f) = f$ for all $f \in \mathcal{B}(X)$ is the 1-element, and the operator $\psi_\emptyset$ defined by $\psi_\emptyset(f) = \top$ for all $f \in \mathcal{B}(X)$ is the 0-element.

Lemma 12. *If* $\psi \in \mathcal{O}_\wedge(\mathcal{B}(X))$, *then* ψ *is monotone.*

Proof. A conjunctive operator ψ is order-preserving(monotone) in the sense that $f \leq g$ implies $\psi(f) \leq \psi(g)$, because $f \leq g \Leftrightarrow f \wedge g = f$ and hence $\psi(f) = \psi(f \wedge g) = \psi(f) \wedge \psi(g) \leq \psi(g)$.

Now we turn to the collection of all binary relations on X, and denote it as $\mathcal{R}(X)$. We recall that the composition $\circ$ of two binary relations $\preceq_1$ and $\preceq_2$ is defined as follows for $x, y \in X$: $x(\preceq_1 \circ \preceq_2)y \Leftrightarrow \exists z \in X$ such that $x \preceq_1 z$ and $z \preceq_2 y$. Since $\preceq_1, \preceq_2 \in \mathcal{R}(X)$ clearly implies $\preceq_1 \circ \preceq_2 \in \mathcal{R}(X)$, $\mathcal{R}(X)$ is a monoid under composition, in which the equality relation $=$ is the 1-element and the empty relation $\emptyset$ (i.e., no $x, y \in X$ satisfies $x\emptyset y$) is the 0-element.

To prove the isomorphism between $\mathcal{R}(X)$ and $\mathcal{O}_\wedge(\mathcal{B}(X))$ as monoids, we introduce the next definitions.

Definition 5. *For a binary relation* $\preceq$ *on* X, *let* $\Box_\preceq$ *be the operator defined by*

$$x \in T(\Box_\preceq(f)) \;\Leftrightarrow\; \uparrow x \leq f,$$

where $\uparrow x$ *was defined in (5). Then* $\Box_\preceq(f) \in \mathcal{B}(X)$ *is called the* positive content *of* f.

Note that, although the function $\uparrow x$ is possibly equal to $\bot$, we always have $x \in T(\Box_\preceq(\uparrow x))$.

Lemma 13. *The operator* $\Box_\preceq$ *is conjunctive.*

Proof. The conditions $\Box_{\preceq}(f \wedge g) = \Box_{\preceq}(f) \wedge \Box_{\preceq}(g)$ and $\Box_{\preceq}(\top) = \top$ are immediate from the definition.

The following definition shows that every conjunctive operator induces a binary relation on X.

Definition 6. *Let ψ be a conjunctive operator on $\mathcal{B}(X)$. Then $\preceq_\psi \in \mathcal{R}(X)$ is defined by $x \preceq_\psi y \Leftrightarrow y \in T(m_x^\psi)$, where*

$$m_x^\psi = \bigwedge\{g \in \mathcal{B}(X) \mid x \in T(\psi(g))\}.$$

In analogy with $\uparrow x$, let the function $\uparrow_\psi x$ be defined by $T(\uparrow_\psi x) = \{y \in X \mid x \preceq_\psi y\}$. Then we have:

$$x \preceq_\psi y \Leftrightarrow y \in T(m_x^\psi) \Leftrightarrow m_x^\psi = \uparrow_\psi x. \tag{10}$$

We now show that the map: $\mathcal{R}(X) \mapsto \mathcal{O}_\wedge(\mathcal{B}(X))$ defined by: $\preceq \mapsto \Box_{\preceq}$, and the inverse map: $\mathcal{O}_\wedge(\mathcal{B}(X)) \mapsto \mathcal{R}(X)$ defined by: $\psi \mapsto \preceq_\psi$, are both bijections.

Lemma 14. *Let $\preceq, \Box_{\preceq}, \psi$ and $\preceq_\psi$ be defined as in Definitions 5 and 6. Then the following properties hold.*

a) $\preceq_{\Box_{\preceq}} = \preceq$, *for any* $\preceq \in \mathcal{R}(X)$,
b) $\psi_{\preceq_\psi} = \psi$, *for any* $\psi \in \mathcal{O}_\wedge(\mathcal{B}(X))$.

Proof. a) Denote $\Box = \Box_{\preceq}$ for simplicity. By definition, we have

$$m_x^\Box = \bigwedge\{g \mid x \in T(\Box(g))\} = \bigwedge\{g \mid \uparrow x \leq g\} = \uparrow x.$$

Therefore, we may conclude: $x \preceq_\Box y \;\Leftrightarrow\; y \in T(\uparrow x) \;\Leftrightarrow\; x \preceq y$.

b) Denote $\Box = \psi_{\preceq_\psi}$ for simplicity. We first show $\Box(f) \leq \psi(f)$ for all $f \in \mathcal{B}(X)$. Assume $x \in T(\Box(f))$. Then $x \in T(\Box(f)) \Leftrightarrow \uparrow_\psi x \leq f$ holds, and hence $\psi(\uparrow_\psi x) \leq \psi(f)$ (since ψ is order preserving). Now, from (10), we have $\psi(\uparrow_\psi x) = \bigwedge\{\psi(g) \mid x \in T(\psi(g))\}$. This implies $x \in T(\psi(\uparrow_\psi x))$ and hence $x \in T(\psi(f))$. Conversely we show $\psi(f) \leq \Box(f)$. Assume $x \in T(\psi(f))$. Then $m_x^\psi = \bigwedge\{g \mid x \in T(\psi(g))\} \leq f$. Now $\uparrow_\psi x = m_x^\psi \leq f$ implies $x \in T(\Box(f))$ by definition.

The next lemma shows that the bijections $\preceq \mapsto \Box_{\preceq}$ and $\psi \mapsto \preceq_\psi$, are both homomorphic in the sense that they preserve the monoid operations on respectively $\mathcal{O}_\wedge(\mathcal{B}(X))$ and $\mathcal{R}(X)$.

