A D-Induced Duality and Its Applications

Jan Brinkhuis * Shuzhong Zhang †

July 19, 2002

Abstract

This paper attempts to extend the notion of duality for convex cones, by basing it on a pre-described conic ordering and a fixed bilinear mapping. This is an extension of the standard definition of dual cones, in the sense that the *nonnegativity* of the inner-product is replaced by a pre-specified conic ordering, defined by a convex cone D, and the inner-product itself is replaced by a general multi-dimensional bilinear mapping. This new type of duality is termed the D-*induced duality* in the paper. Basic properties of the extended duality, including the extended bi-polar theorem, are proven. Examples are given to show the applications of the new results.

Keywords: Convex cones, duality, bi-polar theorem, conic optimization.

AMS subject classification: 46A20, 46N10, 90C25.

*Econometric Institute, Erasmus University, The Netherlands.

†Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, Hong Kong. Research supported by the Foundation 'Vereniging Trustfonds Erasmus Universiteit Rotterdam' in The Netherlands.

1 Introduction

Duality plays a central role in the development of the theory as well as the solution methods for optimization. A good example is the success of the so-called primal-dual interior point methods for conic convex optimization; see, e.g., [4].

A dual object, say, the dual of a convex cone, is defined as the set of all nonnegative linear mappings (functionals) over the cone under consideration. This set itself forms a convex cone, which has a lot of intimate and interesting relationships with the original cone. Duality theory is devoted to reveal the nature of the relationship, and beyond any doubt it has become the foundation of optimization. However, there are circumstances where the concept of nonnegativity needs to be extended in order to better suit some new and interesting applications, arising, e.g., from robust analysis of conic optimization. In this paper we introduce a new type of duality for convex cones, where the nonnegativity is induced by an arbitrary given convex cone, which is obviously a generalization of the usual definition of the dual. This usual definition is the special case that the given convex cone is the nonnegative half-line $\Re_+$. We show that under some conditions, important results such as the bi-polar theorem can be carried over. A key issue is the characterization of the new kind of dual cone, termed as the D-*induced dual cone* in this paper. It is linked naturally to other important issues in convex analysis. For instance, it raises questions such as how to compute the tensor product of two convex cones, and what is the calculus rule for the duality operation (in the ordinary sense) for the tensor product of two convex cones. We believe that this triggers interesting research questions to be answered in the future.

This paper is organized as follows. We shall introduce the new type of duality in Section 2. In the same section we prove some properties of the new duality operation, including the bi-polar theorem and several calculus rules of the new duality operation. The discussion is followed in Section 3 by two applications, one from the robust version of conic convex optimization, and the other from multiple objective conic convex optimization. Then, in Section 4 we continue to discuss how the new type of duality can be characterized and computed. Finally, we conclude the paper in Section 5.

Notations: In most places, letters in calligraphic style, e.g. $\mathcal{X}$, denote vector spaces; $\Re^n$ is n-dimensional Euclidean space; $\Re^n_+$ is the set of all n-dimensional non-negative vectors; $\mathcal{S}^m$ is the space of all m by m symmetric real-valued matrices; $\|\cdot\|$ is the Euclidean norm with appropriate dimension from the context; cl (S) stands for the closure of the set S; conv (S) stands for the convex hull of the set S; epi (f) stands for the epigraph of the function f; SOC(n) is the standard n-dimensional second order cone, i.e., $\mathrm{SOC}(n) = \left\{ [x_1, x_2, \cdots, x_n]^T \mid x_1 \ge \sqrt{\sum_{i=2}^n x_i^2} \right\}$. Finally, vec (A) stands for the vector obtained by stacking together the columns of the matrix A, i.e., vec $(A) = [a_1^T, a_2^T, \cdots, a_m^T]^T$ where $[a_1, a_2, \cdots, a_m] = A$.

2 The D-induced duality

Consider three vector spaces $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{W}$.

Let $\mathsf{D} \subseteq \mathcal{W}$ be a certain fixed convex cone. We assume that D is not a linear subspace; this is equivalent to demanding that there is $d \in \mathsf{D}$ such that $-d \notin \mathsf{D}$. This will be called a *non-flat* cone. Due to the convexity of D, by a separation argument this condition further implies the existence of $d \in \mathsf{D}$ such that $-d \notin \text{cl}\ \mathsf{D}$.

Let

$$\langle x, y \rangle : \ (x, y) \in \mathcal{X} \times \mathcal{Y} \to \mathcal{W}$$

be a given bilinear mapping, i.e., for any fixed $x \in \mathcal{X}$, $\langle x, \cdot \rangle : \ \mathcal{Y} \to \mathcal{W}$ is a linear mapping, and for any fixed $y \in \mathcal{Y}$, $\langle \cdot, y \rangle : \ \mathcal{X} \to \mathcal{W}$ is a linear mapping as well.

In this paper, we assume throughout, for the sake of simplicity, that all vector spaces under consideration are finite dimensional, although some of the results can be easily extended to a more general setting. Moreover we choose for each vector space an inner product: we denote the chosen inner product on $\mathcal{X}$ by $\langle \cdot, \cdot \rangle_{\mathcal{X}}$; similarly for $\mathcal{Y}$ and $\mathcal{W}$. In particular, we assume after suitable choices of coordinates that $\mathcal{X} = \Re^n$, $\mathcal{Y} = \Re^m$, and $\mathcal{W} = \Re^k$ and that the inner products are the usual inner product, viz. the sum of coordinate-wise products, e.g. $\langle x, y \rangle_{\mathcal{X}} = x^T y$ for $x, y \in \mathcal{X}$. In our applications we consider often spaces of matrices; here the choice of coordinates means that we stack the matrices into column-vectors as described before.

Let $U \subseteq \mathcal{X}$ be a cone.

The dual of U as induced by D under $\langle \cdot, \cdot \rangle$ is a convex cone in $\mathcal{Y}$, defined by

$$\begin{aligned} U^*_{\mathsf{D}} &= \{ y \in \mathcal{Y} \mid \langle x, y \rangle \in \mathsf{D} \text{ for all } x \in U \} \\ &= \{ y \in \mathcal{Y} \mid \langle U, y \rangle \subseteq \mathsf{D} \}. \end{aligned}$$

In other words, the D-induced dual cone of U is the collection of all linear mappings that take U to D under $\langle x, \cdot \rangle$. Obviously, U^*_{D} is always a closed convex cone provided that D is closed. However, in general D does not have to be closed.

