Economic lot-sizing games

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Abstract

In this paper we introduce a new class of OR games: economic lot-sizing (ELS) games. There are a number of retailers that have a known demand for a fixed number of periods. To satisfy demand the retailers order products at the same manufacturer. By placing joint orders instead of individual orders, costs can be reduced and a cooperative game arises. In this paper we show that ELS games are balanced. Furthermore, we show that two special classes of ELS games are concave.

Keywords: Game Theory; Lot-sizing; Inventory; Production

1 Introduction

In this paper we consider a new type of inventory/production game. The model underlying the cooperative game is the classical Wagner-Whitin or the (somewhat more general) economic lot-sizing (ELS) problem. Wagner and Whitin (1958) were the first who studied this OR problem and it can be described as follows. Given a known demand for a finite discrete time horizon, one has to find a production plan such that total costs are minimized. Relevant costs include setup cost, production cost and holding cost.

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Consider the following ELS situation in which a number of retailers sell the same product. They order the product at a single manufacturer, store it for some periods and sell it to the customers. Furthermore, assume that the manufacturer charges ordering and production costs for a single order. Now by cooperating the retailers may obtain a cost saving. Namely, if the retailers place one big order instead of placing individual orders total ordering costs will be smaller. Moreover, production cost can also be reduced by merging their production plans. A prime question then becomes how to share the minimal total joint costs. To answer this question we analyze a corresponding TU-game in which the value for each coalition is determined by minimizing its total costs by joint ordering. So the players in this game are the retailers. We will call this game the ELS game related to the underlying ELS situation. A practical situation in which the above model may occur is for example the automobile industry. If different car manufacturers have to order the same components, a cost saving can be obtained by placing joint orders.

In this paper we will focus on the balancedness, the core and concavity of ELS games. The core of a game consists of stable cost divisions of the total joint costs in which no subset of players has a direct incentive to leave the grand coalition, because this will increase the cost for each subset of players. If a cooperative game has a non-empty core, then it is called balanced. A well-known class of balanced games are concave or submodular games. For a concave game it holds that the marginal contribution (in terms of cost) of a single player to a coalition equals at least the marginal contribution of that player to a larger coalition. Concave games are known to have nice properties. Namely, for concave games, the core of the game is equal to the convex hull of the marginal vectors (see Shapley (1971) and Ichiishi (1981)) and, as a result, the Shapley-value is the barycenter of the core. Furthermore, the kernel coincides with the nucleolus and the bargaining set and the core coincide (see Maschler, Peleg and Shapley (1972)).

The first OR games in the area of inventory management were introduced by Meca, Timmer, Garcia-Jurado and Borm (2004). These games arise when there is a possibility of joint ordering in an economic order quantity (EOQ) environment. The underlying OR problem is the well-known EOQ model, which was already introduced by Harris (see Harris (1913)). In this continuous time model with infinite horizon it is assumed that a single firm faces a constant demand rate. The objective of the firm is to minimize its inventory and ordering costs. Meca et al. (2004) extend this analysis to the situation where a collection of firms tries to minimize their total ordering and inventory costs by joint ordering. They analyze a corresponding cooperative game and focus on cost division rules. Our ELS game can be seen as a discrete version of the model by Meca et al. (2004) with a finite horizon.

Besides the area of inventory management, the application of cooperative game theory in other

OR areas has received much attention. In short an OR game consists of multiple players that not only face a joint optimization problem that arises from some OR related problem, but also face the additional problem of how to allocate the cost savings or revenues among the players. A survey of OR games can be found in Borm, Hamers and Hendrickx (2001). Here, next to production and inventory games, OR games on connection (e.g. fixed tree and spanning tree), routing (e.g. Chinese postman and travelling salesman) and scheduling (e.g. sequencing, permutation and assignment) are discussed.

This paper is organized as follows. In section 2 we will describe the ELS model and we recall the well-known Wagner-Whitin algorithm that solves an ELS problem instance. In section 3 we will introduce ELS games and give an example. In section 4 we will prove that ELS games are balanced. The proof is obtained by merging production plans (of balanced collections) of coalitions to production plans of the grand coalition in such a way that we can show the balancedness conditions of the Bondareva-Shapley theorem (see Bondareva (1963) and Shapley (1967)). In section 5 we consider two special classes of ELS games that are concave: the ELS game with equal demand for each player and the 2-period ELS game.

2 The ELS model

In this section we introduce the ELS model. The ELS problem can be described as follows. For a fixed model horizon $\mathcal{T}$ there is a known demand for a single item per time period. This demand has to be satisfied each period by producing in this period or in previous periods, i.e., back-logging is not allowed. Relevant cost typically include production costs and holding (inventory) costs. The production costs consist of a fixed component (setup or ordering cost) and unit production cost. The goal is to determine how much to produce in each period so as to minimize the total costs.

