

Axiomization of the Center Function on Trees

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Abstract

We give a new, short proof that four certain axiomatic properties uniquely define the center of a tree.

For a given connected graph G , a *center vertex* minimizes the maximum distance to the other vertices, while a *median vertex* minimizes the average distance to the other vertices. In particular, the *center of G* , denoted by $C(G)$, consists of all center vertices, i.e., if $e(x) = \max\{d(x, y) \mid y \in V(G)\}$, where $d(u, v)$ denotes the distance between u and v , then $C(G) = \{x \in V(G) \mid e(x) \leq e(z), \text{ for all } z \in V(G)\}$. The *median of G* is defined similarly using sums of distances to the other vertices. These are but two of the many notions of centrality developed for graphs, and in particular for trees (see the references in [3]).

Many centrality notions can be phrased in terms of a location function or consensus function. A *consensus function* is a model to describe a rational way to obtain consensus among a group of agents or clients. The input of the function consists of certain information about the agents, and the output concerns the issue about which consensus should be reached. The rationality of the process is guaranteed by the fact that the consensus function satisfies certain “rational” rules or “consensus axioms”. A typical question in consensus theory is which set of axioms characterizes a given consensus function; for references see [1, 2, 4]. From this viewpoint the *center function* on a graph is the location function for which the input are the locations of the agents in the graph and the output are the vertices that minimize the maximum distance to the locations of the agents. An axiomatic characterization of the median function was done in [1, 4]. We treat the center function in this note, thus providing a short proof of the result in [2].

We first recall some well-known facts about the center of a tree T that we will use. For a vertex x and a subset S of vertices of T , the distance $d(x, S)$ from x to S is the distance from x to a vertex in S closest to x .

Fact 1. $C(T)$ consists of a single vertex or two adjacent vertices.

Fact 2. If a longest path P in T has endvertices x and y , then $C(T) \subseteq V(P)$ and $d(x, C(T)) = d(y, C(T))$.

Fact 3. If Q denotes the path between two vertices x and x' in $V(T)$, and S is a set of one vertex or two adjacent vertices of $V(Q)$ such that $d(x, S) = d(x', S)$, then $C(Q) = S$.

For a given tree T of order n , a *profile on T* is an ordered k -tuple of (not necessarily distinct) vertices of T , for some $k \geq 1$. If π is such a profile on T , then we will write $\{\pi\}$ for the set of distinct vertices that appear in π . A *location function on T* is a function L_T defined on the set of all profiles on T such that $L_T(\pi)$ is a non-empty subset of $V(T)$, for each profile π on T . For vertex x in $V(T)$ and profile $\pi = (y_1, y_2, \dots, y_k)$, we let

$e(T, \pi, x)$ denote the integer $\max\{d(x, y_i) \mid 1 \leq i \leq k\}$. Then an example of a location function on T is the *center function* Cen_T on T . It is defined by

$$Cen_T(\pi) = \{x \in V(T) \mid e(T, \pi, x) \leq e(T, \pi, z), \text{ for every } z \in V(T)\},$$

for every profile π on T . A useful, straightforward fact about this function is:

Fact 4. If π is a profile on T such that $\{\pi\} = V(T)$, then $Cen_T(\pi) = C(T)$.

We will abbreviate L_T by L and Cen_T by Cen .

Inspired by earlier work on axiomatic studies of location and consensus functions, see [1, 4], McMorris, Roberts, and Wang [2] presented four natural axioms for location functions on trees that we list below, and they proved that a location function on a tree T satisfies these four axioms if and only if it is the center function on T . For other references on centrality axiomatics, see the list of references in [2].

As usual, for a tree T , if $S \subseteq V(T)$, then $T[S]$ denotes the tree induced by S , i.e., the intersection of all subtrees of T that contain every vertex in S . And, for a profile π on T , $\pi - x$ denotes the profile obtained from π by deleting all occurrences of x in π .

For a tree T , the four axioms in [2] for a location function on T that we wish to consider are:

Middleness (M): For any $x, y \in V(T)$, if Q denotes the path between x and y , then $L((x, y)) = C(Q)$.

