Approximating the randomized hitting time distribution of a non-stationary gamma process

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Econometric Institute Report 2007–18

May 2007

Abstract

The non-stationary gamma process is a non-decreasing stochastic process with independent increments. By this monotonic behavior this stochastic process serves as a natural candidate for modelling time-dependent phenomena such as degradation. In condition-based maintenance the first time such a process exceeds a random threshold is used as a model for the lifetime of a device or for the random time between two successive imperfect maintenance actions. Therefore there is a need to investigate in detail the cumulative distribution function (cdf) of this so-called randomized hitting time. We first relate the cdf of the (randomized) hitting time of a non-stationary gamma process to the cdf of a related hitting time of a stationary gamma process. Even for a stationary gamma process this cdf has in general no elementary formula and its evaluation is time-consuming. Hence two approximations are proposed in this paper and both have a clear probabilistic interpretation. Numerical experiments show that these approximations are easy to evaluate and their accuracy depends on the scale parameter of the non-stationary gamma process. Finally, we also consider some special cases of randomized hitting times for which it is possible to give an elementary formula for its cdf.

Keywords: non-stationary gamma process, first hitting time, random threshold, condition-based maintenance, approximation.

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1 Introduction

The non-stationary gamma process is a monotone continuous-time non-homogeneous Markov process with independent increments. The increments are gamma distributed with time-dependent shape function and an identical scale parameter. It is a jump process and the number of jumps in any time interval is infinite with probability one ([12]). These properties make the gamma process a suitable candidate to model the temporal variability in monotone phenomena. A recent paper by Van Noortwijk (2006) ([22]) gives an overview of the application of gamma processes within maintenance. Within this field these processes are used to describe time-dependent degradation such as wear, creep and corrosion. Another application of the gamma process outside the field of maintenance, is the aggregate claim process within insurance mathematics [8].

The review by Van Noortwijk ([22]) indicates that many authors have studied the first time a gamma process exceeds a fixed threshold value. However, only a few consider the first time a random threshold value is exceeded. In the remainder of this paper such a time is called a randomized hitting time. The first to propose a randomized hitting time within maintenance was Abdel-Hameed (1975) [2]. He uses the gamma process to model degradation, in particular wear, and the associated randomized hitting time serves as a model for the lifetime of a device. This means that the cumulative distribution function (cdf) of the lifetime of the device is the cdf of a randomized hitting time associated with a given gamma process. However, as indicated by Van Noortwijk, authors proposing this randomized hitting time model do mostly not perform numerical experiments. This is probably due to the complicated structure of the cdf being a two-dimensional integral. Since computing the cdf of this randomized hitting time is a subprocedure, which for some maintenance optimization models needs to be repeated under different parameters, the numerical optimization of such a model is mostly time-consuming. Hence there is a need for a fast numerical procedure giving reliable numerical outcomes for this cdf. As an example we mention a model using randomized hitting times introduced by Nicolai and Frenk ([13]; [14]). They model the duration between two maintenance actions by the time at which the gamma process (representing the deterioration process of a steel structure) exceeds a random reduction resulting from the last imperfect maintenance action. The problem of selecting maintenance actions resulting in the lowest expected cost over a finite horizon is then formulated as a stochastic dynamic programming model. For this model the optimal policy needs to be computed. Due to the above considerations the main purpose of this paper is to investigate in detail the cdf of a randomized hitting time of a (non-)stationary gamma process.

First of all, we show in Section 2 that without loss of generality we may restrict ourselves to the cdf of a randomized hitting time associated with a so-called standard gamma process. In this section we also study in detail the cdf of the fractional and integer part of a randomized hitting time and extend one of the results discussed in [18]. At the same time we simplify the proof technique for this result. Applying now the theoretical results of this section we propose in Section 3 two approximations for the cdf of a randomized hitting time. The first one is based on the approximative assumption that the fractional and the integer part of the randomized hitting

time are independent and that the cdf of the fractional part is uniformly distributed on $(0, 1)$. To justify these approximative assumptions (especially the latter one) we apply the results derived in Section 2. The second approximation is based on the assumption that the sample path of a stationary gamma process can be replaced by a piecewise linear sample path that coincides with the original sample path at integer points. As in both approximations the true cdf at integer time points is required, we also give a method for computing these probabilities. In Section 4 we derive for some random thresholds having a gamma-type cdf elementary formulas for the cdf of the associated randomized hitting times. Finally, in Section 5 numerical experiments are done to assess the quality of both approximations proposed in Section 3. We also compare the computation time for the construction of the approximations and the true cdf's.

2 On randomized hitting times of a non-stationary gamma process

In this section we derive some properties of the hitting time of a non-stationary gamma process exceeding a nonnegative random threshold value. To introduce the definition of a gamma process we first observe ([19]) that a gamma density with scale parameter $\lambda > 0$ and shape parameter $\beta > 0$ is given by

$$f(x) = \Gamma(\beta)^{-1}\lambda^{\beta}x^{\beta-1}\exp(-\lambda x)1_{(0,\infty)}(x)$$

with

$$\Gamma(\beta) := \int_0^{\infty} x^{\beta-1}\exp(-x)\mathrm{d}x$$

the well-known gamma function. In the rest of this paper such a cdf is denoted by gamma(β, λ). Also we mean by $\mathbf{X} \sim F$ that the random variable $\mathbf{X}$ has cdf F.

Definition 1 *Let $\lambda > 0$ and $v : [0, \infty) \to [0, \infty)$ an increasing, right-continuous function satisfying $v(0) = 0$. The stochastic process $\mathbf{X}_{v,\lambda} = \{\mathbf{X}_{v,\lambda}(t) : t \geq 0\}$ is called a gamma process with shape function v and scale parameter $\lambda > 0$ if*

1. *$\mathbf{X}_{v,\lambda}(0) = 0$ with probability 1 and $\mathbf{X}_{v,\lambda}$ is a cadlag process.*
2. *The stochastic process $\mathbf{X}_{v,\lambda}$ has independent increments.*
3. *The random variable $\mathbf{X}_{v,\lambda}(s) - \mathbf{X}_{v,\lambda}(t)$, $s > t$ has a gamma distribution with shape parameter $v(s) - v(t)$ and scale parameter $\lambda > 0$.*

If the function v is linear a gamma process $\mathbf{X}_{v,\lambda}$ is called stationary, otherwise it is called non-stationary. A stationary gamma process with v having slope 1 and scale parameter λ equal to 1 is called standard and denoted by $\mathbf{X}$. For any (non-stationary) gamma process we are now interested in the so-called hitting time $\mathbf{T}_{v,\lambda}(r)$ of level $r > 0$ given by

$$\mathbf{T}_{v,\lambda}(r) := \inf\{t \geq 0 : \mathbf{X}_{v,\lambda}(t) > r\}. \tag{1}$$

If the gamma process is standard the hitting time in (1) is denoted by $\mathbf{T}(r)$. In [14] one is interested in the randomized hitting time $\mathbf{T}_{v,\lambda}(\mathbf{R})$ with $\mathbf{R}$ a nonnegative random variable independent of the (non-stationary) gamma process. This random variable represents the random duration between two imperfect maintenance actions. In particular, the (non-stationary) gamma process models the deterioration of the structure under consideration and the random variable $\mathbf{R}$ having a non-defective cdf $G_{\mathbf{R}}$ satisfying $G_{\mathbf{R}}(0) = 0$ denotes the random reduction in damage due to an imperfect maintenance action. We assume throughout this paper that the random variable $\mathbf{R}$ is independent of the deterioration process $\mathbf{X}_{v,\lambda}$. To relate the properties of the (randomized) hitting time to the hitting time of a standard gamma process we observe by Definition 1 that

$$\mathbf{X}_{v,\lambda}(t) \stackrel{d}{=} \lambda^{-1}\mathbf{X}_{v,1}(t) \stackrel{d}{=} \lambda^{-1}\mathbf{X}(v(t)) \tag{2}$$

where $\stackrel{d}{=}$ is used to indicate that two random variables have the same cdf. Using relation (2) the next result holds.

Lemma 2 *If $\lambda > 0$ and the shape function v is strictly increasing and continuous satisfying $v(0) = 0$ and $v(\infty) = \infty$, then*

$$\mathbf{T}_{v,\lambda}(\mathbf{R}) \stackrel{d}{=} v^{\leftarrow}(\mathbf{T}(\lambda\mathbf{R})) \tag{3}$$

where $v^{\leftarrow}$ denotes the inverse function of v.

Proof. By relations (1) and (2) we obtain

$$\mathbf{T}_{v,\lambda}(\mathbf{R}) \stackrel{d}{=} \inf\{t \geq 0 : \mathbf{X}(v(t)) > \lambda\mathbf{R}\}. \tag{4}$$

Since the function v is strictly increasing and continuous with range $[0, \infty)$ its inverse $v^{\leftarrow}$ is also strictly increasing and continuous and satisfies $v^{\leftarrow}(v(t)) = t$ for every $t \geq 0$. This shows by relation (4) that

$$\mathbf{T}_{v,\lambda}(\mathbf{R}) \stackrel{d}{=} \inf\{v^{\leftarrow}(t) : \mathbf{X}(t) > \lambda\mathbf{R}\} = v^{\leftarrow}(\inf\{t \geq 0 : \mathbf{X}(t) > \lambda\mathbf{R}\})$$

and we have verified the result. □

For a gamma process with continuous shape function it is well-known that it is an increasing jump process with a countably infinite number of jumps over any finite interval (see e.g. [12] or [5]) and so the overshoot at any given level is a non-degenerate random variable. If the overshoot $\mathbf{W}_{v,\lambda}(r)$ of level r of a non-stationary gamma process is given by

$$\mathbf{W}_{v,\lambda}(r) := \mathbf{X}_{v,\lambda}(\mathbf{T}_{v,\lambda}(r)) - r \tag{5}$$

we obtain by Lemma 2 and relation (2) that

$$\mathbf{W}_{v,\lambda}(\mathbf{R}) \stackrel{d}{=} \lambda^{-1}\mathbf{X}(\mathbf{T}(\lambda\mathbf{R})) - \mathbf{R} = \lambda^{-1}\mathbf{W}(\lambda\mathbf{R}) \tag{6}$$

with $\mathbf{W}(\lambda\mathbf{R})$ the overshoot of a standard gamma process beyond level $\lambda\mathbf{R}$. As for the randomized hitting time this shows that the overshoot of a non-stationary gamma process can also be reduced to the overshoot of a standard gamma process.

To investigate in detail the cdf of the hitting time $\mathbf{T}_{v,\lambda}(\mathbf{R})$ it follows by Lemma 2 introducing $H_{\mathbf{R}}(t) := \mathbb{P}\{\mathbf{T}(\mathbf{R}) \leq t\}$ that

$$\mathbb{P}\{\mathbf{T}_{v,\lambda}(\mathbf{R}) \leq t\} = H_{\lambda\mathbf{R}}(v(t)). \tag{7}$$

Hence, to compute or approximate the cdf of the random variable $\mathbf{T}_{v,\lambda}(\mathbf{R})$ for any non-stationary gamma process, it is therefore sufficient to compute or approximate the cdf $H_{\mathbf{R}}$. By relation (7) it is clear for $A_{\lambda\mathbf{R}} : [0,\infty) \to [0,1]$ an approximation of the cdf of $\mathbf{T}(\lambda\mathbf{R})$ that the approximation $A_{v,\lambda}$ of the cdf of the random variable $\mathbf{T}_{v,\lambda}(\mathbf{R})$ is given by

$$A_{v,\lambda}(t) := A_{\lambda\mathbf{R}}(v(t)). \tag{8}$$

For v continuous and increasing satisfying $v(0) = 0$ and $v(\infty) = \infty$ relation (8) implies

$$\|A_{v,\lambda} - H_{v,\lambda}\|_\infty = \|A_{\lambda\mathbf{R}} - H_{\lambda\mathbf{R}}\|_\infty \tag{9}$$

with $H_{v,\lambda}$ denoting the cdf of $\mathbf{T}_{v,\lambda}(\mathbf{R})$ and $\|h\|_\infty := \sup_{0\leq t<\infty} |h(t)|$ the well-known supnorm. Since the nonnegative random variable $\mathbf{R}$ is by definition independent of the gamma process and its cdf satisfies $G_{\mathbf{R}}(0) = 0$ it follows by conditioning on the random variable $\mathbf{R}$ that

$$H_{\mathbf{R}}(t) = \mathbb{P}\{\mathbf{T}(\mathbf{R}) \leq t\} = \mathbb{P}\{\mathbf{X}(t) > \mathbf{R}\} = \int_0^\infty \mathbb{P}\{\mathbf{X}(t) > r\} dG_{\mathbf{R}}(r) \tag{10}$$

for every $t \geq 0$. Moreover, by conditioning on the random variable $\mathbf{X}(t)$ with cdf F_t we obtain for $G_{\mathbf{R}}$ continuous the equivalent representation

$$H_{\mathbf{R}}(t) = \int_0^\infty \mathbb{P}\{\mathbf{R} < x\} dF_t(x) = \mathbb{E}(G_{\mathbf{R}}(\mathbf{X}(t))). \tag{11}$$

Since the gamma process is increasing and continuous in probability relation (10) and a standard application of Lebesque's dominated convergence theorem (see e.g. [6]) imply that the cdf $H_{\mathbf{R}}$ is continuous. Using relation (11) one can also derive for random variables $\mathbf{R}$ having a gamma type cdf an elementary formula for the cdf $H_{\mathbf{R}}$. This will be done in Section 4. For general cdf's this is not possible and so we propose in the next section two elementary approximations. To justify these approximations we first investigate some properties of random variables related to the hitting time.