Symmetrically, for a cone V in the space $\mathcal{Y}$, its D-induced dual under the bilinear mapping $\langle\cdot,\cdot\rangle$ is a convex cone in $\mathcal{X}$, defined by

$$\begin{aligned} V^*_{\mathsf{D}} &= \{x \in \mathcal{X} \mid \langle x, y\rangle \in \mathsf{D} \text{ for all } y \in V\} \\ &= \{x \in \mathcal{X} \mid \langle x, V\rangle \subseteq \mathsf{D}\}. \end{aligned}$$

Therefore, when we speak of the D-induced dual, it is of importance to specify the space in which the cone in question resides.

It is also evident that there are two key factors in this definition, namely the order-defining cone D and the bilinear mapping $\langle\cdot,\cdot\rangle$. We recall that the dual cone in the ordinary sense of a cone $U \subseteq \mathcal{X}$ is defined to be the set of all $x \in \mathcal{X}$ such that $\langle x, u\rangle_{\mathcal{X}} \geq 0$, $\forall u \in U$. As each linear function on $\mathcal{X}$ can be written as $x \to \langle a, x\rangle_{\mathcal{X}}$ for a unique $a \in \mathcal{X}$, the dual cone in the ordinary sense is $\Re_+$-induced with the usual inner product $\langle x, y\rangle_{\mathcal{X}} = x^T y$ as the underlying bilinear mapping. Due to the symmetric form of this bilinear mapping, the usual dual cone need not to be further specified as whether it is in the space $\mathcal{X}$ or in the space $\mathcal{Y}$.

In general, using an appropriate coordinate system, any finite-dimensional bilinear mapping can be specified as

$$\langle x, y\rangle = \begin{bmatrix} x^T A_1 y \\ \vdots \\ x^T A_k y \end{bmatrix},$$

where $A_i \in \Re^{n\times m}$, $i = 1, ..., k$.

We note that since a cone contains the origin, any D-induced dual cone must as well contain the origin; thus it is non-empty. Moreover its closure is identical to the (cl D)-induced counter-part. This is formalized in the following proposition.

Proposition 2.1 *Let* D *be a convex cone. Let* $U \subseteq \mathcal{X}$. *It holds that* $\text{cl } U^*_{\mathsf{D}} = U^*_{\text{cl } \mathsf{D}}$.

Proof. It is obvious that $U^*_{\mathsf{D}} \subseteq U^*_{\text{cl } \mathsf{D}}$. Taking closure on both sides yields $\text{cl } U^*_{\mathsf{D}} \subseteq U^*_{\text{cl } \mathsf{D}}$.

Note that $U^*_{\mathsf{D}} \neq \emptyset$. Take an arbitrary $y \in U^*_{\text{cl } \mathsf{D}}$. We have $\langle U, y\rangle \subseteq \text{cl } \mathsf{D}$. Suppose by contradiction that $y \notin \text{cl } U^*_{\mathsf{D}}$. Let $\hat{y}$ be the projection of y on $\text{cl } U^*_{\mathsf{D}}$, and $\|y - \hat{y}\| = \delta > 0$. Due to the first part of the proof, we know that $\hat{y} \in U^*_{\text{cl } \mathsf{D}}$; that is, $\langle U, \hat{y}\rangle \subseteq \text{cl } \mathsf{D}$. We claim that $(y + \hat{y})/2 \in U^*_{\text{cl } \mathsf{D}}$. Let us check this. Choose an infinite sequence $\{y_n \mid n = 1, 2, ...\}$ in U^*_{D} which tends to $\hat{y}$. Then the sequence $\{\frac{y_n + y}{2} \mid n = 1, 2, ...\}$ is contained in U^*_{D} by the convexity of this set and it tends to $\frac{\hat{y}+y}{2}$. This proves the inclusion $(y+\hat{y})/2 \in U^*_{\mathrm{cl}\ \mathsf{D}}$. Moreover, $\|y-(y+\hat{y})/2)\| = \delta/2$, contradicting the fact that $\hat{y}$ is the projection of y onto cl U^*_{D}. Thus we must have $y \in \mathrm{cl}\ U^*_{\mathsf{D}}$. Hence, $U^*_{\mathrm{cl}\ \mathsf{D}} \subseteq \mathrm{cl}\ U^*_{\mathsf{D}}$. The proposition is proven.

Q.E.D.

A key result concerning the D-induced duality is the extended bi-polar theorem. Before presenting this result, let us first introduce the following notion of surjectivity.

Definition 2.2 *Consider the bilinear mapping*

$$\langle x, y \rangle := \begin{bmatrix} x^T A_1 y \\ \vdots \\ x^T A_k y \end{bmatrix} :\ (x,y) \in \mathcal{X} \times \mathcal{Y} \to \mathcal{W}.$$

We call $\langle \cdot, \cdot \rangle$ dual surjective with respect to D *if for any $a \in \mathcal{X}$ there is a non-flat direction $b \in \mathsf{D}$ ($-b \notin \mathrm{cl}\ \mathsf{D}$) and an element $y \in \mathcal{Y}$ such that the linear equation*

$$[A_1 y, \cdots, A_k y] = ab^T$$

is satisfied. This concept does not depend on the choice of coordinates as can be seen from the following coordinate-free description: $\langle \cdot, \cdot \rangle$ is dual surjective with respect to D *if and only if*

$$\forall a \in \mathcal{X},\ \exists \text{ non-flat } b \in \mathsf{D},\ \exists y \in Y,\ \text{ such that } \langle x, y \rangle = \langle x, a \rangle_{\mathcal{X}} b \text{ for all } x \in \mathcal{X}.$$

Similarly, we call $\langle \cdot, \cdot \rangle$ primal surjective with respect to D *if for any $d \in \mathcal{Y}$ there is a non-flat direction $c \in \mathsf{D}$ ($-c \notin \mathrm{cl}\ \mathsf{D}$) and $x \in \mathcal{X}$ such that the linear equation*

$$\begin{bmatrix} x^T A_1 \\ \vdots \\ x^T A_k \end{bmatrix} = cd^T$$

is satisfied.

We remark here that the coordinate-free description is handy for the purpose of checking the condition in many applications.

Now we are in a position to state the following extended bi-polar theorem for the D-induced duality.

