# On the complexity of the economic lot-sizing problem with remanufacturing options

Wilco van den Heuvel[a] *

[a] *Econometric Institute and Erasmus Research Institute of Management,*

*Erasmus University Rotterdam, P.O. Box 1738, 3000 DR Rotterdam, The Netherlands*

Econometric Institute Report EI 2004-46

October 19, 2004

**Abstract**

In this paper we investigate the complexity of the economic lot-sizing problem with remanufacturing (ELSR) options. Whereas in the classical economic lot-sizing problem demand can only be satisfied by production, in the ELSR problem demand can also be satisfied by remanufacturing returned items. Although the ELSR problem can be solved efficiently for some special cases, we show that the problem is NP-hard in general, even under stationary cost parameters.

*Keywords:* Lot-sizing; Remanufacturing; Complexity

## 1 Introduction

In this paper we consider an extension of the classical economic lot-sizing (ELS) or the Wagner-Whitin model (see Wagner and Whitin (1958)): the economic lot-sizing model with remanufacturing options (ELSR). The classical ELS problem can be described as follows. In each period of a finite and discrete time horizon there is a known demand for a single item. This demand has to be satisfied each period by producing in this period or in previous periods, i.e., back-logging is not allowed. When production occurs in a period, a setup has to be made and unit production costs are incurred. Finally, holding costs are incurred for carrying ending inventory from a period to the next one and it is assumed that all costs are non-negative.

* *E-mail address:* wvandenheuvel@few.eur.nl

In the ELSR problem we assume that demand cannot only be satisfied by production but also by remanufacturing returned items. In each period there is a known amount of returned items that can be remanufactured in this period or in later periods. We assume there is no difference in quality between remanufactured and manufactured items. If remanufacturing occurs in a period, then a setup has to be made and unit remanufacturing costs are incurred. As for the new items (or serviceables) holding costs are incurred for carrying returns from a period to the next period.

Wagner and Whitin (1958) developed an $O(T^2)$ to solve the classical ELS problem. More recently, Federgruen and Tzur (1991), Wagelmans et al. (1992) and Aggarwal and Park (1993) developed $O(T \log T)$ algorithms for the general case and $O(T)$ algorithms for the case of no speculative motives. Richter and Sombrutzki (2000) and Richter and Weber (2001) show that some special cases of the ELSR problem can still be solved in polynomial time. For example, the case with enough returns to satisfy demand, no production and no speculative motives can be solved efficiently. Furthermore, the case with one big return in period 1 (larger than the total demand) and cheaper holding costs for returns than for serviceables can also be solved by a Wagner-Whitin like recursion. In this paper we will show that the ELSR problem is NP-hard in general.

The remainder of the paper is organized as follows. In section 2 we introduce the notation and develop a mathematical model. In section 3 we will show that the general ELSR problem is NP-hard even under stationary cost parameters. We end this paper in section 4 with some concluding remarks.

## 2 Mathematical model

To define the ELSR problem, we use the following notation.

**Parameters:**

$T$ : model horizon,

$D_t$ : demand in period $t$,

$R_t$ : returns in period $t$,

$K_t^m$ : setup cost for manufacturing in period $t$,

$p_t^m$ : unit manufacturing cost in period $t$,

$h_t^s$ : unit cost for holding serviceables in period $t$,

$K_t^r$ : setup cost for remanufacturing in period $t$,

$p_t^r$ : unit remanufacturing cost in period $t$,

$h_t^r$ : unit cost for holding returns in period $t$.

**Decision variables:**

$x_t$ : amount of items manufactured in period $t$,

$I_t^s$ : amount of serviceables in stock at the end of period $t$,

$z_t$ : amount of items remanufactured in period $t$,

$I_t^r$ : amount of returns in stock at the end of period $t$.

If we use the above notation, then the ELSR problem can be modelled as follows:

$$
\begin{array}{lll}
\text{(ELSR)} & \min & \sum_{t=1}^{T} \left( K_t^m \delta(x_t) + p_t^m x_t + I_t^s h_t^s + K_t^r \delta(z_t) + p_t^r z_t + I_t^r h_t^r \right) \\
 & \text{s.t.} & I_t^s = I_{t-1}^s + x_t + z_t - D_t \qquad t = 1, \ldots, T \\
 & & I_t^r = I_{t-1}^r + R_t - z_t \qquad t = 1, \ldots, T \\
 & & x_t, z_t, I_t^s, I_t^r \geq 0 \qquad t = 1, \ldots, T \\
 & & I_0^s = I_0^r = 0,
\end{array}
$$

where

$$
\delta(y) = \begin{cases} 0 & \text{for } y = 0 \\ 1 & \text{for } y > 0. \end{cases}
$$

The first set of constraints models the inventory balance for the serviceables. The amount of serviceables in stock at the end of period $t$ equals the amount of serviceables at the end of period $t-1$ plus the amount of (re)manufactured items in period $t$ minus the demand in period $t$. The second set of constraints models the inventory balance for the returned items. The number of returns in stock at the end of period $t$ equals the number of returns at the end of period $t-1$ plus the new incoming returns in period $t$ minus the amount of remanufactured items in period $t$. We assume that we start with no serviceables and no returns in stock.

## 3 The ELSR problem is NP-hard

Although Richter and Sombrutzki (2000) and Richter and Weber (2001) show that some special cases of the ELSR problem can be solved in polynomial time, the time complexity of the general ELSR problem is not known yet. Richter and Sombrutzki (2000, p. 311) note that "there are probably no simple algorithms to solve that general model". In this section we show that the ELSR problem is NP-hard in general. Therefore we will reduce the NP-complete PARTITION problem to the decision version of the ELSR problem.