Let $\lfloor x \rfloor$ be the largest integer not exceeding x for any $x \geq 0$ and denote by $\mathcal{F}(x)$ its fractional part given by $\mathcal{F}(x) := x - \lfloor x \rfloor$. Clearly

$$\mathbf{T}(\mathbf{R}) = \lfloor \mathbf{T}(\mathbf{R}) \rfloor + \mathcal{F}(\mathbf{T}(\mathbf{R})). \tag{12}$$

For the integer part $\lfloor \mathbf{T}(\mathbf{R}) \rfloor$ it is easy to show the following result.

Lemma 3 *If* $\mathbf{N} := \{\mathbf{N}(t) : t \geq 0\}$ *is a Poisson process with arrival rate* 1*, then* $\lfloor \mathbf{T}(\mathbf{R}) \rfloor \stackrel{d}{=} \mathbf{N}(\mathbf{R})$.

Proof. Since for a standard gamma process

$$\mathbf{X}(n+1) \stackrel{d}{=} \sum\nolimits_{k=1}^{n+1} \mathbf{Y}_k$$

with $\mathbf{Y}_k, 1 \leq k \leq n+1$ independent and exponentially distributed with scale parameter 1 we obtain by the continuity of the cdf of the random variable $\mathbf{T}(\mathbf{R})$ that

$$\begin{aligned} \mathbb{P}\{\lfloor \mathbf{T}(\mathbf{R}) \rfloor \leq n\} &= \mathbb{P}\{\mathbf{T}(\mathbf{R}) < n+1\} = \mathbb{P}\{\mathbf{T}(\mathbf{R}) \leq n+1\} \\ &= \mathbb{P}\{\mathbf{X}(n+1) > \mathbf{R}\} = \mathbb{P}\{\mathbf{N}(\mathbf{R}) \leq n\}. \end{aligned} \tag{13}$$

Hence the desired result follows. □

By the independence of the random variable $\mathbf{R}$ and the standard gamma process $\mathbf{X}$ we obtain by Lemma 3 and the well-known properties of a Poisson process that

$$\mathbb{P}\{\lfloor \mathbf{T}(\mathbf{R}) \rfloor = n\} = \frac{1}{n!}\mathbb{E}\left(\mathbf{R}^n \exp(-\mathbf{R})\right). \tag{14}$$

for every $n \in \mathbb{N} \cup \{0\}$. In the next lemma we give an expression for the cdf of $\mathcal{F}(\mathbf{T}(\mathbf{R}))$. For this expression we introduce the functions $q_t : (0, \infty) \to \mathbb{R}$, $0 < t < 1$ given by

$$q_t(r) := 1 - F_t(r) - \int_r^\infty 1 - F_t(x)dx. \tag{15}$$

Lemma 4 *If* $\mathbf{R}$ *is a nonnegative random variable with cdf* $G_{\mathbf{R}}$ *satisfying* $G_{\mathbf{R}}(0) = 0$ *and this random variable is independent of the standard gamma process* $\mathbf{X}$*, then*

$$\mathbb{P}\{\mathcal{F}(\mathbf{T}(\mathbf{R})) \leq t\} = t + \mathbb{E}\left(q_t(\mathbf{R})\right)$$

for every $0 < t < 1$.

Proof. By the definition of the random variable $\mathcal{F}(\mathbf{T}(\mathbf{R}))$ we obtain

$$\{\mathcal{F}(\mathbf{T}(\mathbf{R})) \leq t\} = \cup_{k=0}^{\infty}\{k \leq \mathbf{T}(\mathbf{R}) \leq k+t\}. \tag{16}$$

Since the cdf of $\mathbf{T}(\mathbf{R})$ is continuous and so $\{k \leq \mathbf{T}(\mathbf{R}) \leq k+t\} \stackrel{a.s}{=} \{k < \mathbf{T}(\mathbf{R}) \leq k+t\}$ it follows for every $k \in \mathbb{N} \cup \{0\}$ that

$$\{k \leq \mathbf{T}(\mathbf{R}) \leq k+t\} \stackrel{a.s}{=} \{\mathbf{X}(k) \leq \mathbf{R} < \mathbf{X}(k+t)\}.$$

This yields by relation (16)

$$\{\mathcal{F}(\mathbf{T}(\mathbf{R})) \leq t\} \stackrel{a.s}{=} \cup_{k=0}^{\infty}\{\mathbf{X}(k) \leq \mathbf{R} < \mathbf{X}(k+t)\}$$

and conditioning on $\mathbf{R}$ implies

$$\mathbb{P}\{\mathcal{F}(\mathbf{T}(\mathbf{R})) \leq t\} = \int_0^{\infty} \mathbb{P}\{\cup_{k=0}^{\infty}\{\mathbf{X}(k) \leq r < \mathbf{X}(k+t)\}\} dG_{\mathbf{R}}(r). \tag{17}$$

Since $\mathbf{X}(k+t) \stackrel{d}{=} \mathbf{X}(k) + \mathbf{Y}(t)$ with $\mathbf{Y}$ a standard gamma process independent of $\mathbf{X}$ and the cdf F_k of $\mathbf{X}(k)$ is the k-fold convolution $F^{k*}$of an exponential distribution with scale parameter 1 we obtain for every $r > 0$

$$\begin{array}{rcl} \mathbb{P}\{\cup_{k=0}^{\infty}\{\mathbf{X}(k) \leq r < \mathbf{X}(k+t)\}\} & = & \sum_{k=0}^{\infty} \mathbb{P}\{\mathbf{X}(k) \leq r < \mathbf{X}(k) + \mathbf{Y}(t)\} \\ & = & 1 - F_t(r) + \int_0^r (1 - F_t(r-x))\, dU(x) \end{array} \tag{18}$$

with $U(x) = \sum_{k=1}^{\infty} F^{k*}(x) = x$ the renewal function associated with a Poisson process having arrival rate 1. Using now

$$\int_0^{\infty} 1 - F_t(x)dx = \mathbb{E}(\mathbf{X}(t)) = t$$

and relations (17) and (18) the desired result follows. □

To rewrite the representation of the function q_t into a more suitable form we introduce a beta cdf on $(0,1)$ with parameters $\alpha, \beta > 0$. Its density is given by

$$b_{\alpha,\beta}(x) := \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} 1_{(0,1)}(x) \tag{19}$$

and the cdf itself is denoted by beta(α, β). If $\beta > \alpha > 0$ and $\tau, \lambda > 0$ it is shown on page 261 of [23] (see also [17]) that

$$\left(\frac{\lambda}{\lambda+\tau}\right)^{\alpha} = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 \left(\frac{\lambda}{\lambda+\tau x}\right)^{\beta} x^{\alpha-1}(1-x)^{\beta-\alpha-1} dx. \tag{20}$$

and this result leads to a well-known probabilistic interpretation of a random variable having a gamma cdf with non-integer shape parameter. Although this result seems to be well-known (see e.g. [7], [19] or [15] for the special case $\beta = 1$) we could not find a detailed proof of this. Hence in the next lemma we list a short proof based on relation (20).

Lemma 5 *If $0 < t < 1$ and $\mathbf{Y}$ and $\mathbf{V}$ are independent nonnegative random variables with $\mathbf{Y} \sim$ gamma(β, λ), $\beta > 0$ and $\mathbf{V} \sim$ beta$(t\beta, (1-t)\beta)$, then $\mathbf{VY} \sim$ gamma$(t\beta, \lambda)$.*

Proof. For every $\tau > 0$ it follows by the independence of $\mathbf{V}$ and $\mathbf{Y}$ that

$$\begin{aligned}\mathbb{E}\left(\exp(-\tau\mathbf{V}\mathbf{Y})\right) &= \frac{\Gamma(\beta)}{\Gamma(t\beta)\Gamma((1-t)\beta)}\int_0^1 \mathbb{E}\left(\exp(-\tau v\mathbf{Y})\right) v^{t\beta-1}(1-v)^{(1-t)\beta-1}dv \\ &= \frac{\Gamma(\beta)}{\Gamma(t\beta)\Gamma(\beta-t\beta)}\int_0^1 \left(\frac{\lambda}{\lambda+\tau v}\right)^{\beta} v^{t\beta-1}(1-v)^{\beta-t\beta-1}dv\end{aligned}$$

Applying relation (20) we obtain using $\alpha = t\beta < \beta$ that $\mathbb{E}(\exp(-\tau\mathbf{V}\mathbf{Y})) = (\frac{\lambda}{\lambda+\tau})^{t\beta}$ and this is the probability Laplace-Stieltjes transform (pLSt) of a gamma distribution with scale parameter λ and shape parameter $t\beta$. □

To rewrite $\mathbb{E}\left(q_t(\mathbf{R})\right)$ in a more suitable form we also introduce a Pareto distribution on $(0,\infty)$ with parameter $\beta > 0$. Observe a Pareto(β) cdf on $(0,\infty)$ with parameter $\beta > 1$ has density ([19])

$$f(x) = (\beta-1)(1+x)^{-\beta}1_{(0,\infty)}(x) \tag{21}$$

and this cdf is denoted by par(β).

Lemma 6 *For every $0 < t < 1$ it follows for any nonnegative random variable $\mathbf{R}$ and $\mathbf{Z} \sim par(2)$ independent of $\mathbf{R}$ that*

$$\mathbb{E}\left(q_t(\mathbf{R})\right) = \pi^{-1}\sin(\pi t)\mathbb{E}\left(\exp(-\mathbf{R}(\mathbf{Z}+1))\mathbf{Z}^{1-t}\right).$$

Proof. It is sufficient to show that

$$q_t(r) = \pi^{-1}\sin(\pi t)\int_0^{\infty}\exp(-r(z+1))z^{1-t}(1+z)^{-2}dz. \tag{22}$$

for every $0 < t < 1$. Since $\mathbf{X}(t) \sim$ gamma$(t,1)$ it follows by relation (15) and Lemma 5 (take $\beta = 1$ and $\lambda = 1$) that

$$q_t(r) = \mathbb{P}\{\mathbf{V}\mathbf{Y} > r\} - \int_r^{\infty}\mathbb{P}\{\mathbf{V}\mathbf{Y} > x\}dx \tag{23}$$

with $\mathbf{Y} \sim$ gamma$(1,1)$ and $\mathbf{V} \sim$ beta$(t, 1-t)$ independent random variables. By Tonelli's theorem ([11])

$$\begin{aligned}\int_r^{\infty}\mathbb{P}\{\mathbf{V}\mathbf{Y} > x\}dx &= \int_r^{\infty}\int_0^1 \mathbb{P}\{v\mathbf{Y} > x\}b_{t,1-t}(v)dvdx \\ &= \int_0^1\int_r^{\infty}\mathbb{P}\{v\mathbf{Y} > x\}dx b_{t,1-t}(v)dv\end{aligned}$$

and this implies by relation (23) that

$$q_t(r) = \int_0^1\left(\mathbb{P}\{v\mathbf{Y} > r\} - \int_r^{\infty}\mathbb{P}\{v\mathbf{Y} > x\}dx\right)b_{t,1-t}(v)dv. \tag{24}$$