Theorem 2.3 *Let the bilinear mapping*

$$\langle x, y\rangle := \begin{bmatrix} x^T A_1 y \\ \vdots \\ x^T A_k y \end{bmatrix} : (x,y) \in \mathcal{X} \times \mathcal{Y} \to \mathcal{W}$$

be fixed. Let $\mathsf{D} \subseteq \mathcal{W}$ *be a given non-linear convex cone. Suppose that* $\langle\cdot,\cdot\rangle$ *is dual surjective with respect to* D*. Let* $U \subseteq \mathcal{X}$ *be a convex cone. Then it holds that*

$$\text{cl } U^{**}_{\mathsf{DD}} = \text{cl } U.$$

Proof. First we prove $U \subseteq U^{**}_{\mathsf{DD}}$.

Take any $x \in U$. Then by definition, $\langle x, y\rangle \in \mathsf{D}$ for all $y \in U^*_{\mathsf{D}}$. Hence, $x \in U^{**}_{\mathsf{DD}}$, and so it follows that $U \subseteq U^{**}_{\mathsf{DD}}$. Consequently, it follows that cl $U \subseteq$ cl U^{**}_{DD}.

Next we shall prove $U^{**}_{\mathsf{DD}} \subseteq$ cl U.

Take any $\hat{x} \in U^{**}_{\mathsf{DD}}$. Thus $\langle \hat{x}, y\rangle \in \mathsf{D}$ for all $y \in U^*_{\mathsf{D}}$.

Let $L(y) := [A_1 y, \cdots, A_k y]$. Observe that

$$\begin{aligned} y \in U^*_{\text{cl } \mathsf{D}} &\iff \langle u, y\rangle \in \text{cl } \mathsf{D} \text{ for all } u \in U \\ &\iff [u^T A_1 y, \cdots, u^T A_k y]v \geq 0 \text{ for all } u \in U \text{ and } v \in \mathsf{D}^* \\ &\iff u^T L(y) v \geq 0 \text{ for all } u \in U \text{ and } v \in \mathsf{D}^*. \end{aligned}$$

In other words,

$$U^*_{\text{cl } \mathsf{D}} = \{y \mid u^T L(y) v \geq 0, \text{ for all } u \in U, v \in \mathsf{D}^*\}. \tag{1}$$

Suppose by contradiction that $\hat{x} \notin$ cl U. Then by the separation theorem, there exists $a \in U^*$ such that $a^T \hat{x} < 0$. By the dual surjectivity of the bilinear mapping, we can find $\hat{y}$ such that

$$L(\hat{y}) = ab^T$$

where $b \in \mathsf{D}$ is a non-flat direction. Hence

$$u^T L(\hat{y}) v = (u^T a)(b^T v) \geq 0$$

for all $u \in U$ and $v \in \mathsf{D}^*$ as $a \in U^*$ and $b \in \mathsf{D}$. Consequently, $\hat{y} \in U^*_{\text{cl } \mathsf{D}}$. This leads us to the following contradiction. On the one hand, $\langle \hat{x}, \hat{y}\rangle \in$ cl D due to the fact that $\hat{x} \in U^{**}_{\mathsf{DD}}$ and $\hat{y} \in U^*_{\text{cl } \mathsf{D}} =$ cl U^*_{D}, where we used Proposition 2.1. On the other hand,

$$\langle \hat{x}, \hat{y}\rangle = (\hat{x}^T ab^T)^T = (a^T \hat{x}) b \notin \text{cl } \mathsf{D},$$

due to the fact that $a^T\hat{x} < 0$ and b is a non-flat direction for D. This proves $U^{**}_{\mathsf{DD}} \subseteq \mathrm{cl}\, U$. The desired result follows by taking closure on both sides.

Q.E.D.

In the same vein, we have an analogue for the dual space.

Theorem 2.4 *Let the bilinear mapping*

$$\langle x, y\rangle := \begin{bmatrix} x^T A_1 y \\ \vdots \\ x^T A_k y \end{bmatrix} : (x, y) \in \mathcal{X} \times \mathcal{Y} \to \mathcal{W}$$

be fixed. Let $\mathsf{D} \subseteq \mathcal{W}$ *be a given non-flat convex cone. Suppose that* $\langle \cdot, \cdot \rangle$ *is primal surjective with respect to* D*. Let* $V \subseteq \mathcal{Y}$ *be a convex cone. Then it holds that*

$$\mathrm{cl}\, V^{**}_{\mathsf{DD}} = \mathrm{cl}\, V.$$

As a matter of notation, let us introduce the tensor product of two convex cones C and D as follows

$$C \otimes D = \mathrm{conv}\, \{uv^T \mid u \in C,\, v \in D\}. \tag{2}$$

We call its dual to be the *bi-positive* cone, denoted by

$$\mathcal{B}(C, D) = \{Z \mid u^T Z v \geq 0 \text{ for all } u \in C,\, v \in D\}. \tag{3}$$

Indeed it is elementary to see that

$$(C \otimes D)^* = \mathcal{B}(C, D). \tag{4}$$

A proof for the above equation and other related equations can also be found in [3].

In the proof for Theorem 2.3 we in fact established the following relation; see (1). Let us formalize it as follows, now using the notion of the bi-positive cone.

Proposition 2.5 *Let* $U \subseteq \mathcal{X}$ *be a cone. Then it holds that*

$$U^*_{\mathrm{cl}\,\mathsf{D}} = \{y \mid L(y) \in \mathcal{B}(U, \mathsf{D}^*)\}.$$

Thus the following result is straightforward.

Proposition 2.6 *Consider convex cones* $U_1, \cdots, U_r \subseteq \mathcal{X}$*. It holds that*

$$(U_1 + \cdots + U_r)^*_{\mathrm{cl}\ \mathsf{D}} = \bigcap_{i=1}^{r} (U_i)^*_{\mathrm{cl}\ \mathsf{D}}.$$

Proof. According to Proposition 2.5, we have

$$\begin{aligned}(U_1 + \cdots + U_r)^*_{\mathrm{cl}\ \mathsf{D}} &= \{y \mid L(y) \in \mathcal{B}(U_1 + \cdots + U_r, \mathsf{D}^*)\} \\ &= \{y \mid L(y) \in \bigcap_{i=1}^{r} \mathcal{B}(U_i, \mathsf{D}^*)\} \\ &= \bigcap_{i=1}^{r} (U_i)^*_{\mathrm{cl}\ \mathsf{D}}.\end{aligned}$$

Q.E.D.