Lemma 15. *Let $\preceq_1, \preceq_2 \in \mathcal{R}(X)$ and $\psi_1, \psi_2 \in \mathcal{O}_\wedge(\mathcal{B}(X))$. Then the following properties hold.*

a) $\Box_{\preceq_1}\Box_{\preceq_2} = \Box_{(\preceq_1 \circ \preceq_2)}$
b) $\preceq_{\psi_1\psi_2} = \preceq_{\psi_1} \circ \preceq_{\psi_2}$.

Proof. a) Denote $\Box_1 = \Box_{\preceq_1}$ and $\Box_2 = \Box_{\preceq_2}$ for simplicity. Let $\preceq = \preceq_1 \circ \preceq_2$. Then $x \in (\Box_\preceq(f)) \Leftrightarrow \uparrow x \leq f$ holds, where $T(\uparrow x) = \{z \mid \ \exists y \text{ such that } x \preceq_1 y \preceq_2 z\}$. Therefore, we prove

$$x \in T(\Box_1\Box_2(f)) \Leftrightarrow \uparrow x \leq f, \tag{11}$$

for all $f \in \mathcal{B}(X)$. For this, we note that $x \in T(\Box_1\Box_2(f)) \Leftrightarrow \uparrow_1 x \leq \Box_2(f)$, where $\uparrow_i x$ is defined by $T(\uparrow_i x) = \{y \mid x \preceq_i y\}$. However, the latter condition is equivalent to saying that, for any y with $x \preceq_1 y$, $y \in T(\Box_2(f))$ (i.e., $\uparrow_2 y \leq f$) holds. The condition $\uparrow_2 y \leq f$ is equivalent to that any z with $y \preceq_2 z$ satisfies $z \in T(f)$. This proves (11).

b) This holds true because the map $\Box$ is bijective by Lemma 14, and is homomorphic by the above property a). Thus the map $\preceq$, which is the inverse of $\Box$ (Lemma 14), is homomorphic.

Combining Lemma 14 and Lemma 15 gives the following result:

Theorem 7. *The monoids $\mathcal{O}_\wedge(\mathcal{B}(X))$ and $\mathcal{R}(X)$ are algebraically isomorphic.*

4.3 Disjunctive $\mathcal{M}$-operators

The results on conjunctive operators can be dualized as follows. Call an operator $\psi : \mathcal{B}(X) \mapsto \mathcal{B}(X)$ *disjunctive* if it satisfies

a) $\psi(f \vee g) = \psi(f) \vee \psi(g)$, for $f, g \in \mathcal{B}(X)$,
b) $\psi(\bot) = \bot$.

The collection of all disjunctive operators ψ is denoted by $\mathcal{O}_\vee(\mathcal{B}(X))$. Note, that the disjunctive operators are just the homomorphism of $\mathcal{B}(X)$ viewed as a bottomed semi-lattice with respect to disjunction. For each binary relation $\preceq$, let the operator $\blacklozenge_\preceq$ be defined by:

$$x \in T(\blacklozenge_\preceq(f)) \Leftrightarrow \quad \uparrow x \wedge f \neq \bot.$$

Then $\blacklozenge_\preceq(f)$ is called the *positive closure* of f. Conversely, given a disjunctive operator ψ, define the following the function

$$M_x^\psi = \bigvee\{f \in \mathcal{B}(X) \mid x \in F(\psi(f))\}, \ x \in X. \tag{12}$$

Then the relation $\preceq_\psi$ induced by ψ is defined by $y \preceq_\psi x \Leftrightarrow y \in F(M_x^\psi)$.

It is possible to show that Lemmas 14 and 15 can be extended to the collection of disjunctive operators and Therefore, the following theorem holds:

Theorem 8. *The monoids $\mathcal{O}_\vee(\mathcal{B}(X))$ and $\mathcal{R}(X)$ are algebraically isomorphic.*

4.4 Properties of $\mathcal{M}$-operators

The properties of $\mathcal{M}$-operators are similar to those of the modal operators known in modal logic [8]. Therefore, we will mention a few properties which are either similar to those in modal logic or easy to prove.

For a binary relation $\preceq$, we have

$$\Box_\preceq = \neg\blacklozenge_\preceq\neg \quad \text{and} \quad \blacklozenge_\preceq = \neg\Box_\preceq\neg. \tag{13}$$

In addition to the positive content $\Box_\preceq$ and the positive closure $\Diamond_\preceq$, we introduce here two more operators $\blacksquare_\preceq$ and $\Diamond_\preceq$ called *negative content* and *negative closure*, respectively. In analogy with (5), define $\downarrow x$ by $x \in T(\downarrow x) = \{y \mid y \preceq x\}$. Then $\blacksquare_\preceq$ and $\Diamond_\preceq$ are defined by

$$x \in \blacksquare_\preceq(f) \Leftrightarrow \downarrow x \le f,$$
$$x \in \Diamond_\preceq(f) \Leftrightarrow \downarrow x \wedge f \neq \bot.$$

Obviously, these operators satisfy properties similar to those of $\Box_\preceq$ and $\blacklozenge_\preceq$.

Lemma 16. *Let $\preceq$ be a relation on X. and $f, g \in \mathcal{B}(X)$. Then:*

a) $\Box(f) \vee \Box(g) \le \Box(f \vee g)$.
b) $\Diamond(f) \wedge \Diamond(g) \le \Diamond(f \wedge g)$.

Lemma 17. *Let $\preceq$ be a relation on X. and $f \in \mathcal{B}(X)$. Then:*

a) $\preceq$ is reflexive $\Leftrightarrow \Box(f) \le f$.
b) $\preceq$ is symmetric $\Leftrightarrow f \le \Box\Diamond(f)$.
c) $f \in \mathcal{M}(X) \Leftrightarrow f \le \Box(f)$.

Lemma 18. *Let $\preceq$ be a relation on X. and $f \in \mathcal{B}(X)$. Then the following assertions are equivalent:*

a) $\preceq$ is transitive.
b) $f \le \Box(f)$.
c) $\Box(f) \in \mathcal{M}(X)$.
d) $\Box(f) \le \Box\Box(f)$.

4.5 Relationship between approximation operators and $\mathcal{M}$-operators

The next lemma shows that if the binary relation $\preceq$ is a quasi-order then the approximation and $\mathcal{M}$-operators are the same.

