

# Optimal social security in a dynastic model with investment externalities and endogenous fertility

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## Abstract

This paper studies optimal pay-as-you-go social security with positive bequests and endogenous fertility. With an investment externality, a competitive solution without social security suffers from under-investment in capital and over-reproduction of population. We show that social security can improve welfare by reducing fertility and increasing capital intensity. We also illustrate numerically that a small degree of this externality is enough to justify the observed high ratios of social security spending to GDP.

*Key words:* Social security; Welfare; Fertility; Bequests; Externality

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## I. Introduction

Pay-as-you-go social security (PAYG) has been practiced in many countries in the last several decades. In industrial nations, payroll tax rates for social security range from 10% to 20% or higher; see the United States Department of Health and Human Services (1999). While most studies of social security have focused on how it affects capital accumulation, much less attention has been paid to how it affects social welfare.

Existing studies on the welfare consequence of social security differ between models with or without altruistic bequests. Without such bequests, social security reduces life-cycle saving and its welfare implication hinges on some form of market failure or on the fact that the competitive solution in the life-cycle model with overlapping generations is typically second-best.<sup>1</sup> In the life-cycle model social security can improve welfare by mitigating the problem of over-accumulation of capital (e.g. Diamond, 1965), or it can emerge from a political equilibrium among different age groups (e.g. Cooley and Soares, 1999).

In a dynastic-family model, by contrast, social security is well known to be neutral concerning households' consumption-saving decision (Barro, 1974), because a rise in social security transfers from workers to the elderly can be fully offset by increasing bequests from parents to children.<sup>2</sup> This neutrality becomes invalid when the rise in the bequest cost of having a child reduces fertility and hence raises capital per worker (e.g., Becker and Barro, 1988; Zhang, 1995). However, the welfare implication of such changes caused by social security remains unclear. In this situation, social security can increase the level, or the growth rate, of per capita output by reducing fertility. Indeed, there is empirical evidence in Zhang and Zhang (2004) and others indicating that social security has a negative effect on

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<sup>1</sup>See, e.g., Samuelson (1958), Diamond (1965), Aaron (1966), Eckstein and Wolpin (1985), Hubbard and Judd (1987), Hansson and Stuart (1989), Cooley and Soares (1999), Kaganovich and Zilcha (1999), and Corneo and Marquardt (2000).

<sup>2</sup>Though there is little doubt about the existence of bequests in the literature, there is no consensus with regard to what motivates bequests; see, e.g., Kotlikoff and Summers (1981), Laitner and Thomas (1996), and Altonji, Hayashi and Kotlikoff (1997).

fertility and a positive effect on the growth rate of per capita income. It is therefore relevant and interesting to explore whether PAYG social security can be justified in terms of welfare when altruistic bequests are operative and fertility is endogenous.

In this paper, we examine the welfare implication of PAYG social security with altruistic bequests and endogenous fertility in a dynastic model of capital accumulation. As in the literature, scaling up PAYG social security has no effect on the saving rate in a dynastic model when altruistic parents respond by leaving more bequests to their children to offset the resultant increase in the future tax burden. However, a rise in the tax rate for social security has opposing effects on fertility. On the one hand, by increasing the bequest cost of having a child, the tax rise tends to reduce fertility. On the other hand, by reducing the after-tax wage rate the tax rise reduces the opportunity cost of spending time rearing a child and thus tends to raise fertility. In addition, if the amount of social security benefits depends on an individual's earnings, the tax rise implies a higher replacement rate that retains a balanced budget of social security. By raising the replacement rate, the tax rise raises the opportunity cost of spending time rearing a child and thus tends to reduce fertility. Given the same saving rate, any change in fertility due to the tax rise must affect capital intensity in an opposite direction. The net effect of scaling up social security on fertility (capital intensity) is found to be negative (positive) if the taste for the welfare, relative to the number, of children is strong enough.

The opposite movements in fertility and capital intensity influence welfare differently. A fall in fertility reduces welfare as the number of children enters utility, while a rise in capital intensity raises labor productivity and hence raises welfare. The net welfare effect of social security depends on the strength of an investment externality in the model.<sup>3</sup> Given this

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<sup>3</sup>The investment externality has been emphasized in the literature on economic growth (e.g. Arrow, 1962; Romer, 1986; Lucas, 1993). The rationale for the externality is that firms and workers enhance their knowledge through learning-by-doing, which also benefits other firms through spillovers to some extent as knowledge is partly a public good in nature. Moreover, Arrow's idea of linking learning by doing to investment builds on empirical observations of large positive effects of experience on productivity, as well

investment externality, private rates of return on investment are lower than the social rate. As a result, competitive solutions without government intervention would suffer from under-investment in capital compared to the social planner's solution. This under-investment in capital lowers the marginal product of labor from its socially optimal level (the opportunity cost of spending time rearing a child), thereby inducing parents to have too many children. We show that social security improves welfare by reducing fertility in the presence of the investment externality.<sup>4</sup> We also illustrate numerically that a small degree of the externality is enough to justify the observed high ratios of social security spending to GDP.

Exploring ideal public policy to deal with the under-investment arising from the investment externality has become an important issue in economic analysis in recent years (e.g. Devarajan, Xie and Zou, 1998; Turnovsky, 2000). A typical result in this literature is to subsidize private investment rather than publicly provide such investment. Our result in this paper indicates that social security can also be an attractive policy instrument to promote per capita accumulation by lowering fertility, because developing countries that have little social security typically have too many children and too little capital per capita. In particular, the mechanism in our analysis differs from the standard infinitely-lived representative agent model used in the related literature in that we incorporate life-cycle savings into a dynastic-family model and treat fertility as an endogenous variable.