If for each period $t = 1, \ldots, \mathcal{T}$ we let

d_t = demand in period t

K_t = setup or ordering cost in period t

h_t = holding cost in period t

p_t = unit production cost in period t

x_t = items produced in period t

I_t = ending inventory in period t,

then the problem can be formulated as follows

$$
\begin{aligned}
C(d) = \min \sum_{t=1}^{\mathcal{T}} & K_t \delta(x_t) + p_t x_t + I_t h_t \\
\text{s.t. } & I_t = I_{t-1} + x_t - d_t, && t = 1, \ldots, \mathcal{T} \\
& x_t, I_t \geq 0, && t = 1, \ldots, \mathcal{T} \\
& I_0 = 0,
\end{aligned}
$$

where

$$
\delta(x) = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0. \end{cases}
$$

So $C(d)$ denotes the total cost when demand equals $d = [d_1, \ldots, d_{\mathcal{T}}]$ in periods $1, \ldots, \mathcal{T}$. Wagner and Whitin (1958) propose a forward dynamic programming algorithm to find the optimal production plan. They show that there exists an optimal solution with the property

$$
I_{t-1} x_t = 0 \text{ for } t = 1, \ldots, \mathcal{T}. \tag{1}
$$

That is, production only takes place if inventory drops down to zero. This is the so-called zero-inventory (ZI) property. Because back-logging is not allowed, this implies that there exists an optimal solution for which

$$
x_t = 0 \text{ or } x_t = \sum_{i=t}^{k} d_i \text{ with } 1 \leq t \leq k \leq \mathcal{T}. \tag{2}
$$

So a production plan is completely determined by its production periods, where a production period is defined as a period with non-zero production.

Let $c(s,t)$ be the cost to satisfy demand in periods $s, \ldots, t$ with a setup only in period s, i.e.,

$$
c(s,t) = K_s + p_s \sum_{i=s}^{t} d_i + \sum_{i=s}^{t-1} h_i \sum_{j=i+1}^{t} d_j.
$$

Furthermore, let $F(t)$ be the minimal cost of the production plan for period 1 through t and let $F(0) = 0$, then

$$
F(t) = \min_{i=1,\ldots,t} \left\{ F(i-1) + c(i,t) \right\}.
$$

Wagner and Whitin show that this algorithm runs in $\mathrm{O}(\mathcal{T}^2)$ time. After applying this recursion (called the Wagner-Whitin algorithm), the optimal production plan can be easily obtained by back-tracking. Note that this recursion formula only works for problem instances with strictly positive demand in period 1, which will be the case in this paper.

More sophisticated algorithms have been developed by Federgruen and Tzur (1991), Wagelmans, Van Hoesel and Kolen (1992) and Aggarwal and Park (1993). They show that a general ELS problem instance can be solved in $O(\mathcal{T} \log \mathcal{T})$ time and a problem instance without speculative motives (i.e., $p_t + h_t \geq p_{t+1}$ for $t = 1, \ldots, \mathcal{T} - 1$) can be solved in $O(\mathcal{T})$ time. For more details on the ELS problem we refer to these papers. We illustrate the ELS problem by the following example.

Example 1 Consider a 6-period ELS problem instance with parameters according to table 1. One can verify that by applying the Wagner-Whitin algorithm the optimal solution of the problem

t	1	2	3	4	5	6
d_t	15	11	12	6	7	16
K_t	40	40	40	40	40	40
p_t	0	0	0	0	0	0
h_t	1	1	1	1	1	1

Table 1: Problem instance of example 1

is to produce 44 items in period 1 and 23 items in period 5 (that is $x_1 = 44, x_5 = 23$ and $x_2 = x_3 = x_4 = x_6 = 0$). Total costs corresponding to this production plan equal 149 (setup cost: $40 + 40 = 80$ and holding cost: $1 \cdot 11 + 2 \cdot 12 + 3 \cdot 6 + 1 \cdot 16 = 69$). Note that for simplicity we have taken stationary cost parameters. Of course, in the case of non-stationary cost parameters the Wagner-Whitin algorithm can also be applied.

3 The economic lot-sizing game

In this section we introduce ELS games. Consider a set of retailers $N = \{1, \ldots, n\}$ that sell the same item. They all have a known demand for a $\mathcal{T}$-period model horizon. All retailers buy the items at the same manufacturer. When a retailer places an order, the manufacturer charges ordering cost and production cost, which is linear in the amount of items ordered. Furthermore, when a retailer carries inventory from one period to the next period to satisfy future demand, holding costs are incurred. We assume that in a single period holding costs are equal for each retailer. Now a single retailer tries to minimize its total ordering, production and holding costs. Note that this is exactly the situation as in the ELS problem, where setup cost in the ELS problem corresponds to ordering cost in the ELS game.

However, if a collection of retailers cooperates, a cost saving may be obtained. Namely, instead of placing individual orders the retailers can place a joint order and save ordering cost. It is not difficult to see that it is always profitable for two (or more) retailers to cooperate. Namely, if we have two optimal production plans we can construct a new production plan, where the amount ordered in each period is the sum of the amount ordered in the two individual plans. That is, if x_t^i and x_t^j are the amounts ordered by retailer i and j in period t, respectively, then the amount ordered in the newly constructed production plan is $x_t^i + x_t^j$. This production plan has total cost equal to the sum of the individual production plans if none of the production periods coincide. If one or more production periods coincide, a gain is obtained, because less setup costs are incurred. Later on we will show that in general even higher cost savings can be obtained.

Let d^i be the demand vector of player (retailer) i and let d^S be the demand vector of coalition $S \subset N$, i.e.,

$$d^S = \sum_{i \in S} d^i.$$

Then the optimal production plan for coalition S can be found by applying the Wagner-Whitin algorithm to demand vector d^S obtaining a total cost of $C(d^S)$. Now an ELS situation can be described by the tuple $\Gamma = \langle N, \mathcal{T}, D, K, p, h \rangle$, where N is the set of players, $\mathcal{T}$ the model horizon, D the $n \times \mathcal{T}$ demand matrix (with row i corresponding to the demand vector d^i of player i), K the vector of setup cost, p the vector of unit production cost and h the vector of holding cost. Note that a vector (and not a matrix) is sufficient to specify the holding cost, because we assume equal holding cost for each player.