Unlike in the usual duality case, the primal and dual status of the D-induced duality is not symmetric in general. For instance, in case that the bilinear mapping is dual surjective, then the 'dual space' $\mathcal{Y}$ is bigger in some sense. Therefore, it can happen that not all convex cones in $\mathcal{Y}$ can be expressed as the dual of some cone in $\mathcal{X}$. However, if the bilinear mapping is both primal and dual surjective, then one may apply the bi-polar theorem on both sides. As a consequence, the following calculus result follows.

Corollary 2.7 *Suppose that* $\langle \cdot, \cdot \rangle$ *is both primal and dual surjective with respect to* D *and that* D *is closed. Let* $U_1, \cdots, U_r$ *be arbitrary convex cones in* $\mathcal{X}$*. Then it holds that*

$$\mathrm{cl}\ \left(\bigcap_{i=1}^{r} \mathrm{cl}\ U_i \right)^*_{\mathsf{D}} = \mathrm{cl}\ \left((U_1)^*_{\mathsf{D}} + \cdots + (U_r)^*_{\mathsf{D}} \right).$$

Proof. The desired result follows immediately if we replace U_i in Proposition 2.6 by $(U_i)^*_{\mathsf{D}}$, $i = 1, ..., r$, and then, on both sides of the resulting identity, take closure and apply Theorem 2.3 from the dual side.

Q.E.D.

It is in fact quite rare that both the primal and the dual surjectivity conditions are satisfied at the same time. This essentially means that we are dealing with the ordinary duality with the usual inner product and $\mathsf{D} = \Re_+$.

Other simple calculus rules for the cone-induced duality are presented below.

Proposition 2.8 *Let U be a convex cone in $\mathcal{X}$.*

(i) *For any convex cones $\mathsf{D}_1, ..., \mathsf{D}_s$ in $\mathcal{W}$, it holds that*

$$U^*_{\mathsf{D}_1\cap\cdots\cap\mathsf{D}_s} = U^*_{\mathsf{D}_1} \cap \cdots \cap U^*_{\mathsf{D}_s}$$

and,

(ii)

$$U^*_{\mathsf{D}_1+\cdots+\mathsf{D}_s} \supseteq U^*_{\mathsf{D}_1} + \cdots + U^*_{\mathsf{D}_s}.$$

(iii) *Suppose that the bilinear mapping is decomposed as*

$$\langle x, y\rangle = \begin{bmatrix} \langle x, y\rangle_1 \\ \vdots \\ \langle x, y\rangle_p \end{bmatrix},$$

where $\langle x, y\rangle_i$ is a bilinear mapping from $\mathcal{X} \times \mathcal{Y}$ to $\mathcal{W}_{k_i}$ where $\mathcal{W}_{k_i} = \Re^{k_i}$ with $\sum_{i=1}^{p} = k$. Furthermore, suppose that convex cones $\mathsf{D}_i \subseteq \mathcal{W}_{k_i}$ are given, $i = 1, ..., p$. Then it holds that

$$U^*_{\mathsf{D}_1\times\cdots\times\mathsf{D}_p} = U^*_{\mathsf{D}_1} \cap \cdots \cap U^*_{\mathsf{D}_p}$$

where the set product is defined as

$$\mathsf{D}_1 \times \cdots \times \mathsf{D}_p = \{(z_1, \cdots, z_p) \mid z_i \in \mathsf{D}_i,\ i = 1, ..., p\}.$$

Proof. To show **(i)** we note that

$$\begin{aligned} y \in U^*_{\mathsf{D}_1\cap\cdots\cap\mathsf{D}_s} &\iff \langle U, y\rangle \in \mathsf{D}_i,\ i = 1, ..., s \\ &\iff y \in U^*_{\mathsf{D}_i},\ i = 1, ..., s \\ &\iff y \in U^*_{\mathsf{D}_1} \cap \cdots \cap U^*_{\mathsf{D}_s}. \end{aligned}$$

For proving **(ii)** we note that if $y \in U^*_{\mathsf{D}_1} + \cdots + U^*_{\mathsf{D}_s}$ then there exist $y_1, \cdots, y_s$ such that $y = y_1 + \cdots + y_s$, and $y_i \in U^*_{\mathsf{D}_i}$, $i = 1, ..., s$. This implies that $\langle U, y_i\rangle \subseteq \mathsf{D}_i$, $i = 1, ..., s$, and so

$$\langle U, y\rangle \subseteq \langle U, y_1\rangle + \cdots + \langle U, y_s\rangle \subseteq \mathsf{D}_1 + \cdots + \mathsf{D}_s.$$

Thus, $U^*_{\mathsf{D}_1+\cdots+\mathsf{D}_s} \supseteq U^*_{\mathsf{D}_1} + \cdots + U^*_{\mathsf{D}_s}$.

Note that the above inclusion is strict in general. For more discussions on this, see Section 4.

Now we prove **(iii)**. Similarly as in **(i)**,

$$\begin{aligned} y \in U^*_{\mathsf{D}_1\times\cdots\times\mathsf{D}_p} &\iff \langle U, y\rangle_i \in \mathsf{D}_i,\ i=1,...,p \\ &\iff y \in U^*_{\mathsf{D}_i},\ i=1,...,p \\ &\iff y \in U^*_{\mathsf{D}_1} \cap \cdots \cap U^*_{\mathsf{D}_p}. \end{aligned}$$

Remark here that the statements in **(i)** and **(iii)** are different: in **(i)**, the D_i's are all contained in $\Re^k$, while in **(iii)**, D_i is in $\Re^{k_i}$, $i=1,...,p$.

Q.E.D.

By **(iii)** of Proposition 2.8 it is clear that one needs only to concentrate on the case that the underlying cone D is non-decomposable, for the duality consideration.

Let us see how the D-induced duality exactly works.

Example 1. Consider

$$\mathcal{X} = \mathcal{S}^n,\ \mathcal{Y} = \left\{ \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12} & Y_{22} \end{bmatrix} \middle| \ Y_{12} \in \mathcal{S}^n \right\},\ \mathcal{W} = \mathcal{S}^2.$$

The bilinear mapping is

$$\langle X, Y\rangle = \begin{bmatrix} X \bullet Y_{11} & X \bullet Y_{12} \\ X \bullet Y_{12} & X \bullet Y_{22} \end{bmatrix}$$

and $\mathsf{D} = \mathcal{S}^2_+$, where $\bullet$ denotes the usual entry-wise inner product for matrices.

In this case, the bilinear mapping is dual surjective.

To see this, let us fix a non-flat direction of D, e.g. $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then, for any $A \in \mathcal{X}$ the choice $Y = \begin{bmatrix} A & A \\ A & A \end{bmatrix}$ satisfies the dual surjective requirement: $\langle X, Y\rangle = (X \bullet A)B$ for all $X \in \mathcal{X}$.