In order to fully characterize the underlying dynamics in this complex model, it is necessary that the tax rate, fertility and proportional allocations of output are constant over time. To deliver this, we assume log preferences and a Cobb-Douglas production function, which are rather standard in the literature on economic growth. We can then solve for the

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as on the evidence that patents — a proxy for learning — closely follow investment in some industries; see Barro and Sala-i-Martin (1995, p. 147). More recent empirical evidence also indicates the presence of the spillovers both within and across industries (e.g. Bernstein and Nadiri, 1989; Nakanishi, 2002).

<sup>4</sup>The impact of an investment externality on the efficiency of private capital accumulation has been analyzed by Belan, Michel and Pestieau (1998) and Corsetti and Schmidt-Hebbel (1997) in a life-cycle model, supporting a switch from unfunded social security to funded saving schemes. For a critical view, see Sinn (2000), pp. 398f.

welfare level that applies not only in the steady state but also in the transitional path. With the solution for the welfare level, we can assess the welfare implication of social security.

The remainder of this paper proceeds as follows. The next section introduces the model. Section III characterizes the equilibrium and derives the solution for the welfare level. Section IV provides the main results. The last section concludes.

## II. The Model

Consider an economy with overlapping generations of identical agents who live for two periods in adulthood. They work when young and live in retirement when old. Every young agent may choose to have  $N_t$  identical children.

A representative dynastic family consists of altruistic members in each generation. The preference of the dynasty is defined over the consumption levels of the young and old members,  $C_t$  and  $D_t$  respectively, as well as over fertility,  $N_t$ , of family members in all generations:

$$U_0 = \sum_{t=0}^{\infty} \alpha^t V(D_t, C_t, N_t),$$

where  $0 < \alpha < 1$  is the discount factor. The household utility function  $V(D_t, C_t, N_t)$  captures what contributes to the dynastic family's welfare within a period: the consumption of coexisting old and young members as well as the number of children. The old-age consumption of a period- $t$  young member will be reflected in  $V(D_{t+1}, C_{t+1}, N_{t+1})$  next period. In this way, we incorporate the life-cycle consumption-saving consideration into a dynastic family model. Moreover, the function  $V(\cdot, \cdot, \cdot)$  is increasing and concave and meets the Inada condition to ensure an interior optimal solution:  $\partial V / \partial x \rightarrow 0$  as  $x \rightarrow \infty$  and  $\partial V / \partial x \rightarrow \infty$  as  $x \rightarrow 0$  for  $x = D, C, N$ .

For tractability, we assume  $V_t = \beta \ln D_t + \alpha(\ln C_t + \rho \ln N_t)$  where  $\beta$  is the taste for utility derived from the consumption of the old parent,  $\alpha$  is the taste for utility from the young-age consumption and the number of children of each working member, and  $\rho$  is the taste for

utility from the number of children relative to that from young-age consumption.<sup>5</sup> We may rewrite the utility function as

$$U_0 = \sum_{t=0}^{\infty} \alpha^t [\beta \ln D_t + \alpha (\ln C_t + \rho \ln N_t)], \quad \alpha \in (0, 1). \quad (1)$$

For an initial old agent in period 0 who had chosen  $N_{-1}$  children, the only remaining decision is the trade-off between his own old-age consumption  $D_0$  and the amount of bequests to children.<sup>6</sup> We assume that  $\alpha$  is large enough such that bequests are operative.

Each young agent devotes one unit of time endowment to rearing children and working. Rearing a child needs  $v$  units of time, implying an upper limit  $1/v$  on  $N$ ; otherwise  $N$  may approach infinity. The amount of working time is equal to  $1 - vN_t$  that earns  $(1 - \tau)(1 - vN_t)W_t$  where  $W$  is the wage rate and  $\tau$  is the tax rate. A young agent also receives a bequest  $B_t$  from his old parent. The young agent allocates the earnings and the received bequest to current consumption  $C_t$  and to retirement savings  $S_t$ . An old agent spends part of his savings plus interest income and social security benefits on consumption and leaves the rest as bequests to children. The budget constraints can be written as

$$C_t = B_t + (1 - vN_t)W_t(1 - \tau) - S_t, \quad (2)$$

$$D_{t+1} = S_t R_{t+1} + T_{t+1} - B_{t+1} N_t, \quad (3)$$

where  $R$  is the interest factor and  $T$  the amount of social security benefits per retiree.

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<sup>5</sup>This specification has an interesting implication: the first-order conditions are the same as those under an alternative specification  $U_t = \ln C_t + \beta \ln D_{t+1} + \rho \ln N_t + \alpha U_{t+1} = \sum_{s=t}^{\infty} \alpha^{s-t} [\ln C_s + \beta \ln D_{s+1} + \rho \ln N_s]$  where  $U_t$  and  $U_{t+1}$  are the welfare of a worker and that of his child, respectively. This equivalence allows us to decompose the welfare of family members across generations:  $U_t = \beta \ln D_t + \alpha U_t$  where  $U_t$  is as defined in (1). From this relationship,  $U_t$  can be regarded as the welfare of an old member who chooses how much bequests to leave, and  $U_t$  the welfare of a young member who takes the amount of bequests from the old parent as given.

