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Logarithmic residues and sums of idempotents in the Banach algebra generated by the compact operators and the identity

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Abstract. A logarithmic residue is a contour integral of the (left or right) logarithmic derivative of an analytic Banach algebra valued function. Logarithmic residues are intimately related to sums of idempotents. The present paper is concerned with logarithmic residues and sums of idempotents in the Banach algebra generated by the compact operators and the identity in the case when the underlying Banach space is infinite dimensional. The situation is more complex than encountered in previous investigations. Logarithmic derivatives may have essential singularities and the geometric properties of the Banach space play a role. The set of sums of idempotents and the set of logarithmic residues have an intriguing topological structure.

1. Introduction

Let $\mathcal{B}$ be a complex Banach algebra with unit element. A logarithmic residue in $\mathcal{B}$ is a contour integral of a logarithmic derivative of an analytic $\mathcal{B}$-valued function F. There is a left version and there is a right version of this notion. The left version corresponds to the left logarithmic derivative $F'(\lambda)F(\lambda)^{-1}$, the right version to the right logarithmic derivative $F(\lambda)^{-1}F'(\lambda)$.

The first to consider integrals of this type in a vector valued context, was L. Mittenthal [Mi]. His goal was to generalize the spectral theory of a single Banach algebra element (i.e., the case where $F(\lambda) = \lambda e - b$ with $b \in \mathcal{B}$ and e being the unit element in $\mathcal{B}$). He succeeded in giving sufficient conditions for a logarithmic residue to be an idempotent. The conditions in question, however, are very restrictive.

Logarithmic residues also appear in the paper [GS1] by I.C. Gohberg and E.I. Sigal. The setting there is $\mathcal{B} = \mathcal{L}(X)$ – the Banach algebra of all bounded linear operators on a complex Banach space – and F is a Fredholm operator valued function. For such functions Gohberg and Sigal introduced the concept of algebraic (or null) multiplicity. It turns out that the algebraic multiplicity of F with respect to a given contour is equal to the trace of the corresponding (left/right) logarithmic residues (see also [BKL2] and [GGK]). For analytic matrix functions, such a result was obtained in [MS].

Further progress was made in [BES2]–[BES6]. In these papers, logarithmic residues are studied from different angles and perspectives. The issues dealt with are of the following type.

Issue 1. *If a logarithmic residue vanishes, does it follows that F takes invertible values inside the integration contour?*
This question was first posed in [B2]. The answer turns out to depend very much on the underlying Banach algebra (see [BES2]). For certain important classes it is positive, for other (equally relevant) classes it is negative. A positive answer is also implied by the results obtained in [GS1] on the basis of the additional assumption that the function F is Fredholm operator valued (cf. [BES5]).

Issue 2. *What kind of elements are logarithmic residues?*
Here a strong connection with (sums of) idempotents appears (see [BES1]). As is the case for Issue 1, the answer here too depends on the Banach algebra under consideration or on special properties of the function F (cf. [BES2]–[BES6]).

Issue 3. *How about left versus right logarithmic residues?*
In all situations where a definite answer could be obtained, the set of left logarithmic residues coincides with the set of right logarithmic residues. In some situations it was possible to identify the pairs of left and right logarithmic residues associated with one single function F and the same integration contour. For details, see [BES4]–[BES6].

Issue 4. *What can be said about the topological properties of the set of logarithmic residues?*
When the underlying Banach algebra is commutative, this set is discrete but in general it is not. For matrix and, more generally, Fredholm operator valued functions, it was possible to identify its connected components. The results exhibit an intriguing connection with Issue 3 (see [BES4] and [BES5]).

The present paper is concerned with logarithmic residues and sums of idempotents in the Banach algebra $\mathcal{L}_{\mathcal{C}}(X)$ generated by the compact operators and the identity operator on a (complex) Banach space X. This important Banach subalgebra of $\mathcal{L}(X)$ has been touched upon in [BES2], Example 4.4, where Issue 1 was already settled in an affirmative fashion. Here we study it in a more systematic way, focusing on Issues 2, 3 and 4. There is an essential difference between the case when the dimension of X is finite and that where it is infinite. The first case (i.e., the matrix case $\mathcal{B} = \mathbb{C}^{n\times n}$), has been studied in [BES4]. *In this paper we concentrate on the infinite dimensional situation.*

Let X be an infinite dimensional complex Banach space, and let F be an analytic function with values in the Banach algebra $\mathcal{L}_{\mathcal{C}}(X)$. This means that F can be written in the form

$$F(\lambda) = f(\lambda)I + C(\lambda)$$

where f (the scalar part of F) is an analytic scalar function and C (the compact part of F) takes compact values. Since X is infinite dimensional, the functions f and C are uniquely determined and $F(\lambda)$ is a Fredholm operator if and only if $f(\lambda) \neq 0$. We are now ready to give an outline of the paper.

Section 2 is partly of a preliminary nature in the sense that it contains definitions and notations. Also it gives a somewhat sharpened formulation of the theorem in [E] on the representation of sums of idempotents as logarithmic residues of entire Banach algebra valued functions. The formulation is such that it is appropriate for the application of the theorem in Sections 4 and 7. In another part, Section 2 deals with Issues 1 and 2 in the situation where the scalar part f of F has no zeros. The values of F are then Fredholm operators and the results of [BES5] apply.

The rest of the paper is concerned with the case where f is allowed to have zeros and F does take compact (non-Fredholm) values. This case is considerably more complicated, as can already be guessed from the possible presence of essential singularities for the logarithmic derivatives. In this connection, the reader is reminded of the role of the origin in the spectral theory of a single compact operator T (i.e., the case where $F(\lambda) = \lambda I - T$).

Section 3 deals with sums of idempotents in the Banach algebra $\mathcal{L}_{\mathcal{C}}(X)$ under consideration. It is first observed that these idempotents are just the projections on X for which either the range or the null space has finite dimension. The sums of idempotents of this type are then characterized in terms of conditions involving ranks, traces and dimensions of null spaces. We also describe the closure of the set of sums of idempotents. As a result the connected components of the set of sums of idempotents in $\mathcal{L}_{\mathcal{C}}(X)$, of its closure and of the set of logarithmic residues in $\mathcal{L}_{\mathcal{C}}(X)$ are identified. The arguments depend crucially on the assumption that the dimension of X is infinite. The section ends with a remark about Issue 3 which suggests that for the specific Banach algebra considered here, there is a connection with Issue 4 too.

In Section 4, the study of sums of idempotents in the Banach algebra generated by the compact operators and the identity is continued. The sums of idempotents are now characterized as the logarithmic residues of those functions F for which the values of the compact part C are finite rank operators on X. In other words, of analytic operator functions F with values in the subalgebra of $\mathcal{L}_{\mathcal{C}}(X)$ generated by the identity and the finite rank operators on X.

Section 5 is concerned with operator valued polynomials with compact non-leading coefficients. It is proved that the logarithmic residues of such operator polynomials are sums of idempotents in $\mathcal{L}_{\mathcal{C}}(X)$, i.e., sums of projections for which either the range or the null space has finite dimension. This generalizes a well known theorem from the spectral theory of a single compact operator. The result on operator polynomials is sharp in the sense that a counterexample is given involving a monic operator polynomial of degree two for which precisely one of the non-leading coefficients is non-compact. The example also shows that for the Banach algebra $\mathcal{L}_{\mathcal{C}}(X)$, the set of sums of idempotents may be strictly contained in the set of logarithmic residues even when the underlying space X is a separable Hilbert space.

In that case – more generally, when X has the approximation property (that is: each compact operator on X is the limit of a sequence of finite rank operators on X) – the logarithmic residues do belong to the closure of the set of idempotents. Without that additional condition on the underlying space X, this need not be true. A counterexample is given in Section 6 which also contains some additional observations on this issue.

Section 7 elaborates on the remark made at the end of Section 3. Its main result contains a necessary and sufficient condition in order that two bounded linear operators on X can be represented as the left and right logarithmic residue with respect to a given Cauchy domain D and one single function F of the type studied in Section 4 (i.e., for which the values of the compact part C are finite rank operators on X). The condition is that after subtracting an appropriate multiple of the identity operator, the resulting finite rank operators should have the same trace.

There are still several unresolved problems concerning logarithmic residues in $\mathcal{L}_{\mathcal{C}}(X)$. Some of them will be indicated in Sections 3, 5 and 6 .

2. Preliminaries and first results

Throughout this section, $\mathcal{B}$ will be a (complex) Banach algebra with unit element. If F is a $\mathcal{B}$-valued function with domain Δ, then F^{-1} stands for the function given by $F^{-1}(\lambda) = F(\lambda)^{-1}$ with domain the set of all $\lambda \in \Delta$ such that $F(\lambda)$ is invertible. If Δ is an open subset of the complex plane $\mathbb{C}$ and $F : \Delta \to \mathcal{B}$ is analytic, then so is F^{-1} on its domain. The derivative of F will be denoted by F'. The *left*, respectively *right*, *logarithmic derivative* of F is the function given by $F'(\lambda)F^{-1}(\lambda)$, respectively $F^{-1}(\lambda)F'(\lambda)$, with the same domain as F^{-1}.

Logarithmic residues are contour integrals of logarithmic derivatives. To make this notion more precise, we shall employ bounded Cauchy domains in $\mathbb{C}$ and their positively oriented boundaries. For a discussion of these notions, see, for instance [TL].

Let D be a bounded Cauchy domain in $\mathbb{C}$. The (positively oriented) boundary of D will be denoted by ∂D. We write $\mathcal{A}_\partial(D;\mathcal{B})$ for the set of all $\mathcal{B}$-valued functions F with the following properties: F is defined and analytic on a neighborhood of the closure $\overline{D} = D \cup \partial D$ of D and F takes invertible values on all of ∂D (hence F^{-1} is analytic on a neighborhood of ∂D). For $F \in \mathcal{A}_\partial(D;\mathcal{B})$, one can define the contour integrals

$$LR_{left}(F;D) = \frac{1}{2\pi i}\int_{\partial D} F'(\lambda)F^{-1}(\lambda)d\lambda, \tag{1}$$

$$LR_{right}(F;D) = \frac{1}{2\pi i}\int_{\partial D} F^{-1}(\lambda)F'(\lambda)d\lambda. \tag{2}$$

The elements of the form (1) or (2) are called logarithmic residues in $\mathcal{B}$. More specifically, we call $LR_{left}(F;D)$ the *left* and $LR_{right}(F;D)$ the *right logarithmic residue of F with respect to D.*

It is convenient to introduce a local version of these concepts too. Given a complex number λ_0, we let $\mathcal{A}(\lambda_0;\mathcal{B})$ be the set of all $\mathcal{B}$-valued functions F with the following properties: F is defined and analytic on an open neighborhood of λ_0 and F takes invertible values on a deleted neighborhood of λ_0. For $F \in \mathcal{A}(\lambda_0;\mathcal{B})$, one can introduce

$$LR_{left}(F;\lambda_0) = \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\varrho} F'(\lambda)F^{-1}(\lambda)d\lambda, \tag{3}$$

$$LR_{right}(F;\lambda_0) = \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\varrho} F^{-1}(\lambda)F'(\lambda)d\lambda, \tag{4}$$

where ϱ is a positive number such that both F and F^{-1} are analytic on an open neighborhood of the punctured closed disc with center λ_0 and radius ϱ. The orientation of the integration contour $|\lambda-\lambda_0| = \varrho$ is, of course, taken positively, that is counterclockwise. Note that the right hand sides of (3) and (4) do not depend on the choice of ϱ. In fact, (3) and (4) are equal to the coefficient of $(\lambda-\lambda_0)^{-1}$ in the Laurent expansion at λ_0 of the left and right logarithmic derivative of F at λ_0, respectively. Obviously, $LR_{left}(F;\lambda_0)$, respectively $LR_{right}(F;\lambda_0)$, is a left, respectively right, logarithmic residue of F in the sense of the definitions given in the preceding paragraph (take for D the open disc with radius ϱ centered at λ_0). We call $LR_{left}(F;\lambda_0)$ the *left* and $LR_{right}(F;\lambda_0)$ the *right logarithmic residue of F at λ_0.*

In certain cases, the study of logarithmic residues with respect to bounded Cauchy domains can be reduced to the study of logarithmic residues with respect to single points. The typical situation is as follows. Let D be a bounded Cauchy domain, let $F \in \mathcal{A}_\partial(D;\mathcal{B})$ and suppose F takes invertible values on D except in a finite number of distinct points $\lambda_1,\ldots,\lambda_n \in D$. Then

$$LR_{left}(F;D) = \sum_{j=1}^{n} LR_{left}(F;\lambda_j), \tag{5}$$

$$LR_{right}(F;D) = \sum_{j=1}^{n} LR_{right}(F;\lambda_j). \tag{6}$$

This occurs, in particular, when F^{-1} is meromorphic on D with a finite number of poles in D, a state of affairs that we will encounter occasionally in what follows.

Sums of idempotents in a Banach algebra with unit element are always logarithmic residues (cf. [BES2]). This is easy to see when one allows Cauchy domains with an arbitrary number of connected components. Things are considerably more complicated when the Cauchy domains are required to be connected. The following theorem, which is a conclusion of a result due to the second author [E], covers this case. It is formulated here in a way appropriate for our needs later in this section.

A Banach algebra valued function is called *entire* when it is defined and analytic on all of $\mathbb{C}$. A pole is said to be *simple* when it has order one.

Theorem 2.1. *Let $\mathcal{B}$ be a complex Banach algebra with unit element e and let $\mathcal{B}_0$ be a subalgebra of $\mathcal{B}$ (possibly non-closed and not necessarily containing e). Let $p_1, \ldots, p_n$ be non-zero idempotents in $\mathcal{B}$ and let $\lambda_1, \ldots, \lambda_n$ be distinct (but otherwise arbitrary) points in $\mathbb{C}$. Assume that for each $j = 1, \ldots, n$, either p_j or $e - p_j$ belongs to $\mathcal{B}_0$. Then there exists an entire function $F : \mathbb{C} \to \mathcal{B}$ such that the following is satisfied:*

(i) *F takes invertible values on $\mathbb{C}$, except in the points $\lambda_1, \ldots, \lambda_n$, where F^{-1} has simple poles;*
(ii) *$LR_{left}(F; \lambda_j) = LR_{right}(F; \lambda_j) = p_j$ for all $j = 1, \ldots, n$;*
(iii) *F admits a representation $F(\lambda) = f(\lambda)e + F_0(\lambda)$, where $f : \mathbb{C} \to \mathbb{C}$ and $F_0 : \mathbb{C} \to \mathcal{B}$ are entire while, moreover, F_0 takes its values in $\mathcal{B}_0$.*

In case all idempotents $p_1, \ldots, p_n$ belong to $\mathcal{B}_0$, the scalar function f can be chosen to be constant with value 1.

The theorem is stated in terms of logarithmic residues at points. In combination with (5) and (6) it can be used to obtain results about logarithmic residues with respect to bounded Cauchy domains. We shall apply Theorem 2.1 in a situation where $e \notin \mathcal{B}_0$. A decomposition of F into f and F_0 as indicated in (iii) is then unique.

Proof. Let $p_1, \ldots, p_n$ be non-zero idempotents in $\mathcal{B}$ and let $\lambda_1, \ldots, \lambda_n$ be distinct (but otherwise arbitrary) points in $\mathbb{C}$. By [E], there exists an entire $\mathcal{B}$-valued function F such that (i) and (ii) are satisfied. The function F as constructed in [E] is a (possibly non-commutative) product of $3n$ functions of the type $e - p + \phi(\lambda)p$, where $p \in \{p_1, \ldots, p_n\}$ and ϕ is an entire scalar function. Now

$$e - p + \phi(\lambda)p = e + (\phi(\lambda) - 1)p = \phi(\lambda)e + (1 - \phi(\lambda))(e - p)$$

and either p or $e - p$ is in $\mathcal{B}_0$. So each of the functions in the product representing F has the form $\alpha(\lambda)e + \beta(\lambda)q$, where α and β are entire scalar functions and $q \in \mathcal{B}_0$ is an idempotent. But then F can be written as a (non-commutative) product

$$F(\lambda) = \prod_{k=1}^{3n} \big(\alpha_k(\lambda)e + \beta_k(\lambda)q_k\big)$$

involving entire scalar functions α_k, β_k and idempotents q_k from $\mathcal{B}_0$. For $\lambda \in \mathbb{C}$, write

$$f(\lambda) = \prod_{k=1}^{3n} \alpha_k(\lambda), \qquad F_0(\lambda) = F(\lambda) - f(\lambda)e.$$

Then f is an entire scalar function. Since $\mathcal{B}_0$ is a subalgebra of $\mathcal{B}$ and $q_1, \ldots, q_n$ belong to $\mathcal{B}_0$, the function F_0 takes its values in $\mathcal{B}_0$ and is entire too. Thus (iii) is satisfied.

The last statement of the theorem follows by observing that, in case all idempotents $p_1, \ldots, p_n$ belong to $\mathcal{B}_0$, one can take $\alpha_1(\lambda) = \cdots = \alpha_{3n}(\lambda) = 1$. $\square$

This paper is concerned with the special Banach algebra

$$\mathcal{L}_{\mathcal{C}}(X) = \{\alpha I + C \mid \alpha \in \mathbb{C},\ C \in \mathcal{C}(X)\}.$$

Here X is a complex Banach space, $\mathcal{C}(X)$ denotes the set of all compact bounded linear operators on X and $I = I_X$ is the identity operator on X. Recall that $\mathcal{C}(X)$ is a closed ideal in $\mathcal{L}(X)$, the Banach algebra of all bounded linear operators on X. Hence $\mathcal{L}_{\mathcal{C}}(X)$ is a Banach subalgebra of $\mathcal{L}(X)$ which contains $\mathcal{C}(X)$ as a closed ideal. Note that $\mathcal{L}_{\mathcal{C}}(X)$ is inverse closed with respect to $\mathcal{L}(X)$. This can most easily be seen from the formula $(\alpha I + C)^{-1} = \alpha^{-1} I - \alpha^{-1} C(\alpha I + C)^{-1}$.

If X has finite dimension n, then $\mathcal{L}(X)$, $\mathcal{C}(X)$ and $\mathcal{L}_{\mathcal{C}}(X)$ coincide and can be identified with $\mathbb{C}^{n\times n}$. In [BES4] the logarithmic residues in this Banach algebra are identified as the sums of idempotent $n \times n$ matrices. *Here we shall investigate the Banach algebra $\mathcal{B} = \mathcal{L}_{\mathcal{C}}(X)$ under the standing assumption that X is infinite dimensional.*

Because of the infinite dimensionality of X, the unit element $I = I_X$ of $\mathcal{L}_{\mathcal{C}}(X)$ is not in $\mathcal{C}(X)$. Hence $\mathcal{C}(X)$ is a complemented closed subspace of $\mathcal{L}_{\mathcal{C}}(X)$ of codimension 1. In fact, for $T \in \mathcal{L}_{\mathcal{C}}(X)$ the representation $T = \alpha I + C$ with $\alpha \in \mathbb{C}$ and $C \in \mathcal{C}(X)$ is unique. Moreover, the mapping $\alpha I + C \in \mathcal{L}_{\mathcal{C}}(X) \mapsto \alpha \in \mathbb{C}$ is Banach algebra homomorphism with kernel $\mathcal{C}(X)$.