Definition 1 *Let $\langle N, \mathcal{T}, D, K, p, h \rangle$ be an ELS situation. Then the corresponding ELS game (N, c) is defined as $c(S) = C(d^S)$ for every $S \subset N$ with $c(\emptyset) = 0$.*

We illustrate the ELS game by the following example.

Example 2 Consider an ELS game with player set $N = \{1, 2, 3\}$. Again we have a 6 period problem with setup cost $K_t = K = 40$, unit production cost $p_t = p = 0$, holding cost $h_t = h = 1$ and demand according to table 2. Applying the Wagner-Whitin algorithm to each possible coalition

period t	1	2	3	4	5	6
player 1	15	11	12	6	7	16
player 2	1	16	17	15	1	9
player 3	18	16	8	20	18	2

Table 2: Demand matrix of example 2

yields the optimal cost for each coalition. In table 3 one can find the optimal cost for each coalition with corresponding production plans. For example, the cost of coalition $S = \{1, 3\}$ is calculated

S	d^S	$c(S)$	x_1	x_2	x_3	x_4	x_5	x_6
$\emptyset$	$(0,0,0,0,0,0)$	0	0	0	0	0	0	0
$\{1\}$	$(15,11,12,6,7,16)$	149	44	0	0	0	23	0
$\{2\}$	$(1,16,17,15,1,9)$	140	17	0	42	0	0	0
$\{3\}$	$(18,16,8,20,18,2)$	134	42	0	0	40	0	0
$\{1,2\}$	$(16,27,29,21,8,25)$	184	43	0	58	0	0	25
$\{1,3\}$	$(33,27,20,26,25,18)$	191	60	0	46	0	43	0
$\{2,3\}$	$(19,32,25,35,19,11)$	186	19	57	0	65	0	0
$\{1,2,3\}$	$(34,43,37,41,26,27)$	223	34	80	0	67	0	27

Table 3: Optimal costs and production plans of example 2

by applying the Wagner-Whitin algorithm to the vector $d^{\{1,3\}} = d^1 + d^3 = (33,27,20,26,25,18)$. Observe that cooperation of player 1 and 3 will lead to a cost saving of $149 + 134 - 191 = 92$.

4 On the balancedness of ELS games

In this section we will provide the balancedness result for ELS games. First we recall the notion of balanced games. Let 2^N be the set of all subsets of N. Then $B \subset 2^N$ is called a balanced collection of 2^N if there exist weights $\lambda_S \in \mathbb{Q}^+$, $S \in B$, such that

$$\sum_{S \in B} \lambda_S e^S = e^N,$$

where e^S is the vector in $\mathbb{R}^n$ with $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ otherwise. A game is balanced if

$$\sum_{S \in B} \lambda_S c(S) \geq c(N)$$

for every balanced collection $B \subset 2^N$ and corresponding weights λ_S, $S \in B$.

To prove balancedness we use the following property of the ELS model. In an ELS model we may assume without loss of generality that holding costs are zero. This can be derived by substituting $I_t = \sum_{i=1}^{t}(x_i - d_i)$. Then the ELS problem can be written as

$$\begin{aligned} \min \sum_{t=1}^{\mathcal{T}} & \left(K_t \delta(x_t) + p_t' x_t - h_t \sum_{i=1}^{t} d_i \right) & \\ \text{s.t.} \ \sum_{i=1}^{t} x_i & \geq \sum_{i=1}^{t} d_i & t = 1, \ldots, \mathcal{T} \\ x_t & \geq 0 & t = 1, \ldots, \mathcal{T}, \end{aligned}$$

where

$$p'_t = p_t + \sum_{i=t}^{\mathcal{T}} h_i \text{ for } t = 1, \ldots, \mathcal{T}. \tag{3}$$

Note that the last term in the objective function is constant and can be omitted. Hence, each ELS problem instance is equivalent to a similar problem instance with zero holding cost and each ELS situation $\Gamma = \langle N, \mathcal{T}, D, K, p, h \rangle$ can be transformed into an ELS situation with holding cost zero. We call this transformed ELS situation the zero ELS situation and we denote it by $\Gamma_0 = \langle N, \mathcal{T}, D, K, p' \rangle$. The following example shows a transformed ELS problem instance.

Example 3 Consider the problem instance of Example 1. Note that $\sum_{t=1}^{6} h_t \sum_{i=1}^{t} d_i = 241$. Substituting $p'_t = p_t + \sum_{i=t}^{6} h_i$ and $h'_t = 0$ leads to the transformed ELS problem instance of table 4. Again the optimal solution of the problem is to produce 44 items in period 1 and 23 items

t	1	2	3	4	5	6
d_t	15	11	12	6	7	16
K_t	40	40	40	40	40	40
p'_t	6	5	4	3	2	1
h'_t	0	0	0	0	0	0

Table 4: Transformed problem instance of example 3

in period 5 (that is $x_1 = 44, x_5 = 23$ and $x_2 = x_3 = x_4 = x_6 = 0$). Total costs corresponding to this production plan equal $80 + 310 = 390$ (setup cost: $40 + 40 = 80$ and production cost: $44 \cdot 6 + 23 \cdot 2 = 310$), so that the optimal costs of the original problem equal $390 - 241 = 149$.