Lemma 19. *Let $\preceq$ be a binary relation on X. Then the following conditions are equivalent.*

a) $\preceq$ *is a quasi-order.*
b) $\triangle_{\preceq} = \lozenge_{\preceq}$.
c) $\triangledown_{\preceq} = \square_{\preceq}$.
d) $\blacktriangledown_{\preceq} = \blacksquare_{\preceq}$.
e) $\blacktriangle_{\preceq} = \blacklozenge_{\preceq}$.

Proof. This follows easily from the definitions.

The preceding Lemma and Theorem 7 imply:

Corollary 3. *There is a one-to-one correspondence between the collection of all quasi-orders $\preceq$ on X and the collection of all approximation operators $\triangledown_{\preceq}$.*

Recall that if $\preceq$ is a binary relation on X, then the reflexive transitive closure of $\preceq$ is denoted by $[\preceq]$. Since, $\mathcal{M}(X_{[\preceq]}) = \mathcal{M}(X_{\preceq})$ we have: $\triangle_{[\preceq]} = \triangle_{\preceq}$. Therefore, the preceding lemma implies that the collection of all approximation operators on $\mathcal{B}(X)$ is a (proper)-subclass of the class of all $\mathcal{M}$-operators:

Theorem 9. *Let $\preceq$ be a binary relation on X. Then:*

a) $\triangle_{\preceq} = \lozenge_{[\preceq]}$.
b) $\triangledown_{\preceq} = \square_{[\preceq]}$.
c) $\blacktriangledown_{\preceq} = \blacksquare_{[\preceq]}$.
d) $\blacktriangle_{\preceq} = \blacklozenge_{[\preceq]}$.

5 Monotone Extensions of Partially Defined Boolean Functions

In this section, we restrict ourselves to Boolean functions, i.e. $X = \{0,1\}^n$. Given a subset $D \subseteq X$. A function

$$f_D : D \mapsto \{0,1\}, \tag{14}$$

is called a *partially defined Boolean function* (pdBf). A pdBf is just a representation of a Boolean data set, and an extension of f is a Boolean function that is consistent with this data set. Extensions of partially defined Boolean functions have been extensively studied in machine learning in general and in logical analysis of data [6, 10] in particular. In machine learning an extension is also called a hypothesis and the collection of all extensions is called the version space [20]. It is easy to that in the case of Boolean functions the version space is a lattice. In this section we will investigate the lattice-structure of version spaces consisting of generalized monotone Boolean functions.

5.1 Preliminaries

Let f_D be a pdBf. Then:

$$\begin{aligned} T_D &= \{x \in D \mid f_D(x) = 1\}, \\ F_D &= \{x \in D \mid f_D(x) = 0\}, \end{aligned} \tag{15}$$

are respectively called the true and false sets of f_D. Two functions f_- and $f_+ : X \mapsto \{0,1\}$ are respectively defined by $T(f_-) = T_D$ and $T(f_+) = X \setminus F_D$, for which $f_- \leq f_+$ clearly holds.

Definition 7. *A Boolean function g is called an* extension *of a pdBf f_D if $f_- \leq g \leq f_+$ holds. The class of all extensions of f_D is denoted by $\mathcal{E}(f_D)$.*

It follows that each extension g agrees with f_D on D : $f(x) = f_D(x)$ for $x \in D$. The following lemma is immediate from the definitions.

Lemma 20. *For a pdBf f_D, $\mathcal{E}(f_D)$ is closed under conjunction and disjunction. Hence $\mathcal{E}(f_D)$ is a finite distributive lattice universally bounded by f_- and f_+.*

5.2 Lattices of generalized monotone extensions

In this subsection we consider version spaces consisting of generalized monotone Boolean functions. Therefore, we assume that $X = \{0,1\}^n$, and that $\preceq$ is an arbitrary relation on X.

Definition 8. *Let $\preceq$ be a binary relation on X. A Boolean function g is a* monotone *extension of a pdBf f_D with respect to $\preceq$ if $g \in \mathcal{E}(f_D) \cap \mathcal{M}(X_\preceq)$ holds. The class of all monotone extensions of f_D is given by $\mathcal{E}_\preceq(f_D) = \mathcal{E}(f_D) \cap \mathcal{M}(X_\preceq)$.*

Assume $\mathcal{E}_{\preceq}(f_D) \neq \emptyset$, and define

$$\begin{aligned} f_{\min} &= \bigwedge\{g \mid g \in \mathcal{E}_{\preceq}(f_D)\}, \\ f_{\max} &= \bigvee\{g \mid g \in \mathcal{E}_{\preceq}(f_D)\}, \end{aligned} \tag{16}$$

and

$$\begin{aligned} m_x &= \bigwedge\{g \in \mathcal{M}(X_{\preceq}) \mid x \in T(g)\}, \\ M_x &= \bigvee\{g \in \mathcal{M}(X_{\preceq}) \mid x \in F(g)\}. \end{aligned}$$

The following theorem shows that $\mathcal{E}_{\preceq}(f_D)$ is a universally bounded distributive lattice under conjunction and disjunction. Therefore, $\mathcal{E}_{\preceq}(f_D)$ is an interval of generalized monotone Boolean functions: $\mathcal{E}_{\preceq}(f_D) = [f_{min}, f_{max}]$.

Theorem 10. *If $\mathcal{E}_{\preceq}(f_D) \neq \emptyset$, then $f_{\min} \leq f_{\max}$, and we have*

$$\mathcal{E}_{\preceq}(f_D) = \{g \in \mathcal{M}(X_{\preceq}) \mid f_{\min} \leq g \leq f_{\max}\}.$$

Proof. The inequality $f_{\min} \leq f_{\max}$ follows from definition (16). The expression for $\mathcal{E}_{\preceq}(f_D)$ also follows from Definition (8) and (16). $\mathcal{E}_{\preceq}(f_D)$ is closed under conjunction and disjunction, since so are $\mathcal{E}(f_D)$ and $\mathcal{M}(X_{\preceq})$. Finally, $\mathcal{E}_{\preceq}(f_D)$ is universally bounded by f_{min} and f_{max}, since $f_{\min}, f_{\max} \in \mathcal{E}_{\preceq}(f_D)$ holds by (16).