By a result in Luo, Sturm and Zhang [3] we know that

$$(\mathcal{S}^n_+)^*_{\mathcal{S}^2_+} = \left\{ \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22} \end{bmatrix} \succeq 0 \ \middle| \ Z_{12} \in \mathcal{S}^n \right\}.$$

The extended bi-polar theorem, Theorem 2.3, thus asserts that

$$\left\{ \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22} \end{bmatrix} \succeq 0 \ \middle| \ Z_{12} \in \mathcal{S}^n \right\}^*_{\mathcal{S}^2_+} = \mathcal{S}^n_+.$$

Example 2. Consider $\mathsf{D} = \mathrm{SOC}(3)$, $\mathcal{X} = \Re^n$, $\mathcal{Y} = \Re^m$, $m = 3n$, and

$$\langle x, y \rangle = \begin{bmatrix} \sum_{i=1}^n x_i y_i \\ \sum_{i=1}^n x_i y_{n+i} \\ \sum_{i=1}^n x_i y_{2n+i} \end{bmatrix}.$$

In this case

$$L(y) = [A_1 y, A_2 y, A_3 y]$$

with $A_1 = [I, 0, 0]$, $A_2 = [0, I, 0]$, $A_3 = [0, 0, I]$. Obviously, $b = [1, 0, 0]^T$ is a non-flat direction for SOC(3). Moreover, for any $a \in \Re^n$ we can let $y = [a^T, 0^T, 0^T]^T \in \Re^m$ to yield $L(y) = ab^T$. This verifies that the dual surjectivity condition is satisfied.

Now let us consider the dual of SOC(n) in the SOC(3)-induced sense. This amounts to consider all $y^T = [y_1^T, y_2^T, y_3^T]$, where $y_i \in \Re^n$, $i = 1, 2, 3$, such that

$$x^T y_1 \geq \sqrt{(x^T y_2)^2 + (x^T y_3)^2}$$

for all $x \in \mathrm{SOC}(n)$. Using the S-procedure result (see e.g. [5]), we obtain that

$$\mathrm{SOC}(n)^*_{\mathsf{D}} = \left\{ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \middle| \; y_1 \in \mathrm{SOC}(n),\ \exists \lambda \geq 0 \text{ such that } y_1 y_1^T - y_2 y_2^T - y_3 y_3^T - \lambda \begin{bmatrix} 1 & 0 \\ 0 & -I \end{bmatrix} \succeq 0 \right\}$$

where $\mathsf{D} = \mathrm{SOC}(3)$.

Although the D-induced dual cone can always be defined, it may be difficult to compute the dual cone explicitly. This situation is different from the usual duality theory. By a famous result of Grötschel, Lovász, and Schrijver [2], if there is a polynomial-time procedure to check the membership for the primal convex cone, then there is also a polynomial-time procedure to check the membership for the dual cone. In the case of the D-induced duality, however, it may happen that both U and D are simple convex cones, but the membership check for U^*_{D} remains a hard task. A theorem leading to this statement is the following result due to Nemirovski and Ben-Tal; see [1]:

Proposition 2.9 *For given symmetric matrices $A_i \in \mathcal{S}^m$, the following decision problem is co-NP complete*

$$A_0 + \sum_{i=1}^n x_i A_i \succeq 0$$

for all $x \in \Re^n$ with $\|x\| \leq 1$.

In the terminology of D-induced duality, this implies the following. Let us consider $\mathcal{X} = \Re^{n+1}$, $\mathcal{Y} = (\mathcal{S}^m)^{n+1}$ (the Cartesian product of $n+1$ copies of the space of $m \times m$ symmetric matrices), and $\mathcal{W} = \mathcal{S}^m$. For $x \in \mathcal{X}$ and $y = (Y_1, ..., Y_{n+1}) \in \mathcal{Y}$, let the bilinear product be defined as

$$\langle x, y \rangle = \sum_{i=1}^{n+1} x_i Y_i.$$

Let $D = \mathcal{S}_+^m$. Then Proposition 2.9 asserts that it is NP-hard to check the membership for the cone U_{D}^* where $U = \mathrm{SOC}(n+1) \subseteq \mathcal{X}$.

3 Applications

To appreciate how the D-duality helps to model and solve optimization problems, we shall first discuss two examples of application in this section, before moving on to discuss more theoretical properties of the D-induced dual cones.

3.1 Robust optimization

The notion of robust conic optimization was studied by Ben Tal and Nemirovski in [1]. Let us now formulate the problem using the newly introduced notion of D-duality. Consider a general conic optimization problem as follows

$$\begin{array}{lll} (P) & \text{minimize} & c^T x \\ & \text{subject to} & Ax + b \in \mathcal{K} \end{array}$$

or equivalently,

$$\begin{array}{lll} (P) & \text{minimize} & c^T x \\ & \text{subject to} & Ax + bx_0 \in \mathcal{K} \\ & & x_0 = 1, \end{array}$$

where $\mathcal{K}$ is a given closed convex cone. For notational convenience, let $\bar{x} = \begin{bmatrix} x_0 \\ x \end{bmatrix}$. In practice, the cone $\mathcal{K}$ can be the first orthant (linear programming), or the product of second order cones (second order cone programming), or the cone of positive semidefinite matrices (semidefinite programming). In [6], it is shown that a standard convex program can also be formulated in the above form in a natural way.

The issue of robust optimization arises when the data of the problem, A and b in the above formulation, are uncertain. In other words, they are perturbed by noises, or 'polluted'. Note that the objective c vector can always be assumed to be certain. This is achieved by reformulating the problem as follows. We introduce one additional variable, x_{n+1}, and one additional constraint, $c^T x - x_{n+1} = 0$, the cone is changed to $\mathcal{K} \times \Re$, and the objective is changed to minimize x_{n+1}.

In many applications, it is crucial to ensure the robustness of the decision x. One way to do this is to guarantee that x should remain feasible for all possible data within an 'uncertainty set'. Let $[\tilde{A}, \tilde{b}]$ be the estimated nominal value of the data. Let us consider the uncertainty set as measured by the 'unit ball' using a given norm $\|\cdot\|_u$ centered around $[\tilde{A}, \tilde{b}]$, i.e.

$$U = \{[A, b] \mid \|[A, b] - [\tilde{A}, \tilde{b}]\|_u \leq 1\}.$$

Note that the norm need not be Euclidean. In fact it can be any suitable gauge. However, for convenience we may choose to use the Euclidean norm if applicable.