As a consequence of the property that an ELS situation can be transformed to a related zero ELS situation, the following proposition follows immediately:

Proposition 1 *Consider an ELS situation Γ and the related zero ELS situation Γ_0. Let (N, c) and (N, c_0) be the corresponding ELS games of Γ and Γ_0, respectively. Then there exists an additive game (N, a) such that*

$$c(S) = c_0(S) + a(S) \text{ for all } S \subset N.$$

Hence, the games (N, c) and (N, c_0) are strategically equivalent. So to prove balancedness we can restrict to games that arise from zero ELS situations.

In the proof we will further use the following convenient notation for production plans. As noted before production plans are completely described by setting the production periods. Let X be a production plan. Then $X_t = 1$ if period t is a production period and $X_t = 0$ otherwise for

$t = 1, \dots, \mathcal{T}$. Further, let $p_t(X)$ be the unit production cost of plan X in period t, i.e., $p_t(X) = p_i$ with $i = \max\{j \leq t | X_j = 1\}$. So the total cost $C(X)$ (assuming that holding cost is zero) of production plan X with demand vector d equals

$$C(X) = \sum_{t=1}^{\mathcal{T}} (X_t K_t + p_t(X) d_t)\,.$$

For further convenience a plan X with $X_t = 0$ for $t = 1, \dots, \mathcal{T}$ is called an empty plan and we define $p_t(X) = \infty$ for every empty plan X.

Theorem 1 *ELS games are balanced.*

Proof By Proposition 1 we may consider an ELS game (N, c) arising from $\Gamma_0 = \langle N, \mathcal{T}, D, K, p \rangle$. Let B be a balanced collection of 2^N and let $\lambda_S \in \mathbb{Q}^+$, $S \in B$, be the corresponding weights. It is sufficient to prove that

$$\sum_{S \in B} \lambda_S c(S) \geq c(N). \tag{4}$$

Note that (4) follows immediately for $B = \{N\}$. Now let $B \neq \{N\}$ and let $q_B \in \mathbb{N}$ be the smallest integer for which $q_B \lambda_S$ is integral for all $S \in B$. Then (4) can be equivalently expressed as

$$\sum_{S \in B} q_B \lambda_S C(d^S) \geq q_B C(d^N). \tag{5}$$

In the summation at the left hand side of (5) we have $q_B \lambda_S$ times the cost of the optimal production plan corresponding to each coalition $S \in B$. So in total we have the cost of $n_B = \sum_{S \in B} q_B \lambda_S$ production plans. For notational convenience let $N_B = \{1, \dots, n_B\}$ be the index set of coalitions (in arbitrary order) and let $S(i)$ be the coalition corresponding to index i. So we have $q_B \lambda_S$ indices for each coalition $S \in B$. Furthermore, let X^i be the optimal production plan corresponding to coalition $S(i)$ $(i = 1, \dots, n_B)$.

The right hand side of (5) denotes q_B times the cost of coalition N. We will show that we can combine plans X^i $(i = 1, \dots, n_B)$ to construct plans Y^j $(j = 1, \dots, q_B)$. Here Y^j denotes a production plan for coalition N with corresponding cost $C(Y^j)$. We will show that the cumulative costs of plans Y^j $(j = 1, \dots, q_B)$ equal at most the cumulative costs of plans X^i $(i = 1, \dots, n_B)$. It sufficient to prove

$$\sum_{i=1}^{n_B} C(X^i) \geq \sum_{j=1}^{q_B} C(Y^j), \tag{6}$$

because then we have

$$\sum_{S \in B} q_B \lambda_S C(d^S) = \sum_{i=1}^{n_B} C(X^i) \geq \sum_{j=1}^{q_B} C(Y^j) \geq q_B C(d^N),$$

where the last inequality follows from the optimality of $C(d^N)$. Note that $n_B = \sum_{S\in B} q_B\lambda_S > q_B$, because $\sum_{S\in B}\lambda_S > 1$ for $B \neq \{N\}$. Furthermore, note that

$$\sum_{S\in B} q_B\lambda_S d^S = q_B d^N,$$

because B is a balanced collection of 2^N and $d^S = \sum_{i\in S} d^i$. This means that the cumulative demands of plans X^i $(i = 1,\ldots,n_B)$ and plans Y^j $(j = 1,\ldots,q_B)$ are equal. In the remainder of the proof we will show how we can construct q_B plans Y^j out of n_B $(> q_B)$ plans X^i and why plans Y^j have cumulative costs at most equal to the cumulative costs of plans X^i. The idea is that we assign setups from plans X^i to plans Y^j. The following procedure shows how plans Y^j are constructed out of the plans X^i.