Let $F : \Delta \to \mathcal{L}_{\mathcal{C}}(X)$ where Δ is a subset of $\mathbb{C}$. Then, as was already indicated in the introduction, there exist unique functions $f : \Delta \to \mathbb{C}$ and $C : \Delta \to \mathcal{C}(X)$ such that

$$F(\lambda) = f(\lambda)I + C(\lambda), \quad \lambda \in \Delta.$$

We call f the *scalar* and C the *compact part* of F. If Δ is an open subset of $\mathbb{C}$ and F is analytic on Δ, then so are f and C. Indeed, for each λ in the domain Δ of F, we obtain that $f(\lambda)$ is the canonical image of $F(\lambda)$ in $\mathcal{L}_{\mathcal{C}}(X)/\mathcal{C}(X)$ where this quotient algebra is identified with $\mathbb{C}$.

Recall that a bounded linear operator $T : X \to X$ is said to be a *Fredholm operator* if its null space $\operatorname{Ker} T$ has finite dimension and its range space $\operatorname{Im} T$ has finite codimension in X (and is therefore closed). The difference of the last and the first number is called the *index* of T. It is well known that if $A \in \mathcal{L}(X)$ is invertible and $C \in \mathcal{C}(X)$, then $A + C$ is a Fredholm operator with index zero. With F, f and C as is the preceding paragraph, we have that the set of zeros of f in Δ coincides with the essential spectrum of F in Δ, i.e., with the set of all λ in Δ for which $F(\lambda)$ is not a Fredholm operator.

Now let D be a bounded Cauchy domain in $\mathbb{C}$ and $F \in \mathcal{A}_\partial(D; \mathcal{L}_{\mathcal{C}}(X))$. Write f for the scalar part of F. Then $f \in \mathcal{A}_\partial(D; \mathbb{C})$. Since X is infinite dimensional and

F takes invertible values on ∂D, the function f has no zeros on ∂D. Consequently, f has only a finite number of zeros (multiplicities counted) in D. The following observation is rather basic.

Proposition 2.2. *Let D be a bounded Cauchy domain in $\mathbb{C}$, let $F \in \mathcal{A}_\partial(D;\mathcal{L}_\mathcal{C}(X))$ and let f be the scalar part of F. Write q for the number of zeros of f in D, multiplicities counted. Then $LR_{left}(F;D)-qI$ and $LR_{right}(F;D)-qI$ are compact.*

In cases as these where it is immaterial whether one considers left or right logarithmic residues, we allow ourselves to suppress the labels *left* and *right*. The conclusion in Proposition 2.2 is then simply written as follows: *Then $LR(F;D)-qI$ is compact.* As a rule, proofs will be given for the left version.

Proof. Let f and C be the scalar and compact part of F, respectively. Since X is (assumed to be) infinite dimensional, f does not vanish on the domain of F^{-1}, the set of all λ in the domain of F such that $F(\lambda)$ is invertible. Writing $F(\lambda) = f(\lambda)G(\lambda)$, we see that the function G, which is given by

$$G(\lambda) = I + \frac{1}{f(\lambda)}C(\lambda),$$

is analytic and invertible on the domain of F^{-1}, so on an open neighborhood of the boundary ∂D. As f is a scalar function, it follows that the left logarithmic derivative of F has the form

$$F'(\lambda)F^{-1}(\lambda) = \frac{f'(\lambda)}{f(\lambda)}I + G'(\lambda)G^{-1}(\lambda). \tag{7}$$

Now observe that the function $G(\lambda)-I$ has compact values. Since $\mathcal{C}(X)$ is a closed subspace of $\mathcal{L}(X)$, the same holds for the derivative $G'(\lambda) - I$. It follows that the left logarithmic derivative $G'(\lambda)G^{-1}(\lambda)$ is an analytic function on a neighborhood of ∂D taking compact values. The statement for the left logarithmic residue of F now follows by integrating the left logarithmic derivative of F given in the form (7) along ∂D. Mutatis mutandis, the same argument works for the right logarithmic residue. □

Theorem 2.3. *Let D be a bounded Cauchy domain in $\mathbb{C}$ and let $F \in \mathcal{A}_\partial(D;\mathcal{L}_\mathcal{C}(X))$. Write F in the form*

$$F(\lambda) = f(\lambda)I + C(\lambda),$$

with f and C the scalar and the compact part of F, respectively. Then the following statements are equivalent:

(i) *F is Fredholm operator valued on D;*
(ii) *f has no zeros in D;*
(iii) *$LR(F;D)$ is compact;*
(iv) *$LR(F;D)$ is of finite rank;*
(v) *$LR(F;D)$ has finite rank and* $\operatorname{rank} LR(F;D) \leq \operatorname{trace} LR(F;D) \in \mathbb{Z}$*;*
(vi) *$LR(F;D)$ is a sum of finite rank projections on X;*
(vii) *$LR(F;D)$ is a sum of rank one projections on X.*

A *projection* on X is an idempotent bounded linear operator on X.

Proof. Clearly, (vii)$\Rightarrow$(vi) and (v)$\Rightarrow$(iv)$\Rightarrow$(iii). The implication (vi)$\Rightarrow$(v) follows from the additivity of the trace, the subadditivity of the rank and the fact that for projections rank and trace coincide. From the observations made in Section 2, it is obvious that (ii)$\Rightarrow$(i). For various proofs of the (non-trivial) implications (i)$\Rightarrow$(v)$\Rightarrow$(vii), we refer to [BES4] and [BES5] (cf. also [HP] and [Wu] for the implication (v)$\Rightarrow$(vii) in the matrix case). It remains to show that (iii) implies (ii). Let q be the number of zeros (multiplicities counted) of f in D. According to Proposition 2.2, we have that $LR(F;D)-qI$ is compact. By assumption, $LR(F;D)$ is compact. Hence qI is compact. Since X is infinite dimensional, this can only happen when $q=0$ which means that f has no zeros in D. $\square$

In the situation of Theorem 2.3 and when the (equivalent) conditions (i)–(vii) are satisfied, the trace of the finite rank operator $LR(F;D)$ is equal to the total algebraic multiplicity of F with respect to D. In other words, it is equal to the number of zeros of F in D counted according to their algebraic multiplicities. Here the notion of algebraic (or null) multiplicity is taken in the sense of [GS1] (cf. [BKL2] and [GGK]). The following corollary is now immediate, taking into account that $\mathcal{L}_{\mathcal{C}}(X)$ is an inverse closed Banach subalgebra of $\mathcal{L}(X)$.

Corollary 2.4. *Let D be a bounded Cauchy domain in $\mathbb{C}$, let $F \in \mathcal{A}_\partial(D;\mathcal{L}_{\mathcal{C}}(X))$ and suppose $LR(F;D)=0$. Then F takes invertible values on all of D.*

There are proofs of this result not using the notion of algebraic multiplicity. Indeed, if $LR(F;D)=0$, then Theorem 2.3 guarantees that the function F is Fredholm operator valued on D and one can apply [BES2], Theorem 3.1, or [BES5], Corollary 3.3. The proof of [BES2], Theorem 3.1, is an application of the state space method in analysis (cf. [BGK1] and [BGK2]); that of [BES5], Corollary 3.3, is based on a factorization result for the function F, partly contained in and partly inspired by [T] and [GS2]. For still another argument, see [BES2], Example 4.4, where a connection is made with Banach algebras of a specific type introduced by S. Roch and B. Silbermann [RS].

Corollary 2.5. *The Banach algebra $\mathcal{L}_{\mathcal{C}}(X)$ has only non-trivial zero sums of idempotents.*

Thus, if $P_1,\dots,P_n$ are idempotents in $\mathcal{L}_{\mathcal{C}}(X)$ and $P_1+\cdots+P_n=0$, then $P_j=0$ for all $j=1,\dots,n$.

Proof. Combine Corollary 2.3 with [BES2], Theorem 5.1. $\square$

A more direct proof will be given in the next section where we study sums of idempotents in $\mathcal{L}_{\mathcal{C}}(X)$.

Corollary 2.5 makes it possible to introduce a partial ordering on the set of idempotents in $\mathcal{L}_{\mathcal{C}}(X)$. For S_1 and S_2 sums of idempotents in $\mathcal{L}_{\mathcal{C}}(X)$, we write $S_1 \preceq S_2$ if $S_2 - S_1$ is again a sum of idempotents in $\mathcal{L}_{\mathcal{C}}(X)$. A straightforward argument shows that $\preceq$ is a partial ordering indeed. In Section 5, this partial ordering will be used to clarify the situation with respect to logarithmic residues of operator polynomials with compact non-leading coefficients.

3. The set of sums of idempotents in $\mathcal{L}_{\mathcal{C}}(X)$: characterization and topological properties

We now turn to the study of sums of idempotents in the Banach algebra $\mathcal{L}_{\mathcal{C}}(X)$, where X is again an infinite dimensional (complex) Banach space. An idempotent in $\mathcal{L}_{\mathcal{C}}(X)$ is a fortiori an idempotent in $\mathcal{L}(X)$. In other words, the idempotents in $\mathcal{L}_{\mathcal{C}}(X)$ are projections on X. Recall that a projection on X is compact if and only if it is of finite rank.

Proposition 3.1. *The idempotents in $\mathcal{L}_{\mathcal{C}}(X)$ are the projections on X for which either the range or the null space has finite dimension.*

In other words, $P \in \mathcal{L}_{\mathcal{C}}(X)$ is an idempotent if and only if either P itself or the complementary projection $I - P$ is a finite rank projection on X.

Proof. Suppose P is an idempotent in $\mathcal{L}_{\mathcal{C}}(X)$ and write $P = \alpha I + C$ with $\alpha \in \mathbb{C}$ and $C \in \mathcal{C}(X)$. Now

$$\alpha I + C = (\alpha I + C)^2 = \alpha^2 I + 2\alpha C + C^2, \tag{8}$$

so $\alpha(\alpha - 1)I$ is compact. Since X is infinite dimensional, it follows that $\alpha = 0$ or $\alpha = 1$. In case $\alpha = 0$, the identity (8) reduces to $C^2 = C$. But then $P = C$ is a compact projection on X, hence of finite rank. In the situation where $\alpha = 1$, we have $I - P = -C$ and the identity (8) becomes $C^2 = -C$; thus $I - P = -C$ is a compact and therefore a finite rank projection on X. □

It is now possible to give a very simple and direct proof of Corollary 2.5. Suppose we have a zero sum of idempotents in $\mathcal{L}_{\mathcal{C}}(X)$. Then there exist non-negative integers n and m and finite rank projections $P_1, \dots, P_{n+m}$ on X such that

$$\sum_{j=1}^{n}(I - P_j) + \sum_{j=n+1}^{n+m} P_j = 0. \tag{9}$$

Clearly, nI is of finite rank and, as X is infinite dimensional, it follows that $n = 0$. But then (9) comes down to

$$\sum_{j=1}^{m} P_j = 0.$$

Taking traces and using that for finite rank projections trace and rank coincide, we see that $P_j = 0$ for all $j = 1, \dots, m$. This concludes the argument.

From Proposition 3.1 we see that a bounded linear operator S on X is a sum of idempotents in $\mathcal{L}_{\mathcal{C}}(X)$ if and only if it can be written in the form

$$S = \sum_{j=1}^{n}(I - P_j) + \sum_{j=n+1}^{n+m} P_j = nI - \left(\sum_{j=1}^{n} P_j - \sum_{j=n+1}^{n+m} P_j\right) \tag{10}$$

with n and m non-negative integers and $P_1, \ldots, P_{n+m}$ finite rank projections on X. Motivated by the last part of (10), we consider the set $\mathcal{P}(X)$ of all bounded linear operator on X of the form $T = V - W$ where V and W are sums of finite rank projections on X. The operators in $\mathcal{P}(X)$ are of finite rank and therefore belong to $\mathcal{L}_{\mathcal{C}}(X)$.

For given $n = 0, 1, 2, \ldots$, let $\mathcal{P}_n(X)$ be the set of all operators T on X that can be written as

$$T = -\left(\sum_{j=1}^{n} P_j - \sum_{j=n+1}^{n+m} P_j\right) \tag{11}$$

with m a (non-fixed) non-negative integer and $P_1, \ldots, P_{m+n}$ finite rank projections on X. Clearly $\mathcal{P}_0(X) \subset \mathcal{P}_1(X) \subset \mathcal{P}_2(X) \subset \ldots$ and $\mathcal{P}(X)$ is the union of the sets $\mathcal{P}_n(X)$. Write $\mathcal{S}(X)$ for the set of sums of projections on X with finite dimensional null space or range. In other words, $\mathcal{S}(X)$ is the set of sums of idempotents in $\mathcal{L}_{\mathcal{C}}(X)$. Clearly, a bounded linear operator S on X belongs to $\mathcal{S}(X)$ if and only if it can be written in the form $S = nI + T$ with n a non-negative integer and $T \in \mathcal{P}_n(X)$. Since X is infinite dimensional, the non-negative integer n in this expression is uniquely determined by S. So,

$$\mathcal{S}(X) = \bigcup_{n=0}^{\infty} \left\{ nI + T \mid T \in \mathcal{P}_n(X) \right\} \tag{12}$$

and this union is disjoint.

This discussion suggests that we deal with the sets $\mathcal{P}_n(X)$ first. We begin by considering $\mathcal{P}_0(X)$. By definition, this is the set of sums of finite rank projections or – what amounts to the same – the set of sums of rank one projections on X. For $\tau = 0, 1, 2, \ldots$, let $\mathcal{P}_{0,\tau}(X)$ denote the set of all $T \in \mathcal{P}_0(X)$ for which $\operatorname{trace} T = \tau$. Obviously, $\mathcal{P}_0(X)$ is the disjoint union of the sets $\mathcal{P}_{0,\tau}(X)$.

The following result is Theorem 4.3 from [BES5], slightly reformulated (cf. also the earlier papers [HP]and [Wu] for the matrix case). As before, X will be an infinite dimensional complex Banach space.

Proposition 3.2. *The following statements are true:*

(i) $\mathcal{P}_0(X)$ *consists of all finite rank operators* T *on* X *for which*

$$\operatorname{rank} T \leq \operatorname{trace} T \in \mathbb{Z};$$

(ii) *The sets* $\mathcal{P}_0(X)$ *and* $\mathcal{P}_{0,\tau}(X)$ *are closed subsets of* $\mathcal{L}_{\mathcal{C}}(X)$*and have empty interior;*

(iii) *The zero operator on X is the unique isolated point of $\mathcal{P}_0(X)$; in fact, for non-zero τ, the set $\mathcal{P}_{0,\tau}(X)$ has no isolated points;*

(iv) *For τ and σ non-negative integers, not both zero,*

$$\operatorname{dist}\left(\mathcal{P}_{0,\tau}(X), \mathcal{P}_{0,\sigma}(X)\right) \geq \frac{|\tau - \sigma|}{\tau + \sigma}$$

where the left hand side in this inequality stands for the distance between $\mathcal{P}_{0,\tau}(X)$ and $\mathcal{P}_{0,\sigma}(X)$;

(v) *The (arcwise) connected components of $\mathcal{P}_0(X)$ are the (different) sets $\mathcal{P}_{0,\tau}(X), \quad \tau = 0, 1, 2, \ldots$.*

It is worthwhile to note that the trace is a continuous function on $\mathcal{P}_0(X)$. For a proof of this (and an even more general result), see [BES5].

Next we turn to $\mathcal{P}_n(X)$ for $n \geq 1$, thereby distinguishing between the cases $n = 1$ and $n \geq 2$. The closure of the set of finite rank operators in $\mathcal{L}(X)$ will be denoted by $\mathcal{C}_{\mathcal{F}}(X)$. Note that $\mathcal{C}_{\mathcal{F}}(X)$ is a (closed) ideal in $\mathcal{L}(X)$ and $\mathcal{L}_{\mathcal{C}}(X)$, which is contained in $\mathcal{C}(X)$. For many important Banach spaces X, the ideals $\mathcal{C}_{\mathcal{F}}(X)$ and $\mathcal{C}(X)$ coincide.

Proposition 3.3. *The following statements are true:*

(i) *$\mathcal{P}_1(X)$ consists of all finite rank operators T on X for which*

$$-\dim \operatorname{Ker}(I + T) \leq \operatorname{trace} T \in \mathbb{Z};$$

(ii) *$\mathcal{P}_1(X)$ has empty interior and no isolated points;*

(iii) *$\mathcal{P}_1(X)$ is arcwise connected;*

(iv) *$\mathcal{P}_1(X)$ is not closed; its closure coincides with $\mathcal{C}_{\mathcal{F}}(X)$ and is a connected (even convex) subset of $\mathcal{L}_{\mathcal{C}}(X)$.*

With regard to (i) we note that the dimension of $\operatorname{Ker}(I + T)$ is finite and equal to the codimension of $\operatorname{Im}(I + T)$ in X. Indeed, as T is of finite rank, $I + T$ is a Fredholm operator of index zero.

Proposition 3.4. *The following statements are true:*

(i) *$\mathcal{P}_n(X) = \mathcal{P}_2(X) = \mathcal{P}_0(X) - \mathcal{P}_0(X) = \mathcal{P}(X)$ for all $n = 2, 3, 4 \ldots$;*

(ii) *$\mathcal{P}(X)$ consists of all finite rank operators T on X for which* $\operatorname{trace} T \in \mathbb{Z}$*;*

(iii) *$\mathcal{P}(X)$ has empty interior and no isolated points;*

(iv) *$\mathcal{P}(X)$ is arcwise connected;*

(v) *$\mathcal{P}(X)$ is not closed; its closure coincides with $\mathcal{C}_{\mathcal{F}}(X)$ and is a connected (even convex) subset of $\mathcal{L}_{\mathcal{C}}(X)$.*

It is convenient to prepare for the proofs of Propositions 3.3 and 3.4 with a lemma. In this lemma the (standing) assumption that X is infinite dimensional is essential.

Lemma 3.5. *Let V be a subset of $\mathcal{C}_{\mathcal{F}}(X)$ containing all finite rank operators on X with zero trace. Then V is arcwise connected.*

An immediate consequence is that V is dense in $\mathcal{C}_{\mathcal{F}}(X)$.

The lemma can be reformulated as follows: *Given $A \in \mathcal{C}_{\mathcal{F}}(X)$, there exists a continuous function $\Phi : [0,1] \to \mathcal{L}(X)$ such that*

(i) *For all t in the half open interval $[0,1)$, the operator $\Phi(t)$ has finite rank and zero trace;*

(ii) $\Phi(0) = 0$ *and* $\Phi(1) = A$.