Since $\mathbf{Y} \sim \text{gamma}(1,1)$ it follows

$$\mathbb{P}\{v\mathbf{Y} > r\} - \int_r^\infty \mathbb{P}\{v\mathbf{Y} > x\}dx = (1-v)\exp(-rv^{-1})$$

for every $0 < v < 1$ and so by by relations (19) and (24) we obtain

$$\begin{aligned} q_t(r) &= \tfrac{\Gamma(1)}{\Gamma(t)\Gamma(1-t)} \int_0^1 v^{t-1}(1-v)^{1-t}\exp(-rv^{-1})dv \\ &= \tfrac{1}{\Gamma(t)\Gamma(1-t)} \int_1^\infty \exp(-rz)(z-1)^{1-t}z^{-2}dz \qquad (25) \\ &= \tfrac{1}{\Gamma(t)\Gamma(1-t)} \int_0^\infty \exp(-r(z+1))z^{1-t}(1+z)^{-2}dz. \end{aligned}$$

Using now Euler's reflection formula for the gamma function (see page 256 of [3]) or [23]) given by $\Gamma(t)\Gamma(1-t) = \pi(\sin \pi t)^{-1}$ for $0 < t < 1$ the desired representation for $q_t(r)$ listed in relation (22) is shown. □

Combining Lemma 4 and 6 we immediately obtain

$$\mathbb{P}\{\mathcal{F}(\mathbf{T}(\mathbf{R})) \le t\} = t + \pi^{-1}\sin(\pi t)\mathbb{E}(\exp(-\mathbf{R}(\mathbf{Z}+1))\mathbf{Z}^{1-t}). \qquad (26)$$

We will now show that the last nonnegative term in relation (26) is small for $\mathbb{E}(\exp(-\mathbf{R}))$ small. Hence we may conclude in this case that the random variable $\mathcal{F}(\mathbf{T}(\mathbf{R}))$ is approximately uniformly distributed. We will use this observation in the next section to justify an approximation for the cdf of $\mathbf{T}(\mathbf{R})$. Since $\mathbf{Z}$ and $\mathbf{R}$ are random variables on $(0,\infty)$ it follows that

$$\exp(-\mathbf{R}(\mathbf{Z}+1))\mathbf{Z}^{1-t} \le \exp(-\mathbf{R})\mathbf{Z}^{1-t}.$$

By the independence of $\mathbf{Z}$ and $\mathbf{R}$ and $\mathbf{Z} \sim par(2)$ this implies for every $0 < t < 1$ that

$$\mathbb{E}(\exp(-\mathbf{R}(\mathbf{Z}+1))\mathbf{Z}^{1-t}) \le \mathbb{E}(\exp(-\mathbf{R}))\mathbb{E}(\mathbf{Z}^{1-t}) = \mathbb{E}(\exp(-\mathbf{R}))O(t^{-1}).$$

Hence by (26) we obtain

$$0 \le \mathbb{P}\{\mathcal{F}(\mathbf{T}(\mathbf{R})) \le t\} - t = O\left(\frac{\sin(\pi t)}{\pi t}\right)\mathbb{E}(\exp(-\mathbf{R})). \qquad (27)$$

A related upperbound for $\mathbf{R}$ a degenerate random variable is derived in [18] using a completely different technique. This upperbound can be derived considering relation (22). In [18] the inversion formula for Laplace transforms is used together with an appropriate choice of the closed contour in the associated complex contour integral. To compute the integer moments of $\mathcal{F}(\mathbf{T}(\mathbf{R}))$ we need to evaluate for every $k \in \mathbb{N} \cup \{0\}$ the function $M_k : \mathbb{R} \to \mathbb{R}$ given by

$$M_k(a) := \int_0^1 \exp(at)t^k \sin(\pi t)dt. \qquad (28)$$

In the next lemma the function M_{-1} denotes the zero function.

Lemma 7 *It follows* $M_0(a) = \frac{\pi}{a^2+\pi^2}(\exp(a)+1)$ *and*

$$M_k(a) = \frac{\pi \exp(a)}{a^2+\pi^2} - \frac{2ka}{a^2+\pi^2} M_{k-1}(a) - \frac{k(k-1)}{a^2+\pi^2} M_{k-2}(a) \tag{29}$$

for every $k \in \mathbb{N}$.

Proof. It is easy to check by differentiation that the antiderivative of the function $t \mapsto \exp(at)\sin(\pi t)$ on $(0,\infty)$ is given by the function

$$t \mapsto \frac{\exp(at)}{a^2+\pi^2}(a\sin(\pi t) - \pi\cos(\pi t)).$$

and so the expression for $M_0(a)$ is verified. To check the recurrence relation (29) we observe by the first part that

$$\pi(\exp(a)+1) = (a^2+\pi^2)M_0(a). \tag{30}$$

Since for every $k \in \mathbb{N}$ it follows that $M_k(a) = M_0^{(k)}(a)$ with $M_0^{(k)}$ denoting the kth derivative of the function M_0 we obtain by differentiation of the identity in (30) the desired result. □

Using a package such as Maple and the above recurrence relation it is possible to give an analytical expression for $M_k(a)$. In the next lemma we give an expression for the first moment of $\mathcal{F}(\mathbf{T}(\mathbf{R}))$.

Lemma 8 *If* $\mathbf{R}$ *is a nonnegative random variable with cdf* $G_{\mathbf{R}}$ *satisfying* $G_{\mathbf{R}}(0) = 0$ *and this random variable is independent of the standard gamma process* $\mathbf{X}$*, then*

$$\mathbb{E}(\mathcal{F}(\mathbf{T}(\mathbf{R})) = \frac{1}{2} - \mathbb{E}\left(\frac{\exp(-\mathbf{R}(\mathbf{Z}+1))(\mathbf{Z}+1)}{(\ln(\mathbf{Z}))^2+\pi^2}\right)$$

with $\mathbf{Z}$ *independent of* $\mathbf{R}$ *and* $\mathbf{Z} \sim$*par*(2).

Proof. Since for any random variable $\mathbf{Y}$ on $(0,1)$ it is well-known that $\mathbb{E}(\mathbf{Y}) = \int_0^1 \mathbb{P}\{\mathbf{Y} > t\}dt$ we obtain by relation (26)

$$\mathbb{E}(\mathcal{F}(\mathbf{T}(\mathbf{R})) = \frac{1}{2} - \pi^{-1}\int_0^1 \sin(\pi t)\mathbb{E}(\exp(-\mathbf{R}(\mathbf{Z}+1)\mathbf{Z}^{1-t})dt. \tag{31}$$

Applying Tonelli's theorem and relation (28) yields

$$\int_0^1 \sin(\pi t)\mathbb{E}(\exp(-\mathbf{R}(\mathbf{Z}+1)\mathbf{Z}^{1-t})dt = \mathbb{E}\left(\exp(-\mathbf{R}(\mathbf{Z}+1))\mathbf{Z}M_0(-\ln(\mathbf{Z}))\right) \tag{32}$$

and by Lemma 7 the desired result follows. □

By relation (12) and Lemma 3 we immediately obtain

$$\mathbb{E}(\mathbf{T}(\mathbf{R})) = \mathbb{E}(\mathbf{R}) + \frac{1}{2} - \mathbb{E}\left(\frac{\exp(-\mathbf{R}(\mathbf{Z}+1))(\mathbf{Z}+1)}{(\ln(\mathbf{Z}))^2+\pi^2}\right). \tag{33}$$

To derive a simple upper and lower bound on the expectation of $\mathbf{T}(\mathbf{R})$ or $\mathcal{F}(\mathbf{T}(\mathbf{R}))$ we observe for any nonnegative random variable $\mathbf{Z}$ that $(\ln(\mathbf{Z}))^2+\pi^2 \geq \pi^2$. This shows by Lemma 8 that

$$\frac{1}{2} - \pi^{-2}\mathbb{E}\left(\exp(-\mathbf{R}(\mathbf{Z}+1))(\mathbf{Z}+1)\right) \leq \mathbb{E}\left(\mathcal{F}(\mathbf{T}(\mathbf{R}))\right) \leq \frac{1}{2} \tag{34}$$

A similar inequality can be derived for the expectation of $\mathbf{T}(\mathbf{R})$. If we want to give a more precise evaluation of these moments we first observe by the inverse transformation method ([16]) and $\mathbf{Z} \sim par(2)$ that

$$\mathbf{Z} \stackrel{d}{=} \mathbf{U}^{-1} - 1$$

with $\mathbf{U}$ a standard uniform random variable. Hence, if it is easy to generate a realization of the random variable $\mathbf{R}$, we can always use Monte Carlo simulation to give an more accurate estimate of the expression (see Lemma 8)

$$\mathbb{E}\left(\frac{\exp(-\mathbf{R}(\mathbf{Z}+1))(\mathbf{Z}+1)}{(\ln(\mathbf{Z}))^2+\pi^2}\right).$$

If we are interested in higher integer moments $\mathbb{E}\left(\mathcal{F}(\mathbf{T}(\mathbf{R}))^{k+1}\right)$ for some $k \in \mathbb{N}$ we observe that

$$\mathbb{E}\left(\mathcal{F}(\mathbf{T}(\mathbf{R}))^{k+1}\right) = (k+1)\int_0^1 t^k \mathbb{P}\{\mathcal{F}(\mathbf{T}(\mathbf{R})) > t\}dt. \tag{35}$$

By the same technique as in Lemma 8 we obtain from relation (35) that

$$\mathbb{E}\left(\mathcal{F}(\mathbf{T}(\mathbf{R}))^{k+1}\right) = \frac{1}{k+2} - \pi^{-1}\mathbb{E}\left(\exp(-\mathbf{R}(\mathbf{Z}+1))\mathbf{Z}M_k(-\ln(\mathbf{Z}))\right). \tag{36}$$

Applying the recurrence relation in Lemma 7 we can again use Monte Carlo integration to estimate the integral in relation (36). In the next section we will apply our findings on $\mathcal{F}(\mathbf{T}(\mathbf{R}))$ and $\lfloor \mathbf{T}(\mathbf{R}) \rfloor$ to justify the first proposed approximation.

3 Approximating the cdf of a randomized hitting time

Since in [14] we use dynamic programming to construct an optimal policy and for such a procedure it is too time-consuming to evaluate the cdf of the randomized hitting time $\mathbf{T}_{v,\lambda}(\mathbf{R})$ exactly (e.g. via numerical integration), we derive in this section two simple approximations of this cdf. By

relation (7) it is sufficient to give an approximation of the cdf $H_{\mathbf{R}}$ of the hitting time $\mathbf{T}(\mathbf{R})$ listed in relation (10). To justify our first approximation we observe by relation (12) that

$$\begin{aligned} H_{\mathbf{R}}(t) &= \mathbb{P}\{\mathbf{T}(\mathbf{R}) \leq t\} \\ &= \mathbb{P}\{\mathbf{T}(\mathbf{R}) - \mathcal{F}(\mathbf{T}(\mathbf{R})) \leq t - \mathcal{F}(\mathbf{T}(\mathbf{R}))\} \\ &= \mathbb{P}\{\lfloor \mathbf{T}(\mathbf{R}) \rfloor \leq t - \mathcal{F}(\mathbf{T}(\mathbf{R}))\} \end{aligned} \tag{37}$$

for every $t > 0$. It is shown in [18] that

$$cov(\mathbf{T}(r), \mathcal{F}(\mathbf{T}(r))) = O(\exp(-r)) \tag{38}$$

and this implies for every nonnegative random variable $\mathbf{R}$ independent of the standard gamma process that

$$cov(\mathbf{T}(\mathbf{R}), \mathcal{F}(\mathbf{T}(\mathbf{R})) = O(\mathbb{E}(\exp(-\mathbf{R}))). \tag{39}$$

Hence for $\mathbb{E}(\exp(-\mathbf{R}))$ small the random variables $\mathbf{T}(\mathbf{R})$ and $\mathcal{F}(\mathbf{T}(\mathbf{R}))$ are practically uncorrelated. Also by the analysis in the previous section we have shown for $\mathbb{E}(\exp(-\mathbf{R}))$ small that the random variable $\mathcal{F}(\mathbf{T}(\mathbf{R}))$ has approximately a uniform distribution on $(0, 1)$. Although uncorrelated random variables are in general not independent (equivalence only holds for normal distributed random variables), it seems by the above observations reasonable (for the purpose to obtain easy expressions) to introduce the following approximation assumption. A similar approximation assumption is also introduced in [18] for a different purpose.