<sup>6</sup>When leisure is added to the utility function  $V$  in the form of  $\eta \ln Z$ , the analysis is more complicated but the results are found to be very similar. Though the inclusion of leisure tends to generate a negative welfare effect of social security, the level of the optimal social security tax rate is found to be just slightly lower than in the case without leisure in the numerical solution for various values of  $\eta$ . We thus exclude leisure in this paper for brevity.

As practiced in many countries such as France and Germany, the amount of social security benefits received by a retiree depends on his own earnings in working age according to a replacement rate  $\phi$ , that is,  $T_{t+1} = \phi(1 - vN_t)W_t$ .<sup>7</sup> With this formula, a worker who has more children (hence less labor time) not only earns less wage income but also expects to receive less social security benefits in old age. The social security program is always balanced in a typical PAYG fashion:  $T_t = \bar{N}_{t-1}\tau(1 - v\bar{N}_t)W_t$  where the bar above a variable indicates its average level. With identical agents in the same generation, in equilibrium we have  $\bar{N} = N$  by symmetry.

The production function is given as

$$Y_t = AK_t^\theta(1 - vN_t)^{1-\theta}\bar{K}_t^\delta, \quad A > 0, \quad \theta \in (0, 1), \quad \delta \in [0, 1 - \theta), \quad (4)$$

where  $Y_t$  and  $K_t$  are output per worker and capital per worker, respectively;  $A$  is the total factor productivity parameter and  $\theta$  the share parameter of capital; and  $\delta$  measures the strength of spillovers from average capital per worker  $\bar{K}_t$ . Factors are paid according to their marginal products:  $R_t = \theta Y_t / K_t$  and  $W_t = (1 - \theta)Y_t / (1 - vN_t)$ . The assumptions of log preferences and Cobb-Douglas production are strong but help to illustrate why social security may raise welfare in the presence of the spillovers in production.

Though the existence of this type of spillover is emphasized in the literature, the exact degree of this externality is unclear. When  $\delta = 1 - \theta$ , the externality is strong enough to generate sustainable growth in the long run, which corresponds to the well known AK-style model of endogenous growth. Using time series data in OECD countries, however, Jones (1995) has found evidence against the AK-style model. We thus limit our attention to  $1 - \theta > \delta \geq 0$ .

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<sup>7</sup>The essence of the results will remain if the amount of social security benefits is less than proportional to individuals' own earnings as in the United States, or is independent of individuals' own earnings, though quantitatively different. As shown in Zhang and Zhang (2003), the more heavily the social security benefits depend on one's own past earnings, the more likely the social security expansion will have a negative (positive) effect on fertility (the growth rate of per capita output).

Finally, in this closed economy the level of capital per worker in the next period is equal to the amount of savings per worker divided by the number of children today:  $K_{t+1} = S_t/N_t$  where the depreciation rate of capital is 100% per generation for tractability. Taking one period as 35 years in this overlapping generations model, the full depreciation of capital per generation is plausible.

### III. Equilibrium

We now solve the family's problem, track down the equilibrium allocation, and derive the solution for the welfare level for the welfare analysis of social security later.

#### *Equilibrium Solution for the Dynastic Family's Problem*

The problem of a dynastic family is to maximize utility in (1) subject to budget constraints (2) and (3), taking the social security tax and replacement rates as given. For  $t \geq 0$ , the first-order conditions are given as follows:

$$B_t : \quad \alpha/C_t = \beta N_{t-1}/D_t, \tag{5}$$

$$S_t : \quad 1/C_t = \beta R_{t+1}/D_{t+1}, \tag{6}$$

$$N_t : \quad \frac{vW_t(1-\tau)}{C_t} + \frac{\beta(\phi W_t v + B_{t+1})}{D_{t+1}} = \frac{\rho}{N_t}. \tag{7}$$

In (5), the marginal loss in the old parent's utility from giving a bequest to each child is equal to the marginal gain in children's utility. In (6), the marginal loss in utility from saving is equal to the marginal gain in utility in old age through receiving the return to saving. In (7), the marginal loss in utility from having an additional child, through giving up a fraction of wage income and earnings-dependent social security benefits and leaving a bequest, is equal to the marginal gain in utility from enjoying the child. These first-order conditions hold for all  $t \geq 0$  due to the recursive structure of the household problem.



**Definition.** Given an initial state  $(N_{-1}, K_0)$ , the competitive equilibrium in this economy with PAYG social security is a sequence of allocations  $\{B_t, C_t, D_t, K_{t+1}, N_t, S_t, \phi, \tau, T_t, Y_t\}_{t=0}^{\infty}$  and prices  $\{R_t, W_t\}_{t=0}^{\infty}$  such that (i) taking prices, the tax and replacement rates  $(\phi, \tau)$  and the average and aggregate variables as given, firms and households optimize and their solutions are feasible, (ii) the social security budget is balanced, and (iii) all markets clear.