Proof. Take A in $\mathcal{C}_{\mathcal{F}}(X)$ and let $A_1, A_2, A_3, \ldots$ be a sequence of finite rank operators on X converging to A. For $n = 1, 2, 3, \ldots$, put $\tau_n = \operatorname{trace} A_n$ and let m_n be a positive integer larger than $n^2\tau_n^2$. From [KS] – see also [Wo], Ch.3.B – we know that there exists a projection P_n on the (infinite dimensional) Banach space X such that $\operatorname{trace} P_n = \operatorname{rank} P_n = m_n$ and $\|P_n\| \le \sqrt{m_n}$. Introduce

$$B_n = A_n - \frac{\tau_n}{m_n} P_n.$$

Then $\operatorname{trace} B_n = 0$ and $B_n \to A$ for $n \to \infty$.

We now define $\Phi : [0,1] \to \mathcal{L}(X)$ as follows. First we put $\Phi(1) = A$, so that the second part of (ii) in the reformulation of the lemma is met. Next we define Φ on the half open intervals

$$\left[1 - \frac{1}{2^{k-1}}, 1 - \frac{1}{2^k}\right), \quad k = 1, 2, 3 \ldots . \tag{13}$$

The definition is

$$\Phi\left(1 - \frac{1+x}{2^k}\right) = xB_{k-1} + (1-x)B_k, \quad x \in (0,1];\ k = 1, 2, 3 \ldots ,$$

where $B_0 = 0$. Then $\Phi(1 - \frac{1}{2^{k-1}}) = B_{k-1}$ for $k = 1, 2, 3, \ldots$; in particular $\Phi(0) = B_0 = 0$. Thus (ii) is satisfied. Clearly, (i) holds too. It remains to prove that Φ is continuous.

Taking limits (from the left) in the right end points of the intervals (13), one sees that Φ is continuous on the half open interval $[0,1)$. To deal with the right end point of the interval $[0,1]$, we note that, for $k = 1, 2, 3 \ldots$ and $x \in (0,1]$,

$$\Phi\left(1 - \frac{1+x}{2^k}\right) - A = x(B_{k-1} - A) + (1-x)(B_k - A).$$

Hence, for $k = 1, 2, 3 \ldots$,

$$\|\Phi(t) - A\| \le \|B_{k-1} - A\| + \|B_k - A\|, \quad 1 - \frac{1}{2^{k-1}} \le t < 1 - \frac{1}{2^k}.$$

But then $\Phi(t) \to A$ for $t \to 1$ (from the left), and the proof is complete. □

Proof of Proposition 3.3. Let $T \in \mathcal{P}_1(X)$ and write T as

$$T = S - P_0, \quad S = \sum_{j=1}^{m} P_j$$

with $P_0, \dots, P_m$ projections of finite rank (see (11)). Taking traces and using that for finite rank projections trace and rank coincide, we see that the trace of T is an integer.

To prove that $\operatorname{trace} T$ is larger than or equal to $-\dim \operatorname{Ker}(I+T)$, we argue as follows. With respect to an appropriately chosen decomposition $X = \widetilde{X} \oplus \widehat{X}$, involving a finite dimensional subspace $\widetilde{X}$ of X and a closed subspace $\widehat{X}$ of X, the finite rank projections $P_0, \dots, P_m$ have the form

$$P_j = \begin{pmatrix} \widetilde{P}_j & 0 \\ 0 & 0 \end{pmatrix}.$$

Here the restrictions $\widetilde{P}_0, \dots, \widetilde{P}_m$ to $\widetilde{X}$ of $P_0, \dots, P_m$, respectively, are projections on $\widetilde{X}$. Now

$$T = \begin{pmatrix} \widetilde{T} & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$\widetilde{T} = \widetilde{S} - \widetilde{P}_0, \quad \widetilde{S} = \sum_{j=1}^{m} \widetilde{P}_j,$$

Clearly, $\operatorname{trace} T = \operatorname{trace} \widetilde{T}$ and $\dim \operatorname{Ker}(I+T) = \dim \operatorname{Ker}(\widetilde{I} + \widetilde{T})$, where $\widetilde{I}$ is the identity operator on $\widetilde{X}$. So it is sufficient to consider $\widetilde{T}$ and $\widetilde{S}$ in place of T and S. This has the advantage that the underlying space $\widetilde{X}$ has finite dimension. Put $d = \dim \widetilde{X}$. Then we get from $\widetilde{I} + \widetilde{T} = \widetilde{S} + \widetilde{I} - \widetilde{P}_0$ that

$$\begin{aligned} d - \dim \operatorname{Ker}(\widetilde{I} + \widetilde{T}) &= \operatorname{rank}(\widetilde{I} + \widetilde{T}) \\ &\le \operatorname{rank} \widetilde{S} + \operatorname{rank}(\widetilde{I} - \widetilde{P}_0) \\ &= \operatorname{rank} \widetilde{S} + d - \operatorname{rank} \widetilde{P}_0 \end{aligned}$$

and so $-\dim \operatorname{Ker}(\widetilde{I} + \widetilde{T}) \le \operatorname{rank} \widetilde{S} - \operatorname{rank} \widetilde{P}_0$. Now $\widetilde{S}$ is a sum of (finite rank) projections on $\widetilde{X}$, hence $\operatorname{rank} \widetilde{S} \le \operatorname{trace} \widetilde{S}$. It follows that

$$-\dim \operatorname{Ker}(\widetilde{I} + \widetilde{T}) \le \operatorname{trace} \widetilde{S} - \operatorname{rank} \widetilde{P}_0 = \operatorname{trace}(\widetilde{S} - \widetilde{P}_0) = \operatorname{trace} \widetilde{T},$$

as desired.

Conversely, assume T has finite rank, integer trace and

$$-\dim \operatorname{Ker}(I+T) \le \operatorname{trace} T.$$

Write $\widetilde{X} = \operatorname{Ker}(I+T)$. Since $I+T$ is a Fredholm operator (of index zero), $\widetilde{X}$ is a finite dimensional space. Let $\widehat{X}$ be a closed complement of $\widetilde{X}$ in X. With respect to the decomposition $X = \widetilde{X} \oplus \widehat{X}$, the operator T has the form

$$T = \begin{pmatrix} -\widetilde{I} & A \\ 0 & \widehat{T} \end{pmatrix}$$

with $\widetilde{I}$ the identity operator on $\widetilde{X}$, $\widehat{T} \in \mathcal{L}(\widehat{X})$ and $A : \widehat{X} \to \widetilde{X}$ a bounded linear operator. Obviously, along with T, the operator $\widehat{T}$ has finite rank. Moreover

$\operatorname{trace} T = -d + \operatorname{trace} \widehat{T}$, where $d = \dim \widetilde{X} = \dim \operatorname{Ker}(I+T)$. It follows that $\operatorname{trace} \widehat{T}$ is a non-negative integer. Besides $\widehat{T}$, the operator A, having its range in $\widetilde{X}$, is of finite rank as well. Hence $\operatorname{Ker} A \cap \operatorname{Ker} \widehat{T}$ is a closed subspace of $\widehat{X}$ with finite codimension in $\widehat{X}$. Let $\widehat{P}$ be a projection of $\widehat{X}$ along $\operatorname{Ker} A \cap \operatorname{Ker} \widehat{T}$. Then $\widehat{P}$ is of finite rank, $A = A\widehat{P}$ and $\widehat{T} = \widehat{T}\widehat{P}$. Define the finite rank projection P on $X = \widetilde{X} \oplus \widehat{X}$ by

$$P = \begin{pmatrix} \widetilde{I} & 0 \\ 0 & \widehat{P} \end{pmatrix}.$$

We claim that $T + P$ is a sum of finite rank projections on X, in other words $T + P \in \mathcal{P}_0(X)$. This is the argument.

From

$$T + P = \begin{pmatrix} 0 & A \\ 0 & \widehat{T} + \widehat{P} \end{pmatrix} = \begin{pmatrix} 0 & A\widehat{P} \\ 0 & \widehat{T}\widehat{P} + \widehat{P} \end{pmatrix} = \begin{pmatrix} 0 & A \\ 0 & \widehat{T} + \widehat{I} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \widehat{P} \end{pmatrix},$$

where $\widehat{I}$ is the identity operator on $\widehat{X}$, it is clear that $T + P$ has finite rank not exceeding that of $\widehat{P}$. Further

$$\operatorname{trace}(T + P) = \operatorname{trace}(\widehat{T} + \widehat{P}) = \operatorname{trace} \widehat{T} + \operatorname{trace} \widehat{P} = \operatorname{trace} \widehat{T} + \operatorname{rank} \widehat{P}.$$

Now $\operatorname{trace} \widehat{T}$ is a non-negative integer, so we may conclude that $T + P$ has integer trace and

$$\operatorname{trace}(T + P) \geq \operatorname{rank} \widehat{P} \geq \operatorname{rank}(T + P).$$

Thus $T + P \in \mathcal{P}_0(X)$ as desired. This finishes the proof of (i).

The trace takes only integer values on $\mathcal{P}_1(X)$. Hence $\mathcal{P}_1(X)$ has empty interior (in the topological space $\mathcal{L}_{\mathcal{C}}(X)$). This covers the first part of (ii). From (i) it is clear that $\mathcal{P}_1(X)$ contains all finite rank operators on X with zero trace. The second part of (ii), (iii) and the first part of (iv) now follow from Lemma 3.5. As the second part of (iv) is obvious, the proof is complete. □

Proof of Proposition 3.4. Mutatis mutandis, the argument for (iii), (iv) and (v) is the same as that for (ii), (iii) and (iv) of Proposition 3.4. Note that again Lemma 3.5 – valid only in an infinite dimensional context – is used. It remains to establish (i) and (ii). For this, we argue as follows.

Take $n \geq 2$. Then, as we observed already, $\mathcal{P}_2(X) \subset \mathcal{P}_n(X)$. From the definitions it is clear that $\mathcal{P}_n(X) \subset \mathcal{P}_0(X) - \mathcal{P}_0(X) = \mathcal{P}(X)$. If T belongs to the latter set, then T is of finite rank and it follows from Proposition 3.2(i) and the linearity of the trace that $\operatorname{trace} T$ is an integer. Now suppose that T fulfills these conditions on the rank and the trace by assumption. We shall prove that $T \in \mathcal{P}_2(X)$. With this (i) and (ii) is established.

Let r be the largest of the integers 0 and $\operatorname{rank} T - \operatorname{trace} T$. Then r is a non-negative integer and $\operatorname{rank} T \leq r + \operatorname{trace} T$. Choose a projection of X having rank (and hence also trace) equal to r. Note that the possibility to do this – regardless of the value of r – stems from the infinite dimensionality of X. Put $H = T + 2P$. Then H has integer trace. Also

$$\operatorname{rank} H \leq r + \operatorname{rank} T \leq 2r + \operatorname{trace} T = \operatorname{trace} H.$$

So $H \in \mathcal{P}_0(X)$ on account of Proposition 3.2(i). Hence $T = H - 2P \in \mathcal{P}_2(X)$, as desired. □

Elaborating on Propositions 3.2–3.4, we note that

$$\mathcal{P}_0(X) \subset \mathcal{P}_1(X) \subset \mathcal{P}_2(X) = \mathcal{P}_3(X) = \cdots = \mathcal{P}(X) = \mathcal{P}_0(X) - \mathcal{P}_0(X) \tag{14}$$

and that the two inclusions in (14) are strict. In fact, $\mathcal{P}_1(X) \setminus \mathcal{P}_0(X)$ consists of all finite rank operators T on X for which $\operatorname{trace} T$ is an integer satisfying

$$-\dim \operatorname{Ker}(I + T) \leq \operatorname{trace} T < \operatorname{rank} T$$

and $\mathcal{P}_2(X) \setminus \mathcal{P}_1(X)$ is the set of all finite rank operators T on X such that $\operatorname{trace} T$ is an integer and

$$\operatorname{trace} T < -\dim \operatorname{Ker}(I + T).$$

So, for instance, when Q is any non-zero finite rank projection on X, then $-Q \in \mathcal{P}_1(X) \setminus \mathcal{P}_0(X)$ and $-2Q \in \mathcal{P}_2(X) \setminus \mathcal{P}_1(X)$.

We now return to $\mathcal{S}(X)$, the set of sums of idempotents in $\mathcal{L}_{\mathcal{C}}(X)$. To facilitate the discussion, we rewrite (12) as

$$\mathcal{S}(X) = \bigcup_{n=0}^{\infty} \mathcal{S}_n(X), \tag{15}$$

where $\mathcal{S}_n(X) = \{nI + T \mid T \in \mathcal{P}_n(X)\}$. Recall that the union in (15) is disjoint. In fact, for n and m non-negative integers,

$$\operatorname{dist}(\mathcal{S}_n(X), \mathcal{S}_m(X)) = |n - m|\,. \tag{16}$$

the left hand side in this identity standing for the distance of $\mathcal{S}_n(X)$ and $\mathcal{S}_m(X)$. To prove (16), we argue as follows. As X is infinite dimensional, there are no finite rank operators T on X such that $\|T - I\| < 1$. This implies that the right hand side of (16) does not exceed the left hand side. On the other hand it is obvious that the left hand side of (16) does not exceed the right hand side, for $nI \in \mathcal{S}_n(X)$ and $mI \in \mathcal{S}_m(X)$.

Since $\mathcal{S}_0(X) = \mathcal{P}_0(X)$, the identity (15) can be rewritten as

$$\mathcal{S}(X) = \left(\bigcup_{\tau=0}^{\infty} \mathcal{P}_{0,\tau}(X) \right) \cup \left(\bigcup_{n=1}^{\infty} \{nI + T \mid T \in \mathcal{P}_n(X)\} \right)$$

which, with the help of Propositions 3.3 and 3.4, can be transformed into

$$\mathcal{S}(X) = \left(\bigcup_{\tau=0}^{\infty} \mathcal{P}_{0,\tau}(X) \right) \cup \{I + T \mid T \in \mathcal{P}_1(X)\} \cup \left(\bigcup_{n=2}^{\infty} \{nI + T \mid T \in \mathcal{P}(X)\} \right).$$

To make the picture complete, we mention that, for $n = 1, 2, 3 \dots,$

$$\operatorname{dist}(\mathcal{S}_n(X), \mathcal{P}_{0,\tau}(X)) = n.$$

This one verifies without difficulty, using that for each T in $\mathcal{P}_{0,\tau}(X)$, the operator $nI + T$ belongs to $\mathcal{S}_n(X) = \{nI + T \mid T \in \mathcal{P}_n(X)\}$.

The next result is now an immediate consequence of Propositions 3.2–3.4.

Theorem 3.6. *A bounded linear operator S on X is a sum of idempotents in $\mathcal{L}_{\mathcal{C}}(X)$ if and only if one of the following three mutually exclusive conditions is satisfied:*

(i) *S is a sum of finite rank projections or – what amounts to the same – rank one projections on X; equivalently, S has finite rank and*

$$\operatorname{rank} S \leq \operatorname{trace} S \in \mathbb{Z};$$

(ii) *$S - I$ has finite rank and*

$$-\dim \operatorname{Ker} S \leq \operatorname{trace}(S - I) \in \mathbb{Z};$$

(iii) *There exists an integer n, $n \geq 2$, such that $S - nI$ has finite rank and integer trace; equivalently, there exists an integer n, $n \geq 2$, such that $S - nI$ is the difference of two operators on X that both can be written as sums of finite rank projections on X.*

Moreover, the zero operator on X is the unique isolated point of $\mathcal{S}(X)$, and $\mathcal{S}(X)$ has empty interior. Finally, the (arcwise) connected components of $\mathcal{S}(X)$ are the (different) sets $\mathcal{P}_{0,\tau}(X)$ and $\mathcal{S}_n(X)$, where $\tau = 0, 1, 2 \dots$ and $n = 1, 2, 3, \dots$.

A few comments are in order. The conditions (i)–(iii) are mutually exclusive, indeed. This corresponds to the fact that the union in (12), written also as (15), is disjoint. With regard to (ii) we note that, since in this case S is a Fredholm operator of index zero, the dimension of $\operatorname{Ker} S$ is finite and equal to the codimension of $\operatorname{Im} S$ in X. For operators on finite dimensional spaces, the conditions (i) and (ii) would amount to the same: in that situation $\operatorname{trace}(S - I) = \operatorname{trace} S - d$ and $-\dim \operatorname{Ker} S = \operatorname{rank} S - d$. For underlying finite dimensional spaces, condition (iii) would mean nothing else than that $\operatorname{trace} S$ is an integer. So the validity of (iii) depends crucially on the assumption that X is infinite dimensional.

In this paper, we are concerned not only with sums of idempotents, but also with logarithmic residues in $\mathcal{L}_{\mathcal{C}}(X)$. Therefore, as we shall soon see, it is also relevant to look at the closure $\overline{\mathcal{S}(X)}$ of $\mathcal{S}(X)$. From (15) and (16), one immediately derives that

$$\overline{\mathcal{S}(X)} = \bigcup_{n=0}^{\infty} \overline{\mathcal{S}_n(X)}$$

and

$$\operatorname{dist}\left(\overline{\mathcal{S}_n(X)}, \overline{\mathcal{S}_m(X)}\right) = |n - m| \,.$$

Now $\overline{\mathcal{S}_n(X)} = \{nI+T \mid T \in \overline{\mathcal{P}_n(X)}\}$. It follows from Propositions 3.3 and 3.4 that $\overline{\mathcal{S}_n(X)} = \{nI+T \mid T \in \mathcal{C}_{\mathcal{F}}(X)\}$, $n = 1, 2, 3\ldots$, where, as before, $\mathcal{C}_{\mathcal{F}}(X)$ stands for the closure of the ideal of all finite rank operators on X. Also, $\mathcal{P}_0(X)$ is closed, so $\overline{\mathcal{S}_0(X)} = \mathcal{S}_0(X) = \mathcal{P}_0(X)$. Thus

$$\begin{aligned}\overline{\mathcal{S}(X)} &= \mathcal{P}_0(X) \cup \bigcup_{n=1}^{\infty} \{nI+T \mid T \in \mathcal{C}_{\mathcal{F}}(X)\} \\ &= \left(\bigcup_{\tau=0}^{\infty} \mathcal{P}_{0,\tau}(X)\right) \cup \left(\bigcup_{n=1}^{\infty} \{nI+T \mid T \in \mathcal{C}_{\mathcal{F}}(X)\}\right).\end{aligned}$$

For completeness we note that, for $n = 1, 2, 3\ldots$ and $\tau = 0, 1, 2\ldots$,

$$\operatorname{dist}\left(\overline{\mathcal{S}_n(X)}, \mathcal{P}_{0,\tau}(X)\right) = n.$$

The following theorem is now obvious.

Theorem 3.7. *A bounded linear operator S on X belongs to the closure $\overline{\mathcal{S}(X)}$ of the set $\mathcal{S}(X)$ of sums of idempotents in $\mathcal{L}_{\mathcal{C}}(X)$ if and only if one of the following two mutually exclusive conditions is satisfied:*

(i) *S is a sum of finite rank projections or – what amounts to the same – rank one projections on X; equivalently, S has finite rank and*

$$\operatorname{rank} S \le \operatorname{trace} S \in \mathbb{Z};$$

(ii) *There exists an integer $n, n \ge 1$, such that $S - nI \in \mathcal{C}_{\mathcal{F}}(X)$, i.e., $S - nI$ is a limit of finite rank operators on X.*

Moreover, the zero operator on X is the unique isolated point of $\overline{\mathcal{S}(X)}$ and $\overline{\mathcal{S}(X)}$ has empty interior. Finally, the (arcwise) connected components of $\overline{\mathcal{S}(X)}$ are the (different) sets $\mathcal{P}_{0,\tau}(X)$ and $\{nI+T \mid T \in \mathcal{C}_{\mathcal{F}}(X)\}$, where $\tau = 0, 1, 2\ldots$ and $n = 1, 2, 3, \ldots$.