Approximation assumption 1. The random variables $\mathbf{T}(\mathbf{R})$ and $\mathcal{F}(\mathbf{T}(\mathbf{R}))$ are independent and $\mathcal{F}(\mathbf{T}(\mathbf{R}))$ is uniformly distributed on $(0, 1)$.

Before investigating the consequence of approximation assumption 1 we list the following result.

Lemma 9 *If the nonnegative random variable* $\mathbf{Y}$ *is independent of* $\mathcal{F}(\mathbf{Y})$, *then* $\lfloor \mathbf{Y} \rfloor$ *is also independent of* $\mathcal{F}(\mathbf{Y})$.

Proof. Since $\lfloor \mathbf{Y} \rfloor = \sum_{n=0}^{\infty} n \mathbf{1}_{\{n \leq \mathbf{Y} < n+1\}}$ and hence $\lfloor \mathbf{Y} \rfloor$ is a function of the random variable $\mathbf{Y}$ the desired result follows. □

If approximation assumption 1 holds we obtain by relation (37) and Lemma 9 that

$$H_{\mathbf{R}}(t) \approx \mathbb{P}\{\lfloor \mathbf{T}(\mathbf{R}) \rfloor \leq t - \mathbf{U}\} \tag{40}$$

with $\mathbf{U}$ a uniformly distributed random variable independent of $\lfloor \mathbf{T}(\mathbf{R}) \rfloor$. Introducing now

$$\mathbb{P}\{\lfloor \mathbf{T}(\mathbf{R}) \rfloor \leq -1\} := 0$$

and using the independence of $\mathbf{U}$ and $\lfloor \mathbf{T}(\mathbf{R}) \rfloor$ it follows for every $t > 0$ that

$$\mathbb{P}\{\lfloor \mathbf{T}(\mathbf{R}) \rfloor \leq t - \mathbf{U}\} = \mathcal{F}(t))\mathbb{P}\{\lfloor \mathbf{T}(\mathbf{R}) \rfloor \leq \lfloor t \rfloor\} + (1 - \mathcal{F}(t))\mathbb{P}\{\lfloor \mathbf{T}(\mathbf{R}) \rfloor \leq \lfloor t \rfloor - 1\}. \quad (41)$$

Also by the continuity of the cdf $H_{\mathbf{R}}$ we obtain for every $t > 0$ that

$$\mathbb{P}\{\lfloor \mathbf{T}(\mathbf{R}) \rfloor \leq \lfloor t \rfloor\} = \mathbb{P}\{\mathbf{T}(\mathbf{R}) < \lfloor t \rfloor + 1\} = H_{\mathbf{R}}(\lfloor t \rfloor + 1).$$

This implies by relation (41)

$$\mathbb{P}\{\lfloor \mathbf{T}(\mathbf{R}) \rfloor \leq t - \mathbf{U}\} = \mathcal{F}(t)H_{\mathbf{R}}(\lfloor t \rfloor + 1) + (1 - \mathcal{F}(t))H_{\mathbf{R}}(\lfloor t \rfloor)$$

and so by relation (40) we obtain

$$H_{\mathbf{R}}(t) \approx \mathcal{F}(t)H_{\mathbf{R}}(\lfloor t \rfloor + 1) + (1 - \mathcal{F}(t))H_{\mathbf{R}}(\lfloor t \rfloor). \quad (42)$$

This shows under approximation assumption 1 that the cdf $H_{\mathbf{R}}$ seems to be well approximated by a linear interpolation of $H_{\mathbf{R}}$ at the integer points.

To evaluate the continuous cdf $H_{\mathbf{R}}$ on its integer points we know by relation (14) that

$$\begin{aligned} H_{\mathbf{R}}(n+1) - H_{\mathbf{R}}(n) &= \mathbb{P}\{n < \mathbf{T}(\mathbf{R}) \leq n+1\} \\ &= \mathbb{P}\{\lfloor \mathbf{T}(\mathbf{R}) \rfloor = n\} \quad (43) \\ &= \tfrac{1}{n!}\mathbb{E}(\mathbf{R}^n \exp(-\mathbf{R})) \end{aligned}$$

for every $n \in \mathbb{N} \cup \{0\}$. Alternatively, if $\widehat{G}_{\mathbf{R}}(\tau) := \mathbb{E}(\exp(-\tau\mathbf{R}))$ is the probability Laplace Stieltjes transform (pLSt) of the cdf $G_{\mathbf{R}}$, then relation (43) is the same as

$$H_{\mathbf{R}}(n+1) - H_{\mathbf{R}}(n) = \frac{(-1)^n}{n!}\widehat{G}_{\mathbf{R}}^{(n)}(1) \quad (44)$$

for every $n \in \mathbb{N} \cup \{0\}$ with $\widehat{G}_{\mathbf{R}}^{(n)}, n \in \mathbb{N}$ denoting the nth derivative of $\widehat{G}_{\mathbf{R}}$ and $\widehat{G}_{\mathbf{R}}^{(0)} := \widehat{G}_{\mathbf{R}}$. If the derivatives of $\widehat{G}_{\mathbf{R}}$ are elementary functions we can directly apply relation (44) as shown in the next example for $\mathbf{R}$ uniformly distributed on $(a, a+b)$. For convenience introduce the sequence p_n, $n \in \mathbb{N} \cup \{0\}$ given by

$$p_n := H_{\mathbf{R}}(n+1) - H_{\mathbf{R}}(n) = \mathbb{P}\{\lfloor \mathbf{T}(\mathbf{R}) \rfloor = n\}. \quad (45)$$

Example 10 *If the random variable* $\mathbf{R}$ *is uniformly distributed on* $(a, a+b)$ *with* $a \geq 0$ *and* $b > 0$ *or* $\mathbf{R} \stackrel{d}{=} a + b\mathbf{U}$ *with* $\mathbf{U}$ *a standard uniformly distributed random variable we obtain*

$$\widehat{G}_{\mathbf{R}}(\tau) = \mathbb{E}(\exp(-\tau(a + b\mathbf{U}))) = b^{-1}(h_a(\tau) - h_{a+b}(\tau)) \quad (46)$$

with the functions $h_d : [0,\infty) \to \mathbb{R}$, $d > 0$ *given by* $h_d(\tau) := \tau^{-1}\exp(-\tau d)$. *If we introduce the function* $e_n : [0,\infty) \to \mathbb{R}$ *given by ([3])*

$$e_n(x) := \sum_{j=0}^{n} \frac{x^j}{j!}$$

it is easy to check using Leibniz formula for the differentiation of the product of two functions that for any positive d *and* τ

$$\begin{aligned}(-1)^n h_d^{(n)}(\tau) &= \exp(-\tau d)\textstyle\sum_{j=0}^{n}\binom{n}{j} d^j \tau^{-(n-j+1)}(n-j)! \\ &= n!\exp(-\tau d)\textstyle\sum_{j=0}^{n}\frac{d^j}{j!}\tau^{-(n-j+1)} \\ &= n!\tau^{-(n+1)}\exp(-\tau d)e_n(\tau d)\end{aligned} \tag{47}$$

with $h_d^{(n)}(\tau)$ *denoting the* n*th derivative of the function* h_d *evaluated in* τ. *This implies*

$$(-1)^n h_d^{(n)}(1) = n!\exp(-d)e_n(d). \tag{48}$$

Hence by relations (44), (45), (46) and (48) we obtain for every $n \in \mathbb{N}\cup\{0\}$

$$\begin{aligned}p_n &= b^{-1}\left(\exp(-a)e_n(a) - \exp(-(a+b))e_n(a+b)\right) \\ &= b^{-1}(\mathbb{P}\{\mathbf{N}(a) \le n\} - \mathbb{P}\{\mathbf{N}(a+b) \le n\}) \\ &= b^{-1}\mathbb{P}\{\mathbf{N}(a) \le n, \mathbf{N}(a+b) > n\}\end{aligned} \tag{49}$$

with $\mathbf{N}$ *a Poisson process with arrival rate* 1. *For* $n = 0$ *this reduces to*

$$\begin{aligned}p_0 &= b^{-1}\mathbb{P}\{\mathbf{N}(a) = 0, \mathbf{N}(a+b) - \mathbf{N}(a) > 0\} \\ &= b^{-1}\mathbb{P}\{\mathbf{N}(a) = 0\}\mathbb{P}\{\mathbf{N}(b) > 0\} \\ &= \exp(-a)b^{-1}\left(1 - \exp(-b)\right).\end{aligned}$$

Using relation (49) we also obtain for every $n \in \mathbb{N}\cup\{0\}$ *the recurrence relation*

$$p_{n+1} = p_n + b^{-1}\left(\frac{\exp(-a)a^{n+1}}{(n+1)!} - \frac{\exp(-(a+b))(a+b)^{n+1}}{(n+1)!}\right). \tag{50}$$

If we consider the special case $a = 0$ *it follows by relation (49) for every* $n \in \mathbb{N}\cup\{0\}$ *that*

$$p_n = b^{-1}\mathbb{P}\{\mathbf{N}(b) > n\}$$

Hence the recurrence relation in (50) reduces to

$$p_{n+1} = p_n - \frac{b^n}{(n+1)!}\exp(-b) \tag{51}$$

with $p_0 = b^{-1}(1-\exp(-b))$.

In some cases we have to use a numerical procedure to evaluate $\widehat{G}_{\mathbf{R}}(1)$. However, as shown by the following example, it is still possible to write down a recurrence relation for the probabilities p_n.

Example 11 *If the random variable* $\mathbf{R} \stackrel{d}{=} b\mathbf{U}^a$ *for some* $b, a > 0$ *and* $\mathbf{U}$ *standard uniformly distributed then clearly the domain of* $\mathbf{R}$ *is* $(0,b)$ *and*

$$\mathbb{P}\{\mathbf{R} \leq x\} = \mathbb{P}\{\mathbf{U} \leq (b^{-1}x)^{a^{-1}}\} = (b^{-1}x)^{a^{-1}}$$

for every $0 \leq x \leq b$. *By relation (43) we obtain*

$$\begin{aligned} p_n &= \tfrac{b^n}{n!}\mathbb{E}(\mathbf{U}^{an}\exp(-b\mathbf{U}^a)) \\ &= \tfrac{b^n}{n!}\int_0^1 x^{an}\exp(-bx^a)dx \\ &= \tfrac{a^{-1}b^n}{n!}\int_0^1 y^{n-1+a^{-1}}\exp(-by)dy. \end{aligned} \tag{52}$$

Applying Tonelli's theorem it follows for every $\zeta > 0$

$$\begin{aligned} \int_0^1 y^\zeta \exp(-by)dy &= \zeta\int_0^1\int_0^y x^{\zeta-1}dx\exp(-by)dy \\ &= \zeta\int_0^1\int_x^1 \exp(-by)dyx^{\zeta-1}dx \\ &= b^{-1}\zeta\int_0^1 \exp(-bx)x^{\zeta-1}dx - b^{-1}\exp(-b). \end{aligned}$$

By this recurrence relation and relation (52) it is easy to see for every $n \in \mathbb{N}\cup\{0\}$ *that*

$$p_{n+1} = \frac{n+a^{-1}}{n+1}p_n - \frac{a^{-1}b^n}{(n+1)!}\exp(-b) \tag{53}$$

and so for $a = 1$ *we recover relation (51). To compute the values* p_n *we first need to compute*

$$p_0 = \mathbb{E}\left(\exp(-b\mathbf{U}^a)\right)$$

and this can be done by some numerical integration method (see e.g. [20]).

A generalization of a uniformly distributed random variable $\mathbf{R}$ is given in the following example.