Quasi-Consistency (QC): For any two profiles π and π' on T , if $L(\pi) = L(\pi')$, then $L(\pi\pi') = L(\pi)$. (Here, $\pi\pi'$ denotes the profile on T consisting of the terms of π followed by the terms of π' .)

Redundancy (R): For any profile π on T , if x is a vertex of the subtree $T[\{\pi - x\}]$, then $L(\pi - x) = L(\pi)$.

Population Invariance (PI): For any two profiles π and π' on T , if $\{\pi\} = \{\pi'\}$, then $L(\pi) = L(\pi')$.

It is not difficult to show that, for a tree T , the center function Cen satisfies the four axioms. The purpose of this note is to give a new, short proof that, for a tree T , any location function on T that satisfies the four axioms must be the center function on T .

Theorem. Let T be a tree. If L is a location function on T that satisfies Middleness (M), Quasi-Consistency (QC), Redundancy (R), and Population Invariance (PI), then $L(\pi) = Cen(\pi)$, for every profile π on T .

Proof. Fix a tree and a profile π on T . Let T_π denote $T[\{\pi\}]$. Clearly, $\{\pi\} \subseteq V(T_\pi)$. In fact, every pendant vertex of T_π is in $\{\pi\}$, and, without loss of generality because of (R), we may assume that $\{\pi\} = V(T_\pi)$.

Let a_1 and a_2 be the endpoints of a longest path P in T_π . By Facts 4 and 2, $Cen(\pi) = C(T_\pi) \subseteq V(P)$. For any $x \in V(T_\pi)$, let Q_x be the path from x to the set $Cen(\pi)$, and let z denote the first vertex of Q_x that is also on P (proceeding from x to P). Without loss of generality, we may assume that $d(a_1, z) \leq d(a_2, z)$. As P is a longest path, $d(x, z) \leq d(a_1, z)$. So, $d(x, Cen(\pi)) = d(x, z) + d(z, Cen(\pi)) \leq d(a_1, z) + d(z, Cen(\pi)) = d(a_1, Cen(\pi)) = d(a_2, Cen(\pi))$. The last equality follows from Fact 2. Now $Cen(\pi) = C(T_\pi)$ is a set of one or two adjacent vertices in $V(P)$ by Fact 1. As $d(\cdot, Cen(\pi))$ yields every integer value between 0 and $d(a_2, Cen(\pi))$ on the path P' from $Cen(\pi)$ to a_2 , there must be a vertex x' on P' (and hence $x' \neq x$) so that $d(x, Cen(\pi)) = d(x', Cen(\pi))$. Consequently, as $Cen(\pi) = C(T_\pi)$ is a set of one or two adjacent vertices, if P_x denotes the path between x and x' , Fact 3 implies that $C(P_x) = Cen(\pi)$. Axiom (M) implies that $L((x, x')) = C(P_x)$. Let α_x denote any profile consisting of exactly the vertices of P_x in any fixed order, and let z be any interior vertex of P_x (i.e., any vertex of P_x other than x or x'). By Axiom (R), as z is not a vertex of $T[\{\alpha_x - z\}]$, we have $L(\alpha_x - z) = L(\alpha_x)$. By similar repeated use of Axiom (M), we deduce that $L((x, x')) = L(\alpha_x)$. Combining this with the above yields $L(\alpha_x) = C(P_x) = Cen(\pi)$. So, for every vertex x in $V(T)$, we have $L(\alpha_x) = Cen(\pi)$. In particular, for any $x_1, x_2 \in V(T_\pi)$, we have $L(\alpha_{x_1}) = L(\alpha_{x_2}) = Cen(\pi)$. By Axiom (QC), $L(\alpha_{x_1} \alpha_{x_2}) = Cen(\pi)$. If $V(T_\pi) = \{x_1, x_2, \dots, x_n\}$, then repeated use of Axiom (QC) yields $L(\alpha_{x_1} \alpha_{x_2} \dots \alpha_{x_n}) = Cen(\pi)$. Note that $\{\alpha_{x_1} \alpha_{x_2} \dots \alpha_{x_n}\} = \{\pi\}$, so by Axiom (PI), $L(\alpha_{x_1} \alpha_{x_2} \dots \alpha_{x_n}) = L(\pi)$. Consequently, $L(\pi) = Cen(\pi)$, as desired. $\square\square\square$

References

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