Now we consider when $\mathcal{E}_{\preceq}(f_D) \neq \emptyset$ holds.

Lemma 21. *Let f_D, T_D, F_D and $\preceq$ be defined as above.*

a) $\mathcal{E}_{\preceq}(f_D) \neq \emptyset \Leftrightarrow T(\bigvee\{m_x \mid x \in T_D\}) \bigcap F_D = \emptyset \Leftrightarrow T_D \subseteq T(\bigwedge\{M_x \mid x \in F_D\})$.

b) If $\preceq$ is a quasi-order, then: $\mathcal{E}_{\preceq}(f_D) \neq \emptyset \Leftrightarrow$ no $x \in T_D$ and $y \in F_D$ satisfy $x \preceq y$.

Proof. a) Assume $\mathcal{E}_{\preceq}(f_D) \neq \emptyset$. Suppose $f \in \mathcal{E}_{\preceq}(f_D)$ and $x \in T_D$. Then $m_x \leq f$ holds since $f(x) = 1$. As f is an extension, we have $T(f) \cap F_D = \emptyset \Rightarrow T(m_x) \cap F_D = \emptyset$. Thus $\bigvee\{m_x \mid x \in T_D\} \wedge F_D = \emptyset$. Conversely, if $\bigvee\{m_x \mid x \in T_D\} \wedge F_D = \emptyset$, then $g = \bigvee\{m_x \mid x \in T_D\}$ is a monotone extension of f_D, proving that $\mathcal{E}_{\preceq}(f_D) \neq \emptyset$. This proves the first equivalence. The second equivalence can be proved in a similar manner just by dualizing the argument.

b) In this case, $m_x = \uparrow x$ by Lemma 6. Also $T(\uparrow x) \cap F_D = \emptyset \Leftrightarrow \exists y \in F_D$ such that $x \preceq y$. Thus, b) follows from the first part of a).

Note that the condition in b) of the above lemma can be checked in polynomial time in terms of the input length $n(|T_D| + |F_D|)$, assuming that the condition $y \preceq x$ can be checked in polynomial time. This was discussed in [4].

To derive other explicit formulas for $f_{\min}$ and $f_{\max}$, we further define

$$\min T_D = \{x \in T_D \mid \text{no } y \in T_D \text{ satisfies } y \preceq x \text{ and } x \not\preceq y\},$$
$$\max F_D = \{x \in F_D \mid \text{no } y \in F_D \text{ satisfies } x \preceq y \text{ and } y \not\preceq x\}.$$

Lemma 22. *If f_D is pdBf such that $\mathcal{E}_{\preceq}(f_D) \neq \emptyset$, then $f_{\min}$ and $f_{\max}$ are given by and*

$$f_{\min} = \bigvee \{m_x \mid x \in \min T_D\},$$
$$f_{\max} = \bigwedge \{M_x \mid x \in \max F_D\}.$$

Proof. Denote the right hand side of $f_{\min}$ by G. We first note that $G = \bigvee\{m_x \mid x \in T_D\}$ holds, since $x \preceq y \Rightarrow m_y \leq m_x$ holds by Lemma 2. Then $G \in \mathcal{E}_{\preceq}(f_D)$ as noted in the proof of Lemma 21 a). This implies $f_{\min} \leq G$ by the definition of $f_{\min}$. Conversely, note that $m_x \leq f_{\min}$ holds for all $x \in T_D$ because $f_{\min}(x) = 1$. This implies $m_x \leq f_{\min}$ for all $x \in T_D$ and hence $G \leq f_{\min}$.

The proof for $f_{\max}$ can be done similarly by dualizing the argument.

Example 7. Consider the case of positive functions, i.e., $\preceq = \leq$. In this case, m_x is obtained from the minterm of x by deleting negative literals, as discussed in Example 3. Similarly, M_x is obtained from the maxclause of $\bar{x}$ by deleting negative literals. For example, $x = (10011)$ has $m_x = x_1x_4x_5$ and $M_x = (x_2 \vee x_3)$. Let $T_D = \{(10011), (11001), (01111)\}$ and $F_D = \{(10010), (01010), (10101)\}$. Obviously $\min T_D = T_D$ and $\max F_D = F_D$ hold in this case. Then it follows that

$$\begin{aligned} f_{\min} &= x_1x_4x_5 \vee x_1x_2x_5 \vee x_2x_3x_4x_5, \\ f_{\max} &= (x_2 \vee x_3 \vee x_5)(x_1 \vee x_3 \vee x_5)(x_2 \vee x_4) \\ &= x_1x_2 \vee x_2x_3 \vee x_2x_5 \vee x_3x_4 \vee x_4x_5. \end{aligned}$$

5.3 The structure of the lattice $\mathcal{E}_{\preceq}(f_D)$

We will now study the structure of the lattice $\mathcal{E}_{\preceq}(f_D)$ in more detail. With a function $g \in \mathcal{M}(X_{\preceq})$, we can associate the monotone extension $\pi(g)$ of the pdBf f_D as follows:

$$\pi(g) = f_{\min} \vee g f_{\max}. \tag{17}$$

By Theorem 10, it is easy to see that π is a map from $\mathcal{M}(X_{\preceq})$ onto $\mathcal{E}_{\preceq}(f_D)$, and that respectively $\pi(g) = g$ if and only if $g \in \mathcal{E}_{\preceq}(f_D$, and π is idempotent, i.e., $\pi^2 = \pi$. It is also important to observe the following property.

Lemma 23. *The map π is a lattice homomorphism from $\mathcal{M}(X_{\preceq})$ onto $\mathcal{E}_{\preceq}(f_D)$.*

Proof. It was already noted that π maps $\mathcal{M}(X_{\preceq})$ onto $\mathcal{E}_{\preceq}(f_D)$. To show that π is homomorphism, we note that for all $g_1, g_2 \in \mathcal{M}(X_{\preceq})$:

$$\begin{aligned} \pi(g_1 \wedge g_2) &= \pi(g_1) \wedge \pi(g_2) \\ \pi(g_1 \vee g_2) &= \pi(g_1) \vee \pi(g_2). \end{aligned}$$

The first relation holds because

$$\begin{aligned} \pi(g_1) \wedge \pi(g_2) &= (f_{\min} \vee g_1 f_{\max})(f_{\min} \vee g_2 f_{\max}) \\ &= f_{\min} \vee g_1 f_{\min} f_{\max} \vee g_2 f_{\min} f_{\max} \vee g_1 g_2 f_{\max} \\ &= f_{\min} \vee g_1 g_2 f_{\max} \\ &= \pi(g_1 \wedge g_2) \end{aligned}$$

by Theorem 10. Similarly for the second relation.