Specifically, these equilibrium conditions correspond to the first-order conditions of firms and households, the budget constraints of households and the government, the technology, the capital market clearing condition, and the amount of labor supply per worker equal to  $1 - vN_t$ , for  $t \geq 0$ . In addition, as mentioned earlier, we have  $X = \bar{X}$  for  $X = K, N$  in equilibrium by symmetry. Moreover, with the log utility, the Cobb-Douglas technology and the full depreciation of capital in one period, we expect that the proportional allocations of time and output are constant over time, given any initial state and any time-invariant tax rate, as was known in the literature.<sup>8</sup>

Letting the fraction of output spent on item  $X_t$  be a time-invariant lower-case variable  $x = X_t/Y_t$ , we transform the variables in the budget constraints and first-order conditions to their relative ratios to output. The transformed budget constraints take the form:  $c = b + (1 - \theta)(1 - \tau) - s$ ; and  $d = N[\theta + \tau(1 - \theta) - b]$  for  $t > 0$  and  $d_0 = N_{-1}[\theta + \tau(1 - \theta) - b]$  for a predetermined  $N_{-1}$ . Similarly, the transformed first-order conditions are:  $\alpha/c = \beta N/d$  for  $t > 0$  and  $\alpha/c = \beta N_{-1}/d_0$ ;  $1/c = \beta K_{t+1} R_{t+1} Y_t / (D_{t+1} K_{t+1}) = \beta \theta N / (sd)$  under  $K_{t+1} = (s/N)Y_t$ . Also, combining  $\alpha/c = \beta N/d$  and the social security constraint  $\phi(1 - vN)W_t = N\tau(1 - vN)W_{t+1}$  with (7) and arranging the resulting equation, we have

$$\frac{v(1 - \tau)(1 - \theta)}{c(1 - vN)} + \frac{\alpha v \tau (1 - \theta)}{c(1 - vN)} + \frac{\alpha b}{cN} = \frac{\rho}{N}. \quad (8)$$

Similar to (7), equation (8) keeps a balance between the costs (the left-hand side) and the

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<sup>8</sup>For example, Devereux and Love (1994) have noted that in a Cobb-Douglas world with full depreciation of capital, even a two-sector model of endogenous growth has time-invariant proportional allocations for every period (in and outside the balanced equilibrium path), of which our current one-sector model is a special case.

benefit (the right-hand side) of having a child. The first cost component in (8) is the forgone wage income of spending time rearing a child, which falls with the social security tax rate. The second cost component is the forgone social security benefit of spending time rearing a child, which rises with the tax rate through the linkage between the replacement rate and the tax rate under a balanced social security budget. Thus, when the tax rate rises, the rise in the second cost component partially offsets the fall in the first cost component, and the overall time cost of having a child is likely to fall. The third cost component is the bequest cost of having a child, which is an increasing function of the social security tax rate because parents respond to a rise in the social security tax by giving more bequests to reduce the subsequent tax burden on children. There are thus opposing effects of a rise in the tax rate on fertility: The fall in the time cost of having a child tends to raise fertility, while the rise in the bequest cost of having a child tends to reduce fertility. The net effect on fertility will depend on two taste parameters involved in (8):  $\alpha$  (part of the discount factor on utility from own old-age consumption and the taste for the welfare of children) and  $\rho$  (the taste for utility from the number of children). According to (8), a larger  $\alpha$  means higher weights for the cost components of having a child that rise with the tax rate, and thus makes it more likely that fertility falls with the tax rate.

From these equilibrium conditions, we obtain the following constant allocation rules:

$$c = C_t/Y_t = \frac{\alpha(1 - \alpha\theta)}{\alpha + \beta}, \quad (9)$$

$$s = S_t/Y_t = \alpha\theta, \quad (10)$$

$$b = B_t/Y_t = \theta + \tau(1 - \theta) - (\beta/\alpha)c, \quad (11)$$

$$N = \frac{\rho c - \alpha b}{v\{(1 - \theta)[1 - \tau(1 - \alpha)] + \rho c - \alpha b\}}, \quad (12)$$

$$d = D_t/Y_t = N[\theta + \tau(1 - \theta) - b] = \frac{\beta c N}{\alpha} \text{ for } t > 0, \quad (13)$$

$$d_0 = D_0/Y_0 = N_{-1}[\theta + \tau(1 - \theta) - b] = \frac{\beta c N_{-1}}{\alpha}. \quad (14)$$

Observe that if  $\rho c > ab$  then fertility  $N$  is positive in the solution. The possibility of non-positive fertility follows the presence of non-convexity in the form of  $B_{t+1}N_t$  in the budget constraint (3). However, as shown in Zhang, Zhang and Lee (2001) and Zhang (1995), the sufficient condition for the solution to be optimal is a sufficiently large taste parameter for the number of children ( $\rho$ ) such that fertility is positive. Further, we have assumed a strong enough taste for the welfare of children  $\alpha$  such that bequests are positive. Under these restrictions on  $\rho$  and  $\alpha$ , there is a unique optimal interior solution in this model.

Some features of the solution merit attention. First, it can be verified that these constant proportional allocations satisfy the equilibrium conditions for  $t \geq 0$ , given any initial state and any constant tax rate. The government budget constraint implies that for any given tax rate, there is a corresponding replacement rate. Second, these proportional allocations are consistent across generations, in the sense that agents in any generation will choose these optimal proportional allocations when other generations do so, because they have the same first-order conditions and budget constraints in this recursive structure. As a result, these proportional allocations are the equilibrium solution on the entire equilibrium path of the economy. These features allow us to obtain an analytical solution for the levels of the variables of interest in every period, starting from the initial period. Thus, we can analyze how social security affects the economy and what is its optimal scale to maximize welfare.