Example 12 *Let $G_{\mathbf{R}}$ be a concave cdf. It is well-known ([19]) that a cdf $G_{\mathbf{R}}$ of a nonnegative random variable $\mathbf{R}$ is concave on $(0,\infty)$ if and only if $\mathbf{R} \stackrel{d}{=} \mathbf{UY}$ with $\mathbf{U}$ a standard uniformly distributed random variable, $\mathbf{Y}$ a nonnegative random variable and $\mathbf{U}$ and $\mathbf{Y}$ independent. Using this representation we obtain for every $\tau > 0$ that*

$$\widehat{G}_{\mathbf{R}}(\tau) = \int_0^\infty \mathbb{E}\left(\exp(-\tau b\mathbf{U})\right) dG_{\mathbf{Y}}(b)$$

and so by relations (44), (45) and (50) (take $a = 0$) it follows $p_0 = \mathbb{E}(\exp(-\mathbf{UY}))$ and

$$p_{n+1} = p_n - \frac{1}{(n+1)!}\mathbb{E}(\mathbf{Y}^n \exp(-\mathbf{Y})) \tag{54}$$

for every $n \in \mathbb{N} \cup \{0\}$. If we introduce the sequence

$$r_n := \mathbb{P}\{\lfloor \mathbf{T}(\mathbf{Y}) \rfloor = n\}$$

we obtain by relations (14) and (54) that

$$p_{n+1} = p_n - \frac{1}{n+1} r_n.$$

This shows that one can evaluate $\mathbb{P}\{\lfloor \mathbf{T}(\mathbf{UY}) \rfloor = n\}$ once it is possible to compute the probabilities $\mathbb{P}\{\lfloor \mathbf{T}(\mathbf{Y}) \rfloor = n\}$.

If $\mathbf{R} \sim$ gamma(β, ρ) we obtain by relation (44) and (45) the recurrence relation

$$p_{n+1} = \frac{\beta + n}{(n+1)(\rho+1)} p_n,\ n \in \mathbb{N} \cup \{0\}, \tag{55}$$

with starting value $p_0 = (\frac{\rho}{1+\rho})^\beta$. In the next example we discuss the case of $\mathbf{R}$ having an infinitely divisible distribution thus generalizing $\mathbf{R}$ having a gamma distribution. A cdf $G_{\mathbf{R}}$ on $(0,\infty)$ is called infinitely divisible if for every $n \in \mathbb{N}$ there exists a sequence of independent, identically distributed and nonnegative random variables $\mathbf{R}_i$, $1 \le i \le n$ such that

$$\mathbf{R} \stackrel{d}{=} \mathbf{R}_1 + \ldots + \mathbf{R}_n$$

Examples of infinitely divisible cdf's on $(0,\infty)$ are discussed in [19] and include gamma distributions and scale mixtures of gamma distributions with shape parameter $\alpha \le 2$. It is also shown in [19] that all infinitely divisible cdf's on $(0,\infty)$ can be obtained as weak limits of compound-Poisson distributions.

Example 13 *Let the cdf $G_{\mathbf{R}}$ satisfying $G_{\mathbf{R}}(0) = 0$ be infinitely divisible. Necessarily the non-degenerate random variable $\mathbf{R}$ has unbounded support (cf. [19]) and so this excludes nonnegative random variables $\mathbf{R}$ with bounded support. Also by Theorem 4.10 on page 95 and Theorem 4.14 on page 97 of [19] it follows that $G_{\mathbf{R}}$ satisfying $G_{\mathbf{R}}(0) = 0$ is an infinitely divisible cdf if and only if there exists some (Laplace-Stieltjes transform) LSt-able function K (the so-called canonical function) satisfying $\int_{(0,\infty)} x^{-1} dK(x) = \infty$ such that*

$$\int_0^x r dG_{\mathbf{R}}(r) = (G_{\mathbf{R}} * K)(x) \tag{56}$$

for every $x \geq 0$ with $*$ the well-known convolution operator. A function K is called a LSt-able function ([19]) if K is right continuous and non-decreasing with $K(x) = 0$ for $x < 0$ and

$$\widehat{K}(\tau) := \int_{0-}^{\infty} \exp(-\tau x) dK(x)$$

is finite for every $\tau > 0$. By Tonelli's theorem and the binomial theorem applied to $(u+v)^{n-1}$ we obtain for $n \in \mathbb{N}$ using relations (43), (45) and (56) that

$$\begin{aligned} n! p_n &= \int_0^{\infty} \exp(-r) r^n dG_{\mathbf{R}}(r) \\ &= \int_0^{\infty} \exp(-r) r^{n-1} d(G_{\mathbf{R}} * K)(r) \\ &= \int_{\mathbb{R}_+^2} \exp(-(u+v))(u+v)^{n-1} dG_{\mathbf{R}}(u) dK(v) \\ &= \sum_{j=0}^{n-1} \binom{n-1}{j} \int_0^{\infty} \exp(-u) u^j dG_{\mathbf{R}}(u) \int_0^{\infty} \exp(-v) v^{n-1-j} dK(v) \\ &= (n-1)! \sum_{j=0}^{n-1} \frac{1}{(n-1-j)!} p_j \int_0^{\infty} \exp(-v) v^{n-1-j} dK(v). \end{aligned} \tag{57}$$

Introducing the constants

$$r_k := \frac{1}{k!} \int_0^{\infty} x^k \exp(-x) dK(x),\ k \in \mathbb{N} \cup \{0\}$$

this implies by relation (57) and (43) that

$$p_0 = \widehat{G}_{\mathbf{R}}(1) \text{ and } p_n = \frac{1}{n} \sum_{j=0}^{n-1} p_j r_{n-1-j},\ n \in \mathbb{N}. \tag{58}$$

From relation (56) it is easy to see (see also [19]) that

$$\widehat{K}(\tau) = \frac{-\widehat{G}_{\mathbf{R}}^{(1)}(\tau)}{\widehat{G}_{\mathbf{R}}(\tau)}$$

and so by relation (58) one can evaluate p_n, $n \in \mathbb{N} \cup \{0\}$ *if it is possible to compute the constants* r_k, $k \in \mathbb{N}$. *Examples are given by Gamma distributions with shape parameter* β *and scale parameter* ρ. *For this case the canonical function* K *has derivative* $k(x) = \beta \exp(-\rho x)$ *and so* $r_n = \beta(\rho+1)^{-(n+1)}$, $n \in \mathbb{N} \cup \{0\}$. *This yields* $p_n = \frac{\beta}{n} \sum_{j=0}^{n-1} p_j (\rho+1)^{-(n-j)}$, $n \in \mathbb{N}$, *with* $p_0 = (\frac{\rho}{1+\rho})^{\beta}$. *This actually defines the same sequence as relation (55) does.*

Finally, if it is not possible to derive elementary expressions for the nth derivative of the pLSt $\widehat{G}_{\mathbf{R}}$ we observe by relation (14) that the generating function (gP) P of the sequence p_n, $n \in \mathbb{N} \cup \{0\}$ is given by

$$P(z) := \mathbb{E}\left(z^{\lfloor \mathbf{T}(\mathbf{R}) \rfloor}\right) = \int_0^{\infty} \exp(-r) \sum\nolimits_{n=0}^{\infty} \frac{(zr)^n}{n!} dG_{\mathbf{R}}(r) = \widehat{G}_{\mathbf{R}}(1-z) \tag{59}$$

for every $z \in \mathbb{C}$ with $|z| \leq 1$. Hence we can apply the (numerical) FFT method ([1]) to evaluate the sequence p_n, $n \in \mathbb{N}$.

A second approximation of the cdf $H_{\mathbf{R}}$ is motivated by the following idea. Replace (see Figure 1) each sample path of the standard gamma process by a piecewise linear sample path coinciding with the original sample path at integer points and consider this stochastic process $\mathbf{X}_a$ instead of the original standard gamma process $\mathbf{X}$. Clearly by construction

$$\mathbf{X}_a(n) \stackrel{a.s.}{=} \mathbf{X}(n)$$

and

$$\mathbf{X}_a(n+s) = \mathbf{X}(n) + s(\mathbf{X}(n+1) - \mathbf{X}(n))$$

for every every $n \in \mathbb{N} \cup \{0\}$ and $0 < s < 1$. Moreover, it follows that $\mathbb{E}(\mathbf{X}_a(t)) = \mathbb{E}(\mathbf{X}(t))$ for every $t > 0$ and so this new stochastic process has the same expectation at each point as the original one. Alternatively by Lemma 5 we know for every non-integer $t > 0$ that

$$\mathbf{X}(t) \stackrel{d}{=} \mathbf{X}(\lfloor t \rfloor) + \mathbf{V}_{\mathcal{F}(t)} \mathbf{Y} \tag{60}$$

with $\mathbf{V}_{\mathcal{F}(t)} \sim$ beta $(\mathcal{F}(t), 1-\mathcal{F}(t))$, $\mathbf{Y} \sim$ gamma$(1,1)$, $\mathbf{X}(\lfloor t \rfloor) \sim$ gamma$(\lfloor t \rfloor, 1)$ independent random variables. Replacing now in relation (60) the random variable $\mathbf{V}_{\mathcal{F}(t)}$ by its expectation $\mathcal{F}(t)$ we obtain the proposed stochastic process $\mathbf{X}_a = \{\mathbf{X}_a(t) : t \geq 0\}$. We now introduce the following approximation assumption.

Approximation assumption 2. $H_{\mathbf{R}}(t) = \mathbb{P}\{\mathbf{X}(t) > \mathbf{R}\} \approx \mathbb{P}\{\mathbf{X}_a(t) > \mathbf{R}\}$.

To evaluate the probability $\mathbb{P}\{\mathbf{X}_a(t) > \mathbf{R}\}$ we first need the following result.

Lemma 14 *For every non-integer* $t > 1$ *and* $x > 0$

$$(1-\mathcal{F}(t))\mathbb{P}\{\mathbf{X}_a(t) \leq x\} = \mathbb{P}\{\mathbf{X}_a(\lfloor t \rfloor) \leq x\} - \mathcal{F}(t)\mathbb{P}\{\mathbf{X}_a(t-1) \leq x\}.$$

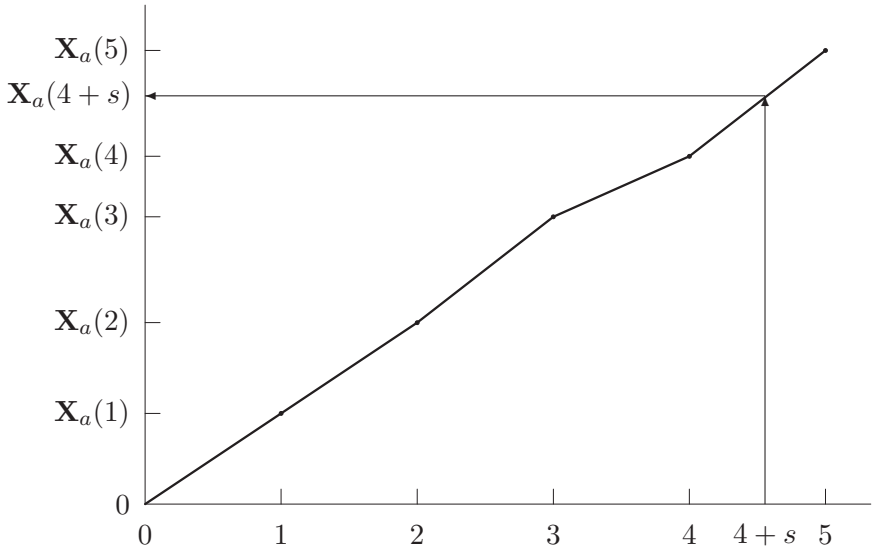

Figure 1: Piecewise linear sample path of the process $\mathbf{X}_a$.