Now define an equivalence relation θ on $\mathcal{M}(X_{\preceq})$ by

$$g_1 \theta g_2 \Leftrightarrow \pi(g_1) = \pi(g_2).$$

It is easy to see that

$$g_1 \theta g_2 \Leftrightarrow g_1 \oplus g_2 \leq f_{\min} \vee \bar{f}_{\max},$$

i.e., $g_1(x)$ and $g_2(x)$ can differ only if $x \in T(f_{\min}) \cup F(f_{\max})$. Let $[g]_\theta$ denote the equivalence class of g. Then according to standard lattice theory (e.g., [11]) θ is a so-called *congruence relation*, i.e. θ is an equivalence relation such that $\forall f \in \mathcal{M}(X_{\preceq})$: $f_1 \theta f_2 \Rightarrow f_1 f \theta f_2 f$, and we have:

Lemma 24. *Let f_D be a pdBf on X and let π and θ be defined as above. Furthermore, let $g \in \mathcal{M}(X_{\preceq})$. Then*

(a) $[g]_\theta$ is a sublattice of lattice $\mathcal{M}(X_{\preceq})$,
(b) $\mathcal{M}(X_{\preceq})/\theta \cong \mathcal{E}_{\preceq}(f_D)$ (where $\cong$ denotes isomorphism),
(c) π is order preserving, i.e., $g_1 \leq g_2 \Rightarrow \pi(g_1) \leq \pi(g_2)$.

5.4 Minimal representations of extensions

Let f_D be a partially defined Boolean function and $\preceq$ a binary relation on $X = \{0,1\}^n$. Since $\mathcal{M}_{\preceq}(X) = \mathcal{M}_{[\preceq]}(X)$, where $[\preceq]$ denotes the reflexive transitive closure of $\preceq$, we may assume that $\preceq$ is a quasi-order on X. Let $g \in \mathcal{M}_{\preceq}(X)$ Then according to section 3.3 we have the following irredundant and unique representation of g :

$$g = \bigvee_{x \in R(\min T(g))} m_x. \tag{18}$$

Recall that

$$\min T(g) = \{x \in T(g) \mid \text{ no } y \in T(g) \text{ satisfies } y \preceq x \text{ and } x \not\preceq y\},$$

and that $R(\min T(g))$ denotes a fixed set of representatives of the equivalence classes $[x]_\mu$ contained in $\min T(g)$. The equivalence relation μ on X was defined by $x\mu y \Leftrightarrow m_x = m_y$. Equivalently we have: $x\mu y \Leftrightarrow (\uparrow x = \uparrow y) \Leftrightarrow (x \preceq y \text{ and } y \preceq x)$.

Definition 9. *Let $x \in X$. Then the extension induced by x is defined by:*

$$e_x = \pi(m_x) = f_{\min} \vee m_x f_{\max}. \tag{19}$$

Let $x \in f_{\max}$. Since we assume that $\preceq$ is a quasi-order on X, we have $x \in T(m_x) \subseteq T(f_{\max})$. Therefore, in this case $e_x = f_{\min} \vee m_x$, and from definition (19) it follows that e_x is the smallest extension of f_D that contains x:

Lemma 25. *Let $x \in f_{\max}$. Then*

$$e_x = \bigcap \{g \in \mathcal{E}_{\preceq}(f_D) \mid x \in T(g)\}. \tag{20}$$

Now, let $f \in \mathcal{E}_{\preceq}(f_D)$. Then $\pi(f) = f$, and equation (18) implies:

$$f = \bigvee \{e_x \mid x \in R(minT(f)) \setminus minT(f_{\min})\}. \tag{21}$$

Although it can be easily verified that this representation is unique and irredundant, it is not minimal. To minimize the representation in equation (21) we will use induced extensions $e_x \leq f$, where x is *not* restricted to $x \in f_{\max}$. Therefore, we introduce the universal bounds of the lattice $[g]_\theta$ discussed in the preceding subsection.

Definition 10. *Let $g \in \mathcal{M}(X_{\preceq})$. Then $\hat{g}$ and $\check{g}$ denote respectively the smallest and the greatest element in the sublattice $[g]_\theta$.*

The determination of these bounds will be discussed in the next subsection. Here, we will use the minimal vectors of $\check{f}$ to minimize the representation (21) of an extension f of f_D. Note, that $\hat{g}$ and $\check{g}$ are respectively the smallest and largest function in $\mathcal{M}_{\preceq}$ such that $\pi(\hat{g}) = g$ and $\pi(\check{g}) = g$.

Lemma 26. *Suppose $f \in \mathcal{E}_{\preceq}(f_D)$. Then $\forall x, y \in X$:*

a) $e_y \leq f \Leftrightarrow m_y \leq \check{f} \Leftrightarrow \exists x \in \min T(\check{f})$ such that $x \preceq y$.
b) Let $x \in \min T(\check{f})$ and let $y \preceq x$. Then $e_y \leq f$ implies $m_x = m_y$, (or equivalently $x \mu y$).

Proof. a) From $\pi(\check{f} \vee m_y) = \pi(\check{f}) \vee \pi(m_y) = f \vee e_y$ and the definition of $\check{f}$ it follows that $e_y \leq f \Leftrightarrow m_y \leq \check{f}$. The second equivalence follows from the assumption that $\preceq$ is a quasi-order, so that $y \in T(m_y)$.
b) Let $x \in \min T(\check{f})$ and let $y \preceq x$. According to a) $e_y \leq f$ implies $\exists z \in \min T(\check{f})$ with $z \preceq y$. Since $y \preceq x$, we have by transitivity $z \preceq x$. From the minimality of x we conclude that $z \mu x \mu y$.