$Y^j_t := 0$ $(j = 1,\ldots,q_B,\ t = 1,\ldots,\mathcal{T})$

$p_{max} := \infty$

For $t = 1,\ldots,\mathcal{T}$

    For $i = 1,\ldots,n_B$

        If $X^i_t = 1$ and $p_t(X^i) < p_{max}$ Then

            $k := \arg\max_{j=1,\ldots,q_B}\{p_t(Y^j)\}$

            $Y^k_t := 1$

            $p_{max} := \max_{j=1,\ldots,q_B}\{p_t(Y^j)\}$

        End If

    Next i

Next t

In words the procedure works as follows. We start with empty plans Y^j $(j = 1,\ldots,q_B)$ and we iterate over all periods t and over all plans X^i. If we encounter a setup in plan X^i, we assign it to some plan Y^j. But we only assign a new setup if the production cost is strictly smaller than the maximum current production cost of all plans Y^j $(j = 1,\ldots,q_B)$. If the latter is true, then the setup will be assigned to plan Y^j with maximum current production cost $p_t(Y^j)$. This means that in period t the q_B smallest production costs have been assigned to plans Y^j $(j = 1,\ldots,q_B)$.

Now we will show why this procedure works. That is, we will show why the cumulative costs of plans Y^j $(j = 1,\ldots,q_B)$ equal at most the cumulative costs of plans X^i $(i = 1,\ldots,n_B)$. First note that the cumulative setup costs of plans Y^j equal at most the cumulative setup costs of plans X^i, because by applying the procedure, we use every setup in each plan X^i at most once in constructing plans Y^j. So we obtain the inequality

$$\sum_{j=1}^{q_B}\sum_{t=1}^{\mathcal{T}} Y_t^j K_t \leq \sum_{i=1}^{n_B}\sum_{t=1}^{\mathcal{T}} X_t^i K_t. \tag{7}$$

It is now sufficient to show that the cumulative production costs of plans Y^j $(j = 1, \ldots, q_B)$ equal at most the cumulative production costs of plans X^i $(i = 1, \ldots, n_B)$.

Because a setup is assigned to plan Y^j with the maximum value $p_t(Y^j)$ in period t and no setup is assigned if the production cost is larger than the maximum value $p_t(Y^j)$, the set $\{p_t(Y^j)\}_{j=1,\ldots,q_B}$ consists of the q_B smallest production costs of the setups in plans X_k^i $(i = 1, \ldots, n_B)$ for periods $k = 1, \ldots, t$. Assume without loss of generality that $p_t(Y^1) \leq p_t(Y^2) \leq \cdots \leq p_t(Y^{q_B})$. Now let A be an arbitrary subset of N_B with $|A| = q_B$. That is, we choose q_B arbitrary coalitions out of N_B with corresponding production plans X^i $(i \in A)$. Let $\{p_t(X^i)\}_{i\in A}$ be the set of corresponding production costs. We may assume without loss of generality that $A = \{1, \ldots, q_B\}$ and that $p_t(X^1) \leq p_t(X^2) \leq \cdots \leq p_t(X^{q_B})$. But then it holds $p_t(Y^j) \leq p_t(X^j)$ for $j = 1, \ldots, q_B$. Otherwise $\{p_t(Y^j)\}_{j=1,\ldots,q_B}$ is not the set with the q_B smallest production costs. But this implies

$$\sum_{j=1}^{q_B} p_t(Y^j) \leq \sum_{i\in A} p_t(X^i) \tag{8}$$

for every $A \subset N_B$ with $|A| = q_B$.

Now consider some player $m \in N$ and some period t. We will compare the production cost of player m in period t in production plans X^i $(i = 1, \ldots, n_B)$ and plans Y^j $(j = 1, \ldots, q_B)$. Let $I_m \subset N_B$ be the index set of coalitions in which player m is contained. Note that we must have $|I_m| = q_B$. Let $\{p_t(X^i)\}_{i\in I_m}$ be the set of corresponding production costs. Then the total production costs for player m in plans X^i $(i \in I_m)$ in period t equal

$$d_t^m \sum_{i\in I_m} p_t(X^i).$$

Furthermore, the total production costs for player m in plans Y^j $(j = 1, \ldots, q_B)$ in period t equal

$$d_t^m \sum_{j=1}^{q_B} p_t(Y^j).$$

But by (8) we have

$$d_t^m \sum_{j=1}^{q_B} p_t(Y^j) \leq d_t^m \sum_{j\in I_m} p_t(X^j). \tag{9}$$

So the production costs for each player m in each period t are smaller in plans Y^j than in plans X^i.

To complete the proof note that

$$\sum_{m=1}^{n} d_t^m \sum_{j=1}^{q_B} p_t(Y^j) = \sum_{j=1}^{q_B} d_t^N p_t(Y^j) \text{ and } \sum_{m=1}^{n} d_t^m \sum_{j\in I_m} p_t(X^j) = \sum_{i=1}^{n_B} d_t^{S(i)} p_t(X^i) \tag{10}$$

and by combining (9) and (10) we have

$$\sum_{j=1}^{q_B} d_t^N p_t(Y^j) \leq \sum_{i=1}^{n_B} d_t^{S(i)} p_t(X^i). \tag{11}$$

But then we have

$$\begin{aligned}
\sum_{j=1}^{q_B} C(Y^j) &= \sum_{j=1}^{q_B}\sum_{t=1}^{\mathcal{T}} \left(Y_t^j K_t + p_t(Y^j) d_t^N\right) \\
&= \sum_{j=1}^{q_B}\sum_{t=1}^{\mathcal{T}} Y_t^j K_t + \sum_{j=1}^{q_B}\sum_{t=1}^{\mathcal{T}} p_t(Y^j) d_t^N \\
&\leq \sum_{i=1}^{n_B}\sum_{t=1}^{\mathcal{T}} X_t^i K_t + \sum_{i=1}^{n_B}\sum_{t=1}^{\mathcal{T}} p_t(X^i) d_t^{S(i)} \\
&= \sum_{i=1}^{n_B}\sum_{t=1}^{\mathcal{T}} \left(X_t^i K_t + p_t(X^i) d_t^{S(i)}\right) \\
&= \sum_{i=1}^{n_B} C(X^i),
\end{aligned}$$

where the inequality follows from (7) and (11). Hence we have proved (6) and hence the theorem. □

The following example shows how the procedure of constructing new plans out of existing plans works.