Note that the sets $\{nI+T \mid T \in \mathcal{C}_{\mathcal{F}}(X)\}$ are even convex.

We now make a first connection with logarithmic residues. This connection will be further elaborated on in the subsequent sections.

To facilitate the discussion, we introduce $\mathcal{LR}_{\mathcal{C}}(X)$ as the set of all logarithmic residues in $\mathcal{L}_{\mathcal{C}}(X)$. Thus $L \in \mathcal{LR}_{\mathcal{C}}(X)$ if and only if there exist a bounded Cauchy domain D in $\mathbb{C}$ and a function $F \in \mathcal{A}_\partial(D; \mathcal{L}_{\mathcal{C}}(X))$ such that $L = LR(F; D)$. Recall from the paragraph between Proposition 2.2 and its proof that we can read $L = LR(F; D)$ as the *left* but also as the *right* variant of the logarithmic residue. So there is a *left* version and a *right* version of $\mathcal{LR}_{\mathcal{C}}(X)$. We do not yet know whether these two versions coincide. A positive answer would be in line with the results obtained in [BES4] and [BES5].

In each complex Banach algebra (with unit element), the sums of idempotents are logarithmic residues (cf. [BES2] and [E]). As a particular case of this result we have $\mathcal{S}(X) \subset \mathcal{LR}_{\mathcal{C}}(X)$. This inclusion may be strict and, in general, $\mathcal{LR}_{\mathcal{C}}(X)$ need not even be contained in $\overline{\mathcal{S}(X)}$ (see Examples 5.4 and 6.3 below). The inclusion $\mathcal{LR}_{\mathcal{C}}(X) \subset \overline{\mathcal{S}(X)}$ does hold, however, when X has the *approximation property*, by which we mean here that $\mathcal{C}(X) = \mathcal{C}_{\mathcal{F}}(X)$, that is: each compact operators on X is the limit of a sequence of finite rank operators on X. This follows immediately from Proposition 2.2, Theorem 2.3 and Theorem 3.7 (cf. also Theorem 6.1 below).

In the remainder of this section, we will assume that the Banach space X has the approximation property. Then we can rewrite the expressions for $\overline{\mathcal{S}(X)}$ given just before Theorem 3.7 as

$$\overline{\mathcal{S}(X)} = \left(\bigcup_{\tau=0}^{\infty} \mathcal{P}_{0,\tau}(X) \right) \cup \left(\bigcup_{n=1}^{\infty} \{ nI + T \mid T \in \mathcal{C}(X) \} \right). \tag{17}$$

For $n = 0, 1, 2 \ldots$, let $\mathcal{LR}_{\mathcal{C},n}(X)$ be the set of all logarithmic residues L such that $L - nI \in \mathcal{C}(X)$. In view of Theorem 2.3, we have $\mathcal{LR}_{\mathcal{C},0}(X) = \mathcal{P}_0(X) = \mathcal{S}_0(X)$. Combining this with Proposition 2.2, one gets

$$\begin{aligned}
\mathcal{LR}_{\mathcal{C}}(X) &= \bigcup_{n=0}^{\infty} \mathcal{LR}_{\mathcal{C},n}(X) \\
&= \mathcal{P}_0(X) \cup \bigcup_{n=1}^{\infty} \mathcal{LR}_{\mathcal{C},n}(X) \\
&= \left(\bigcup_{\tau=0}^{\infty} \mathcal{P}_{0,\tau}(X) \right) \cup \left(\bigcup_{n=1}^{\infty} \mathcal{LR}_{\mathcal{C},n}(X) \right)
\end{aligned}$$

and these unions are disjoint. In fact,

$$\mathcal{S}_n(X) \subset \mathcal{LR}_{\mathcal{C},n}(X) \subset \overline{\mathcal{S}_n(X)}, \quad n = 0, 1, 2 \ldots$$

(where the second inclusion is based on the present assumption that X has the approximation property), and hence

$$\begin{aligned}
\operatorname{dist}\left(\mathcal{LR}_{\mathcal{C},n}(X), \mathcal{LR}_{\mathcal{C},m}(X)\right) &= |n - m|, \qquad && n, m = 0, 1, 2 \ldots \\
\operatorname{dist}\left(\mathcal{LR}_{\mathcal{C},n}(X), \mathcal{P}_{0,\tau}(X)\right) &= n, && n, \tau = 0, 1, 2 \ldots
\end{aligned}$$

while, as we saw already in Proposition 3.2,

$$\operatorname{dist}\left(\mathcal{P}_{0,\tau}(X), \mathcal{P}_{0,\sigma}(X)\right) \geq \frac{\sigma - \tau}{\sigma + \tau}, \quad \sigma, \tau = 0, 1, 2 \ldots ; \ \sigma > \tau.$$

From Proposition 2.2 we also see that if a logarithmic residue L is given in the form (1), i.e.,

$$L = LR_{left}(F; D) = \frac{1}{2\pi i} \int_{\partial D} F'(\lambda) F^{-1}(\lambda) d\lambda,$$

then L belongs to $\mathcal{LR}_{\mathcal{C},n}(X)$ if and only if the scalar part f of F has precisely n zeros in the Cauchy domain D, multiplicities counted. The analogous remark for right logarithmic residues (i.e., those of the form (2)) is, of course, valid too.

Theorem 3.8. *Suppose X has the approximation property. Then $\mathcal{LR}_{\mathcal{C}}(X) \subset \overline{\mathcal{S}(X)}$, the zero operator on X is the unique isolated point in $\mathcal{LR}_{\mathcal{C}}(X)$, and $\mathcal{LR}_{\mathcal{C}}(X)$ has empty interior. Moreover, the (arcwise) connected components of $\mathcal{LR}_{\mathcal{C}}(X)$ are the (different) sets $\mathcal{P}_{0,\tau}(X)$ and $\mathcal{LR}_{\mathcal{C},n}(X)$, where $\tau = 0, 1, 2\ldots$ and $n = 1, 2, 3, \ldots$.*

Proof. Given the remarks made prior to the theorem, we only need to show that, for $n = 1, 2, 3, \ldots$, the set $\mathcal{LR}_{\mathcal{C},n}(X)$ is arcwise connected. Put

$$V_n = \{T - nI \mid T \in \mathcal{LR}_{\mathcal{C},n}(X)\}\,.$$

Since $\mathcal{S}_n(X) \subset \mathcal{LR}_{\mathcal{C},n}(X)$, we have $\mathcal{P}_n(X) \subset V_n$. In particular, $\mathcal{LR}_{\mathcal{C},n}(X)$ contains all finite rank operators on X with zero trace (see Propositions 3.3 and 3.4). By assumption, X has the approximation property, that is $\mathcal{C}(X) = \mathcal{C}_{\mathcal{F}}(X)$. So, as observed already above, $\mathcal{LR}_{\mathcal{C},n}(X) \subset \overline{\mathcal{S}_n(X)} = \{nI + T \mid T \in \mathcal{C}_{\mathcal{F}}(X)\}$. Hence V_n is a subset of $\mathcal{C}_{\mathcal{F}}(X)$. Now apply Lemma 3.5. □

Theorem 3.8 remains true when $\mathcal{LR}_{\mathcal{C}}(X)$ is thought of as the set of *all* logarithmic residues in $\mathcal{L}_{\mathcal{C}}(X)$ – *left or right* – and $\mathcal{LR}_{\mathcal{C},n}(X)$ is defined accordingly. In that case, for each bounded Cauchy domain D and each $F \in \mathcal{A}_{\partial}(D; \mathcal{L}_{\mathcal{C}}(X))$, the logarithmic residues

$$LR_{left}(F; D) = \frac{1}{2\pi i}\int_{\partial D} F'(\lambda)F^{-1}(\lambda)d\lambda$$

and

$$LR_{right}(F; D) = \frac{1}{2\pi i}\int_{\partial D} F^{-1}(\lambda)F'(\lambda)d\lambda$$

belong to the same connected component of $\mathcal{LR}_{\mathcal{C}}(X)$. This is clear from Proposition 2.2 and the results obtained in [BES5], Section 5. Since these components are arcwise connected, this amounts to saying that $LR_{left}(F; D)$ and $LR_{right}(F; D)$ can be connected via a continuous curve lying completely inside $\mathcal{LR}_{\mathcal{C}}(X)$.

It is an open question whether or not two operators L and R from $\mathcal{LR}_{\mathcal{C}}(X)$ that can be connected via a continuous curve lying completely inside $\mathcal{LR}_{\mathcal{C}}(X)$ can be written in the form $L = LR_{left}(F; D)$ and $R = LR_{right}(F; D)$ for a suitable (or given) Cauchy domain D and $F \in \mathcal{A}_{\partial}(D; \mathcal{L}_{\mathcal{C}}(X))$. A positive answer would be in line with results obtained in [BES4] and [BES5]. A partial solution, dealing with the type of functions considered in the next section, will be given in Section 7.

4. Sums of idempotents in $\mathcal{L}_{\mathcal{C}}(X)$: a characterization as logarithmic residues

We continue the study of sums of idempotents in the Banach algebra $\mathcal{L}_{\mathcal{C}}(X)$, elaborating on the connection with logarithmic residues. Notations are as before and – as all the time in this paper – the Banach space X is assumed to be infinite dimensional.

We shall now prove that the set $\mathcal{S}(X)$ of sums of idempotents in $\mathcal{L}_{\mathcal{C}}(X)$ coincides with the set of logarithmic residues of functions taking their values in the subalgebra of $\mathcal{L}_C(X)$ generated by the identity operator and the finite rank operators on X.

Theorem 4.1. *Let D be a bounded Cauchy domain in $\mathbb{C}$ and let S be a bounded linear operator on the infinite dimensional Banach space X. The following statements are equivalent:*

(i) *S is a sum of idempotents in $\mathcal{L}(X)$, in other words $S \in \mathcal{S}(X)$;*
(ii) *S is the left logarithmic residue with respect to D of a function F in $\mathcal{A}_\partial(D; \mathcal{L}_C(X))$ whose values on D belong to the subalgebra of $\mathcal{L}_C(X)$ generated by the identity operator and the finite rank operators on X;*
(iii) *S is the right logarithmic residue with respect to D of a function F in $\mathcal{A}_\partial(D; \mathcal{L}_C(X))$ whose values on D belong to the subalgebra of $\mathcal{L}_C(X)$ generated by the identity operator and the finite rank operators on X.*

In this result the Cauchy domain D is given. It may or may not be connected. In connection with this, we note that the proof of the implications (i)⇒(ii) and (i)⇒(iii) will provide additional information about the freedom one has in choosing the function F. Among other things it will become clear that F can always be chosen to be an entire function such that F^{-1} has only a finite number of poles which are all simple.

We prepare for the proof with the following simple lemma in which $\mathcal{B}$ is a complex Banach algebra with unit element and λ_0 is a complex number.

Lemma 4.2. *Let $F \in \mathcal{A}(\lambda_0; \mathcal{B})$ and assume F^{-1} has a pole at λ_0 of (positive) order p. Let G be a $\mathcal{B}$-valued function which is defined and analytic on an open neighborhood of λ_0, and suppose that $F - G$ has a zero at λ_0 of order at least $2p$. Then $G \in \mathcal{A}(\lambda_0; \mathcal{B})$, G^{-1} has a pole at λ_0 of order p and $LR(F; \lambda_0) = LR(G; \lambda_0)$.*

The logarithmic residues $LR(F; \lambda_0)$ and $LR(G; \lambda_0)$ are the coefficients of the term $(\lambda - \lambda_0)^{-1}$ in the Laurent expansion at λ_0 of the appropriate left or right logarithmic derivative of F or G. In fact, as we shall see, under the assumptions of the lemma, the principal parts of the Laurent expansion at λ_0 of the left, respectively right, logarithmic derivatives of F and G coincide.

Proof. We denote the unit element in $\mathcal{B}$ by e. For λ in a deleted neighborhood of λ_0, put $H(\lambda) = e - (F(\lambda) - G(\lambda))F^{-1}(\lambda)$ and write $H(\lambda_0) = e$. Then H is analytic on a neighborhood of λ_0 and the function $H(\lambda) - e$ has a zero at λ_0 of order at least p. Hence, for λ in a neighborhood of λ_0, $H(\lambda)$ is invertible and the function $H(\lambda)^{-1} - e$ also has a zero at λ_0 of order at least p. From the identity $G(\lambda) = H(\lambda)F(\lambda)$ it is now clear that $G \in \mathcal{A}(\lambda_0; \mathcal{B})$ and that the principal part of the Laurent expansion of G^{-1} at λ_0 coincides with that of F^{-1}. So, in particular, G^{-1} has a pole at λ_0 of order p. Observe that $F' - G'$ has a zero at λ_0 of order at least $2p - 1$. It follows that the principal parts of the Laurent expansion at λ_0 of the left logarithmic derivatives of F and G coincide and the same conclusion holds for the right logarithmic derivatives. With this, the proof is complete. □

Proof of Theorem 4.1. We begin by proving the implications (i)⇒(ii) and (i)⇒(iii). The complexity of the argument depends on the "shape" of D.

Let $P_1, \ldots, P_n$ be idempotents in $\mathcal{L}_{\mathcal{C}}(X)$ and let $D_1, \ldots, D_k$ be the connected components of D. When $k \geq n$, the situation is rather simple and the argument is just a slight modification of the proof of [BES3], Proposition 2.1. Indeed, choose distinct points $\lambda_1, \ldots, \lambda_n$ in $D_1, \ldots, D_n$ respectively, and let $F \in \mathcal{A}_\partial(D; B)$ be such that

$$F(\lambda) = \begin{cases} I - P_j + (\lambda - \lambda_j)P_j, & \lambda \in \overline{D}_j;\, j = 1, \ldots, n, \\ I, & \lambda \in \overline{D}_j;\, j = n+1, \ldots, k. \end{cases}$$

Then one verifies without difficulty that

$$LR_{left}(F; \lambda_j) = LR_{right}(F; \lambda_j) = P_j, \qquad j = 1, \ldots, n$$

and hence, see (5) and (6),

$$LR_{left}(F; D) = LR_{right}(F; D) = \sum_{j=1}^{n} P_j. \tag{18}$$

For each j, either the projection P_j or the complementary projection $I - P_j$ is of finite rank. Consequently, the function F has its values in the subalgebra of $\mathcal{L}_{\mathcal{C}}(X)$ generated by the identity operator and the finite rank operators on X.

Things are considerably more complicated when $k < n$. The key to the solution is then Ehrhardt's theorem as formulated in Section 2. Indeed, applying Theorem 2.1 to the situation where $\mathcal{B} = \mathcal{L}_{\mathcal{C}}(X)$ and $\mathcal{B}_0$ is the subalgebra of $\mathcal{L}_{\mathcal{C}}(X)$ consisting of all finite rank operators on X, one immediately gets the following result. *Let $P_1, \ldots, P_n$ be non-zero idempotents in $\mathcal{L}_{\mathcal{C}}(X)$ and let $\lambda_1, \ldots, \lambda_n$ be distinct (but otherwise arbitrary) points in $\mathbb{C}$. Then there exists an entire function $F : \mathbb{C} \to \mathcal{L}_{\mathcal{C}}(X)$ with the following properties:*

(a) *F takes invertible values on $\mathbb{C}$, except in the points $\lambda_1, \ldots, \lambda_n$, where F^{-1} has simple poles;*
(b) *$LR_{left}(F; \lambda_j) = LR_{right}(F; \lambda_j) = P_j$, for all $j = 1, \ldots, n$;*
(c) *The values of F on $\mathbb{C}$ belong to the subalgebra of $\mathcal{L}_{\mathcal{C}}(X)$ generated by the identity operator and the finite rank operators on X.*

Taking into account (5) and (6) and choosing the points $\lambda_1, \ldots, \lambda_n$ in the given Cauchy domain D, one gets the identities (18). This settles the implications (i)⇒(ii) and (i)⇒(iii).

Next we prove that (ii) implies (i). Let $S = LR_{left}(F; D)$ be the left logarithmic residue with respect to D of a function $F \in \mathcal{A}_\partial(D; \mathcal{L}_{\mathcal{C}}(X))$ whose values on D belong to the subalgebra of $\mathcal{L}_{\mathcal{C}}(X)$ generated by the identity operator and the finite rank operators on X. Write f and C for the scalar and the compact part of F, respectively. Then $f \in \mathcal{A}_\partial(D; \mathbb{C})$ and $C(\lambda)$ is of finite rank for each $\lambda \in D$. The function f does not vanish on ∂D and so f has only a finite number of zeros in D. We denote these zeros by $\mu_1, \ldots, \mu_k$. Since X is infinite dimensional, the operators

$F(\mu_1), \dots, F(\mu_k)$ are not invertible. For λ satisfying $f(\lambda) \neq 0$, define $H(\lambda)$ by

$$H(\lambda) = \frac{1}{f(\lambda)} F(\lambda) = I + \frac{1}{f(\lambda)} C(\lambda)$$

and put $D_0 = D \setminus \{\mu_1, \dots, \mu_k\}$. Then H is analytic on D_0 and has poles or removable singularities at the points $\mu_1, \dots, \mu_k$. We shall prove that there exist $\mu_{k+1}, \dots, \mu_l \in D_0$ such that $H(\lambda)$ is invertible for λ in $D_0 \setminus \{\mu_{k+1}, \dots, \mu_l\} = D \setminus \{\mu_1, \dots, \mu_l\}$ and that the function H^{-1} has poles or removable singularities at the points $\mu_1, \dots, \mu_l$. The argument – which draws heavily on and [Ho] and [B1] (cf. also [BKL1], Section 7) – is as follows.

By assumption, $C(\lambda)$ is a finite rank operator for each $\lambda \in D$. Maybe somewhat surprising at first sight, this implies that there exists a finite upper bound for the rank of $C(\lambda)$ when λ ranges through D. To be precise, this holds on each connected component of D. The extension to all of D follows by noting that D, being a Cauchy domain, has only a finite number of connected components. As a consequence of the boundedness of the rank (and using the lower semi-continuity of the rank), we have that for each $\lambda \in D$, the values $C'(\lambda)$ of the derivative of C are of finite rank again and the same conclusion holds for the higher order derivatives of C.

Thus the coefficients in the Taylor expansions of C at points of D are always of finite rank. It follows that the Laurent expansion of H at a point in D has a constant term which is a Fredholm operator of index zero while all other coefficients are of finite rank. In particular, H is what is called finitely meromorphic on D (cf. [GGK]). Along with F, the function H takes invertible values on the boundary of D, and hence also on a neighborhood of ∂D. Such a neighborhood has a nonempty intersection with each component of D. Thus we may conclude that H^{-1} is also finitely meromorphic on D and that H^{-1} has a finite number of poles in D (see [GGK], Section XI.8). In particular there exist $\mu_{k+1}, \dots, \mu_l$ in D_0 with the properties indicated above.