Let $\delta A := A - \tilde{A}$ and $\delta b := b - \tilde{b}$. We introduce a homogenizing variable t, and let

$$\mathcal{C}(U) := \left\{ \left[\begin{array}{c} t \\ \text{vec}\ (\delta A) \\ \delta b \end{array} \right] \middle| \ t \geq \left\| \left[\begin{array}{c} \text{vec}\ (\delta A) \\ \delta b \end{array} \right] \right\|_u \right\}.$$

Now it is easy to see that $Ax + bx_0 \in \mathcal{K}$ for all $[A, b] \in U$ if and only if

$$t(\tilde{A}x + \tilde{b}x_0) + (\delta A)x + (\delta b)x_0 \in \mathcal{K}$$

for all $\left[\begin{array}{c} t \\ \text{vec}\ (\delta A) \\ \delta b \end{array} \right] \in \mathcal{C}(U)$.

Consider the bilinear mapping defined by

$$\left\langle \left[\begin{array}{c} t \\ \text{vec}\ (\delta A) \\ \delta b \end{array} \right], \left[\begin{array}{c} x_0 \\ x \end{array} \right] \right\rangle := t(\tilde{A}x + \tilde{b}x_0) + (\delta A)x + (\delta b)x_0.$$

Thus, the robust version of (P) is

$$\begin{array}{lll} (RP) & \text{minimize} & \bar{c}^T \bar{x} \\ & \text{subject to} & \bar{x} \in \mathcal{C}(U)^*_{\mathcal{K}} \\ & & x_0 = 1, \end{array}$$

where $\bar{c}^T = [0, c^T]$. Obviously, (RP) is also a conic optimization problem itself. Now it is clear that the crux is the ability to characterize the cone $\mathcal{C}(U)^*_{\mathcal{K}}$ in a tangible way.

If $\mathcal{K} = \Re^n_+$, and $\|\cdot\|_u$ is Euclidean, then $\mathcal{C}(U)^*_{\mathcal{K}}$ is the product of second order cones.

3.2 Multiple objective conic programming

Let us consider the following ordering relation based on a convex cone $\mathsf{D} \subseteq \Re^k$:

$$x \succeq_{\mathsf{D}} y \text{ if and only if } x - y \in \mathsf{D}.$$

The multiple objective convex conic program is now given as

$$\begin{array}{lll} (P)_{\mathsf{D}} & \min_{\mathsf{D}} & Cx \\ & \text{s.t.} & Ax = b \\ & & x \in \mathcal{K} \end{array}$$

where $C \in \Re^{k\times n}$, $A \in \Re^{m\times n}$, $b \in \Re^m$ and $\mathcal{K} \subseteq \Re^n$ is a convex cone. The derivation of its dual can be done in a similar way as for usual linear programming. We wish to establish a lower bound for the objective in $(P)_{\mathsf{D}}$. Let Y be a linear mapping $\Re^m \to \Re^k$ to be applied on both sides of $Ax = b$, leading to $YAx = Yb$. In order for YAx to be a lower bound for the objective vector, we need to have $(C - YA)x \in \mathsf{D}$ for all primal feasible x.

Let us consider the bilinear mapping defined as

$$\langle x, S\rangle = Sx \in \Re^k.$$

Clearly, the above bilinear mapping is dual surjective, because $L(S) = S^T$ and so the equation $L(S) = ab^T$ is always satisfiable.

In the notion of the D-induced duality, the condition $(C - YA)x \in \mathsf{D}$ for all $x \in \mathcal{K}$ is simply $C - YA \in \mathcal{K}^*_{\mathsf{D}}$. Now we wish to optimize over all the bounds obtained this way. This leads to a dual problem given as follows:

$$\begin{array}{lll} (D)_{\mathsf{D}} & \max_{\mathsf{D}} & Yb \\ & \text{s.t.} & YA + S = C \\ & & S \in \mathcal{K}^*_{\mathsf{D}}. \end{array}$$

By this construction we naturally have the following weak duality theorem.

Theorem 3.1 *If x is a feasible solution for $(P)_{\mathsf{D}}$ and (Y, S) is a feasible solution for $(D)_{\mathsf{D}}$, then it holds that*

$$Cx \succeq_{\mathsf{D}} Yb.$$

By the extended bipolar theorem, if D and $\mathcal{K}$ are closed, then the dual of $(D)_{\mathsf{D}}$ is precisely $(P)_{\mathsf{D}}$.

It is well known that, if the ordering '$\succeq_{\mathsf{D}}$' is incomplete, there might be multiple, incomparable, optimal solutions from either the primal or dual point of view, known as the *Pareto* optimal solutions. If we insist on the strong duality (complementarity), then the optimality can be defined as follows.

Definition 3.2 *We call x^* and (Y^*, S^*) complementary optimal solutions for $(P)_{\mathsf{D}}$ and $(D)_{\mathsf{D}}$ respectively if they are feasible and*

$$Cx^* = Y^* b$$

or, equivalently,

$$S^* x^* = 0 \in \mathsf{D}.$$

Obviously, the complementary optimality is a stringent condition. Consider the following simple multiple objective linear program

$$(P_1)_{\mathsf{D}} \quad \min_{\mathsf{D}} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\text{s.t.} \qquad x_1 + x_2 = 1$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \Re^2_+.$$

Let us first consider the case where $\mathsf{D} = \Re^2_+$. Due to the binding relation, no feasible solution of $(P_1)_{\mathsf{D}}$ can be dominated by any other feasible solutions. Therefore, they are all Pareto optimal. However, from the complementary optimality point of view, no optimal solution exists. To see this, consider its dual problem

$$(D_1)_{\mathsf{D}} \quad \max_{\mathsf{D}} \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\text{s.t.} \qquad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} [1,1] + S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S \in (\Re^2_+)^*_{\mathsf{D}},$$

with $\mathsf{D} = \Re^2_+$.

It is easy to see that $(\Re^2_+)^*_{\mathsf{D}} = \Re^{2\times 2}_+$. In particular, if we let $S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}$, then we have $s_{11} > s_{12} \geq 0$ and $s_{22} > s_{21} \geq 0$ due to the dual feasibility condition. This implies that the equation

$Sx = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ cannot have a solution among feasible x and S, i.e., no complementary optimal solution exists.