Corollary 4.
a) If $e_y \leq f$, then $\exists x \in \min T(\check{f})$ such that $e_y \leq e_x \leq f$.
b) Let $x \in \min T(\check{f})$ and let $y \preceq x$. Then $e_y = e_x$ implies $x \mu y$.

Now, let $f \in \mathcal{E}_{\preceq}(f_D)$. Then according to Corollary (4a) we can rewrite equation (21) as:

$$f = \bigvee \{e_x \mid x \in R(minT(\check{f})) \setminus minT(f_{\min})\}. \tag{22}$$

However, we cannot conclude from Lemma (26) that the representation in equation (22) is irredundant. For, if x, y and z are pairwise incomparable (with respect to $\preceq$) minimal vectors of $\check{f}$, then e.g. the following may occur: $e_x < e_y$ or $e_x \leq e_y \vee e_z$, as is shown in the following example.

Example 8. Consider the case of standard positive functions, i.e., $\preceq = \leq$.
Let $T_D = \{(11010), (01111)\}$ and $F_D = \{(11100), (11001), (01010)\}$.
Then it follows that $f_{\min} = x_1x_2x_4 \vee x_2x_3x_4x_5$ and $f_{\max} = x_1x_4 \vee x_3x_4 \vee x_3x_5 \vee x_4x_5$.
First consider the extension: $f = x_1x_2x_4 \vee x_2x_3x_4 \vee x_3x_5$. Then it is easy to verify that (11000) and (01100) are minimal vectors of $\check{f}$, and that $e_{12} < e_{23}$, where $e_{12} = \pi(x_1x_2)$ and $e_{23} = \pi(x_2x_3)$. Subsequently, consider the extension: $f = x_3x_4 \vee x_4x_5 \vee x_1x_2x_4 \vee x_1x_3x_5 \vee x_2x_3x_5$. In this case it can be verified that (10001), (10100) and (00011) are minimal vectors of $\check{f}$ and that $e_{15} < e_{13} \vee e_{45}$, see also the next example.

The problem of generating irredundant expressions of the form (22) can be formulated as a set-covering problem. Let $\{v_1, v_2, \cdots, v_n\}$ and $\{w_1, w_2, \cdots, w_m\}$ respectively denote the set of minimal vectors of an extension f and of $\check{f}$. Then according to Lemma (26) $\forall v_i \exists w_j$ such that $v_i \in T(e_{w_j})$. Therefore, the set $C_i = \{j \mid v_i \in T(e_{w_j})\}$ is non-empty, and $v_i \in \bigwedge\{T(e_{w_j}) \mid j \in C_i\}$. Define the positive Boolean function F by:

$$F(y_1, y_2, \cdots y_m) = \bigvee_{i=1}^{n} \bigwedge\{y_j \mid j \in C_i\}. \tag{23}$$

Let $t = y_{i_1} y_{i_2} \cdots y_{i_k}$ be a prime implicant of the dual of F. Then, as is well known, the term t has at least one literal in common with every prime implicant of F (transversal property). From the definition of F it follows that:

$$f = \bigvee\{e_{w_j} \mid j \in \{i_1, i_2, \cdots i_k\}\}. \tag{24}$$

Since t is a prime implicant of F^d equation (24) is an irredundant expression of f. Therefore, the irredundant expressions of f of the form (24) are in one-one correspondence with the minimal vectors of F^d. Examples will be given in subsection (5.7)

5.5 Universal bounds of the sublattice $[g]_\theta$

We now focus on the sublattice $[g]_\theta$. In this subsection, we characterize its universal bounds, and in the next subsection we discuss how to compute them. Subsequently, these results will be applied to the case of standard positive functions. In the following lemma $\Box(f)$ denotes the largest monotone minor of f with respect to a binary relation $\preceq$, so according to Theorem **??** $\Box(f) = \Box_{[\preceq]}(f) = \nabla_{\preceq}(f)$. Similarly, $\blacklozenge = \blacklozenge_{[\preceq]}(f) = \blacktriangle_{\preceq}(f)$ denotes the largest monotone major of f.

Theorem 11. *Let $g \in \mathcal{M}(X_{\preceq})$, and let $\hat{g}$ and $\check{g}$ denote the smallest and the greatest elements in the sublattice $[g]_\theta$, respectively. Then we have:*

(a) $\check{g} = \Box(f_{\min} \vee g \vee \bar{f}_{\max})$,
(b) $\hat{g} = \blacklozenge(g f_{\max} \bar{f}_{\min})$

Proof. a) Let G denote the right hand side, i.e.,

$$G = \Box(f_{\min} \vee g \vee \bar{f}_{\max}) = \Box(f_{\min} \vee g f_{\max} \vee \bar{f}_{\max}) = \Box(\pi(g) \vee \bar{f}_{\max}).$$

Since $\pi(g)$ is monotone, this implies $\pi(g) \leq G$. Furthermore $G \leq \pi(g) \vee \bar{f}_{\max}$ by the definition of $\square$. Next, since π is order-preserving (Lemma 24) and idempotent, it follows that

$$\begin{aligned}\pi(g) = \pi^2(g) &\leq \pi(G) \leq \pi(\pi(g) \vee \bar{f}_{\max}) \\ &= \pi^2(g) \vee \pi(\bar{f}_{\max}) \text{ (by Lemma 23)} \\ &= \pi(g) \vee (f_{\min} \vee \bar{f}_{\max} f_{\max}) = \pi(g).\end{aligned}$$