Let us return to the function F. Clearly

$$F(\lambda) = f(\lambda) H(\lambda), \qquad \lambda \in D \setminus \{\mu_1, \dots, \mu_k\}$$

and the scalar function f does not vanish on $D \setminus \{\mu_1, \dots, \mu_k\}$. Further H takes invertible values on $D \setminus \{\mu_1, \dots, \mu_l\}$ which is a subset of $D \setminus \{\mu_1, \dots, \mu_k\}$. It follows that F takes invertible values on $D \setminus \{\mu_1, \dots, \mu_l\}$ and

$$F^{-1}(\lambda) = \frac{1}{f(\lambda)} H^{-1}(\lambda), \qquad \lambda \in D \setminus \{\mu_1, \dots, \mu_l\}.$$

As H^{-1} has poles or removable singularities at the points $\mu_1, \dots, \mu_l$, so does F^{-1}.

The upshot of all of this is that F takes invertible values on D except in a finite number of distinct points $\lambda_1, \dots, \lambda_n$ where F^{-1} has poles. In view of the identities (5) and (6), things can now be reduced to the case $n = 1$, where D contains only one point λ_0 at which F is not invertible and $S = LR_{left}(F; \lambda_0)$. This also means that f has at most one zero in D which is then located at λ_0.

If f does not vanish at λ_0, then F is Fredholm operator valued on D and we know from Theorem 2.3 that S is a sum of finite rank projections; in particular, S is sum of idempotents in $\mathcal{L}_{\mathcal{C}}(X)$. It remains to tackle the (more interesting) situation where $f(\lambda_0) = 0$ and $F(\lambda_0)$ is of finite rank. We shall first show that it suffices to consider the case when F is a function of polynomial type.

Let p be the order of λ_0 as a pole of F^{-1} and let q be the order of λ_0 as a zero of f. Since X is infinite dimensional, a compact operator can not cancel the identity. Hence $q \leq p$. Introduce

$$G(\lambda) = \sum_{k=0}^{2p-1} (\lambda - \lambda_0)^k F_k,$$

where F_k stands for the coefficient of $(\lambda - \lambda_0)^k$ in the Taylor expansion of F at λ_0. So G is the $(2p-1)$-th order approximation of F at λ_0. The scalar part of G is then the $(2p-1)$-th order approximation of f at λ_0 and has therefore a zero at λ_0 of order q where $q \leq 2p-1$. Obviously, the function $F - G$ has a zero at λ_0 of order at least $2p$. Thus, by Lemma 4.2, G takes invertible values in a deleted neighborhood of λ_0, G^{-1} has a pole at λ_0 of order p and

$$LR_{left}(G; \lambda_0) = LR_{left}(F; \lambda_0) = S.$$

So, as claimed above, we may assume F to be a function of polynomial type.

Now, if F is a function of polynomial type, then so are its scalar and compact part. Write the compact part C as

$$C(\lambda) = \sum_{k=0}^{m} (\lambda - \lambda_0)^k C_k,$$

where $C_0, \ldots, C_m$ are of finite rank. Let $X = \widehat{X} \oplus \widetilde{X}$ be a direct sum decomposition of X, with $\widehat{X}$ finite dimensional and $\widetilde{X}$ closed, such that the operators C_j have a representation of the form

$$C_j = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} : \widehat{X} \oplus \widetilde{X} \to \widehat{X} \oplus \widetilde{X}.$$

Then F can be written as

$$F(\lambda) = \begin{pmatrix} \widehat{F}(\lambda) & 0 \\ 0 & f(\lambda)\widetilde{I} \end{pmatrix} : \widehat{X} \oplus \widetilde{X} \to \widehat{X} \oplus \widetilde{X},$$

where $\widetilde{I}$ is the identity operator on $\widetilde{X}$ and $\widehat{F} \in \mathcal{A}(\lambda_0; \mathcal{L}(\widehat{X}))$. It follows that

$$LR_{left}(F; \lambda_0) = \begin{pmatrix} LR_{left}(\widehat{F}; \lambda_0) & 0 \\ 0 & q\widetilde{I} \end{pmatrix} : \widehat{X} \oplus \widetilde{X} \to \widehat{X} \oplus \widetilde{X},$$

where q is the order of λ_0 as a zero of f. Since $\widehat{X}$ is finite dimensional (hence $\widehat{F}$ may be identified with a matrix function), we know from [BES4] that $LR_{left}(\widehat{F}; \lambda_0)$ is a sum of projections on $\widehat{X}$, say

$$LR_{left}(\widehat{F};\lambda_0) = \sum_{j=1}^{k} \widehat{P}_j.$$

But then, with respect to the decomposition $X = \widehat{X} \oplus \widetilde{X}$, the left logarithmic residue of F at λ_0 has the matrix representation

$$LR_{left}(F;\lambda_0) = q \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{I} \end{pmatrix} + \sum_{j=1}^{n} \begin{pmatrix} \widehat{P}_j & 0 \\ 0 & 0 \end{pmatrix}.$$

On account of Proposition 3.1, we may now conclude that $S = LR_{left}(F;\lambda_0)$ is a sum of idempotents in $\mathcal{L}_{\mathcal{C}}(X)$. As was to be expected (cf. Proposition 2.2), precisely q of these idempotents have a finite rank complementary projection.

With this we have established the implication (ii)⇒(i). Mutatis mutandis, the same argument can be used for (iii)⇒(i). □

5. Logarithmic residues in $\mathcal{L}_{\mathcal{C}}(X)$: operator polynomials with compact non-leading coefficients

The material in the previous section suggests that we should look at the simplest instances of entire operator functions, the operator polynomials. We begin with some rather straightforward observations.

Let A be an operator polynomial with coefficients in $\mathcal{L}(X)$ where, as before, X is an infinite dimensional complex Banach space. By the *spectrum* of A, written $\mathrm{Sp}\,A$, we mean the set of all $\lambda \in \mathbb{C}$ such that $A(\lambda)$ is not invertible. Clearly, $\mathrm{Sp}\,A$ is a closed subset of $\mathbb{C}$.

Proposition 5.1. *Let A be an operator polynomial with coefficients in $\mathcal{L}(X)$ and suppose* $\mathrm{Sp}\,A$ *is not all of* $\mathbb{C}$. *Assume, in addition, that the non-leading coefficients of A are compact. Then the leading coefficient of A is a Fredholm operator of index zero and* $\mathrm{Sp}\,A$ *is either a finite set or a countable set with zero as its only accumulation point.*

In particular, $\mathrm{Sp}\,A$ is a compact subset of $\mathbb{C}$.

Proof. Write A in the form

$$A(\lambda) = \lambda^m A_m + \lambda^{m-1} A_{m-1} + \cdots + \lambda A_1 + A_0$$

and suppose $A_0, \ldots, A_{m-1}$ are compact. Then $\lambda^m A_m$ is a Fredholm operator with index zero whenever $A(\lambda)$ is invertible. Since the complement of $\mathrm{Sp}\,A$ of in $\mathbb{C}$ is open and – by assumption – non-empty, it follows that, for some non-zero λ in $\mathbb{C}$, the operator $\lambda^m A_m$ is Fredholm with zero index. But then so is A_m. Consider the reversed polynomial B, given by

$$B(\lambda) = \lambda^m A_0 + \lambda^{m-1} A_1 + \cdots + \lambda A_{m-1} + A_m.$$

The values of B are all Fredholm operators (of index zero) and $\mathrm{Sp}\, B$ is not all of $\mathbb{C}$. Hence the theory of analytic Fredholm operator valued functions guarantees that each compact subset of $\mathbb{C}$ contains only a finite number of points at which A takes a non-invertible value (see [GGK], Section XI.8). This means that either $\mathrm{Sp}\, B$ is a finite set or a countable set with infinity as its only accumulation point. The conclusion of the proposition is now immediate from the identity $A(\lambda) = \lambda^m B(\lambda^{-1})$. $\square$

Let A be as in Proposition 5.1. Then A_m is Fredholm with index zero. Hence there exists a finite rank operator F on X such that $E_m = A_m - F_m$ is invertible. Thus A is of the form

$$A(\lambda) = \lambda^m E_m + \lambda^m F_m + \lambda^{m-1} A_{m-1} + \cdots + \lambda A_1 + A_0$$

with E_m invertible. Put

$$\widetilde{A}(\lambda) = \lambda^m I + \lambda^m F_m E_m^{-1} + \lambda^{m-1} A_{m-1} E_m^{-1} + \cdots + \lambda A_1 E_m^{-1} + A_0 E_m^{-1},$$

$$\widehat{A}(\lambda) = \lambda^m I + \lambda^m E_m^{-1} F_m + \lambda^{m-1} E_m^{-1} A_{m-1} + \cdots + \lambda E_m^{-1} A_1 + E_m^{-1} A_0.$$

Then $\widetilde{A}$ and $\widehat{A}$ are operator polynomials with coefficients in $\mathcal{L}_{\mathcal{C}}(X)$ and scalar part given by λ^m. Both these operator polynomials are *monic*, i.e., as leading coefficient they have the identity operator.

Suppose D is a bounded Cauchy domain in $\mathbb{C}$. Then

$$LR_{left}(A; D) = LR_{left}(\widetilde{A}; D), \tag{19}$$

$$LR_{right}(A; D) = LR_{right}(\widehat{A}; D), \tag{20}$$

provided that at least one of these – and hence all these – expressions make sense. Note that the right hand sides of (19) and (20) may be viewed as logarithmic residues in $\mathcal{L}_{\mathcal{C}}(X)$, which allows us to invoke the analysis presented in Sections 2 and 3. For that reason, from now on, we shall consider operator polynomials with coefficients in $\mathcal{L}_{\mathcal{C}}(X)$.

Theorem 5.2. *Let A be an operator polynomial of degree m with coefficients in $\mathcal{L}_{\mathcal{C}}(X)$ and suppose $A \in \mathcal{A}_\partial(D; \mathcal{L}_{\mathcal{C}}(X))$ where D is a bounded Cauchy domain in $\mathbb{C}$. Assume, in addition, that the non-leading coefficients of A are compact. Then the* (*left or right*) *logarithmic residue $LR(A; D)$ of A with respect to D is a sum of idempotents in $\mathcal{L}_{\mathcal{C}}(X)$ and $LR(A; D) \preceq mI$. Moreover,*

(i) *if $0 \notin D$, then $LR(A; D) \in \mathcal{P}_0(X)$,*
(ii) *if $0 \in D$, then $mI - LR(A; D) \in \mathcal{P}_0(X)$.*

Recall from the last paragraph of Section 2 that the partial ordering on the set $\mathcal{S}(X)$ of sums of idempotents in $\mathcal{L}_{\mathcal{C}}(X)$ is defined as follows: if S_1 and S_2 are in $\mathcal{S}(X)$, then $S_1 \preceq S_2$ if and only if $S_2 - S_1$ is in $\mathcal{S}(X)$ again.

We prepare for the proof of Theorem 5.2 with some remarks and a proposition. For the degree zero case, Theorem 5.2 is trivial; we then have $LR(A; D) = 0$.

In view of Theorem 3.6, one may expect that the degree one case, although non-trivial, is somewhat exceptional too. It is indeed, as the following proposition shows. The proposition is a slight reformulation of material from [S].

Proposition 5.3. *Let T and S be bounded linear operators on X and consider the pencil E given by $E(\lambda) = \lambda S - T$. Suppose D is a bounded Cauchy domain in $\mathbb{C}$ such that $E \in \mathcal{A}_\partial(D; \mathcal{L}(X))$. Assume, in addition, that T is compact. Then the following statements are true:*

(i) *If $0 \notin D$, then $LR(E; D)$ is a finite rank projection on X;*
(ii) *If $0 \in D$, then $I - LR(E; D)$ is a finite rank projection on X.*

Proof of Theorem 5.2. As was already observed, the case of zero degree is trivial and the degree one situation is covered by Proposition 5.3. So we assume $m \geq 2$.

Since $A \in \mathcal{A}_\partial(D; \mathcal{L}_\mathcal{C}(X))$, the sets ∂D and $\operatorname{Sp} A$ are disjoint. In particular, $\operatorname{Sp} A$ is not all of $\mathbb{C}$. So, by Proposition 5.1, the leading coefficient of A is Fredholm (with index zero). By assumption, the non-leading coefficients of A are compact. Hence $A(\lambda)$ is Fredholm for all non-zero λ. We also see from Proposition 5.1 that $\operatorname{Sp} A$ is either a finite set or a countable set with zero as its only accumulation point. In particular, $\operatorname{Sp} A$ is a compact subset of $\mathbb{C}$.

Suppose $0 \notin D$. Then A is Fredholm operator valued on D and – taking for $LR(A; D)$ the left variant of the logarithmic residue – we know from Theorem 2.3 that

$$LR(A; D) = \frac{1}{2\pi i} \int_{\partial D} A'(\lambda) A^{-1}(\lambda) d\lambda$$

is a sum of finite rank projections on X. So $LR(A; D) \in \mathcal{P}_0(X) \subset \mathcal{S}(X)$. Since $m \geq 2$, it also follows from Theorem 3.6 that $mI - LR(A; D)$ is a sum of idempotents in $\mathcal{L}_\mathcal{C}(X)$ and so $LR(A; D) \preceq mI$.

Next we consider the (more challenging) case when $0 \in D$. So at (precisely) one point in D, namely the origin, A has a compact (non-Fredholm) value and the logarithmic derivative of A possibly has an essential singularity there. First we shall deal with the situation where A is monic. An approximation argument will then be used to cover the general situation. For the monic case we choose to follow an approach which avoids the use of Proposition 5.1 and which is interesting in its own right. The approach in question is suggested by [Ha] (cf. also [Ma]) and uses linearization involving operator companion matrices.

Write

$$A(\lambda) = \lambda^m I + \lambda^{m-1} A_{m-1} + \cdots + \lambda A_1 + A_0,$$

with $A_0, \dots, A_{m-1}$ compact, and let $\boldsymbol{X} = X \oplus \cdots \oplus X$ be the direct sum of m copies of X. Introduce

$$\boldsymbol{L} = \begin{pmatrix} 0 & \dots & 0 & 0 & -A_0 \\ I & 0 & \dots & 0 & -A_1 \\ 0 & I & & & -A_2 \\ \vdots & & \ddots & \vdots & \\ 0 & \dots & 0 & I & -A_{m-1} \end{pmatrix} : \boldsymbol{X} \to \boldsymbol{X},$$

i.e., $\boldsymbol{L}$ is the (second) companion operator matrix associated with the monic operator polynomial A. It is well-known that $\boldsymbol{L}$ is a linearization of A in the sense of, for example, [BGK1] and [GKL]. In fact,

$$\begin{pmatrix} L(\lambda) & 0 & 0 & \dots & 0 \\ 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & 0 & \dots & 0 & I \end{pmatrix} = E(\lambda)(\lambda I_{\boldsymbol{X}} - \boldsymbol{L})F(\lambda) \tag{21}$$

where E and F are operator polynomials taking invertible values on all of $\mathbb{C}$. Thus, for each $\lambda \in \mathbb{C}$

$$\dim \operatorname{Ker} A(\lambda) = \dim \operatorname{Ker}(\lambda I_{\boldsymbol{X}} - \boldsymbol{L}), \quad \operatorname{codim} \operatorname{Im} A(\lambda) = \operatorname{codim} \operatorname{Im}(\lambda I_{\boldsymbol{X}} - \boldsymbol{L}).$$

It follows that $A(\lambda)$ is invertible if and only if $\lambda I_{\boldsymbol{X}} - \boldsymbol{L}$ is invertible, so the set of all $\lambda \in \mathbb{C}$ for which $A(\lambda)$ is not invertible coincides with the spectrum of the single operator $\boldsymbol{L}$. In other words, $\operatorname{Sp} A = \sigma(\boldsymbol{L})$. Also $A(\lambda)$ is a Fredholm operator if and only if the same is true for $\lambda I_{\boldsymbol{X}} - \boldsymbol{L}$.

Now compute $\boldsymbol{L}^2$:

$$\boldsymbol{L}^2 = \begin{pmatrix} 0 & \dots & \dots & 0 & -A_0 & A_0 A_{m-1} \\ 0 & & & & -A_1 & -A_0 + A_1 A_{m-1} \\ I & & & & -A_2 & -A_1 + A_2 A_{m-1} \\ 0 & I & & & \vdots & \vdots \\ \vdots & \ddots & \ddots & & -A_{m-2} & -A_{m-3} + A_{m-2} A_{m-1} \\ 0 & \dots & 0 & I & -A_{m-1} & -A_{m-2} + A_{m-1}^2 \end{pmatrix} : \boldsymbol{X} \to \boldsymbol{X}.$$

Clearly, the last two columns in this matrix representation contain compact operators only. Proceeding in this way (and as a matter of fact by finite induction), one sees that all operator entries in $\boldsymbol{L}^m$ are compact. It follows that $\boldsymbol{L}^m$ itself is a compact operator on $\boldsymbol{X}$. Hence $\boldsymbol{L}^m$ belongs to the class of the so called Riesz operators. These are the operators with quasi-nilpotent canonical image in the Calkin algebra (in fact that of $\boldsymbol{L}$ is even nilpotent). As is well known, such operators have the same spectral properties as compact operators. But then we can draw a similar conclusion for A. In particular we recover what was already observed before, namely that $A(\lambda)$ is Fredholm for all non-zero $\lambda \in \mathbb{C}$ and that $\operatorname{Sp} A$ is either a finite set or a countable set with zero as its only accumulation point. As $A(0) = A_0$ is a compact operator on an infinite dimensional Banach space, it is not invertible and so $0 \in \operatorname{Sp} A$.

In spite of the promising expression (21), the relationship of linearization that exists between A and $\boldsymbol{L}$ is not well behaved with respect to the logarithmic residues of the operator polynomial A and the spectral projections of the single operator $\boldsymbol{L}$. It is on this point that we now proceed.

Let $r = r(\boldsymbol{L})$ be the spectral radius of $\boldsymbol{L}$. Then $\operatorname{Sp} A$ is contained in the closed disc $|\lambda| \leq r$. Hence A has a Laurent expansion on $|\lambda| > r$. This expansion is readily seen to have the form

$$A'(\lambda)A^{-1}(\lambda) = \frac{m}{\lambda}I + \frac{1}{\lambda^2}L_1 + \frac{1}{\lambda^3}L_2 + \cdots$$

and it follows that

$$\frac{1}{2\pi i}\int_{|\lambda|=R} A'(\lambda)A^{-1}(\lambda)d\lambda = mI.$$

Here R is any real number larger than r.