For the problem instance $(P_1)_{\mathsf{D}}$, let us now consider the conic ordering induced by the lexicographic ordering of the coordinates. In this case, since the lexicographic ordering is complete and the feasible set of $(P_1)_{\mathsf{D}}$ is compact, an optimal solution for $(P_1)_{\mathsf{D}}$ exists. In particular, the optimal solution for $(P_1)_{\mathsf{D}}$ is obviously $[x_1^*, x_2^*] = [0, 1]$. Furthermore, we shall see below that in fact a pair of primal-dual complementary optimal solutions exists.

In the two-dimensional case, the lexicographic ordering on (x_1, x_2) corresponds to the conic ordering which is defined by the cone

$$\mathsf{D} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \,\middle|\, x_1 > 0 \right\} \bigcup \left(\Re_+ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \tag{5}$$

which is a non-flat and non-closed convex cone.

Let us compute the cone $(\Re^2_+)^*_{\mathsf{D}}$. Take an arbitrary $S \in (\Re^2_+)^*_{\mathsf{D}}$, that is $Sx \in \mathsf{D}$ for all $x \in \Re^2_+$. In other words, $s_{11}x_1 + s_{12}x_2 \geq 0$ for all $x_1, x_2 \geq 0$, and if $s_{11}x_1 + s_{12}x_2 = 0$ then $s_{21}x_1 + s_{22}x_2 \geq 0$. This leads to

$$(\Re^2_+)^*_{\mathsf{D}} = \left\{ \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \,\middle|\, \begin{array}{l} s_{11}, s_{12} > 0, \text{ or } s_{11} = 0, s_{12} > 0, s_{21} \geq 0, \\ \text{or } s_{11} > 0, s_{12} = 0, s_{22} \geq 0, \text{ or } s_{11} = s_{12} = 0, s_{21} \geq 0, s_{22} \geq 0 \end{array} \right\}.$$

The dual problem $(D_1)_{\mathsf{D}}$, where D is given as in (5), has an optimal solution $y_1^* = 0$, $y_2^* = 1$, and $S^* = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$. Moreover, this optimal solution is complementary to the primal optimal solution $[x_1^*, x_2^*] = [0, 1]$ as $S^*x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. A remarkable fact is that $(D_1)_{\mathsf{D}}$ does not even have a closed feasible set.

4 Further properties of the D-dual cone

In the previous section we saw some applications of the newly introduced D-induced cone. We also recognized that computing the D-induced cone can be a difficult task in general; see Proposition 2.9. Nevertheless, it is possible to further characterize the conditions under which the D-induced cone can be computed.

As Proposition 2.5 asserts, we have for D closed

$$\begin{aligned} U_{\mathsf{D}}^{*} &= \{y \mid L(y) \in \mathcal{B}(U, \mathsf{D}^{*})\} \\ &= \{y \mid L(y) \in (U \otimes \mathsf{D}^{*})^{*}\}. \end{aligned}$$

It is therefore clear that it is crucial to analyze the dual (in the usual sense) of the tensor product (or the Kronecker product in the matrix form) of two convex cones, if we wish to characterize the D-induced dual cone.

Consider two closed and solid - that is, full-dimensional - convex cones, $C \in \mathcal{X}$ and $D \in \mathcal{W}$. First of all, C and D may not be pointed cones. In that case we can decompose them in the following way

$$C = P_c + L_c \text{ and } D = P_d + L_d$$

where P_c and P_d are closed pointed convex cones, and L_c and L_d are linear subspaces.

Then,

$$C \otimes D = P_c \otimes P_d + P_c \otimes L_d + L_c \otimes P_d + L_c \otimes L_d. \tag{6}$$

Lemma 4.1 *Let $K \in \mathcal{X}$ be a solid convex cone, namely,* $\text{span } K = \mathcal{X}$, *and L be a linear space. Then, $K \otimes L$ is a linear space, in fact*

$$K \otimes L = (\text{span } K) \otimes L.$$

Proof.

Consider any $\sum_i x_i y_i^T \in K \otimes L$ with $x_i \in K$ and $y_i \in L$ for all i. Since L is a linear space, we have

$$-\sum_i x_i y_i^T = \sum_i x_i (-y_i)^T \in K \otimes L.$$

Therefore this concludes that $K \otimes L$ is a linear space, and furthermore $K \otimes L = (\text{span } K) \otimes L$.

Q.E.D.

Lemma 4.1 shows that $C \otimes D$ can be written as the sum of $P_c \otimes P_d$ and a linear subspace. Therefore, in order to analyze the tensor product, without losing generality we may assume C and D to be pointed and solid convex cones.

Now, for a closed, pointed, convex cone C in $\mathcal{X}$, it is convenient to view C, by a suitable choice of coordinates in $\mathcal{X}$, as the epigraph of a sublinear function ϕ, i.e.,

$$C \sim \left\{ \left[\begin{array}{c} \phi(u) + r \\ u \end{array} \right] \middle| \, r \in \Re_+, \, u \in \mathcal{U} \right\}$$

where '$\sim$' stands for a certain linear bijective transformation, and $\mathcal{U}$ is a vector space which has dimension equal to the dimension of $\mathcal{X}$ minus one. Remember that by definition a sublinear function $\phi(\cdot)$ is finite-valued and satisfies the following conditions:

$$\phi(tu) = t\phi(u) \text{ and } \phi(u+v) \leq \phi(u) + \phi(v)$$

for all $t \geq 0$, and $u, v \in \mathcal{U}$.

For instance, if C is a second order cone, then the corresponding sublinear function is simply the Euclidean norm $\phi(u) = \|u\|$.

Similarly, since D is also a closed pointed cone, let us assume that

$$D \sim \left\{ \left[\begin{array}{c} \psi(w) + s \\ w \end{array} \right] \middle| \, s \in \Re_+, \, w \in \mathcal{W} \right\}$$

where $\mathcal{W}$ is a vector space and ψ is a sublinear function on $\mathcal{W}$.

One may be tempted to conjecture that

$$(C \otimes D)^* = C^* \otimes D^*.$$

Unfortunately, this is not the case, although it is obvious that

$$C^* \otimes D^* \subseteq (C \otimes D)^*.$$

As an example, let us consider $C = D = \mathrm{SOC}(n+1)$. Then, by the Cauchy-Schwartz inequality one easily checks that

$$\left[\begin{array}{cc} 1 & 0 \\ 0 & -I \end{array} \right] \in (\mathrm{SOC}(n+1) \otimes \mathrm{SOC}(n+1))^*.$$

However,

$$\left[\begin{array}{cc} 1 & 0 \\ 0 & -I \end{array} \right] \not\in \mathrm{SOC}(n+1)^* \otimes \mathrm{SOC}(n+1)^* = \mathrm{SOC}(n+1) \otimes \mathrm{SOC}(n+1).$$

Note that in the argument above we have used the self-duality of the second order cone.