Therefore $\pi(g) = \pi(G)$ or equivalently $G \in [g]_\theta$. This establishes the inequality $G \leq \check{g}$. To prove the converse, note that $\pi(\check{g})$ $(= f_{\min} \vee \check{g} f_{\max}) = \pi(g)$. This implies $\check{g} f_{\max} \leq \pi(g)$, or equivalently $\check{g} = \check{g}(f_{\max} \vee \bar{f}_{\max}) \leq \pi(g) \vee \bar{f}_{\max}$. Since $\check{g}$ is monotone, this implies $\check{g} = \square\check{g} \leq \square(\pi(g) \vee \bar{f}_{\max}) = G$. Thus we conclude $\check{g} = G$. b) Denote the right hand side as $H = \blacklozenge(g f_{\max} \bar{f}_{\min})$. First note that $\pi(g) = f_{\min} \vee g f_{\max} = \pi(g f_{\max})$ by definition. This says that $\hat{g} \leq g f_{\max} \leq f_{\max}$, i.e., $\pi(\hat{g}) = f_{\min} \vee \hat{g}$. Hence $g f_{\max} \bar{f}_{\min} \leq (f_{\min} \vee g f_{\max})\bar{f}_{\min} = \pi(g)\bar{f}_{\min} = \pi(\hat{g})\bar{f}_{\min} \leq \hat{g}$. Since $\hat{g}$ is monotone, applying $\blacklozenge$ to both sides, we then have $H \leq \blacklozenge\hat{g} = \hat{g}$. Next, we shall show $\pi(H) = \pi(g)$. (This means $\hat{g} \leq H$ and proves b).) Since $\pi(g) = f_{\min} \vee g f_{\max} \geq g f_{\max} \bar{f}_{\min}$ and $\pi(g)$ is monotone, we have $\pi(g) = \blacklozenge(\pi(g)) \geq \blacklozenge(g f_{\max} \bar{f}_{\min}) = H$ implying $\pi^2(g) = \pi(g) \geq \pi(H)$. Furthermore, $\pi(g) = f_{\min} \vee g f_{\max} = f_{\min} \vee g f_{\max} \bar{f}_{\min} \leq f_{\min} \vee H f_{\max} = \pi(H)$ follows from $g f_{\max} \bar{f}_{\min} \leq H$ (by definition of $\blacklozenge$). Thus $\pi(H) = \pi(g)$.

Finally, using the fact derived in Lemma 10 that $\square = d\blacklozenge d$, we conclude:

Corollary 5. *If $X = \{0,1\}^n$ and $\preceq$ is a self-dual relation on X, then $\hat{g}^d = \square(f^*_{\min} \vee g^d \vee f^d_{\max}))$.*

5.6 Computation of the universal bounds

We will now first show that for Boolean functions and the standard partial order it is possible to compute the DNFs/CNFs of the universal bounds of the sublattice $[g]_\theta$. Subsequently, we will indicate how these results can be extended to the case of an arbitrary relation $\preceq$ on $X = \{0,1\}^n$. We first note that according to Theorem 11 the function $\check{g}$ is the largest positive minor of a non-positive function. However, we can take advantage of the fact that the functions $f_{\min}$ and $f_{\max}$ are monotone. So, consider the function $g \in \mathcal{E}_{\leq}(f_D)$. Then, since $f_{\min} \leq g \leq f_{\max}$, we have

$$\check{g} = \square(f_{\min} \vee g \vee \bar{f}_{\max}) = \square(g \vee \bar{f}_{\max}) = d\blacklozenge(g^d f^*_{\max}), \tag{25}$$

where the last equality follows from Lemma 10. Therefore, an essential step is computing the least monotone major of $g^d f^*_{\max}$. In this case, $g \leq f_{\max}$ implies $g^d \geq f^d_{\max}$. In [3], for Boolean functions and the standard partial order we have proved the following lemma.

Lemma 27. *Let f and g be positive functions such that $f \leq g$ Then:*

$$\min T(g\bar{f}) = \min T(g) \setminus \min T(f).$$

Lemma 28. *Let h be a not necessarily positive Boolean function. Then:*

$$\min T(\blacklozenge(h)) = \min T(h).$$

Proof. Since $\blacklozenge(h)$ is the positive closure of h we have by definition: $y \in T(\blacklozenge(h)) \Leftrightarrow \exists x \leq y$, where $x \in T(h)$. Therefore, if $y \in T(\blacklozenge(h))$, then $\exists z \in \min T(h)$ such that $z \leq y$. This implies $\min T(h) \subseteq \min T(\blacklozenge(h))$. To prove the converse note that $h \leq \blacklozenge(h)$. This implies: if $y \in \min T(\blacklozenge(h))$ then $y = z$. Therefore, $\min T(\blacklozenge(h)) \subseteq \min T(h)$.

Theorem 12. *Suppose f_D is a pdBf and et $g \in \mathcal{E}_{\leq}(f_D)$. Then:*

$$\min T(\blacklozenge(g^d f^*_{\max}) = \min T(g^d) \setminus \min T(f^d_{\max}).$$

Proof. This follows from Lemma 27 and Lemma 28.

Noting that $\breve{g}$ is the dual of the positive closure $\blacklozenge(g^d f^*_{\max})$, we now have the following algorithm to compute all the prime implicants in the DNF of $\breve{g}$.

Algorithm: MAX($[g]_\theta$)
Input: A monotone extension $g \in \mathcal{E}_{\preceq}(f_D)$.
Output: All prime implicants in the DNF of $\breve{g}$.

1. Dualize g and $f_{\max}$ to compute all prime implicants of g^d and $f^d_{\max}$, respectively.
2. Remove all prime implicants of g^d that are also prime implicants of $f^d_{\max}$. According to Lemma 12, the resulting set gives all prime implicants of $\blacklozenge(g^d f^*_{\max})$.
3. Dualize the DNF obtained in step 2. This yields the DNF of $\breve{g}$.

The complexity of this algorithm is open, since the complexity of dualizing a monotone function is one of the well known open problems [2, 12, 15], and may not be done in polynomial time even though there is a pseudo-polynomial algorithm (hence it is unlikely to be NP-hard) [15].

Generalized monotone Boolean functions. We will now indicate how Theorem (12) can be generalized to the case that $\preceq$ is an arbitrary binary relation on $X = \{0,1\}^n$. It is easy to see that Lemmas (27) and (28) also hold for an arbitrary binary relation $\preceq$. Note, that in this case the minimal vectors of a function f are just the true vectors of f that are minimal with respect to the relation $\preceq$. If $\preceq$ is self-dual, then we use the property $\Box = d\blacklozenge d$, to show that Theorem 12 still holds. However, if $\preceq$ is not self-dual then we can use the property $\Box = \neg\Diamond\neg$, to prove a theorem similar to Theorem 12. Note, that in this case Lemma 27 has to be reformulated for the case of negative functions.