Now take $R > r$ so large that $\overline{D}$ is contained in the open disc Δ with center the origin and radius R. Clearly, A takes invertible values on $\Delta\diagdown\overline{D}$ except in a finite number of (different) points, $\lambda_1, \ldots, \lambda_n$, say. For ϱ positive and sufficiently small, we now have

$$\frac{1}{2\pi i}\int_{\partial D} A'(\lambda)A^{-1}(\lambda)d\lambda + \sum_{j=1}^{n}\frac{1}{2\pi i}\int_{|\lambda-\lambda_j|=\varrho} A'(\lambda)A^{-1}(\lambda)d\lambda$$

$$= \frac{1}{2\pi i}\int_{|\lambda|=R} A'(\lambda)A^{-1}(\lambda)d\lambda.$$

In other words

$$LR(A;D) + \sum_{j=1}^{n} LR(A;\lambda_j) = mI.$$

Since $0 \in D$, the operator polynomial A is Fredholm operator valued on $\Delta\diagdown\overline{D}$. Hence the logarithmic residues $LR(A;\lambda_j)$ are sums of finite rank projections and have integer trace (cf. Theorem 2.3). But then the same is true for $mI - LR(A;D)$. Since $m \geq 2$, Theorem 3.6(iii) applies and we see that $LR(A;D)$ belongs to $\mathcal{S}(X)$ and $LR(A;D) \preceq mI$.

This covers the monic case. Let us now deal with the general (possibly) non-monic situation. The approach will be based on an approximation argument.

Let $F_1, F_2, F_3, \ldots$ be a sequence of finite rank operators on X, converging to the zero operator on X, such that all operators $A_m + F_k$ are invertible. For $k = 1, 2, 3\ldots$, introduce the operator polynomial A_k by

$$A_k(\lambda) = \lambda^m(A_m + F_k) + \lambda^{m-1}A_{m-1} + \cdots + \lambda A_1 + A_0.$$

Then $A_k(\lambda) \to A(\lambda)$ and $A'(\lambda) \to A'(\lambda)$ uniformly on compact subsets of $\mathbb{C}$. A routine argument yields that, for k sufficiently large – and hence without loss of generality for all k – the operator polynomial $A_k \in \mathcal{A}_\partial(D; \mathcal{L}_\mathcal{C}(X))$ and $A_k^{-1}(\lambda) \to A^{-1}(\lambda)$, where the convergence is uniform on ∂D. Thus $A_k'(\lambda)A_k^{-1}(\lambda) \to A'(\lambda)A^{-1}(\lambda)$ uniformly on ∂D and so $LR(A_k; D) \to LR(A; D)$ for $k \to \infty$. Considering, as usual, the left version of the logarithmic residue, we have $LR(A_k; D) = LR(\widetilde{A}_k; D)$ where $\widetilde{A}_k$, given by $\widetilde{A}_k(\lambda) = A_k(\lambda)(A_m + F_k)^{-1}$, is a monic operator polynomial with compact non-leading coefficients. Let D_k be a bounded Cauchy domain such that $\overline{D}_k$ is disjoint from $\overline{D}$ and $\operatorname{Sp} \widetilde{A}_k$ is contained in $D \cup D_k$. Then, as we saw above,

$$LR(\widetilde{A}_k; D) + LR(\widetilde{A}_k; D_k) = mI.$$

Since $0 \in D$, we have $0 \notin D_k$ and $LR(\widetilde{A}_k; D_k) \in \mathcal{P}_0(X)$. Also

$$LR(\widetilde{A}_k; D_k) = mI - LR(\widetilde{A}_k; D) \to mI - LR(A; D)$$

for $k \to \infty$. As $\mathcal{P}_0(X)$ is closed (see [BES5] and Proposition 3.2), it follows that $mI - LR(A; D)$ is in $\mathcal{P}_0(X)$. But then $LR(A; D) \in \mathcal{S}(X)$ by Theorem 3.6(iii) and $LR(A; D) \preceq mI$. With this, the proof is complete. □

Elaborating on the proof of Theorem 5.2, we note that if A is a degree m monic operator polynomial, $D_1, \ldots, D_n$ are pairwise disjoint Cauchy domains in $\mathbb{C}$ and A takes invertible values on the boundaries $\partial D_1, \ldots, \partial D_n$ of $D_1, \ldots, D_n$, respectively, then

$$\sum_{j=1}^{n} LR(A; D_j) = mI \tag{22}$$

provided that $D_1 \cup \cdots \cup D_n$ contains all points λ for which $A(\lambda)$ is not invertible. Under the additional assumption that the non-leading coefficients of A are compact, the converse of this is also true, as can be easily seen from Corollary 2.4.

What happens when we drop the condition that A is monic, but instead impose the conditions of Proposition 5.1. Then $\operatorname{Sp} A$ is compact and we can define $LR_{max}(A)$ by $LR_{max}(A) = LR(A; \Delta)$, where Δ is a disc centered at the origin which is so large that $\operatorname{Sp} A$ is contained in it. Obviously, this definition does not depend on the choice of such a Δ, so $LR_{max}(A)$ is well-defined. Now the statements of the preceding paragraph remain true if one replaces the left hand side of (22) by $LR_{max}(A)$. With respect to the partial ordering $\preceq$, $LR_{max}(A)$ is the (unique) maximal element of the set of all logarithmic residues of A (where one should keep in mind that there may be a difference between the left version and the right version). Also $LR_{max}(A) \preceq mI$. This inequality may be strict. For instance, when A is given by $A(\lambda) = \lambda^m(I - P) + P$, where P is a non-zero finite rank projection on X, then $LR_{max}(A) = m(I - P)$.

Further elaborating on the situation of Theorem 5.2, we observe that the conclusion $LR(A; D) \preceq mI$ means a serious restriction. It implies that not every sum of idempotents in $\mathcal{L}_{\mathcal{C}}(X)$ can be obtained as a logarithmic residue of an operator polynomial with coefficients in $\mathcal{L}_{\mathcal{C}}(X)$ and compact non-leading coefficients. To see this, let k be a positive integer, let R be a rank one projection on X and consider $T = kI + R$. Clearly T belongs to $\mathcal{S}(X)$ and $kI \preceq T$. Suppose T is a logarithmic residue of an operator polynomial A with coefficients in $\mathcal{L}_{\mathcal{C}}(X)$ and compact non-leading coefficients, say $T = LR(A; D)$ for some bounded Cauchy domain D. Then $T \preceq mI$, where m is the degree of A. As T is not of finite rank, the origin must belong to D and $mI - T$ is of finite rank. Since $mI - T = (m-k)I - R$ and X is infinite dimensional, it follows that $m = k$. So $T \preceq kI$. But this is incompatible with $kI \preceq T$ and the fact that $R \neq 0$.

Write $\mathcal{Q}(X)$ for the set of all projections Q on X with finite dimensional null space. From the material presented above it is clear that the set of logarithmic residues – left or right – of operator polynomials with coefficients in $\mathcal{L}_{\mathcal{C}}(X)$ and compact non-leading coefficients is contained in the set

$$\mathcal{P}_0(X) \cup \mathcal{Q}(X) \cup \bigcup_{m=2}^{\infty} \{mI - T \mid T \in \mathcal{P}_0(X)\}, \tag{23}$$

where this union is disjoint. We can rewrite (23) as

$$\left(\bigcup_{\tau=0}^{\infty} \mathcal{P}_{0,\tau}(X)\right) \cup \left(\bigcup_{r=0}^{\infty} \{I - P \mid P^2 = P, \operatorname{rank} P = r\}\right)$$

$$\cup \left(\bigcup_{m=2}^{\infty} \bigcup_{\tau=0}^{\infty} \{mI - T \mid T \in \mathcal{P}_{0,\tau}(X)\}\right).$$

This union is again disjoint. As a matter of fact, it exhibits the decomposition of the set (23) in its connected components (cf. [BES5], Theorem 4.3 and its proof). We do not know whether or not each operator $T \in \mathcal{S}(X)$ which belongs to the set (23) can be written as a logarithmic residue of an operator polynomial with compact non-leading coefficients.

The second term in the union (23) is the set $\mathcal{Q}(X)$. At first sight, one might have expected the set $\{I - T \mid T \in \mathcal{P}_0(X)\}$ there instead. In this context it is illustrative to note that the intersection of the latter set with $\mathcal{S}(X)$ is precisely the set $\mathcal{Q}(X)$. The point to show is that $T \in \mathcal{P}_0(X)$ and $I - T \in \mathcal{S}(X)$ implies $I - T \in \mathcal{Q}(X)$. Given that $I - T$ is in $\mathcal{S}(X)$ and that T is of finite rank, we can write $I - T$ as

$$I - T = I - P + \sum_{j=1}^{k} P_j$$

where $P, P_1, \ldots, P_k$ are finite rank projections on X (see Proposition 3.1). The operator T, being a member of $\mathcal{P}_0(X)$, can be expressed as a sum of finite rank projections on X, say $T = P_{k+1} + \cdots + P_m$. It follows that

$$P = \sum_{j=1}^{m} P_j,$$

so here we have a sum of finite rank projections which is a (finite rank) projection again. In the finite dimensional (matrix) case, it is an amusing exercise to show that this implies that $P_1, \ldots, P_m$ are mutually disjoint, i.e., $P_i P_j = P_j P_i = 0$ for $i \neq j$. The infinite dimensional case can be reduced to the finite dimensional situation in the standard way by employing a suitable decomposition of the underlying space X (cf. the proof of Proposition 3.3). It follows that T itself is a finite rank projection and so $I - T \in \mathcal{Q}(X)$ as desired.

The assumption in Theorem 5.2 concerning the non-leading coefficients of A is essential. This appears from the following example which also shows that the set $\mathcal{S}(X)$ of sums of idempotents in $\mathcal{L}_{\mathcal{C}}(X)$ can be a proper subset of $\mathcal{LR}_{\mathcal{C}}(X)$, the set of logarithmic residues in $\mathcal{L}_{\mathcal{C}}(X)$, even when the underlying space X is a Hilbert space. In the latter case – more generally when X has the approximation property – we do have the inclusion $\mathcal{LR}_{\mathcal{C}}(X) \subset \overline{\mathcal{S}(X)}$ (see Theorems 3.8 and 6.1), but is is not known whether equality holds.

Example 5.4. Let Y be an infinite dimensional Banach space and suppose $N \in \mathcal{L}(Y)$ is a compact operator on Y for which $N^3 = 0$ and N^2 has infinite rank. Concrete instances of such situations involving the sequence spaces ℓ_p, $1 \leq p \leq \infty$, are easy to produce (see below). Put $X = Y \oplus Y$ and introduce

$$A(\lambda) = \lambda^2 I + \lambda A_1 + A_0$$

where $A_0, A_1 : Y \oplus Y \to Y \oplus Y$ are given by

$$A_0 = \begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix}, \qquad A_1 = -\begin{pmatrix} I_Y & N \\ N & I_Y \end{pmatrix}.$$

Then A is a monic operator polynomial with coefficients in $\mathcal{L}_{\mathcal{C}}(X)$ and

$$A(\lambda) = \begin{pmatrix} \lambda(\lambda-1)I_Y & -\lambda N \\ -(\lambda-1)N & \lambda(\lambda-1)I_Y \end{pmatrix}. \tag{24}$$

Note that A_0 is compact but A_1 is not. In fact A_1 is a Fredholm operator with index zero. The operators $A(0)$ and $A(1)$ are compact, hence not invertible. For λ different from 0 and 1, the operator $A(\lambda)$ is invertible with inverse

$$A(\lambda)^{-1} = \begin{pmatrix} \frac{1}{\lambda(\lambda-1)} I_Y + \frac{1}{\lambda^2(\lambda-1)^2} N^2 & \frac{1}{\lambda(\lambda-1)^2} N \\ \frac{1}{\lambda^2(\lambda-1)} N & \frac{1}{\lambda(\lambda-1)} I_Y + \frac{1}{\lambda^2(\lambda-1)^2} N^2 \end{pmatrix}.$$

A straightforward computation now yields the following expressions for the logarithmic derivatives of A:

$$A'(\lambda)A^{-1}(\lambda) = \begin{pmatrix} \frac{2\lambda-1}{\lambda(\lambda-1)} I_Y + \frac{1}{\lambda(\lambda-1)^2} N^2 & \frac{1}{(\lambda-1)^2} N \\ \frac{1}{\lambda^2} N & \frac{2\lambda-1}{\lambda(\lambda-1)} I_Y + \frac{1}{\lambda^2(\lambda-1)} N^2 \end{pmatrix},$$

$$A^{-1}(\lambda)A'(\lambda) = \begin{pmatrix} \frac{2\lambda-1}{\lambda(\lambda-1)}I_Y + \frac{1}{\lambda^2(\lambda-1)}N^2 & \frac{1}{(\lambda-1)^2}N \\ \frac{1}{\lambda^2}N & \frac{2\lambda-1}{\lambda(\lambda-1)}I_Y + \frac{1}{\lambda(\lambda-1)^2}N^2 \end{pmatrix}.$$

For the left and right logarithmic residues of A at 0 it follows that

$$LR_{left}(A;0) = \frac{1}{2\pi i}\int_{|\lambda|=\frac{1}{2}} A'(\lambda)A^{-1}(\lambda)d\lambda = \begin{pmatrix} I_Y + N^2 & 0 \\ 0 & I_Y - N^2 \end{pmatrix},$$

$$LR_{right}(A;0) = \frac{1}{2\pi i}\int_{|\lambda|=\frac{1}{2}} A^{-1}(\lambda)A'(\lambda)d\lambda = \begin{pmatrix} I_Y - N^2 & 0 \\ 0 & I_Y + N^2 \end{pmatrix}.$$

So $LR_{left}(A;0)-I$ and $LR_{right}(A;0)-I$ are compact but not of finite rank. Hence these logarithmic residues do not belong to $\mathcal{S}(X)$.

To make this example more explicit, we specialize to the case when the underlying Banach space Y is the Hilbert space ℓ_2 or, more generally, the sequence space ℓ_p with $1 \le p \le \infty$ (so $X = \ell_p \oplus \ell_p$ can be identified with ℓ_p). Let $N : \ell_p \to \ell_p$ be defined by

$$N(x_1, x_2, x_3, x_4, x_5, x_6, \dots) = \left(\frac{x_2}{2}, \frac{x_3}{3}, 0, \frac{x_5}{5}, \frac{x_6}{6}, 0, \frac{x_8}{8}, \frac{x_9}{9}, \dots\right).$$

Then, indeed, $N^3 = 0$ and N^2 has infinite rank. Also N is compact. This is clear from the fact that N is the limit of the sequence of finite rank operators $N_1, N_2, N_3, \dots$, where N_k is given by

$$N_k(x_1, x_2, x_3, x_4, x_5, x_6, \dots) = \left(\frac{x_2}{2}, \frac{x_3}{3}, 0, \frac{x_5}{5}, \frac{x_6}{6}, \dots, 0, \frac{x_{3k-1}}{3k-1}, \frac{x_{3k}}{3k}, 0, 0, \dots\right).$$

In this concrete case we can extract some additional information. Let A_k be the monic operator polynomial determined by the right hand side of (24) with N replaced by N_k. As $N_k^3 = 0$, one can repeat the argument presented above. Hence

$$LR_{left}(A_k;0) = \begin{pmatrix} I_Y + N_k^2 & 0 \\ 0 & I_Y - N_k^2 \end{pmatrix},$$

$$LR_{right}(A_k;0) = \begin{pmatrix} I_Y - N_k^2 & 0 \\ 0 & I_Y + N_k^2 \end{pmatrix}.$$

The compact part of C_k of A_k is given by

$$C_k(\lambda) = \begin{pmatrix} 0 & -\lambda N_k \\ -(\lambda-1)N_k & 0 \end{pmatrix}.$$

From this we see that the values of C_k are of finite rank. It follows from Theorem 4.1 that $LR_{left}(A_k;0)$ and $LR_{right}(A_k;0)$ belong to $\mathcal{S}(X)$, something which can also be obtained from Theorem 3.6(ii) by observing that $LR_{left}(A_k;0) - I$ and $LR_{right}(A_k;0) - I$ have finite rank and zero trace. The sequence $N_1, N_2, N_3, \dots$ converges to N. Thus, for $k \to \infty$,

$$LR_{left}(A_k;0) \to LR_{left}(A;0), \quad LR_{right}(A_k;0) \to LR_{right}(A;0)$$

and we may conclude that the logarithmic residues $LR_{left}(A;0)$ and $LR_{right}(A;0)$ belong to the closure of $\mathcal{S}(X)$. Since Hilbert spaces have the approximation property, this conclusion corroborates Theorem 3.8 (cf. also Theorem 6.1). □

Note that Example 5.4 involves a monic operator polynomial of degree two with one non-compact and one compact non-leading coefficient. In fact, the constant term is compact and the coefficient of λ is not. It is easy to adapt the example in such a way that the coefficient of λ is compact and the constant term is not (replace λ by $\frac{\lambda+1}{2}$).

6. Logarithmic residues and the closure of the set of sums of idempotents in $\mathcal{L}_{\mathcal{C}}(X)$

There exist Banach algebras $\mathcal{B}$ allowing for logarithmic residues that do not belong to the closure of the set of sums of idempotents in $\mathcal{B}$. For an example involving a subalgebra of $\mathbb{C}^{3\times 3}$, see [BES6], Example 4.5. The example in question shows that a logarithmic residue need not even belong to the closure of the algebra generated by the idempotents in $\mathcal{B}$. This is in sharp contrast to the existence of situations where the logarithmic residues can be identified as the sums of idempotents (cf. [BES3], [BES4] and [BES5]).

In this section, we consider the case $\mathcal{B} = \mathcal{L}_{\mathcal{C}}(X)$ and address the following question: *Under what conditions (on X or F) is a logarithmic residue in $\mathcal{L}_{\mathcal{C}}(X)$ contained in the closure of $\mathcal{S}(X)$?*

We begin with a simple observation. Write $\mathcal{L}_{\mathcal{F}}(X)$ for the Banach subalgebra of $\mathcal{L}(X)$ generated by the finite rank operators on X and the identity operator on X. So $\mathcal{L}_{\mathcal{F}}(X) = \{\alpha I + C \mid \alpha \in \mathbb{C},\ C \in \mathcal{C}_{\mathcal{F}}(X)\}$ where, as before, $\mathcal{C}_{\mathcal{F}}(X)$ is the closed ideal generated by the finite rank operators on X. Note that $\mathcal{C}_{\mathcal{F}}(X)$ is a complemented (closed) subspace of $\mathcal{L}_{\mathcal{F}}(X)$ with codimension 1 and $\mathcal{L}_{\mathcal{F}}(X)$ is inverse closed with respect to $\mathcal{L}(X)$. As is easily verified, the results in the preceding sections remain true when $\mathcal{L}_{\mathcal{C}}(X)$ is replaced by $\mathcal{L}_{\mathcal{F}}(X)$. From the thus modified version of Theorem 3.6 it is clear that $\mathcal{S}(X)$ is not only the set of sums of idempotents in $\mathcal{L}_{\mathcal{C}}(X)$, but also the set of sums of idempotents in $\mathcal{L}_{\mathcal{F}}(X)$. The following theorem now results from Theorem 3.7 and the modified version of Proposition 2.2.