We now ask how the solution responds to a rise in the social security tax rate:

**Lemma 1.** *A rise in the social security tax rate has a positive effect on the ratio of bequests per child to output per worker but has no effect on the fractions of output spent on young-age consumption and saving. Also, defining  $\bar{\rho} \equiv \alpha(1 + \beta)/(1 - \alpha)$ , a rise in the tax rate reduces fertility if  $\rho < \bar{\rho}$ , increases fertility if  $\rho > \bar{\rho}$ , and has no effect on fertility if  $\rho = \bar{\rho}$ .*

*Proof.* The first part of the lemma is straight forward from differentiating (9), (10) and (11) respectively with respect to  $\tau$ . For the second part, we have

$$\text{sign} \frac{dN}{d\tau} = \rho(1 - \alpha) - \alpha(1 + \beta). \quad (15)$$

The claim on how fertility responds to the tax rate change follows. ■

Both the unresponsiveness of the fractions of output spent on young-age consumption and saving and the positive response of bequests to social security are standard results in dynastic-family models with operative bequests.<sup>9</sup> In response to a rise in the social security tax rate, the rise in bequests per child as a fraction of output per worker helps to offset the increased tax burden on future generations. Also, a rise in the tax rate may reduce, increase, or have no effect on fertility, depending on the relative strength of the taste for the welfare versus the number of children ( $\alpha$  versus  $\rho$ ). On the left-hand side of (7) or (8), there are three cost factors through which social security may influence the fertility decision. First, a rise in the social security tax rate reduces the opportunity cost of spending time rearing a child (the after-tax wage rate), thereby tending to raise fertility. Second, the rise in the tax rate also raises the earnings-dependent social security benefit via the replacement rate  $\phi$  and hence raises the cost of having a child, thereby tending to reduce fertility. Third, the rise in the tax rate raises the bequest cost of having a child, thereby tending to reduce fertility. As mentioned earlier, when  $\alpha$  becomes larger (a stronger taste for the welfare of children and a larger discount factor), then the last two cost components of having a child become more important relative to the first. Thus, if  $\alpha$  is large enough relative to  $\rho$ , then a rise in the tax rate will reduce fertility. This negative net effect of social security on fertility is consistent with empirical evidence in the literature (e.g., Cigno and Rosati, 1992; Zhang and Zhang, 2004).

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<sup>9</sup>Empirical evidence on how social security affects the saving rate has been inconclusive. For example, the estimated effect of social security on the saving rate is negative in Ehrlich and Zhong (1998), positive in Cigno and Rosati (1992), and statistically insignificant in Zhang and Zhang (2004).

As noted above, fertility is positive if  $\rho$  is sufficiently large (i.e. the taste for the number of children is strong enough). Given the fact that the fraction of output spent on young-age consumption should be much greater than the fraction of output spent on bequests, the condition  $\rho > \alpha$  would guarantee positive fertility according to the signing factor  $\rho c - \alpha b$  in the fertility equation (12). Specifically, fertility is positive if  $\rho > \alpha b/c = (\alpha + \beta)\theta / (1 - \alpha\theta) - \beta \equiv \underline{\rho}$  at  $\tau = 0$ , from equations (9) and (11). It is easy to verify that  $\text{sign}[\bar{\rho} - \underline{\rho}] = 1 - \theta > 0$ , i.e.  $\bar{\rho} > \underline{\rho}$ . Combining this with Lemma 1, for  $\rho \in (\underline{\rho}, \bar{\rho})$  fertility is positive and decreasing with the social security tax rate.

### *Dynamic Equilibrium Path*

Most existing studies of social security focus on steady-state solutions (e.g., Zhang, 1995). To fully capture the welfare impact of social security, we also need to track down the entire dynamic equilibrium path starting from any initial level of capital per worker  $K_0$ . Substituting the solutions for  $s$  and  $N$  into  $K_{t+1} = S_t/N_t$ , we have

$$K_{t+1} = (s/N)A(1 - vN)^{1-\theta} K_t^{\theta+\delta} \equiv \Theta K_t^{\theta+\delta}. \quad (16)$$

Under the assumption  $\theta + \delta < 1$ , the solution for the log of  $K_t$  is found to be

$$\ln K_t = \frac{[1 - (\theta + \delta)^t]}{1 - (\theta + \delta)} \ln \Theta + (\theta + \delta)^t \ln K_0. \quad (17)$$

According to this,  $\ln K_t$  converges monotonically to its steady-state level  $\ln \bar{K} \equiv [1 - (\theta + \delta)]^{-1} \ln \Theta$  that is increasing with the saving rate but decreasing with fertility by the definition of  $\Theta$  in (16). From this solution and that for fertility, we can also solve for the log of  $Y_t$ :

$$\ln Y_t = \ln A(1 - vN)^{1-\theta} + (\theta + \delta) \ln K_t \quad (18)$$

which rises with initial capital and the saving rate but falls with fertility by (17) and (18).

### *Solution for the Welfare Level $U_0(\tau)$*

With the full characterization of the equilibrium path of the model, we can now solve for the welfare level. Using the solution for  $(b, c, d, N)$  and for the sequence  $\{\ln K_t, \ln Y_t\}_0^\infty$  given an initial state  $(N_{-1}, K_0)$ , we can obtain the solution for the welfare level as<sup>10</sup>

$$\begin{aligned} U_0 &= \beta(\ln \beta N_{-1} c / \alpha + \ln Y_0) + \alpha \sum_{t=0}^{\infty} \alpha^t [\ln c + \ln Y_t + \beta \ln \beta N c / \alpha + \beta \ln Y_{t+1} + \rho \ln N] \\ &= \Phi_0 + F(\tau) + \Phi_1 \ln K_0, \quad \text{with} \end{aligned} \quad (19)$$