5.7 Application to standard positive functions

In this subsection we show how the results of the previous subsections can be applied to the case of standard positive functions.

Example 9. Consider the pdBf f_D of Example (8).
So $f_{\min} = x_1x_2x_4 \vee x_2x_3x_4x_5$ and $f_{\max} = x_1x_4 \vee x_3x_4 \vee x_3x_5 \vee x_4x_5$.
Let f be the extension $f = x_3x_4 \vee x_4x_5 \vee x_1x_2x_4 \vee x_1x_3x_5 \vee x_2x_3x_5$.
Note, that f is self-dual: $f^d = f$. To compute $\check{f}$ we first determine $f^d_{\max}$ as follows. $f^d_{\max} = (x_1 \vee x_4)(x_3 \vee x_4)(x_3 \vee x_5)(x_4 \vee x_5) = x_3x_4 \vee x_4x_5 \vee x_1x_3x_5.$. Applying Steps 2 and 3 of algorithm MAX$([f]_\theta)$ yields $\check{f}^d = x_1x_2x_4 \vee x_2x_3x_5$.
So, $\check{f} = x_2 \vee x_1x_3 \vee x_1x_5 \vee x_3x_4 \vee x_4x_5$. Therefore, we have:

$$f = e_2 \vee e_{13} \vee e_{15} \vee e_{34} \vee e_{45} \tag{26}$$

To minimize expression (26) we note that:

$$\begin{aligned}
e_2 &= x_1x_2x_4 \vee x_2x_3x_4 \vee x_2x_3x_5 \vee x_2x_4x_5\\
e_{13} &= x_1x_2x_4 \vee x_1x_3x_4 \vee x_1x_3 \vee x_2x_3x_4x_5\\
e_{15} &= x_1x_2x_4 \vee x_1x_3x_5 \vee x_1x_4x_5 \vee x_2x_3x_4x_5\\
e_{34} &= x_1x_2x_4 \vee x_3x_4 \vee x_2x_3x_4x_5\\
e_{45} &= x_1x_2x_4 \vee x_4x_5.
\end{aligned}$$

Now equation (23) yields: $F(y_1, \cdots, y_5) = y_1 \vee y_2y_3 \vee y_4 \vee y_5$. By dualizing the function F it appears that f has the following two irredundant expressions:

$$\begin{aligned}
f &= e_2 \vee e_{13} \vee e_{34} \vee e_{45},\\
f &= e_2 \vee e_{15} \vee e_{34} \vee e_{45}.
\end{aligned}$$

Basic DNF representations of extensions Let f_D be a pdBF, then $e_i := \pi(x_i) = f_{\min} \vee x_i f_{\max}$ is a monotone extension of f_D. We call this extension a *basic*-extension. Recall that two basic-extensions e_i and e_j are the same if and only if $x_i \oplus x_j \leq f_{\min} \vee \bar{f}_{\max}$ holds. Furthermore, if g is an arbitrary monotone function, then Lemma 23 says that

$$\pi(g(x_1, x_2, \ldots, x_n)) = g(\pi(x_1), \pi(x_2), \ldots, \pi(x_n)) = g(e_1, e_2, \ldots, e_n).$$

Therefore, every extension $\pi(g)$ is a monotone function of the basic-extensions e_i. Furthermore, if $\breve{g}$ has the DNF $\breve{g} = \bigvee_{(i_1, i_2, \ldots, i_m) \in I} x_{i_1} x_{i_2} \cdots x_{i_m}$, which uses only uncomplemented literals (recall that such DNF is unique (e.g., [16, 21] and Theorem 5)), then $\pi(g)$ can be represented by the following *basic* DNF

$$\pi(g) = \pi(\breve{g}) = \bigvee_{(i_1, i_2, \ldots, i_m) \in I} e_{i_1} e_{i_2} \cdots e_{i_m}. \tag{27}$$

However, as we have seen this expression is in general not irredundant. However, by using the results of section (5.4) we can obtain irredundant representations by using the minimal vectors of $\breve{g}$.

Example 10. We continue with extensions of the pdBf f_D of Example (9). Again let f be the extension
$f = x_3 x_4 \vee x_4 x_5 \vee x_1 x_2 x_4 \vee x_1 x_3 x_5 \vee x_2 x_3 x_5$. Using the functions $f_{\min}$ and $f_{\max}$ obtained in this Example we have found:

$$f = e_2 \vee e_{13} \vee e_{34} \vee e_{45},$$
$$f = e_2 \vee e_{15} \vee e_{34} \vee e_{45}.$$

Therefore, f has the following two basic minimal representations:

$$f = e_2 \vee e_1 e_3 \vee e_3 e_4 \vee e_4 e_5,$$
$$f = e_2 \vee e_1 e_5 \vee e_3 e_4 \vee e_4 e_5.$$

6 Conclusion and Further Research

We studied generalized monotone functions from the lattice theoretic point of view. Moreover, we studied the properties of conjunctive and disjunctive operators on characteristic functions of the form $f : X \mapsto \{0, 1\}$. Subsequently, we investigated the relationship between these operators and the monoid of binary relations on X. The results were then applied to the problem of finding (generalized) monotone extensions of a given partially defined Boolean function. The problem of extensions is an important subject in such fields as data mining, knowledge discovery and logical analysis of data. As there are many important classes of generalized monotone functions, as noted in Section (2.2), the results in this paper will find places in various applications.

It should be pointed out, however, that many algorithmic and complexity issues related to generalized monotone functions are not answered yet. For example, such problems as listed below may be of interest: how to compute m_x and M_x, how to compute the positive content and positive closure of a given function f, how to compute $\check{g}$, how to compute $f_{\min}$ and $f_{\max}$ of a pdBf f_D, and how to compute basic extensions in Subsection (5.7).

An important omission from the generalized monotone functions is the class of Horn functions and related functions [13, 14, 17, 19]. As the class of Horn functions is a topped $\wedge$-semilattice (but not closed under disjunction), it may also be an interesting challenge to extend the results in this paper to such semilattices.

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