Proof. If $1 < t < 2$ then $\mathbf{X}_a(t) \stackrel{d}{=} \mathbf{X}(1) + \mathcal{F}(t)\mathbf{Y}_1$ with the random variables $\mathbf{X}(1)$ and $\mathbf{Y}_1$ independent and both exponentially distributed with parameter 1. After some calculations this implies for every $x > 0$ that

$$\begin{aligned}(1-\mathcal{F}(t))\mathbb{P}\{\mathbf{X}_a(t) \le x\} &= (1-\mathcal{F}(t))\mathbb{P}\{\mathbf{X}(1) + \mathcal{F}(t)\mathbf{Y}_1 \le x\} \\ &= \mathbb{P}\{\mathbf{X}(1) \le x\} - \mathcal{F}(t)\mathbb{P}\{\mathcal{F}(t)\mathbf{Y}_1 \le x\}\end{aligned} \tag{61}$$

Since

$$\mathbf{X}_a(t-1) \stackrel{d}{=} \mathcal{F}(t)\mathbf{Y}_1 \text{ and } \mathbf{X}(1) = \mathbf{X}_a(1)$$

we have verified the desired formula for $0 < t < 1$. To verify the formula for arbitrary non-integer $t > 2$ it follows by the definition of the process $\mathbf{X}_a$ that

$$\mathbf{X}_a(t) = \mathbf{X}_a(\lfloor t \rfloor) + \mathcal{F}(t)\mathbf{Y}_1 \stackrel{d}{=} \mathbf{X}_a(\lfloor t \rfloor - 1) + \mathbf{X}(1) + \mathcal{F}(t)\mathbf{Y}_1 \tag{62}$$

with the random variables $\mathbf{X}(1)$, $\mathbf{Y}_1$ and $\mathbf{X}_a(\lfloor t \rfloor - 1)$ independent and $\mathbf{X}(1)$, $\mathbf{Y}_1$ exponentially distributed with parameter 1. This implies by relations (62) and (61) that

$$\begin{aligned}(1-\mathcal{F}(t))\mathbb{P}\{\mathbf{X}_a(t) \le x\} &= (1-\mathcal{F}(t))\mathbb{P}\{\mathbf{X}_a(\lfloor t \rfloor - 1) + \mathbf{X}(1) + \mathcal{F}(t)\mathbf{Y}_1 \le x\} \\ &= \mathbb{P}\{\mathbf{X}_a(\lfloor t \rfloor - 1) + \mathbf{X}(1) \le x\} - \mathcal{F}(t)\mathbb{P}\{\mathbf{X}_a(\lfloor t \rfloor - 1) + \mathcal{F}(t)\mathbf{Y}_1 \le x\}.\end{aligned}$$

Since $\mathbf{X}_a(\lfloor t \rfloor) \stackrel{d}{=} \mathbf{X}_a(\lfloor t \rfloor - 1) + \mathbf{X}(1)$ and $\mathbf{X}_a(t-1) = \mathbf{X}_a(\lfloor t \rfloor - 1) + \mathcal{F}(t)\mathbf{Y}_1$, the desired result follows. $\square$

By Lemma 14 it follows for $\mathbf{R}$ independent of the standard gamma process $\mathbf{X}$ and hence independent of $\mathbf{X}_a$ that

$$(1 - \mathcal{F}(t))\mathbb{P}\{\mathbf{X}_a(t) \leq \mathbf{R}\} = \mathbb{P}\{\mathbf{X}_a(\lfloor t \rfloor) \leq \mathbf{R}\} - \mathcal{F}(t)\mathbb{P}\{\mathbf{X}_a(t-1) \leq \mathbf{R}\}. \tag{63}$$

This implies the recurrence relation

$$(1 - \mathcal{F}(t))\mathbb{P}\{\mathbf{X}_a(t) > \mathbf{R}\} = \mathbb{P}\{\mathbf{X}_a(\lfloor t \rfloor) > \mathbf{R}\} - \mathcal{F}(t)\mathbb{P}\{\mathbf{X}_a(t-1) > \mathbf{R}\}. \tag{64}$$

To compute in relation (63) the probability $\mathbb{P}\{\mathbf{X}_a(\lfloor t \rfloor) \leq \mathbf{R}\}$ we observe by the continuity of $H_{\mathbf{R}}$ and $\mathbf{X}_a(\lfloor t \rfloor) = \mathbf{X}(t)$ that

$$\mathbb{P}\{\mathbf{X}_a(\lfloor t \rfloor) > \mathbf{R}\} = \mathbb{P}\{\mathbf{T}(\mathbf{R}) \leq \lfloor t \rfloor\} = \mathbb{P}\{\lfloor \mathbf{T}(\mathbf{R}) \rfloor \leq \lfloor t \rfloor - 1\}$$

and this implies by relations (43) and (45) that

$$\mathbb{P}\{\mathbf{X}(\lfloor t \rfloor) > \mathbf{R}\} = \sum\nolimits_{j=0}^{\lfloor t \rfloor - 1} \frac{\mathbb{E}(\mathbf{R}^j \exp(-\mathbf{R}))}{j!} = \sum\nolimits_{j=0}^{\lfloor t \rfloor - 1} p_j \tag{65}$$

for $t > 1$. Finally, for $0 < t < 1$, we obtain using $\mathbf{X}_a(t) \stackrel{d}{=} \mathcal{F}(t)\mathbf{Y}_1$ with $\mathbf{Y}_1 \sim \text{gamma}(1,1)$ that

$$\mathbb{P}\{\mathbf{X}_a(t) > \mathbf{R}\} = \mathbb{E}(\exp(-\mathcal{F}(t)^{-1}\mathbf{R})) = \widehat{G}_{\mathbf{R}}(\mathcal{F}(t)^{-1}). \tag{66}$$

By relation (64) up to (66) we can compute recursively the value $\mathbb{P}\{\mathbf{X}_a(t) > \mathbf{R}\}$ in case the pLSt of the random variable $\mathbf{R}$ is an elementary expression. In Appendix A an algorithm for computing $\mathbb{P}\{\mathbf{X}_a(t) > \mathbf{R}\}$ for $t > 0$ is given. In the next section we consider some special cases for which the cdf of $\mathbf{T}(\mathbf{R})$ has an elementary expression.

4 On the cdf of the randomized hitting time for some special cases

In this section we consider some special cases for which one can give an analytical and/or a simpler probabilistic interpretation of the cdf $H_{\mathbf{R}}$. We start with the simplest case of $\mathbf{R}$ having a degenerate distribution at $r > 0$, i.e. the threshold $\mathbf{R}$ is not random but deterministic.

Example 15 *If $\mathbf{R}$ has a degenerate distribution at $r > 0$, then*

$$H_{\mathbf{R}}(t) = \mathbb{P}\{\mathbf{X}(t) \geq r\} = \frac{\Gamma(t, r)}{\Gamma(t)}, \tag{67}$$

where $\Gamma(a, x) = \int_{z=x}^{\infty} z^{a-1} \exp(-z) \mathrm{d}z$ is the incomplete gamma function for $a \geq 0$ and $x > 0$. It follows that

$$\mathbb{P}\{\mathbf{T}_{v,\lambda}(\mathbf{R}) \leq t\} = \frac{\Gamma(v(t), r\lambda)}{\Gamma(v(t))}. \tag{68}$$

Another simple case occurs when $\mathbf{R}$ has an exponential distribution.

Example 16 *If the random variable $\mathbf{R}$ has an exponential cdf with scale parameter $\rho > 0$ given by $G_{\mathbf{R}}(r) = 1 - \exp(-\rho r)$, $r > 0$ then by relation (11)*

$$H_{\mathbf{R}}(t) = 1 - \mathbb{E}\left(\exp(-\rho \mathbf{X}(t))\right) = 1 - (1+\rho)^{-t}. \tag{69}$$

This implies

$$\mathbf{T}(\mathbf{R}) \stackrel{d}{=} \frac{\mathbf{Y}}{\ln(1+\rho)}. \tag{70}$$

with $\mathbf{Y}$ exponentially distributed with parameter 1. Since the random variable $\lambda\mathbf{R}$ has an exponential cdf with scale parameter $\rho\lambda^{-1}$ it follows by Lemma 2 and relation (69) that

$$\mathbf{T}_{v,\lambda}(\mathbf{R}) \stackrel{d}{=} v^{\leftarrow}\left(\frac{\mathbf{Y}}{\ln(1+\rho\lambda^{-1})}\right). \tag{71}$$

and this shows

$$\mathbb{P}\{\mathbf{T}_{v,\lambda}(\mathbf{R}) \leq t\} = 1 - (1+\rho\lambda^{-1})^{-v(t)}.$$

For the power function $v(t) = t^q$, $q > 0$ we obtain that $v^{\leftarrow}(u) = u^{q^{-1}}$ and so by relation (71) this yields

$$\mathbf{T}_{v,\lambda}(\mathbf{R}) \stackrel{d}{=} \frac{\mathbf{Y}^{q^{-1}}}{(\ln(1+\rho\lambda^{-1}))^{q^{-1}}}.$$

Since $\mathbf{Y}$ is exponentially distributed with parameter 1 and hence $\mathbf{Y}^{q^{-1}}$ is Weibull distributed with shape parameter q and scale parameter 1 ([4]) we obtain that $\mathbf{T}_{v,\lambda}(\mathbf{R})$ is a Weibull distributed random variable with shape parameter q and scale parameter $(\log(1+\rho\lambda^{-1}))^{q^{-1}}$.

A generalization of the exponential distribution is given in the following example.

Example 17 *If the positive random variable $\mathbf{R}$ has density function $g_{\mathbf{R}}$ on $(0,\infty)$ belonging to the class CM of completely monotone densities g, i.e. g is nonnegative and $(-1)^n g^{(n)}(x) \geq 0$ for $x > 0$ and $n \in \mathbb{N}$ with $g^{(n)}(\cdot)$ denoting the nth derivative of g, then by Bernstein's theorem ([9, 19]) we obtain*

$$g_{\mathbf{R}}(r) = \int_0^\infty y\exp(-yr)dG(y),\ r > 0$$

for some cdf G on $(0,\infty)$. This shows $g_{\mathbf{R}} \in CM$ if and only if $\mathbf{R} \stackrel{d}{=} \mathbf{ZY}$ with $\mathbf{Y}$ exponentially distributed with parameter 1, $\mathbf{Z}$ a nonnegative random variable and $\mathbf{Z}$ independent of $\mathbf{Y}$. By relation (70) and $z\mathbf{Y}$ has an exponential distribution with scale parameter z^{-1} this yields

$$\mathbf{T}(\mathbf{R}) \stackrel{d}{=} \frac{\mathbf{Y}}{\ln(1+\mathbf{Z}^{-1})} \tag{72}$$

and so for $\mathbf{R}$ *having a mixture of exponential distributions also the cdf* $H_{\mathbf{R}}$ *has a mixture of exponential distributions. In general it is not possible to give a nice analytical expression for the cdf* $H_{\mathbf{R}}$ *unless the cdf* $G_{\mathbf{R}}$ *has a finite mixture of exponential distributions given by*

$$G_{\mathbf{R}}(r) = \sum\nolimits_{k=1}^{q} p_k(1 - \exp(-\lambda_k r))$$

with $0 < \lambda_1 < \ldots < \lambda_q$ *and* $p_i > 0$, $\sum_{k=1}^{q} p_k = 1$. *This means that the discrete distribution of* $\mathbf{Z}$ *is given by*

$$\mathbb{P}\{\mathbf{Z} = \lambda_k^{-1}\} = p_k.$$

and by relation (72) we obtain

$$H_{\mathbf{R}}(t) = \sum\nolimits_{k=1}^{q} p_k(1 - (1+\lambda_k)^{-t}) = 1 - \sum\nolimits_{k=1}^{q} p_k(1+\lambda_k)^{-t}. \tag{73}$$

Similarly it follows by Lemma 2, $\lambda\mathbf{R} \stackrel{d}{=} \lambda\mathbf{Z}\mathbf{Y}$ *and relation (72) that*

$$\mathbf{T}_{v,\lambda}(\mathbf{R}) \stackrel{d}{=} v^{\leftarrow}\left(\frac{\mathbf{Y}}{\ln(1+(\lambda\mathbf{Z})^{-1})}\right) \tag{74}$$

and this implies for $v(t) = t^q$, $q > 0$ *that*

$$\mathbf{T}_{v,\lambda}(\mathbf{R}) \stackrel{d}{=} \frac{\mathbf{Y}^{q^{-1}}}{\ln(1+(\lambda\mathbf{Z})^{-1})^{q^{-1}}}.$$

As before for $\mathbf{R}$ *having a finite mixture of exponential distributions it is easy to check that the cdf* $H_{\mathbf{R}}$ *has a nice analytical expression given by a finite mixture of Weibull distributions. Moreover, if it is possible to generate a realization of the random variable* $\mathbf{Z}$ *it is easy by relation (74) to estimate the cdf of* $\mathbf{T}_{v,\lambda}(\mathbf{R})$ *by Monte Carlo simulation.*

If the positive random variable $\mathbf{R}$ has a gamma distribution with shape parameter $0 < \beta < 1$ and scale parameter 1 it follows by Lemma 5 that $\mathbf{R}$ has a completely monotone density. In particular one can show the following.