Example 4 Consider a 3-player ELS game with 9 periods. Let $B = \{\{1,2\},\{1,3\},\{2,3\}\}$ with $\{\lambda_S\}_{S\in B} = \{\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$. In this example we have $q_B = 2$, $n_B = 3$, $N_B = \{1,2,3\}$, $S(1) = \{1,2\}$, $S(2) = \{1,3\}$ and $S(3) = \{2,3\}$. That means, we construct 2 plans Y^1 and Y^2 out of the 3 plans X^1, X^2 and X^3. In table 5 the values of X_t^i, Y_t^j, $p_t(X^i)$ and $p_t(Y^j)$ are presented after applying the procedure described in the proof. Note that in this example we save setup costs in periods 1

t	1	2	3	4	5	6	7	8	9
K_t	10	10	10	10	10	10	10	10	10
p_t	10	8	9	7	4	3	5	2	1
X_t^1 $(p_t(X^1))$	1 (10)	0 (10)	1 (9)	0 (9)	0 (9)	1 (3)	0 (3)	0 (3)	0 (3)
X_t^2 $(p_t(X^2))$	1 (10)	0 (10)	0 (10)	1 (7)	0 (7)	0 (7)	1 (5)	0 (5)	1 (1)
X_t^3 $(p_t(X^3))$	1 (10)	1 (8)	0 (8)	1 (7)	1 (4)	0 (4)	0 (4)	1 (2)	0 (2)
Y_t^1 $(p_t(Y^1))$	1 (10)	1 (8)	0 (8)	1 (7)	1 (4)	0 (4)	0 (4)	1 (2)	0 (2)
Y_t^2 $(p_t(Y^2))$	1 (10)	0 (10)	1 (9)	1 (7)	0 (7)	1 (3)	0 (3)	0 (3)	1 (1)

Table 5: Illustration of the procedure described in the balancedness proof

and 7 both once. Furthermore, note that for every period t and for every $A \subset N_B$ with $|A| = 2$, we have that production costs for plans Y^j $(j = 1, 2)$ are pairwise smaller than for plans X^i $(i \in A)$ after ordering them. For example, for $t = 7$ and $A = \{2, 3\} = I_3$ we have $p_7(Y^2) = 3 \leq 4 = p_7(X^3)$ and $p_7(Y^1) = 4 \leq 5 = p_7(X^2)$.

The proof and the example show that we can not only save setup costs by cooperation, but we can also save production costs. Furthermore, because the holding costs are incorporated in the (transformed) production costs (see equation (3)), this means that by cooperation also holding cost can be reduced.

Using theorem 1 and the Bondareva-Shapley theorem (see Bondareva (1963) and Shapley (1967)) the following corollary holds:

Corollary 2 *ELS games have a non-empty core.*

Recall that the core of a cost game (N, c) is defined as the set

$$\{z \in \mathbb{R}^N | \sum_{i \in N} z_i = c(N) \text{ and } \sum_{i \in S} z_i \leq c(S) \text{ for all } S \subset N, S \neq \emptyset\}.$$

5 On the concavity of ELS games

In the previous section it was shown that ELS games are balanced. In this section we will give two classes of ELS games for which we can prove a stronger result. In particular, we will show that games within those two classes are concave. A cost game (N, c) is concave if for aech $i \in N$

$$c(S \cup \{i\}) - c(S) \geq c(T \cup \{i\}) - c(T) \text{ for all } S \subset T \subset N \setminus \{i\}. \tag{12}$$

Concave games have the following nice property. As already mentioned a game is concave if and only if its core is the convex hull of all marginal vectors (see Shapley (1971) and Ichiishi (1981)), where a marginal vector is defined as follows. Let $\Pi(N)$ the set of all orders of N. Let $\sigma \in \Pi(N)$ be an order on the players, i.e., a bijection $\sigma : N \to \{1, \dots, n\}$, where $\sigma(i) = j$ means that player i is at position j. A marginal vector $m^\sigma(c) \in \mathbb{R}^N$ with respect to the order σ in the game (N, c) is defined as

$$m^\sigma_{\sigma^{-1}(k)}(c) = c(\{j \in N : \sigma(j) \leq \sigma(k)\}) - c(\{j \in N : \sigma(j) < \sigma(k)\})$$

for every $k \in \{1, \dots, n\}$. The following example shows that ELS games are not concave in general.