Theorem 6.1. *Let D be a bounded Cauchy domain in $\mathbb{C}$ and let $F \in \mathcal{A}_{\partial}(D;\mathcal{L}_{\mathcal{F}}(X))$. Then the left and right logarithmic residues of F with respect to D belong to $\overline{\mathcal{S}(X)}$, the closure of the set of idempotents in $\mathcal{L}_{\mathcal{F}}(X)$.*

Note that $\mathcal{L}_{\mathcal{F}}(X) = \mathcal{L}_{\mathcal{C}}(X)$ if and only if $\mathcal{C}(X) = \mathcal{C}_{\mathcal{F}}(X)$, i.e., if and only if X has the approximation property, that is each compact operator on X is the limit of a sequence of finite rank operators on X. Thus, for Banach spaces X with this property (and so in particular for Hilbert spaces X), Theorem 6.1 is true when $\mathcal{L}_{\mathcal{F}}(X)$ is replaced by $\mathcal{L}_{\mathcal{C}}(X)$, a fact which was already revealed in Section 3.

Next, we present a necessary and sufficient condition for the logarithmic residues in $\mathcal{L}_{\mathcal{C}}(X)$ to belong to $\overline{\mathcal{S}(X)}$. It is formulated in terms of certain specific operator polynomials. We will say that an operator polynomial with coefficients in $\mathcal{L}(X)$ is *pseudo monic with compact secondary coefficients* if all its coefficients are compact, except one, which is equal to the identity operator on X. Monic operator polynomials with compact non-leading coefficients are pseudo monic with compact secondary coefficients. So are co-monic operator polynomials (this means that the constant term is the identity operator) such that the coefficients of the non-constant terms are compact. For both these "extreme" cases, the logarithmic residues belong to $\mathcal{S}(X)$. An operator polynomial which is pseudo monic with compact secondary coefficients automatically has its coefficients in $\mathcal{L}_{\mathcal{C}}(X)$.

Theorem 6.2. *The following two statements are equivalent:*

(i) *The logarithmic residues in $\mathcal{L}_{\mathcal{C}}(X)$ belong to $\overline{\mathcal{S}(X)}$;*
(ii) *The logarithmic residues of pseudo monic operator polynomials with compact secondary coefficients are in $\overline{\mathcal{S}(X)}$.*

Note that there are two versions of the theorem, depending on whether one deals with the left or with the right variant of the logarithmic residue.

In the context of Theorem 6.2, only those logarithmic residues of pseudo monic operator polynomials with compact secondary coefficients are relevant that are associated with Cauchy domains containing the origin. Indeed, if D is a Cauchy domain not containing the origin and A is a pseudo monic operator polynomial with compact secondary coefficients, then A is Fredholm operator valued on D. Hence a logarithmic residue of A with respect to D is in this case a sum of finite rank idempotents (see Theorem 2.3) and therefore belongs to $\mathcal{S}(X)$.

Proof. The implication (i)$\Rightarrow$(ii) is trivial. So we will concentrate on (ii)$\Rightarrow$(i). As usual, we will work with the left version of the logarithmic residue.

Let D be a bounded Cauchy domain in $\mathbb{C}$ and let $F \in \mathcal{A}_\partial(D;\mathcal{L}_{\mathcal{C}}(X))$. Write f and C for the scalar and compact part of F, respectively. Then $f \in \mathcal{A}_\partial(D;\mathbb{C})$ and $C(\lambda) \in \mathcal{C}(X)$ for each $\lambda \in D$. The function f does not vanish on ∂D and so f has only a finite number of zeros in D. We denote these by $\mu_1, \dots, \mu_k$. Since X is infinite dimensional, the operators $F(\mu_1), \dots, F(\mu_k)$ are not invertible. Put $D_0 = D \setminus \{\mu_1, \dots, \mu_k\}$. Then D_0 is an open subset of $\mathbb{C}$ and f does not vanish on D_0. Hence F is Fredholm operator valued on D_0.

The connected components of D_0 are just the connected components of D, possibly with a finite number of points deleted. The function F takes invertible values on the boundary of D, and hence even on a neighborhood of ∂D. Such a neighborhood has a non-empty intersection with each connected component of D_0. Thus each connected component of D_0 contains points where F takes invertible values. It follows that F^{-1} is (finitely) meromorphic on D_0 ([GGK], Section XI.8). In particular, the set Γ of points in D_0 where F takes non-invertible values has no accumulation point in D_0. Clearly, Γ has no accumulation point on ∂D. So the only accumulation points Γ can have are the zeros $\mu_1, \ldots, \mu_k$ of f in D. From here on we shall assume that $\mu_1, \ldots, \mu_k$ are distinct.

Choose a positive real number ϱ such that all closed discs $|\lambda - \mu_j| \le \varrho,\ j = 1, \ldots, k$ are contained in D and F takes invertible values on their boundaries $|\lambda - \mu_j| = \varrho$. This can be done because Γ is at most countable. Denote the union of the open discs $|\lambda - \mu_j| < \varrho,\ j = 1, \ldots, k$ by D_1. Then Γ has only a finite number of distinct points $\mu_{k+1}, \ldots, \mu_l$ in $D_0 \diagdown D_1$. We may assume that ϱ has been taken so small that the closed discs $|\lambda - \mu_j| \le \varrho,\ j = 1, \ldots, k$ are mutually disjoint. Things being arranged this way, we have

$$LR_{left}(F; D) = \sum_{j=1}^{k} \frac{1}{2\pi i} \int_{|\lambda - \mu_j| = \varrho} F'(\lambda) F^{-1}(\lambda) d\lambda + \sum_{j=k+1}^{l} LR_{left}(F; \mu_j). \quad (25)$$

Since F is Fredholm operator valued on D_0, the terms $LR_{left}(F; \mu_j)$ in the second sum are sums of finite rank projections on X (see Theorem 2.3). In particular this second sum belongs to $\mathcal{S}(X)$. It remains to prove that, subject to (ii), the terms in the first sum in (25) are in the closure of $\mathcal{S}(X)$. For this, we argue as follows.

It is sufficient to consider only one term, say

$$L = \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \varrho} F'(\lambda) F^{-1}(\lambda) d\lambda$$

where λ_0 is one of the complex numbers $\mu_1, \ldots, \mu_k$, so $f(\lambda_0) = 0$. Take $R > \varrho$ such that the open disc $|\lambda - \lambda_0| < R$ is contained in D and contains no zeros of F other than λ_0. Let p be the order of λ_0 as a zero of f and write $f(\lambda) = (\lambda - \lambda_0)^p g(\lambda)$ where g is analytic and does not vanish on $|\lambda - \lambda_0| < R$. Define G on $|\lambda - \lambda_0| < R$ by $G(\lambda) = g(\lambda)^{-1} F(\lambda)$. Then

$$\begin{aligned} L &= \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \varrho} F'(\lambda) F^{-1}(\lambda) d\lambda \\ &= \left(\frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \varrho} \frac{g'(\lambda)}{g(\lambda)} d\lambda \right) I + \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \varrho} G'(\lambda) G^{-1}(\lambda) d\lambda \\ &= \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \varrho} G'(\lambda) G^{-1}(\lambda) d\lambda \end{aligned}$$

because the logarithmic residue of the scalar function g with respect to the Cauchy domain $|\lambda - \lambda_0| < R$ vanishes. The scalar part of G is given by $g(\lambda)^{-1} f(\lambda) = (\lambda - \lambda_0)^p$.

Consider

$$G(\lambda) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n G_n,$$

the Taylor expansion of G on $|\lambda - \lambda_0| < R$. Since the scalar part of G is given by $(\lambda - \lambda_0)^p$, the coefficients G_n with $n \ne p$ are compact. Also G_p is the sum of a compact operator and the identity operator, hence Fredholm with index zero. Let $F_1, F_2, F_3, \ldots$ be a sequence of finite rank operators on X, converging to the zero operator on X, such that all operators $G_p + F_k$ are invertible. For $k = 0, 1, 2, \ldots$, we now introduce the operator polynomials G_k and $\widetilde{G}_k$ as follows:

$$G_k(\lambda) = \sum_{n=0}^{k} (\lambda - \lambda_0)^n G_n,$$

$$\widetilde{G}_k(\lambda) = (\lambda - \lambda_0)^p F_k + G_k(\lambda) = (\lambda - \lambda_0)^p F_k + \sum_{n=0}^{k} (\lambda - \lambda_0)^n G_n.$$

Note that G_k is the k-th order approximation of G at λ_0. Thus $G_k(\lambda) \to G(\lambda)$ and $G_k'(\lambda) \to G'(\lambda)$ uniformly on $|\lambda - \lambda_0| \leq \varrho$. As $F_k \to 0$ for $k \to \infty$, it follows that $\widetilde{G}_k(\lambda) \to G(\lambda)$ and $\widetilde{G}_k'(\lambda) \to G'(\lambda)$ uniformly on $|\lambda - \lambda_0| \leq \varrho$ too. A routine argument shows that for k sufficiently large, $\widetilde{G}_k$ (along with G) takes invertible values on $|\lambda - \lambda_0| = \varrho$ and on this circle $\widetilde{G}_k^{-1}(\lambda) \to G^{-1}(\lambda)$ where the convergence is again uniform. But then $\widetilde{G}_k'(\lambda)\widetilde{G}_k^{-1}(\lambda) \to G'(\lambda)G^{-1}(\lambda)$ uniformly on $|\lambda - \lambda_0| = \varrho$. Hence

$$\frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \varrho} \widetilde{G}_k'(\lambda)\widetilde{G}_k^{-1}(\lambda) d\lambda \to \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \varrho} G'(\lambda)G^{-1}(\lambda) d\lambda = L \tag{26}$$

for $k \to \infty$.

Take k so large that $\widetilde{G}_k$ takes invertible values on $|\lambda - \lambda_0| = \varrho$ and, in addition, $k \geq p$. The latter means that $\widetilde{G}_k(\lambda)$ can be written as

$$\widetilde{G}_k(\lambda) = \left(\sum_{n=0}^{p-1} (\lambda - \lambda_0)^n G_n \right) + (\lambda - \lambda_0)^p (G_p + F_k) + \left(\sum_{n=p+1}^{k} (\lambda - \lambda_0)^n G_n \right).$$

Recall that $(G_p + F_k)$ is invertible and write

$$A_k(\lambda) = \widetilde{G}_k(\lambda + \lambda_0)(G_p + F_k)^{-1}.$$

Then H_k, along with $\widetilde{G}_k$, takes invertible values on $|\lambda| = \varrho$ and, with L_k given by the left hand side of (26),

$$L_k = \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \varrho} \widetilde{G}_k'(\lambda)\widetilde{G}_k^{-1}(\lambda) d\lambda = \frac{1}{2\pi i} \int_{|\lambda| = \varrho} A_k'(\lambda) A_k^{-1}(\lambda) d\lambda.$$

Now

$$A_k(\lambda)) = \left(\sum_{n=0}^{p-1} \lambda^n G_n (G_p + F_k)^{-1} \right) + \lambda^p I + \left(\sum_{n=p+1}^{k} \lambda^n G_n (G_p + F_k)^{-1} \right).$$

So the operator polynomials A_k are pseudo monic with compact secondary coefficients. Thus, assuming (ii), the operators L_k belong to $\overline{\mathcal{S}(X)}$. But then it follows from (26) that L belongs to the closure of $\mathcal{S}(X)$ too, as desired. □

The following example shows that in general the logarithmic residues in $\mathcal{L}_{\mathcal{C}}(X)$ are not contained in the closure of $\mathcal{S}(X)$. The example is a modification of Example 5.4 and – as is to be expected on the basis of the material presented in Section 3 – involves a Banach space that does not have the approximation property.

Example 6.3. Let Y be a Banach space and suppose $N \in \mathcal{L}(Y)$ is a compact operator on Y such that $N^3 = 0$ and N^2 is not the limit of a sequence of finite rank operators in $\mathcal{L}(Y)$. That such a situation can occur will be made clear later on.

We now could follow the path taken in Example 5.4. This then would lead to an example featuring logarithmic residues not belonging to the closure of $\mathcal{S}(X)$ and involving a degree 2 monic operator polynomial with one compact and one non-compact leading coefficient. In light of Theorem 6.2, however, we prefer a slightly different approach which results in an example involving a pseudo monic operator polynomial with compact secondary coefficients. All we have to do is to adapt Example 5.4 along the lines suggested by the proof of Theorem 6.2.

Put $X = Y \oplus Y$ and introduce the operator polynomial A with coefficients in $\mathcal{L}_{\mathcal{C}}(X)$ by stipulating that

$$A(\lambda) = \lambda^2 C_2 + \lambda I + C_0$$

where $C_0, C_2 : Y \oplus Y \to Y \oplus Y$ are given by

$$C_0 = \begin{pmatrix} N^2 & 0 \\ -N & 0 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}.$$

Then A is pseudo monic with compact secondary coefficients. It is convenient to present A also in the form

$$A(\lambda) = \begin{pmatrix} \lambda I_Y + N^2 & \lambda^2 N \\ -N & \lambda I_Y \end{pmatrix}.$$

The operator $A(0) = C_0$ is compact, hence not invertible. For $\lambda \neq 0$, the operator $A(\lambda)$ is invertible with inverse

$$A(\lambda)^{-1} = \begin{pmatrix} \frac{1}{\lambda}(I_Y - N^2) - \frac{1}{\lambda^2}N^2 & -N \\ \frac{1}{\lambda^2}N & \frac{1}{\lambda}(I_Y - N^2) \end{pmatrix}.$$

A straightforward computation now yields the following identities for the logarithmic derivatives of A:

$$A'(\lambda)A^{-1}(\lambda) = \begin{pmatrix} \frac{1}{\lambda}(I_Y + N^2) - \frac{1}{\lambda^2}N^2 & N \\ \frac{1}{\lambda^2}N & \frac{1}{\lambda}(I_Y - N^2) \end{pmatrix},$$

$$A^{-1}(\lambda)A'(\lambda) = \begin{pmatrix} \frac{1}{\lambda}(I_Y - N^2) - \frac{1}{\lambda^2}N^2 & N \\ \frac{1}{\lambda^2}N & \frac{1}{\lambda}(I_Y + N^2) \end{pmatrix}.$$

For the left and right logarithmic residues of A at 0 it follows that they are given by the same expressions as we had in Example 5.4, namely

$$LR_{left}(A;0) = \frac{1}{2\pi i}\int_{|\lambda|=\frac{1}{2}} A'(\lambda)A^{-1}(\lambda)d\lambda = \begin{pmatrix} I_Y + N^2 & 0 \\ 0 & I_Y - N^2 \end{pmatrix},$$

$$LR_{right}(A;0) = \frac{1}{2\pi i}\int_{|\lambda|=\frac{1}{2}} A^{-1}(\lambda)A'(\lambda)d\lambda = \begin{pmatrix} I_Y - N^2 & 0 \\ 0 & I_Y + N^2 \end{pmatrix}.$$

Clearly, $LR_{left}(A;0) - I$ and $LR_{right}(A;0) - I$ are compact. However these operators do not belong to $\mathcal{C}_{\mathcal{F}}(X)$, the closure of the set of finite rank operators in $\mathcal{L}(X)$. Indeed, otherwise the operator N^2 would appear as the limit of a sequence of finite rank operators in $\mathcal{L}(Y)$. On account of Theorem 3.7(ii), we may conclude that $LR_{left}(A;0)$ and $LR_{right}(A;0)$ do not belong to the closure of $\mathcal{S}(X)$.

One thing remains. We still have to produce a situation as was indicated in the first paragraph of this example. So we have to come up with a Banach space Y and a compact operator $N \in \mathcal{L}(Y)$ such that $N^3 = 0$ and N^2 is not the limit of a sequence of finite rank operators on Y. The construction will make use of a factorization result for compact operators which was brought to our attention by A. Pietsch, whose help is hereby gratefully acknowledged (cf. [P], Subsection 3.1.7.).

Let W be a complex Banach space that fails to have the approximation property. Then there is a complex Banach space U and a compact bounded linear operator $H : W \to U$ such that H is not the limit of a sequence of bounded linear finite rank operators acting from W into U. By the factorization result referred to above, there exist a complex Banach space V and compact operators $E : V \to U$ and $F : W \to V$ such that $H = EF$. Put $Y = U \oplus V \oplus W$ and define $N : U \oplus V \oplus W \to U \oplus V \oplus W$ by

$$N = \begin{pmatrix} 0 & E & 0 \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $N \in \mathcal{L}(Y)$ and

$$N^2 = \begin{pmatrix} 0 & 0 & EF \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & H \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So, in view of our choice of H, the operator N^2 can not be the limit of a sequence of finite rank operators in $\mathcal{L}(Y)$. Since obviously $N^3 = 0$, we have arrived at the desired situation. □

We conclude this section by indicating some open problems with regard to the relationships between the sets $\mathcal{S}(X), \mathcal{LR}_{\mathcal{C}}(X)$ and $\overline{\mathcal{S}(X)}$.

Recall that $\mathcal{LR}_{\mathcal{C}}(X)$ denotes the set of logarithmic residues in $\mathcal{L}_{\mathcal{C}}(X)$. As mentioned in Section 3, three versions of $\mathcal{LR}_{\mathcal{C}}(X)$ can be distinguished, depending on whether one works with *left* logarithmic residues, with *right* logarithmic residues or with *all* (left or right) logarithmic residues. We do not know how these versions are related to each other. For what follows, it is immaterial which interpretation one chooses.

The inclusion $\mathcal{S}(X) \subset \mathcal{LR}_{\mathcal{C}}(X)$ is just a special case of a general Banach algebra result. Example 5.4 shows that for the specific Banach algebra investigated here the inclusion may be strict, even when X is a Hilbert space. In view of the actual form of the example, we conjecture that there are no (infinite dimensional) Banach spaces for which $\mathcal{S}(X)$ and $\mathcal{LR}_{\mathcal{C}}(X)$ coincide.

We do not know whether $\mathcal{LR}_{\mathcal{C}}(X)$ is always closed. A positive answer would imply that $\overline{\mathcal{S}(X)} \subset \mathcal{LR}_{\mathcal{C}}(X)$, an inclusion which so far we have not been able to prove. We have reasons to believe, however, that when X is a separable Hilbert space, one even has $\mathcal{LR}_{\mathcal{C}}(X) = \overline{\mathcal{S}(X)}$. Indeed, in that situation, we have been able to show that the sets $\{nI + T \mid T \in \mathcal{L}(X),\ T\,\text{compact}\}$ are contained in $\mathcal{LR}_{\mathcal{C}}(X)$ whenever $n \geq 16$. The latter restriction can probably be removed which, in light of (17) and Theorem 3.8, would give the desired equality of the two sets. It is our intention to return to this point in the future.

Regardless of the outcome on this point, in the Hilbert space case or, more generally, when X has the approximation property, $\mathcal{LR}_{\mathcal{C}}(X)$ is contained in $\overline{\mathcal{S}(X)}$. This is part of Theorem 3.8. Here the extra condition on X is essential, as is seen from Example 6.3. It is an intriguing question whether or not in an example of this type *any* Banach space lacking the approximation property could serve as the underlying space X. In other words: Theorem 3.8 and Example 6.3 suggest the following problem: does $\mathcal{LR}_{\mathcal{C}}(X) \subset \overline{\mathcal{S}(X)}$ imply that X has the approximation property?