$$\begin{aligned} \Phi_0 &= \beta \ln \frac{\beta N_{-1} c}{\alpha} A + \frac{\alpha}{1 - \alpha} \left[ \ln c + \beta \ln \frac{\beta c}{\alpha} A s^{\theta + \delta} \right] + \left[ \frac{1 + \beta(\theta + \delta)}{1 - \alpha} \right] \times \\ &\quad \left[ \ln A + \frac{\alpha(\theta + \delta)}{1 - \alpha(\theta + \delta)} \ln A s \right], \end{aligned}$$

$$\Phi_1 = \beta(\theta + \delta) + \frac{[1 + \beta(\theta + \delta)](\theta + \delta)\alpha}{1 - \alpha(\theta + \delta)},$$

$$F(\tau) = \Pi_0 \ln N + \Pi_1 \ln(1 - vN),$$

where

$$\begin{aligned} \Pi_0 &= \frac{\alpha(\beta + \rho)[1 - \alpha(\theta + \delta)] - \alpha(\alpha + \beta)(\theta + \delta)}{(1 - \alpha)[1 - \alpha(\theta + \delta)]}, \\ \Pi_1 &= \frac{(1 - \theta)[\alpha + \beta]}{(1 - \alpha)[1 - \alpha(\theta + \delta)]} > 0. \end{aligned}$$

While  $\Phi_0$ ,  $\Phi_1$ ,  $\Pi_0$  and  $\Pi_1$  are all constant and unresponsive to a tax rate change,  $F(\tau)$  is a function of the tax rate via fertility. Obtaining this solution for the welfare level in a dynastic model with multi-period life-cycle consumption appears to be an interesting result in its own right.

Obviously, if fertility were exogenous as in most studies of social security, a change in the tax rate would have no impact on welfare in this dynastic model as in Barro (1974). The

<sup>10</sup>Note that  $\ln Y_{t+1} = \ln A(s/N)^{\theta + \delta}(1 - vN)^{1 - \theta} + (\theta + \delta) \ln Y_t$  by  $Y_{t+1} = AK_{t+1}^{\theta + \delta}(1 - vN)^{1 - \theta}$  and  $K_{t+1} = sY_t/N$ .

main task next is to investigate how social security affects welfare with endogenous fertility and what is the optimal social security tax rate to maximize welfare.

## IV. Welfare Implications

For comparison purposes, we begin with the unrealistic case without externality  $\delta = 0$ . This case corresponds to a typical neoclassical growth model without market friction.

### *No Investment Externality*

Absent externalities with  $\delta = 0$ , the welfare implication of social security is given below:

**Proposition 1.** *For  $\delta = 0$ , the competitive solution without social security is Pareto optimal under  $\rho > \underline{\rho} \equiv (\alpha + \beta)\theta/(1 - \alpha\theta) - \beta$ . For  $\rho \neq \bar{\rho} \equiv \alpha(1 + \beta)/(1 - \alpha)$ , social security always reduces welfare, whereas for  $\rho = \bar{\rho}$ , social security is neutral.*

*Proof.* It is sufficient to focus on  $F(\tau)$  in dealing with the relationship between the welfare  $U_0$  and the tax rate. According to (19), we have

$$F'(\tau) = \frac{dN}{d\tau} \left( \frac{\Pi_0}{N} - \frac{v\Pi_1}{1 - vN} \right). \quad (20)$$

Using (12), it can be verified that  $\Pi_0/N = v\Pi_1/(1 - vN)$  if  $\tau = 0$  and  $\delta = 0$ . So  $F'(\tau) = 0$  at  $\tau^* = 0$  under these conditions. Also, fertility is positive under  $\rho > (\alpha\theta - \beta c)/c$ .

According to Lemma 1, for  $\rho \neq \bar{\rho}$ , we have  $dN/d\tau \neq 0$ . For  $\tau > 0$ , if  $dN/d\tau > 0$ , then  $\Pi_0/N < v\Pi_1/(1 - vN)$ , implying  $F'(\tau) < 0$ . Similarly, if  $dN/d\tau < 0$ , then  $\Pi_0/N > v\Pi_1/(1 - vN)$ , implying again  $F'(\tau) < 0$ . Thus, for  $\delta = 0$ ,  $\tau^* = 0$  maximizes  $F(\tau)$ . Finally, for  $\rho = \bar{\rho}$ ,  $dN/d\tau = 0$  under Lemma 1, and therefore  $F'(\tau) = 0$  for  $1 > \tau > 0$  by (20). ■

Without externality, Proposition 1 provides the condition for an interior solution and describes the first-best nature of the competitive solution without social security. With endogenous fertility, social security is not neutral in general as opposed to the Ricardian equivalence hypothesis in Barro (1974). In the absence of the externality, social security

reduces welfare by changing fertility from its first-best level. This conclusion is consistent with the standard view against social security in the literature. In the rest of this section, we will see how the investment externality can justify social security.

### *With Investment Externality*

With the externality, the optimal tax rate of social security is given as

**Proposition 2.** *For  $0 < \delta < 1 - \theta$ , the optimal tax rate of social security is given by*

$$\tau^* = \frac{\alpha \Pi_1 [(\beta + \rho)(1 - \alpha\theta) - \theta(\alpha + \beta)] - \Pi_0(1 - \theta)(\alpha + \beta)}{(1 - \theta)(\alpha + \beta)[\alpha \Pi_1 - \Pi_0(1 - \alpha)]}.$$

*In addition,  $\tau^* > 0$  for  $\rho \in (\underline{\rho}, \bar{\rho})$ . Also, the stronger the externality, the higher the optimal tax rate of social security.*

*Proof.* According to (20),  $F'(\tau) = 0$  when  $vN/(1 - vN) = \Pi_0/\Pi_1$ , which is the necessary condition for the optimal tax rate. Substituting (12) for  $N$  in this equation yields  $\tau^*$  whose actual level depends on the value of  $\delta$  via  $(\Pi_0, \Pi_1)$ . As a starting position, note that  $\tau^* = 0$  at  $\delta = 0$  as implied by Proposition 1 (or confirmed by checking the value of  $\tau^*$  at  $\delta = 0$  directly). The remaining step for  $\tau^* > 0$  is to derive the conditions under which  $d\tau^*/d\delta > 0$ .