Example 5 Consider the 3-player 6-period ELS game according to table 6. The optimal costs and corresponding production plans for each coalition can be found in table 7. One can verify

t	1	2	3	4	5	6
d_t^1	15	5	14	8	20	11
d_t^2	1	1	11	17	3	14
d_t^3	20	8	1	11	1	19
K_t	1	1	11	17	3	14
h_t	5	3	5	4	2	1
p_t	1	11	4	7	8	8

Table 6: Problem instance of example 5

S	d^S	$c(S)$	x_1	x_2	x_3	x_4	x_5	x_6
$\emptyset$	$(0,0,0,0,0,0)$	0	0	0	0	0	0	0
$\{1\}$	$(15,5,14,8,20,12)$	644	43	0	0	0	31	0
$\{2\}$	$(1,1,11,17,3,14)$	511	13	0	0	34	0	0
$\{3\}$	$(20,8,1,11,1,19)$	483	29	0	0	31	0	0
$\{1,2\}$	$(16,6,25,26,23,25)$	1029	22	0	51	0	48	0
$\{1,3\}$	$(35,13,15,20,21,30)$	1004	63	0	0	20	51	0
$\{2,3\}$	$(21,9,12,28,4,33)$	869	42	0	0	32	0	33
$\{1,2,3\}$	$(36,14,26,37,24,44)$	1393	76	0	0	37	68	0

Table 7: Optimal costs and production plans of example 5

that

$$c(\{1,2,3\}) + c(\{1\}) = 1393 + 644 = 2037 > 2033 = 1029 + 1004 = c(\{1,2\}) + c(\{1,3\})$$
$$c(\{1,2,3\}) + c(\{2\}) = 1393 + 511 = 1904 > 1898 = 1029 + 869 = c(\{1,2\}) + c(\{2,3\})$$
$$c(\{1,2,3\}) + c(\{3\}) = 1393 + 483 = 1876 > 1873 = 1004 + 869 = c(\{1,3\}) + c(\{2,3\}),$$

which even implies that the core contains none of the marginal vectors. For example, the marginal vector z corresponding to the order $\sigma = (1,2,3)$ equals

$$z = m^\sigma(c) = (c(\{1\}), c(\{1,2\}) - c(\{1\}), c(\{1,2,3\}) - c(\{1,2\}) = (644, 385, 364).$$

But then for coalition $\{1,3\}$ it holds that

$$z_1 + z_3 = c(\{1\}) + c(\{1,2,3\}) - c(\{1,2\}) = 644 + 364 = 1008 > 1004 = c(\{1,3\}).$$

Clearly $m^\sigma(c)$ is not a core element, because coalition $\{1,3\}$ prefers to leave the grand coalition obtaining a cost of 1004 instead of 1008.

In this section we will give two classes of ELS games that are concave. In the proofs of the concavity theorems again we assume that holding costs are zero.

Theorem 2 *Let* $\Gamma = \langle N, \mathcal{T}, D, K, p, h\rangle$ *be an ELS situation with* $d^1 = d^2 = \cdots = d^n = d$*. Then the corresponding ELS game* (N, c) *is concave.*

Proof By proposition 1 it is sufficient to consider an ELS game (N, c) arising from a zero ELS situation. First, we will proof a property of the function $C(\delta d)$ with $\delta > 0$, that is, $C(\delta d)$ is the optimal cost for some positive multiple of the demand vector d. Let P be a production plan represented by its production periods (in increasing order) and let $P(\mathcal{T})$ be the set of all possible production plans. Furthermore, let $s(t)$ be the first production period after period t in plan P and if period t has no successor, then let $s(t) = \mathcal{T} + 1$. Then the cost $C^P(\delta d)$ of production plan $P \in P(\mathcal{T})$ and demand δd equals

$$C^P(\delta d) = \sum_{t\in P} K_t + \sum_{t\in P} p_t \sum_{i=t}^{s(t)-1} \delta d_i = \sum_{t\in P} K_t + \delta \sum_{t\in P} p_t \sum_{i=t}^{s(t)-1} d_i.$$

So for a fixed production plan $C^P(\delta d)$ is a linear function of δ. Furthermore, because

$$C(\delta d) = \min_{P\in P(\mathcal{T})} C^P(\delta d)$$

and $|P(\mathcal{T})|$ is finite ($|P(\mathcal{T})| = 2^{\mathcal{T}-1}$ if $d_1 > 0$), it follows that $C(\delta d)$ is the minimum of a finite number of linear functions, which implies that $C(\delta d)$ is piecewise linear concave.

To complete the proof let $i \in N$ and $S \subset T \subset N \setminus \{i\}$. Using the concavity of the function $C(\delta d)$ and $|S| \leq |T|$, we have

$$c(S \cup \{i\}) - c(S) = C((|S| + 1)d) - C(|S|d) \geq C((|T| + 1)d) - C(|T|d) = c(T \cup \{i\}) - c(T),$$

where the equalities are justified by the fact that $d = d^1 = \cdots = d^n$.

□

Besides the ELS game with equal demands for all players, every 2-period ELS game is also concave. For a 2-period ELS problem with demand d we have two possible production plans:

- Production plan $(1, 0)$:

  Produce all demand $(d_1 + d_2)$ in period 1 incurring costs

$$K_1 + p_1(d_1 + d_2). \tag{13}$$

- Production plan $(1, 1)$:

  Produce d_1 in period 1 and produce d_2 in period 2 incurring costs

$$K_1 + p_1 d_1 + K_2 + p_2 d_2. \tag{14}$$

It is easy to verify from (13) and (14) that we have the following proposition:

Proposition 3 *Plan* $(1,0)$ *is preferred over plan* $(1,1)$ *if and only if* $p_1 d_2 \leq K_2 + p_2 d_2$.