Problem PARTITION: Given $n$ positive integer $a_1, \ldots, a_n$. Does there exist a set $S \subset N = \{1, \ldots, n\}$ such that $\sum_{i \in S} a_i = \sum_{i \in N \setminus S} a_i = A$?

**Theorem 1** *The decision version of the ELSR problem is NP-complete.*

**Proof** Given an instance of PARTITION we define the cost parameters of the ELSR instance as follows:

$T = n$,

$D_t = a_t$ for $t = 1, \dots, T$,

$R_1 = A$, $R_t = 0$ for $t = 2, \dots, T$,

$K_t^m = K_t^r = 1$ for $t = 1, \dots, T$,

$p_t^m = 1$ for $t = 1, \dots, T$,

$p_t^r = 0$ for $t = 1, \dots, T$,

$h_t^s = 3$ for $t = 1, \dots, T$,

$h_t^r = 0$ for $t = 1, \dots, T$.

Clearly, this reduction can be done in polynomial time. We will show that the answer to PARTITION is positive if and only if the ELSR instance has a solution with cost at most $T + A$.

- if part:

  Assume that we have a solution for the ELSR instance with cost at most $T + A$. First, note that (re)manufactured items will never be held in stock. Namely, if $t$ is a (re)manufacturing period with at least one item in stock at the end of period $t$, then reducing the number of items being (re)manufactured by one in period $t$ and increasing the number of items being (re)manufactured by one in period $t + 1$ will reduce total cost by at least 1. So it is never optimal to hold (re)manufactured items in stock and hence demand is satisfied by either manufacturing or remanufacturing (and not by serviceables in stock) in each period.

  Furthermore, because at most $A$ items can be remanufactured and all demand has to be satisfied, we incur at least cost $A$ for manufactured items and we incur exactly cost $A$ if all returns are remanufactured. Finally, there can not be both remanufacturing and manufacturing in a single period, because then the total setup costs will exceed $T$. Because total cost is at most $T + A$, this means that the demand in the remanufacturing periods is satisfied by $A$ returns and not by manufacturing. But then the total demand in the remanufacturing periods exactly equals $A$ and hence the remanufacturing periods form the set $S$.

- only if part:

  Let $S$ be the set for which $\sum_{i \in S} a_i = \sum_{i \in N \setminus S} a_i = A$. It is easy to verify that by remanufacturing $a_t$ items in each period $t \in S$ and manufacturing $a_t$ items in each period $t \in N \setminus S$ all demand is satisfied and total costs equal $T + A$.

□

Note that changing the order of the integers $a_1, \ldots, a_n$ in the PARTITION instance does not change the optimal cost of the corresponding ELSR problem (as in the NP-completeness proof of the capacitated lot-sizing problem (see Florian et al. (1980))). This shows that the ELSR problem is NP-hard even in the case of increasing (or decreasing) demand over time and stationary cost parameters.

# 4 Concluding remarks

We note that the reduced ELSR problem instance in the proof of section 3 has quite natural assumptions on the cost parameters. It is natural to assume that holding serviceables is at least as costly as holding returns (i.e., $h_t^s \geq h_t^r$), that manufacturing is at least as expensive as remanufacturing (i.e., $p_t^m \geq p_t^r$) and that total demand in periods $1, \ldots, T$ equals at least the total returns in periods $1, \ldots, T$ (i.e., $\sum_{t=1}^{T} D_t \geq \sum_{t=1}^{T} R_t$).

Furthermore, the NP-hardness result immediately proves the result of Golany et al. (2001). Golany et al. (2001) considers the ELSR problem with an additional option of disposing returned items and shows that this problem is NP-hard. Because the ELSR problem is a special case of this problem, the result of Golany et al. (2001) follows immediately from our result. Moreover, our result shows that the ELSR problem with disposal options is NP-hard even in the case of stationary cost parameters. This result is stronger than the result of Golany et al. (2001), who do not assume stationary cost parameters in their proof.

# References

A. Aggarwal and J.K. Park. Improved algorithms for economic lot-size problems. *Operations Research*, 14:549–571, 1993.

A. Federgruen and M. Tzur. A simple forward algorithm to solve general dynamic lot sizing models with $n$ periods in $O(n \log n)$ or $O(n)$ time. *Management Science*, 37, 1991.

M. Florian, J.K. Lenstra, and A.H.G. Rinnooy Kan. Deterministic production planning: algorithms and complexity. *Management Science*, 26:669–679, 1980.

B. Golany, J. Yang, and G. Yu. Economic lot-sizing with remanufacturing options. *IIE Transactions*, 33:995–1003, 2001.

K. Richter and M. Sombrutzki. Remanufacturing planning for the reverse wagner/whitin models. *European Journal of Operational Research*, 121:304–315, 2000.

K. Richter and J. Weber. The reverse wagner/whitin model with variable manufacturing and remanufacturing cost. *International Journal of Production Economics*, 71:447–456, 2001.

A.P.M. Wagelmans, C.P.M. van Hoesel, and A. Kolen. Economic lot sizing: An $O(n \log n)$ algorithm that runs in linear time in the Wagner-Whitin case. *Operations Research*, 40:S145–S156, 1992.

H.M. Wagner and T.M. Whitin. Dynamic version of the economic lot size model. *Management Science*, 5:89–96, 1958.