Example 18 *If the random variable* $\mathbf{R}$ *has a gamma distribution with shape parameter* $0 < \beta < 1$ *and scale parameter* 1*, then by Lemma 5 we obtain that*

$$\mathbf{R} \stackrel{d}{=} \mathbf{Z}\mathbf{Y}$$

with $\mathbf{Y}$ *and* $\mathbf{Z}$ *independent,* $\mathbf{Y} \sim$ *gamma*$(1,1)$ *and* $\mathbf{Z} \sim$ *beta*$(\beta, 1-\beta)$. *Hence by relation (72) we obtain*

$$\mathbf{T}(\mathbf{R}) \stackrel{d}{=} \frac{\mathbf{Y}}{\ln(1+\mathbf{Z}^{-1})} \tag{75}$$

Since $\mathbf{Z} \sim$ *beta*$(\beta, 1-\beta)$ *it is well-known ([10], [24]) that* $\mathbf{Z} \stackrel{d}{=} \mathbf{Z}_\beta(\mathbf{Z}_\beta + \mathbf{Z}_{1-\beta})^{-1}$ *with* $\mathbf{Z}_\beta$, $\mathbf{Z}_{1-\beta}$ *independent and* $\mathbf{Z}_\beta \sim$ *gamma*$(\beta, 1)$ *and* $\mathbf{Z}_{1-\beta} \sim$ *gamma*$(1-\beta, 1)$. *This implies by relation (75) that*

$$\mathbf{T}(\mathbf{R}) \stackrel{d}{=} \frac{\mathbf{Y}}{\ln(2 + \mathbf{Z}_{1-\beta}\mathbf{Z}_\beta^{-1})}.$$

By Lemma 2 it follows similarly

$$\mathbf{T}_{v,\lambda}(\mathbf{R}) \stackrel{d}{=} v^{\leftarrow}\left(\frac{\mathbf{Y}}{\ln(1 + \lambda + \mathbf{Z}_{1-\beta}\mathbf{Z}_\beta^{-1}) - \ln(\lambda)}\right)$$

and so for $v(t) = t^q$, $q > 0$ *we obtain*

$$\mathbf{T}_{v,\lambda}(\mathbf{R}) \stackrel{d}{=} \frac{\mathbf{Y}^{q^{-1}}}{\left(\ln(1 + \lambda + \mathbf{Z}_{1-\beta}\mathbf{Z}_\beta^{-1}) - \ln(\lambda)\right)^{q^{-1}}}.$$

Another generalization of the exponential cdf is given by the following.

Example 19 *If the positive random variable* $\mathbf{R}$ *follows a gamma distribution with shape parameter* $m \in \mathbb{N}$ *and scale parameter* $\rho > 0$ *we obtain*

$$G_{\mathbf{R}}(r) = 1 - \exp(-\rho r) \sum_{j=0}^{m-1} \frac{(\rho r)^j}{j!}.$$

Hence by relation (11) it follows that

$$H_{\mathbf{R}}(t) = 1 - \sum_{j=0}^{m-1} \frac{\rho^j}{j!} \mathbb{E}(\mathbf{X}(t)^j \exp(-\rho \mathbf{X}(t))). \tag{76}$$

To evaluate the expressions in relation (76) we observe using $\mathbf{X}(t) \sim$ *gamma*$(t, 1)$ *that*

$$\mathbb{E}(\mathbf{X}(t)^j \exp(-\rho \mathbf{X}(t))) = \frac{1}{\Gamma(t)} \int_0^\infty \exp(-(1+\rho)x) x^{j+t-1} dx \tag{77}$$

It is now easy to show by its relation with a gamma distribution with scale parameter $\lambda > 0$ *and shape parameter* $j + t$ *that*

$$\frac{1}{j!\Gamma(t)} \int_0^\infty \exp(-\lambda r) r^{j+t-1} dr = \frac{\Gamma(j+t)}{\Gamma(t) j! \lambda^{j+t}} = \binom{t+j-1}{j} \lambda^{-(j+t)} \tag{78}$$

with

$$\binom{k}{j} := \begin{cases} \frac{\prod_{p=0}^{j-1}(k-p)}{j!} & j \in \mathbb{N} \\ 1 & j = 0 \end{cases}, \; k \in \mathbb{R},$$

for every $\lambda > 0$, $t > 0$ and $j \in \mathbb{N} \cup \{0\}$. Hence it follows for every $j \in \{0, ..., m-1\}$

$$\mathbb{E}(\mathbf{X}(t)^j \exp(-\rho \mathbf{X}(t))) = j! \binom{t+j-1}{j} (1+\rho)^{-(j+t)}$$

and this implies using relation (76) that

$$H_{\mathbf{R}}(t) = 1 - (1+\rho)^{-t} \sum_{j=0}^{m-1} \binom{t+j-1}{j} \left(\frac{\rho}{1+\rho}\right)^j. \tag{79}$$

Moreover, since $\lambda\mathbf{R}$ has a gamma distribution with shape parameter $m \in \mathbb{N}$ and scale parameter $\lambda^{-1}\rho$ we obtain by relation (79) that

$$H_{\lambda\mathbf{R}}(t) = 1 - (\lambda^{-1}\rho + 1)^{-t} \sum_{j=0}^{m-1} \binom{t+j-1}{j} \left(\frac{\rho}{\lambda+\rho}\right)^j$$

and applying lemma 2 yields

$$\mathbb{P}\{\mathbf{T}_{v,\lambda}(\mathbf{R}) \leq t\} = 1 - (\lambda^{-1}\rho + 1)^{-t} \sum_{j=0}^{m-1} \binom{v(t)+j-1}{j} \left(\frac{\rho}{\lambda+\rho}\right)^j.$$

5 Numerical study

This section presents the computational results of the two proposed approximations of the randomized hitting times discussed in Section 3. We first focus on the accuracy of the approximations of the cdf of the random variable $\mathbf{T}(\mathbf{R})$. Next we also compare the computing time of the approximations and the exact (numerical) evaluation of the cdf of $\mathbf{T}_{v,\lambda}(\mathbf{R})$. To serve our purposes we have written a computer program using MATLAB 7.2 on a Pentium III–2 GHz personal computer.

In our first experiments we consider four distributions for the random variable $\mathbf{R}$ and these are given in Table 1. The discrete mixture of two exponentials in Table 1 is also referred to as a hyperexponential-2 with balanced means ([21]). For these four distributions analytical expressions for the cdf $H_{\mathbf{R}}$ exist and this enables us to assess the accuracy of the approximations.

The accuracy of the approximations is measured by the maximum absolute difference between the approximative and the true cdf on its entire domain. By relation (9) we only have to focus on the accuracy of the used approximation of the cdf $H_{\lambda\mathbf{R}}$. To give an approximation of the supnorm error we first evaluate the true cdf $H_{\lambda\mathbf{R}}$ and the used approximation for values of t on a grid $\{ih\}_{i=1,2,...,N}$, with $h = 0.005$ and

$$N = \inf\{n \in \mathbb{N} : \ \mathbb{P}\{\mathbf{T}(\lambda\mathbf{R}) \leq \lceil nh \rceil\} > 1 - 10^{-7}\}.$$

Subsequently, the right-hand side of (9) is estimated by the maximum absolute difference between the approximations and the true cdf on this grid, i.e.

$$\max_{i=1,2,...,N} |A_{\lambda\mathbf{R}}(ih) - H_{\lambda\mathbf{R}}(ih)|, \ h > 0.$$

Distribution	**Parameters**
Deterministic (det)	$r > 0$
Uniform (unif)	$a, a+b,\ a, b > 0$
Erlang (erlang)	$m \in \mathbb{N},\ \rho > 0$
Mixture of two exponentials (hyp-2)	$p_1, \lambda_1, p_2, \lambda_2 > 0,\ p_1 + p_2 = 1,\ \frac{p_1}{\lambda_1} = \frac{p_2}{\lambda_2}$

Table 1: The four distributions for $\mathbf{R}$ under consideration.

Since for the first proposed approximation we have to compute the cdf of $\mathbf{T}(\mathbf{R})$ at integer points and then interpolate, it is straightforward to obtain the approximation of this cdf at non-integer points. The algorithms presented in Appendix A are implemented to compute the second approximation of $H_{\mathbf{R}}$. Since it is possible to vary the scale parameter λ of a stationary gamma process in such a way that the first hitting time of $\mathbf{R}$ can take a wide range of values we fix the value of the expectation of $\mathbf{R}$ to 100. This means for a degenerate random variable that $r = 100$. Next, we vary the coefficient of variation $c_{\mathbf{R}}$ of $\mathbf{R}$, given by $c_{\mathbf{R}} = (\text{Var}(\mathbf{R}))^{1/2} \ / \ \mathbb{E}(\mathbf{R})$, from 0 ($\mathbf{R}$ deterministic) to 1.2 ($\mathbf{R}$ mixture of two exponentials) by steps of 0.2. The parameters of the three non-degenerate distributions are now determined by means of a two-moment fit ([21]). The coefficient of variation of a uniform random variable is bounded from above by $\sqrt{3}/3$. The discrete mixture of two exponentials with balanced means has only two free parameters and its coefficient of variation is bounded from below by 1. The coefficient of variation of the Erlang distribution can take any value greater than 0.

In Table 2 the maximum absolute differences between the approximations and the true cdf of $\mathbf{T}(\lambda\mathbf{R})$ are shown for a range of values of λ. The higher the value of λ the higher the accuracy of the approximations. Both methods appear to be quite accurate for $\lambda \geq 0.05$. Since for λ increasing the expectation $\mathbb{E}(\exp(-\lambda\mathbf{R}))$ becomes smaller this is to be expected from our theoretical results in Section 2. The smaller this expectation the more the fractional part of $T_{\lambda\mathbf{R}}$ has a uniform cdf and at the same time the correlation between the fractional and integer part of $T_{\lambda\mathbf{R}}$ vanishes. This makes our approximative assumption more accurate. The effect of the coefficient of variation of the random variable $\mathbf{R}$ on precision is not unambiguous. In Figure 2 the accuracy of the approximations is plotted against λ (on a double-logarithmic scale) for $\mathbf{R}$ Erlang distributed with $c_{\mathbf{R}} = 1.2$. We observe that the maximum absolute error is smaller than 0.01 for $\lambda > 0.02$. In Figure 3 the exact and approximative cdf's of $\mathbf{T}(\lambda\mathbf{R})$ are plotted against time t for uniformly distributed $\mathbf{R}$ with $c_{\mathbf{R}} = 0.2$ and $\lambda = 0.02$. It appears that the left tail of the distribution is not approximated very well.

We have seen that the scale parameter of the gamma process rules the accuracy of the approximations. However, the computing time of the true cdf of the random variable $\mathbf{T}_{v,\lambda}(\mathbf{R})$ is affected by both the shape function and the scale parameter. In order to assess this effect, we focus on

					λ		
$\mathbf{R}$	$c_{\mathbf{R}}$	Approx	0.0005	0.005	0.05	0.5	5
Deterministic	0	1	2.85E-01	4.32E-02	7.31E-03	6.35E-04	6.14E-05
		2	3.98E-01	5.57E-02	7.83E-03	6.38E-04	6.15E-05
Uniform	0.2	1	2.86E-01	4.24E-02	6.66E-03	2.52E-04	7.97E-06
		2	3.98E-01	5.53E-02	7.12E-03	2.53E-04	7.97E-06
	0.4	1	2.92E-01	3.98E-02	5.72E-03	1.84E-04	5.81E-06
		2	3.98E-01	5.96E-02	6.52E-03	1.86E-04	5.81E-06
Erlang	0.4	1	2.96E-01	4.08E-02	6.33E-03	1.19E-04	1.32E-06
		2	4.05E-01	6.41E-02	6.74E-03	1.20E-04	1.33E-06
	0.6	1	2.90E-01	3.67E-02	6.09E-03	8.43E-05	8.85E-07
		2	3.87E-01	5.88E-02	6.46E-03	8.44E-05	8.87E-07
	0.8	1	2.79E-01	3.45E-02	5.14E-03	1.11E-04	1.21E-06
		2	3.61E-01	5.07E-02	5.32E-03	1.11E-04	1.21E-06
Hyp-2	1	1	3.24E-01	9.01E-02	3.80E-03	4.85E-05	4.99E-07
		2	4.04E-01	9.43E-02	3.80E-03	4.85E-05	4.99E-07
	1.2	1	3.31E-01	1.01E-01	5.35E-03	7.40E-05	7.67E-07
		2	4.13E-01	1.07E-01	5.36E-03	7.40E-05	7.67E-07

Table 2: The maximum absolute difference between the approximation and the true cdf for different values of λ and various choices of the random variable $\mathbf{R}$.