Theorem 3 *Let* $\Gamma = \langle N, \mathcal{T}, D, K, p, h \rangle$ *be an ELS situation with* $\mathcal{T} = 2$. *Then the corresponding ELS game* (N, c) *is concave.*

Proof By Proposition 1 it is sufficient to consider an ELS game (N, c) arising from a zero ELS situation. We will show that (12) holds. Let $S \subset T \subset N \setminus \{i\}$. We will consider the cases $p_1 \leq p_2$ and $p_1 > p_2$.

- $p_1 \leq p_2$:

  It immediately follows from Proposition 3 that plan $(1,0)$ is preferred over plan $(1,1)$ for all possible coalitions. So we have

$$c(S \cup \{i\}) - c(S) = p_1(d_1^i + d_2^i) = c(T \cup \{i\}) - c(T).$$

- $p_1 > p_2$:

  For this case the possible combinations of production plans can be found in table 8. Note

coalition	S	$S \cup \{i\}$	T	$T \cup \{i\}$
Case 1	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$
Case 2	$(1,0)$	$(1,0)$	$(1,0)$	$(1,1)$
Case 3	$(1,0)$	$(1,0)$	$(1,1)$	$(1,1)$
Case 4	$(1,0)$	$(1,1)$	$(1,0)$	$(1,1)$
Case 5	$(1,0)$	$(1,1)$	$(1,1)$	$(1,1)$
Case 6	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$

Table 8: Possible combinations of production plans for $\mathcal{T} = 2$ and $p_1 > p_2$

that if the optimal plan for e.g. coalition S is $(1,1)$, then it follows from Proposition 3 that the optimal plan of coalition $S \cup \{i\}$ also must be $(1,1)$, because $p_1 > p_2$. The same argument applies to T and $T \cup \{i\}$ too.

Case 1: Clearly

$$c(S \cup \{i\}) - c(S) = p_1(d_1^i + d_2^i) = c(T \cup \{i\}) - c(T).$$

Case 2: It follows from Proposition 3 that

$$p_1(d_2^T + d_2^i) \geq K_2 + p_2(d_2^T + d_2^i).$$

So we have

$$\begin{aligned} c(S \cup \{i\}) - c(S) &= p_1(d_1^i + d_2^i) \\ &= p_1 d_1^i + p_1(d_2^T + d_2^i) - p_1 d_2^T \\ &\geq p_1 d_1^i + K_2 + p_2(d_2^T + d_2^i) - p_1 d_2^T \\ &= c(T \cup \{i\}) - c(T). \end{aligned}$$

Case 3: We now have

$$c(S \cup \{i\}) - c(S) = p_1(d_1^i + d_2^i) \geq p_1 d_1^i + p_2 d_2^i = c(T \cup \{i\}) - c(T).$$

Case 4: Clearly $(p_2 - p_1)d_2^S \geq (p_2 - p_1)d_2^T$ and hence

$$\begin{aligned} c(S \cup \{i\}) - c(S) &= p_1 d_1^i + K_2 + p_2(d_2^S + d_2^i) - p_1 d_2^S \\ &\geq p_1 d_1^i + K_2 + p_2(d_2^T + d_2^i) - p_1 d_2^T = c(T \cup \{i\}) - c(T). \end{aligned}$$

Case 5: It follows from Proposition 3 that

$$K_2 + p_2 d_2^S - p_1 d_2^S \geq 0$$

implying that

$$\begin{aligned} c(S \cup \{i\}) - c(S) &= p_1 d_1^i + K_2 + p_2(d_2^S + d_2^i) - p_1 d_2^S \\ &\geq p_1 d_1^i + p_2 d_2^i = c(T \cup \{i\}) - c(T). \end{aligned}$$

Case 6: It immediately follows that

$$c(S \cup \{i\}) - c(S) = p_1 d_1^i + p_2 d_2^i = c(T \cup \{i\}) - c(T).$$

□

References

Aggarwal, A., Park, J.K., 1993. Improved algorithms for economic lot-size problems, Operations Research 14 549–571.

Bondareva, O.N., 1963. Some applications of linear programming methods to the theory of cooperative games [in Russian], Problemy Kibernetiky 10 119–139.

Borm, P., Hamers, H., Hendrickx, R., 2001. Operations research games: A survey, TOP, 9(2) 139–216.

Federgruen, A., Tzur, M., 1991. A simple forward algorithm to solve general dynamic lot sizing models with n periods in $O(n \log n)$ or $O(n)$ time, Management Science 37 909–925.

Harris, F.W., 1913. How many parts to make at once, Factory, The magazine of management 10 135–136.

Ichiishi, T., 1981. Super-modularity: applications to convex games and the greedy algorithm for LP, Journal of Economic Theory 25 283–286.

Maschler, M., Peleg, B., Shapley, L.S., 1972. The kernel and bargaining set of convex games, International Journal of Game Theory 2 73–93.

Meca, A., Timmer, J., Garcia-Jurado, I., Borm, P., 2004. Inventory games, European Journal of Operational Research 156 127–139.

Shapley, L.S., 1967. On balanced sets and cores, Naval Research Logistics 14 453–460.

Shapley, L.S., 1971. Cores of convex games, International Journal of Game Theory 1 11–26.

Wagelmans, A.P.M., Van Hoesel, C.P.M., Kolen, A., 1992. Economic lot sizing: An $O(n \log n)$ algorithm that runs in linear time in the Wagner Whitin case, Operations Research 40 S145-S156.

Wagner, H.M., Whitin, T.M., 1958. Dynamic version of the economic lot size model, Management Science 5 89–96.