Let us proceed to analyzing the structure of $C \otimes D$ using their respective generating sublinear functions ϕ and ψ.

For simplicity, let us assume that the linear transformations '$\sim$' are simply the identity ones. This leads to

$$C = \text{epi } (\phi) \text{ and } D = \text{epi } (\psi).$$

Then, we may explicitly write the cone of the bi-positive mappings as

$$\mathcal{B}(C,D) = \\ \left\{ \begin{bmatrix} t & b^T \\ a & M \end{bmatrix} \middle| \begin{bmatrix} \phi(u)+r \\ u \end{bmatrix}^T \begin{bmatrix} t & b^T \\ a & M \end{bmatrix} \begin{bmatrix} \psi(w)+s \\ w \end{bmatrix} \ge 0, \forall u \in \mathcal{U}, w \in \mathcal{W}, r,s \in \Re_+ \right\}. \quad (7)$$

We now aim at a procedure to check the membership for the cone $\mathcal{B}(C,D)$.

Consider

$$\begin{bmatrix} t & b^T \\ a & M \end{bmatrix} \in \mathcal{B}(C,D).$$

By (7) we know that

$$t(\phi(u)+r)(\psi(w)+s) + (\phi(u)+r)b^T w + (\psi(w)+s)a^T u + u^T M w \ge 0 \quad (8)$$

for all $u \in \mathcal{U}$, $w \in \mathcal{W}$, $r \ge 0$ and $s \ge 0$.

It follows immediately from (8) that $t \ge 0$.

If $t = 0$ then

$$(\phi(u)+r)b^T w + (\psi(w)+s)a^T u + u^T M w \ge 0$$

for all $u \in \mathcal{U}$, $w \in \mathcal{W}$, $r \ge 0$ and $s \ge 0$. This leads to $a^T u \ge 0$ for all $u \in \mathcal{U}$, and $b^T w \ge 0$ for all $w \in \mathcal{W}$. Since $\mathcal{U}$ and $\mathcal{W}$ are vector spaces, we conclude that $a = 0$ and $b = 0$. Therefore the inequality reduces further to:

$$u^T M w \ge 0, \forall u \in \mathcal{U} \text{ and } \forall w \in \mathcal{W}.$$

Since $\mathcal{U}$ and $\mathcal{W}$ are vector spaces, we get $M = 0$.

Now we consider the situation $t > 0$. Without losing generality let us scale the value of t and assume $t = 1$. As $\mathcal{B}(C,D)$ is a cone, the scaling does not change the nature of the membership checking procedure.

We now rewrite (8) as

$$\begin{aligned} & (\phi(u)+r)(\psi(w)+s) + (\phi(u)+r)b^T w + (\psi(w)+s)a^T u + u^T M w \\ = \; & \phi(u)\psi(w) + u^T M w + \phi(u) b^T w + \psi(w) a^T u + (\phi(u) + a^T u)s + (\psi(w) + b^T w)r + rs \\ \ge \; & 0 \end{aligned} \quad (9)$$

for all $u \in \mathcal{U}$, $w \in \mathcal{W}$, $r \geq 0$ and $s \geq 0$. It is evident that (9) is equivalent to the following three conditions:

$$\phi(u) + a^T u \geq 0 \text{ for all } u \in \mathcal{U} \tag{10}$$

$$\psi(w) + b^T w \geq 0 \text{ for all } w \in \mathcal{W} \tag{11}$$

$$\phi(u)\psi(w) + u^T M w + \phi(u) b^T w + \psi(w) a^T u \geq 0 \text{ for all } u \in \mathcal{U}, w \in \mathcal{W}. \tag{12}$$

Conditions (10) and (11) are equivalent to

$$\begin{bmatrix} 1 \\ a \end{bmatrix} \in C^* \tag{13}$$

and

$$\begin{bmatrix} 1 \\ b \end{bmatrix} \in D^*. \tag{14}$$

Finally, condition (12) is equivalent to the following two statements

$$\begin{bmatrix} \psi(w) + b^T w \\ \psi(w) a + M w \end{bmatrix} \in C^* \text{ for all } w \in \mathcal{W} \tag{15}$$

$$\begin{bmatrix} \phi(u) + a^T u \\ \phi(u) b + M^T u \end{bmatrix} \in D^* \text{ for all } u \in \mathcal{U}. \tag{16}$$

We note that it may or may not be a simply task to verify conditions (15) and (16).

For instance, when C and D are second order cones, then conditions (15) and (16) can be reduced to verifying

$$1 + b^T w \geq \|Mw + a\| \text{ for all } \|w\| = 1$$

and

$$1 + a^T u \geq \|M^T u + b\| \text{ for all } \|u\| = 1.$$

Using the S-procedure result (see also Sturm and Zhang [5]), this can be achieved by checking two *Linear Matrix Inequalities*.

For general convex cones, however, as Proposition 2.9 reveals, the membership checking is a hard task.

5 Concluding remarks

In this paper we extended the definition of duality using a pre-described conic ordering relation. The new duality is shown to be useful in several applications. It also brings up interesting questions such as how to characterize the bi-positive cones. A good understanding of this subject appears to be important both for the theory and practice of optimization and for convex analysis in general.

References

[1] A. Ben-Tal, and A. Nemirovski, *Robust Convex Optimization*, Mathematics of Operations Research 23, pp. 769 – 805, 1998.

[2] M. Grötschel, L. Lovász, and A. Schrijver, Geometric Algorithms and Combinatorial Optimization, Springer-Verlag, 1993.

[3] Z.Q. Luo, J.F. Sturm, and S. Zhang, *Cones of nonnegative mappings – complexity and applications*, Working Paper, 2001.

[4] Yu. Nesterov and A. Nemirovsky, Interior point polynomial methods in convex programming, Studies in Applied Mathematics 13, SIAM, Philadelphia, PA, 1994.

[5] J.F. Sturm, and S. Zhang, *On cones of nonnegative quadratic functions*, SEEM2001-01, The Chinese University of Hong Kong, 2001.

[6] S. Zhang, *A new self-dual embedding method for convex programming*, SEEM2001-09, The Chinese University of Hong Kong, 2001.