Note that  $\Pi_0/\Pi_1$  can be written as

$$\frac{\Pi_0}{\Pi_1} = \frac{\alpha(\beta + \rho)[1 - \alpha(\theta + \delta)] - \alpha(\alpha + \beta)(\theta + \delta)}{(1 - \theta)(\alpha + \beta)}.$$

Obviously,  $d(\Pi_0/\Pi_1)/d\delta < 0$ . Differentiating the expression of  $\tau^*$  with respect to  $\delta$  provides:

$$\text{sign} \frac{d\tau^*}{d\delta} = [\rho(1 - \alpha) - \alpha(1 + \beta)] \frac{d(\Pi_0/\Pi_1)}{d\delta}. \quad (21)$$

Therefore,  $d\tau^*/d\delta > 0$  and hence  $\tau^* > 0$  under  $\rho < \bar{\rho} = \alpha(1 + \beta)/(1 - \alpha)$ .

Now we give the sufficient condition for the optimal tax rate. If  $\tau > \tau^*$ , then (20) implies

$$F'(\tau) = \frac{dN}{d\tau} \left( \frac{\Pi_0}{\Pi_1} - \frac{vN}{1 - vN} \right) \frac{\Pi_1}{N} < 0$$



under  $\rho \in (\underline{\rho}, \bar{\rho})$  since then  $N > 0$ ,  $\Pi_1 > 0$ ,  $dN/d\tau < 0$ , and  $\Pi_0/\Pi_1 > vN/(1 - vN)$  where  $\Pi_i$  for  $i = 0, 1$  is not a function of  $\tau$  but  $N/(1 - vN)$  decreases with  $\tau$  under  $\rho < \bar{\rho}$ . On the other hand, if  $\tau < \tau^*$ , then  $\Pi_0/\Pi_1 < vN/(1 - vN)$  and hence  $F'(\tau) > 0$  under the same condition for  $\rho$ . ■

The intuition here lies in the fact that atomic individuals cannot internalize the externality themselves individually. Thus, unlike in Proposition 1, the competitive solution is no longer Pareto optimal with the externality. From (9)-(14), we can easily see that the degree of the externality measured by  $\delta$  does not influence the competitive solution for time and income allocations. But a social planner, who can internalize the externality, will respond to the value of  $\delta$ . The social planner maximizes (1) subject to  $D_t + N_{t-1}C_t + N_{t-1}N_tK_{t+1} = N_{t-1}Y_t$  and  $Y_t = AK_t^{\theta+\delta}(1 - vN_t)^{1-\theta}$ , having the following solution:

$$c = \frac{C_t}{Y_t} = \frac{\alpha[1 - \alpha(\theta + \delta)]}{\alpha + \beta}, \quad (22)$$

$$s = \frac{NK_{t+1}}{Y_t} = \alpha(\theta + \delta), \quad (23)$$

$$N = \frac{\alpha(\rho + \beta)[1 - \alpha(\theta + \delta)] - \alpha(\alpha + \beta)(\theta + \delta)}{v\{(\alpha + \beta)(1 - \theta) + \alpha(\rho + \beta)[1 - \alpha(\theta + \delta)] - \alpha(\alpha + \beta)(\theta + \delta)\}}, \quad (24)$$

$$d = \frac{D_t}{Y_t} = N(1 - c - s) \quad \text{for } t > 0; \quad d_0 = \frac{D_0}{Y_0} = N_{-1}(1 - c - s). \quad (25)$$

The welfare function in the social planner's solution is the same as that in the competitive solution. We now have:

**Proposition 3.** *For  $\delta > 0$ , fertility is higher but the saving rate is lower in the competitive solution without social security ( $\tau = 0$ ) than in the social planner's. Also, the optimal social security tax rate is only a second-best solution.*

*Proof.* Fertility in the competitive solution without social security (i.e.  $\tau = 0$ ) can be rewritten as

$$N = \frac{\alpha[(\rho + \beta)(1 - \alpha\theta) - \theta(\alpha + \beta)]}{v\{(1 - \theta)(\alpha + \beta) + \alpha[(\rho + \beta)(1 - \alpha\theta) - \theta(\alpha + \beta)]\}}$$

which is always greater than fertility in the social planner's solution (24) for  $\delta > 0$ . The two are equal when  $\delta = 0$ . The difference in the saving rates is obvious when comparing (10) with (23). Since the saving rate in the optimal social security solution in Proposition 2 is below the saving rate in the social planner's solution, the optimal social security solution is a second-best solution. ■

The reason for the difference between the social planner's solution and the competitive solutions is as follows. First, there exists under-investment in the competitive solution because with the externality the private rate of return on investment is lower than the social rate. The under-investment in capital also means that the marginal product of labor (hence the wage rate or the opportunity cost of spending time rearing a child) will be lower than its socially optimal level. By (7), a lower wage cost of having a child (the first term on the left) should be matched with a rise in fertility so as to reduce the marginal benefit of having a child (the right-hand side). In other words, the externality engenders not only too little capital but also too many children compared to the social planner's solution.