7. Left versus right logarithmic residues in $\mathcal{L}_{\mathcal{C}}(X)$

We now return to the problem posed at the end of Section 3. More specifically, we shall deal with the following question. Under what circumstances can two operators L and R in $\mathcal{S}(X)$, the set of sums of idempotents in $\mathcal{L}_{\mathcal{C}}(X)$, be represented in the form

$$LR_{left}(F;D) = \frac{1}{2\pi i}\int_{\partial D} F'(\lambda)F^{-1}(\lambda)d\lambda$$

$$LR_{right}(F;D) = \frac{1}{2\pi i}\int_{\partial D} F^{-1}(\lambda)F'(\lambda)d\lambda$$

where D is a bounded Cauchy domain in $\mathbb{C}$ and F is a function in $\mathcal{A}_{\partial}(D;\mathcal{L}_{\mathcal{C}}(X))$ whose values on D belong to the subalgebra of $\mathcal{L}_{\mathcal{C}}(X)$ generated by the identity operator and the finite rank operators on X.

We begin with a further analysis of the sets $\mathcal{P}_n(X)$ introduced and studied in Section 3. Write $\mathcal{P}_n(X)$ as a union

$$\mathcal{P}_n(X) = \bigcup_{\tau=-\infty}^{\infty} \mathcal{P}_{n,\tau}(X),$$

where, for $n = 0, 1, 2 \ldots$ and $\tau \in \mathbb{Z}$, $\mathcal{P}_{n,\tau}(X) = \{T \in \mathcal{P}_n(X) \mid \operatorname{trace} T = \tau\}$. Note that for non-negative τ, the expression $\mathcal{P}_{0,\tau}(X)$ has the same meaning as before, while $\mathcal{P}_{0,\tau}(X)$ is empty whenever τ is negative.

It is convenient to have the following lemma.

Lemma 7.1. *Let P and Q be rank one projections on X. Then there exists a finite rank operator E on X such that $Q = \exp(-E)P\exp(E)$.*

Since $\exp(-E) = \exp(E)^{-1}$, the identity $Q = \exp(-E)P\exp(E)$ comes down to a similarity between P and Q of a specific type. From the series expansion of $\exp(E)$, one sees that the range of $\exp(E) - I$ is contained in that of E, hence $\exp(E) - I$ is of finite rank and $\exp(E)$ belongs to $\mathcal{L}_{\mathcal{C}}(X)$.

Proof. The lemma is a slightly sharpened reformulation of [BES5], Lemma 4.2. The proof of that lemma shows that P and Q are similar and that there exists a similarity operator S – with $Q = S^{-1}PS$ – which is the sum of the identity operator and a finite rank operator on X. With respect to an appropriately chosen decomposition $X = \widetilde{X} \oplus \widehat{X}$, involving a finite dimensional subspace $\widetilde{X}$ of X and a closed subspace $\widehat{X}$ of X, S has the form

$$S = \begin{pmatrix} \widetilde{S} & 0 \\ 0 & \widehat{I} \end{pmatrix},$$

where $\widehat{I}$ is the identity operator on $\widehat{X}$. Clearly, $\widetilde{S}$ is an invertible operator on the finite dimensional space $\widetilde{X}$, so $\widetilde{S}$ has a logarithm. In other words, $\widetilde{S}$ can be written as an exponential, $\widetilde{S} = \exp(\widetilde{E})$ say. With

$$E = \begin{pmatrix} \widetilde{E} & 0 \\ 0 & 0 \end{pmatrix},$$

we have $\exp(E) = S$, where F is of finite rank, as desired. □

Proposition 7.2. *The sets $\mathcal{P}_{n,\tau}(X)$ are arcwise connected.*

Here $n = 0, 1, 2 \ldots$ and $\tau \in \mathbb{Z}$.

Proof. Take S and T in $\mathcal{P}_{n,\tau}(X)$, and write these operators as

$$S = -\sum_{j=1}^{n} S_j + \sum_{j=1}^{k} P_j, \qquad T = -\sum_{j=1}^{n} T_j + \sum_{j=1}^{l} Q_j,$$

where all S_j, P_j, T_j, Q_j appearing in the right hand side of these expressions are finite rank projections on X. Here k and l are non-negative integers and we may assume that the projections P_j and Q_j have rank one.

With respect to an appropriately chosen decomposition $X = \widetilde{X} \oplus \widehat{X}$, involving a finite dimensional subspace $\widetilde{X}$ of X and a closed subspace $\widehat{X}$ of X, the projections S_j and T_j have the form

$$S_j = \begin{pmatrix} \widetilde{S}_j & 0 \\ 0 & 0 \end{pmatrix}, \qquad T_j = \begin{pmatrix} \widetilde{T}_j & 0 \\ 0 & 0 \end{pmatrix}.$$

Denote the projection of X onto $\widetilde{X}$ along $\widehat{X}$ by P. Then P is of finite rank and looks like

$$P = \begin{pmatrix} \widetilde{I} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\widetilde{I}$ is the identity operator on $\widetilde{X}$. But then the operators

$$P - S_j = \begin{pmatrix} \widetilde{I} - \widetilde{S}_j & 0 \\ 0 & 0 \end{pmatrix}, \qquad P - T_j = \begin{pmatrix} \widetilde{I} - \widetilde{T}_j & 0 \\ 0 & 0 \end{pmatrix}$$

are finite rank projections on X which can be written as a sum of rank one projections on X. Also

$$S = -nP + \sum_{j=1}^{n} (P - S_j) + \sum_{j=1}^{k} P_j,$$

$$T = -nP + \sum_{j=1}^{n} (P - T_j) + \sum_{j=1}^{l} Q_j.$$

Thus we obtain S and T in the form

$$S = -nP + \sum_{j=1}^{s} P_j, \qquad T = -nP + \sum_{j=1}^{t} Q_j,$$

where s and t are non-negative integers and $P_1, \ldots, P_s, Q_1, \ldots, Q_t$ are rank one projections on X. Taking traces, we get

$$\operatorname{trace} S = s - n \operatorname{trace} P, \qquad \operatorname{trace} T = t - n \operatorname{trace} P$$

and it follows that $s = t$. So

$$S = -nP + \sum_{j=1}^{r} P_j, \qquad T = -nP + \sum_{j=1}^{r} Q_j$$

where $r = s = t$ is a non-negative integer, P is a finite rank projection on X and $P_1, \ldots, P_r, Q_1, \ldots, Q_r$ are rank one projections on X.

For $j = 1, \ldots, r$, choose finite rank operators F_j such that

$$Q_j = \exp(-F_j) P_j \exp(F_j).$$

Lemma 7.1 guarantees that this is possible. Now define $\Psi : [0,1] \to \mathcal{L}(X)$ by

$$\Psi(u) = -nP + \sum_{j=1}^{r} \exp(-uF_j) P_j \exp(uF_j).$$

Then $\Psi(0) = S$ and $\Psi(1) = T$. Also Ψ has its values in $\mathcal{P}_{n,\tau}(X)$. Finally, Ψ is continuous, and the proof is complete. $\square$

Recall that

$$\mathcal{S}(X) = \bigcup_{n=0}^{\infty} \mathcal{S}_n(X)$$

where $\mathcal{S}_n(X) = \{nI + T \mid T \in \mathcal{P}_n(X)\}$. With the help of the sets $\mathcal{P}_{n,\tau}(X)$, we can rewrite this as

$$\mathcal{S}(X) = \bigcup_{n=0}^{\infty} \bigcup_{\tau=-\infty}^{\infty} \mathcal{S}_{n,\tau}(X) \tag{27}$$

with $\mathcal{S}_{n,\tau}(X) = \{nI + T \mid T \in \mathcal{P}_{n,\tau}(X)\}$. For $\tau < 0$, the set $\mathcal{S}_{0,\tau}(X) = \mathcal{P}_{0,\tau}(X)$ is empty. By virtue of the results obtained in Section 3, the other sets $\mathcal{S}_{n,\tau}(X)$ can be described as follows:

$n = 0,\ \tau \geq 0$:
$\mathcal{S}_{0,\tau}(X) = \{S \in \mathcal{L}(X) \mid S \text{ of finite rank, } \operatorname{rank} S \leq \operatorname{trace} S = \tau\}$,

$n = 1,\ \tau < 0$:
$\mathcal{S}_{1,\tau}(X) = \{S \in \mathcal{L}(X) \mid S - I \text{ of finite rank, } -\dim \operatorname{Ker} S \leq \operatorname{trace}(S - I) = \tau\}$,

$n = 1,\ \tau \geq 0$:
$\mathcal{S}_{1,\tau}(X) = \{S \in \mathcal{L}(X) \mid S - I \text{ of finite rank, } \operatorname{trace}(S - I) = \tau\}$,

$n \geq 2,\ \tau \in \mathbb{Z}$:
$\mathcal{S}_{n,\tau}(X) = \{S \in \mathcal{L}(X) \mid S - nI \text{ of finite rank, } \operatorname{trace}(S - nI) = \tau\}$.

Along with the sets $\mathcal{P}_{n,\tau}(X)$, the sets $\mathcal{S}_{n,\tau}(X)$ are arcwise connected. So (27) is a disjoint union of arcwise connected sets.

Theorem 7.3. *Let D be a bounded Cauchy domain in $\mathbb{C}$ and let L and R be bounded linear operators on X. The following statements are equivalent:*

(i) *There exists a function F in $\mathcal{A}_\partial(D; \mathcal{L}(X))$, whose values on D belong to the subalgebra of $\mathcal{L}_\mathcal{C}(X)$ generated by the identity operator and the finite rank operators on X, such that L is the left and R is the right logarithmic residue of F with respect to D, i.e.,*

$$L = LR_{left}(F; D) = \frac{1}{2\pi i} \int_{\partial D} F'(\lambda) F^{-1}(\lambda) d\lambda,$$

$$R = LR_{right}(F; D) = \frac{1}{2\pi i} \int_{\partial D} F^{-1}(\lambda) F'(\lambda) d\lambda;$$

(ii) *There exist integers n and τ, $n \geq 0$, such that L and R both belong to the (arcwise connected) set $\mathcal{S}_{n,\tau}(X)$.*

Loosely speaking, (ii) says that L and R belong to one and the same (arcwise connected) "constituent" in the decomposition (27) of $\mathcal{S}(X)$.

The proof, especially the part dealing with the implication (ii)⇒(i) will provide additional information about the freedom one has in choosing the function F. As we shall see, F can be chosen to be an entire function such that F^{-1} has only a finite number of poles which are all simple.

Proof. Suppose we have (i) and let f be the scalar part of F. Write n for the number of zeros of F in D, multiplicities counted. Then $L - nI$ and $R - nI$ are compact by Proposition 2.2. Also L and R are in $\mathcal{S}(X)$ by Theorem 4.1. Hence $L - nI$ and $R - nI$ belong to $\mathcal{P}_n(X)$. It remains to prove that $L - nI$ and $R - nI$ have the same trace. Introduce

$$H(\lambda) = \frac{1}{f(\lambda)} F(\lambda) = I + \frac{1}{f(\lambda)} C(\lambda).$$

As we saw in the proof of Theorem 4.1, the function H is finitely meromorphic on D. Also, the constant terms in the Laurent expansions of H at the points of D are Fredholm (with index zero). Finally, in each connected component of D there are points at which H takes invertible values. It follows that the integrals

$$\frac{1}{2\pi i} \int_{\partial D} H'(\lambda) H^{-1}(\lambda) d\lambda \tag{28}$$

and

$$\frac{1}{2\pi i} \int_{\partial D} H^{-1}(\lambda) H'(\lambda) d\lambda \tag{29}$$

are of finite rank. Comparing Laurent expansions and using the commutativity property of the trace, one sees that (28) and (29) have the same trace. Actually, these coinciding traces are equal to the total algebraic multiplicity of the meromorphic Fredholm operator valued function H with respect to D (see [GS1], [GGK] and [BKL2]). The desired result is now clear from the fact that (28) and (29) are equal to $L - nI$ and $R - nI$, respectively. This settles the implication (i)⇒(ii).

Assume L and R satisfy (ii). We shall prove the following more elaborate version of (i). *There exists an entire function $F : \mathbb{C} \to \mathcal{L}_{\mathcal{C}}(X)$ with the following properties:*

(a) *F takes invertible values on all of $\mathbb{C}$, except in a finite number of points, all lying in D, where F^{-1} has simple poles;*
(b) *The values of F on $\mathbb{C}$ belong to the subalgebra of $\mathcal{L}_{\mathcal{C}}(X)$ generated by the identity operator and the finite rank operators on X.*
(c) *L is the left and R is the right logarithmic residue of F with respect to D.*

The argument is a modification of a part of the proof of [BES5], Theorem 5.1.

With n and τ as in (ii), put $S = L - nI$ and $T = R - nI$. Then S and T belong to $\mathcal{P}_{n,\tau}(X)$. From the proof of Proposition 7.2, we know that S and T can be written in the form

$$S = -nP + \sum_{k=1}^{r} P_k, \qquad T = -nP + \sum_{k=1}^{r} Q_k$$

where r is a non-negative integer, P is a finite rank projection on X and $P_1, \dots, P_r$, $Q_1, \dots, Q_r$ are rank one projections on X. Hence

$$L = n(I - P) + \sum_{k=1}^{r} P_k, \qquad R = n(I - P) + \sum_{k=1}^{r} Q_k$$

and in this way, both L and R are written as a sums of $r + n$ non-zero idempotents in $\mathcal{L}_{\mathcal{C}}(X)$.

Choose distinct points $\lambda_1, \dots, \lambda_{r+n}$ in D and apply Theorem 2.1 to the situation where $\mathcal{B} = \mathcal{L}_{\mathcal{C}}(X)$ and $\mathcal{B}_0$ is the subalgebra of $\mathcal{L}_{\mathcal{C}}(X)$ consisting of all finite rank operators on X. One then obtains an entire $\mathcal{L}(X)$-valued function G such that G takes invertible values on $\mathbb{C}$, except in the points $\lambda_1, \dots, \lambda_{r+n}$ where G^{-1} has simple poles,

$$LR_{left}(G; \lambda_k) = LR_{right}(G; \lambda_k) = \begin{cases} P_k, & k = 1, \dots, r, \\ I - P, & k = r + 1, \dots, r + n, \end{cases}$$

while, in addition, the values of G on $\mathbb{C}$ belong to the subalgebra of $\mathcal{L}_{\mathcal{C}}(X)$ generated by the identity operator and the finite rank operators on X. Clearly, $G \in \mathcal{A}_{\partial}(D; \mathcal{L}(X))$ and, taking into account (5) and (6),

$$LR_{left}(G; D) = LR_{right}(G; D) = n(I - P) + \sum_{k=1}^{r} P_k = L.$$

We shall now modify G in such a way that the left residue of the resulting function remains L, but the right logarithmic residue becomes R. For this we shall use an interpolation argument.

By Lemma 7.1, there exist finite rank operators $F_1, \dots, F_r$ such that

$$Q_k = \exp(-F_k) P_k \exp(F_k), \qquad k = 1, \dots, r.$$

Choose scalar polynomials $r_1, \dots, r_{r+n}$ with

$$r_j(\lambda_k) = \delta_{jk}, \quad r_j'(\lambda_k) = 0, \qquad j, k = 1, \dots, r + n$$

(δ_{jk} is the Kronecker delta) and, for $j = 1, \dots, r$, put

$$H_j(\lambda) = \exp(r_j(\lambda) F_j).$$

Then $H_j : \mathbb{C} \to \mathcal{L}_(X)$ is analytic and takes invertible values on all of $\mathbb{C}$. Also

$$\begin{aligned} H_j(\lambda_k) &= I, && j = 1, \dots, r;\ k = 1, \dots, r + n; j \neq k, \\ H_j(\lambda_j) &= \exp(F_j), && j = 1, \dots, r, \\ H_j'(\lambda_k) &= 0, && j = 1, \dots, r;\ k = 1, \dots, r + n. \end{aligned}$$

From the definition of H_j and the power series expansion of the exponential function, it is obvious that the ranges of the operators $H_j(\lambda) - I$ are contained in the range of F_j. Thus the functions $H_1, \ldots, H_r$ have their values in the subalgebra of $\mathcal{L}_{\mathcal{C}}(X)$ generated by the identity operator and the finite rank operators on X.

Write $H(\lambda) = H_1(\lambda) \cdots H_r(\lambda)$. Then $H : \mathbb{C} \to \mathcal{L}_{\mathcal{C}}(X)$ is analytic and takes invertible values on all of $\mathbb{C}$. Also

$$\begin{aligned} H(\lambda_k) &= \exp(F_k), && k = 1, \ldots, r, \\ H(\lambda_k) &= I, && k = r+1, \ldots, r+n, \\ H'(\lambda_k) &= 0, && k = 1, \ldots, r+n. \end{aligned}$$

Finally, along with $H_1, \ldots, H_r$, the function H takes its values in the subalgebra of $\mathcal{L}_{\mathcal{C}}(X)$ generated by the identity operator and the finite rank operators on X.

Put $F(\lambda) = G(\lambda)H(\lambda)$. Then $F : \mathbb{C} \to \mathcal{L}_{\mathcal{C}}(X)$ clearly has the properties (a) and (b). It remains to prove that (c) is satisfied too. For this, we argue as follows.

For ϱ positive and sufficiently small, we have

$$LR_{left}(F; \lambda_k) = LR_{left}(G; \lambda_k) + \frac{1}{2\pi i} \int_{|\lambda - \lambda_k| = \varrho} G(\lambda)H'(\lambda)H^{-1}(\lambda)G^{-1}(\lambda)d\lambda.$$

The first term in the right hand side is equal to P_k when $k = 1, \ldots, r$ and to $I - P$ when $k = r+1, \ldots, r+n$. The second vanishes because G^{-1} has a simple pole at λ_k and $H'(\lambda_k) = 0$. So

$$LR_{left}(F; \lambda_k) = \begin{cases} P_k, & k = 1, \ldots, r, \\ I - P, & k = r+1, \ldots, r+n. \end{cases}$$

Thus $LR_{left}(F; D) = P_1 + \cdots + P_k + n(I - P) = L$.

Analogously we have

$$LR_{right}(F; \lambda_k) = LR_{right}(H; \lambda_k) + \frac{1}{2\pi i} \int_{|\lambda - \lambda_k| = \varrho} H^{-1}(\lambda)G^{-1}(\lambda)G'(\lambda)H(\lambda)d\lambda.$$

The first term in the right hand side vanishes. The second is equal to

$$H^{-1}(\lambda_k)LR_{right}(G; \lambda_k)H(\lambda_k).$$

Now $H(\lambda_k)$ is equal to $\exp(F_k)$ for $k = 1, \ldots, r$ and to I for $k = r+1, \ldots, r+n$. Further, $LR_{right}(G; \lambda_k)$ is equal to P_k when $k = 1, \ldots, r$ and to $I - P$ when $k = r+1, \ldots, r+n$. Hence

$$LR_{right}(F; \lambda_k) = \begin{cases} Q_k, & k = 1, \ldots, r, \\ I - P, & k = r+1, \ldots, r+n, \end{cases}$$

It follows that $LR_{right}(F; D) = Q_1 + \cdots + Q_k + n(I - P) = R$ and the proof is complete. $\square$

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