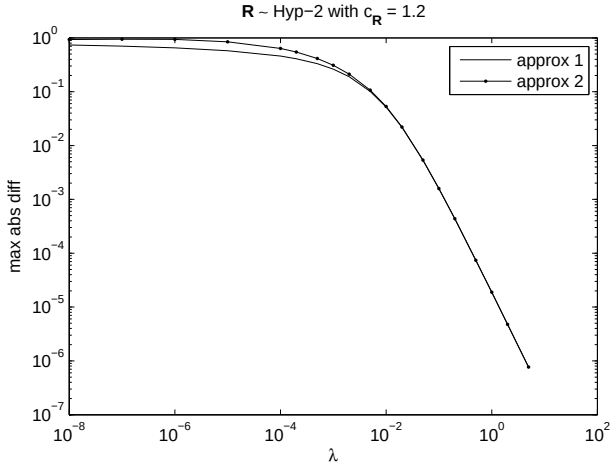

Figure 2: Maximum absolute difference between $\mathbb{P}\{\mathbf{T}(\lambda\mathbf{R}) \leq t\}$ and the two approximations for different values of λ and $\mathbf{R} \sim$ hyp-2 with $c_{\mathbf{R}} = 1.2$.

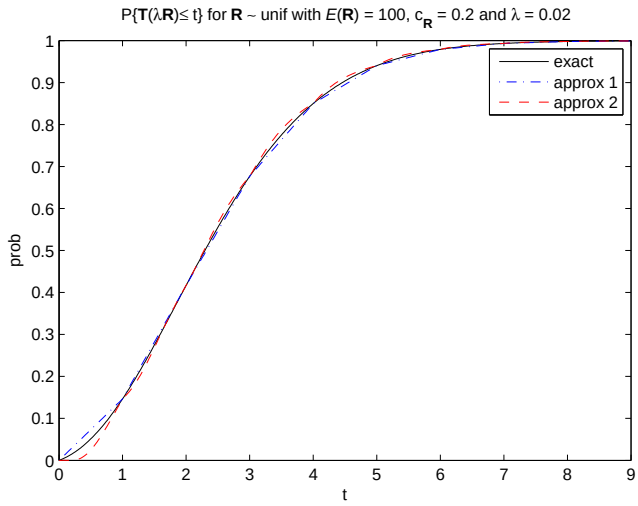

Figure 3: Exact and approximative cumulative distribution functions of $\mathbf{T}(\lambda\mathbf{R})$, where $\mathbf{R} \sim$ unif with $c_{\mathbf{R}} = 0.2$ and $\lambda = 0.02$.

non-standard gamma processes $\mathbf{X}_{v,\lambda}$ having a shape function v proportional to a power of time, i.e. $v(t) = \nu t^q$ with $\nu > 0$, and scale parameter $\lambda > 0$. We set the value of the power q at 0.5, 1, 2 and 4 and thus cover concave as well as convex shape functions. The mean of the non-stationary gamma process at time t_0 is fixed at $x = \lambda^{-1}\nu t_0^q = 100$ and for t_0 we consider the values 10, 20, 50 and 100. The coefficient of variation of the gamma process at time t_0 is varied from 0.1 to 0.7 by steps of 0.2. For given values of q, t_0 and $c = c_{\mathbf{X}_{v,\lambda}(t_0)} = \nu^{-1/2} {t_0}^{-q/2}$, the parameters ν and λ are now determined by a two-moment fit. This yields $\nu = c^{-2} t_0^{-q}$ and $\lambda = c^{-2} x^{-1}$. Observe a high value of c implies a small value of λ. Therefore, the approximation of the cdf of a randomized hitting time associated with non-stationary gamma processe is accurate whenever the variability of the gamma process is not too high. For obvious reasons we are only interested in the computing times of good approximations. In the scenarios defined by the experimental settings in Table 3 the value of λ ranges from 0.020 ($c = 0.7$) to 1 ($c = 0.1$) and hence by our previous findings the approximations are accurate.

The computing time is defined as the time needed to evaluate or approximate the cdf of $\mathbf{T}_{v,\lambda}(\mathbf{R})$ on a grid $\{ih\}_{i=1,2,\ldots,N}$, where $h = 0.02$ and N is the first integer such that the cdf in the integer point $\lceil v(Nh) \rceil$ exceeds 0.999. We only focus on $\mathbf{R}$ deterministic and $\mathbf{R}$ uniform, since for these random variables the cdf of $\mathbf{T}_{v,\lambda}(\mathbf{R})$ has no nice analytical expression.

In Table 4 the computing times of the approximations and the true cdf are given for one particular scenario of Table 3. Note that the value of λ is constant and so the hitting times of the

Settings $\mathbf{X}_{v,\lambda}$	Values
$t_0 = \{t > 0 : \mathbb{E}(\mathbf{X}_{v,\lambda}(t)) = 100\}$	10, 20, 50, 100 time units
q	0.5, 1, 2, 4
$c_{\mathbf{X}_{v,\lambda}(t_0)}$	0.1, 0.3, 0.5, 0.7

Table 3: Experimental settings.

			q			
$\mathbf{R}$	$c_{\mathbf{R}}$	Evaluation	0.5	1	2	4
Deterministic	0	exact	4.14	1.86	1.23	1.04
		approx1	1.10E-02	3.80E-03	2.94E-03	5.99E-03
		approx2	9.63E-02	4.24E-02	5.58E-02	5.68E-02
Uniform	0.2	exact	19.07	8.06	5.20	4.17
		approx1	3.64E-02	2.87E-02	2.75E-02	3.15E-02
		approx2	1.99E-01	1.39E-01	1.46E-01	1.46E-01
Uniform	0.4	exact	24.14	8.89	5.60	4.55
		approx1	4.84E-02	3.69E-02	3.60E-02	3.96E-02
		approx2	2.29E-01	1.40E-01	1.48E-01	1.89E-01

Table 4: Computing times for different values of q and different random variables $\mathbf{R}$. Here, $t_0 = 50$ and $c_{\mathbf{X}_{v,\lambda}(t_0)} = 0.3$ yielding $\lambda = \frac{1}{9}$.

(non-stationary) gamma processes are all related to the same random variable $\mathbf{T}(\lambda\mathbf{R})$. The results show that the efforts of obtaining both approximations is much less than the effort of computing the true cdf. In all cases the first approximation can be obtained faster than the second approximation. Also, increasing values of q are negative related to the effort of computing the true cdf. This is caused by the fact that the right tail of the hitting time distribution is smaller for larger values of q. Thus for fixed values of $\mathbb{E}(\mathbf{R})$ and $c_{\mathbf{R}}$ the cdf is computed in fewer points for q large. This effect is not that apparent for the approximations since their computing times are mainly affected by the number of *integer* time points in which the cdf needs to be computed and this again depends on the value of λ. The experiments with the other scenarios yield similar results and are available upon request.

6 Conclusions

The gamma process plays an important role in maintenance optimization. In particular, the first time at which this process exceeds a random threshold is often used to model the lifetime of structures subject to degradation. In this article we have investigated in detail the cdf of this random variable. We have first shown that the cdf of a randomized hitting time associated with a non-stationary gamma processes is easily derived from the cdf of a similar hitting time of a standard gamma process. Secondly, we have extended an existing result on the cdf of the fractional part of a randomized hitting time.

Explicit formulas for the cdf of a randomized hitting time have been derived for some special cases. In general however, the evaluation of the cdf of a randomized hitting time for a standard gamma process is time-consuming. Therefore, we have proposed two approximations having a clear probabilistic interpretation. The first approximation comes down to a linear interpolation of the exact hitting time cdf at integer points and it is justified by above-mentioned result on the cdf of the fractional part of the hitting time. We have shown that due to the structure of a standard gamma process it is possible to compute the probability distribution of a randomized hitting time at integer time points. The second approximation is obtained by replacing each sample path of a standard gamma process by a piecewise linear sample path coinciding with the original sample path at integer points.

Numerical experiments show that both approximation formulas are quite accurate when the random threshold is not too small and the variability of the (non-stationary) gamma process is not too high. The second approximation method may be somewhat prohibitive from a numerical point of view. On the other hand, the first approximation is quite efficient and appears to be a good replacement for the exact distribution in time-consuming optimization algorithms.

Acknowledgements

The authors are grateful to Jan Brinkhuis who proposed to use a backward algorithm for the numerical computation of approximation 2. We would also like to thank Rommert Dekker for valuable comments and suggestions.

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A Algorithm for approximation 2

The algorithm below computes $\mathbb{P}\{\mathbf{X}_a(t) > \mathbf{R}\}$ for $t > 0$.

Algorithm 20 (Forward approximation algorithm)
Input: time $t > 0$. *Output:* $\mathbb{P}\{\mathbf{X}_a(t) > \mathbf{R}\}$.

(1) For $k = 0$ *to* $\lfloor t \rfloor$
Let cdfint(k) $= \sum_{j=0}^{k-1} p_j$, *where* $\sum_{j=0}^{-1} p_j = 0$ *(cdfint(k)* $= \mathbb{P}\{\mathbf{X}_a(k) > \mathbf{R}\}$*).*
Next k
Here p_j *is computed using the right-hand side of* (44). *If* $\mathcal{F}(t) = 0$, *then return cdfint(*$\lfloor t \rfloor$*). Otherwise, proceed with step (2).*

(2) Let cdf $= \widehat{G}_{\mathbf{R}}(\mathcal{F}(t)^{-1})$ *(cdf* $= \mathbb{P}\{\mathbf{X}_a(\mathcal{F}(t)) > \mathbf{R}\}$*).*

(3) For j = 1 to $\lfloor t \rfloor$
Let cdf $= (cdfint(j) - cdf \cdot \mathcal{F}(t))/(1 - \mathcal{F}(t))$ *(cdf* $= \mathbb{P}\{\mathbf{X}_a(t) > \mathbf{R}\}$*,* $j < t < j + 1$*).*
Next j
Return cdf.

Unfortunately, when the values of cdf and cdfint are close to 1 and $\mathcal{F}(t) > 0.5$, the repeated subtraction in step (3) leads to loss of precision for $t > 1$. These numerical difficulties are circumvented by employing a backward version of algorithm 20.

Algorithm 21 (Backward approximation algorithm)
Input: time $t > 1$ *with* $\mathcal{F}(t) > 0.5$*. Output:* $\mathbb{P}\{\mathbf{X}_a(t) > \mathbf{R}\}$*.*
Let M *be a large integer, say* $M = 100$*.*

(1) For $k = 0$ *to* $\lfloor t + M \rfloor$ *+1*
Let cdfint(k) $= \sum_{j=0}^{k-1} p_j$*, where* $\sum_{j=0}^{-1} = 0$ *(cdfint(k)* $= \mathbb{P}\{\mathbf{X}_a(k) > \mathbf{R}\}$*).*
Next k
Here p_j *is computed using expression* (44)*.*

(2) Let cdf=cdfint$(\lfloor t + M \rfloor + 1)\mathcal{F}(t)$*+ cdfint*$(\lfloor t + M \rfloor)(1 - \mathcal{F}(t))$*. (cdf here represents the approximate value of the cdf at time* $M + \mathcal{F}(t)$ *according to method* 1*).*

(3) For $j = \lfloor t \rfloor + C$ *to* $\lfloor t \rfloor + 1$ *step* (-1)
Let cdf $= (cdfint(j) - cdf \cdot (1 - \mathcal{F}(t)))/\mathcal{F}(t)$ *(cdf* $= \mathbb{P}\{\mathbf{X}_a(t) > \mathbf{R}\}$*,* $j < t < j + 1$*).*
Next j
Return cdf.

In step (2) we estimate the cdf at time $M + \mathcal{F}(t)$ by linear interpolation at the surrounding integers (approximation method 1). The estimate does not even have to be very accurate, because the backwards algorithm gains precision in every step (as opposed to the forward algorithm).

Finally, note that in order to compute the cdf $\mathbb{P}\{\mathbf{X}_a(t) > \mathbf{R}\}$ one needs to compute the cdf at time points $\mathcal{F}(t), \mathcal{F}(t) + 1, \ldots, t - 1$ (forward algorithm) or $M + \mathcal{F}(t), M - 1 + \mathcal{F}(t), \ldots, t + 1$ (backward algorithm). So, if one wants to compute the cdf at equidistant points ih, $i = 1, \ldots, N$, for some $N \in \mathbb{N}$ and $h > 0$, one only has to compute the cdf at the greatest (smallest) time points with different fractional parts in the forward (backward) algorithm. The other cdf values are obtained for free.