Social security can improve on the competitive solution by reducing fertility. The reduction in the number of workers will increase labor productivity via increasing capital per worker for the same saving rate or for a given level of initial capital stock. With the externality from average capital, the increase in capital per worker helps to improve welfare given the under-investment in the first place. Thus, for  $\rho < \bar{\rho}$ , social security reduces fertility and increases capital intensity, moving the economy closer to the social planner's solution.<sup>11</sup> Further, the stronger the externality, the greater the deviation of the competitive solution from the social planner's. Hence, a stronger externality leads to a higher optimal social security tax rate. However, because social security is unable to alter the saving rate, it cannot achieve the first-best solution.

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<sup>11</sup>The effect of social security on welfare may be similar if we consider the case in which working children support retired parents, rather than receive bequests from parents, because in this case social security can still reduce fertility and increase capital intensity as shown in Zhang (1993).

One important feature of this result is that all generations could gain from social security, unlike those based on generational conflicts in the absence of altruism whereby young workers are usually the losers (e.g. Cooley and Soares, 1999). In other words, in our model all adult agents, who share the same household utility, prefer the optimal social security tax rate to a zero tax rate and therefore are willing to vote for social security in a democratic society. The fact that PAYG social security has been instituted in so many countries with very different demographic features seems to favor the wider support for social security by all generations, particularly in developing countries where young work-age population usually accounts for the majority of adult population.

### *Numerical Results*

It is also interesting to know how strong the externality should be to generate realistic ratios of social security taxes to wage income, e.g. 10%-20% in industrial nations (or 7%-14% of GDP) for plausible parameterizations. As widely accepted, the income share parameters are set at 0.25 and 0.75, respectively, for capital and labor. The values of other parameters are chosen to produce realistic values for the saving and fertility rates. First, if the saving rate  $s$  is equal to 20% given that capital's share in output  $\theta$  equals 0.25, then  $\alpha = 0.8$  according to  $s = \alpha\theta$ . The remaining parameterization  $\rho = 0.84$ ,  $\beta = 0.3$ , and  $v = 0.3$  can then deliver  $N = 1.273$  in the absence of social security (i.e. at  $\tau = 0$ ). Moreover, we set  $A = 20$ ,  $K_0 = 1$  and  $N_{-1} = 1.5$ , which are non-essential for the result. Also, we set the value of  $\delta$  at 0.05, 0.1 and 0.2, respectively, because of the lack of empirical evidence in this regard. For the purpose of making a better comparison, we also consider the case without externality (i.e.  $\delta = 0$ ).

The simulation results are reported in Table 1, with rising tax rates and corresponding solutions for the saving rate, the fertility rate, output per worker (in the long run), and the welfare level. In the case without externality, a rise in the social security tax rate reduces

fertility, raises output per worker, but always reduces welfare. This indicates that focusing on whether social security is conducive to capital accumulation alone does not necessarily justify social security. Much of the literature on social security has indeed focused on the impact of social security on capital accumulation (e.g., Feldstein, 1974; Zhang, 1995).

In Cases 2.1 to 2.3 with  $\delta = 0.05, 0.12$  and  $0.20$ , respectively, the effects of social security on fertility and output per worker are similar to those without externality, but the welfare level increases with the tax rate until reaching the optimal levels of the tax rate (14%, 27%, and 52%, correspondingly). According to these cases, the stronger the externality (a larger  $\delta$ ), the higher the optimal tax rate. When  $\delta$  is equal to 0.2, the resultant optimal tax rate becomes too high to be realistic. Also, when the externality is stronger, the rate of fertility at the optimal tax rate becomes smaller and may lie below its replacement level of  $N = 1$ . However, the population will not be sustainable when the rate of fertility falls below its replacement level. According to Table 1, the sustainable social security tax rate is at 20% or lower. Also, with this parameterization, bequests are positive in all the simulations carried out. It is now clear that even a weak investment externality can justify the observed high social security contribution rates and can cause significant changes in fertility and output per worker. Finally, the social planner's solution is reported at the end of each case in Table 1, having a higher saving rate but lower fertility compared to the competitive solution. Also note that in the social planner's solution, the rate of fertility may lie below its replacement level as well.

## V. Conclusion

In this paper we have examined the welfare implication of social security by incorporating life-cycle savings, bequests, and fertility in a dynastic model. It is the first such welfare analysis in this type of model whereby life-cycle and dynastic-family decisions are determined together. To achieve this, we overcame the difficulty in tracking down the entire equilibrium

path of capital accumulation and deriving an explicit solution for the welfare level.

We have shown that scaling up social security improves welfare under the same condition it reduces fertility and raises capital intensity, until reaching an optimal tax rate. In terms of underlying driving forces, our results emerge from the combination of altruistic bequests, learning-by-doing spillovers and endogenous fertility. Such a combination has not been used in dealing with the justification of social security. In this sense, our results are complementary to Cooley and Soares (1999) that justifies social security in an overlapping generations model with selfish agents. We say this partly because, as some observers would argue, altruistic bequests may not be operative in all families.

Quantitatively, we have also illustrated that realistic social security tax rates can be justified in this simple model. For rather weak externalities, the optimal tax rates for social security are found to range from 10% to 20%. However, our results would be over-stated if they were taken to reflect the full picture of the functioning of social security in reality. Nevertheless, they do highlight the importance of the investment externality and endogenous fertility in the welfare assessment of unfunded social security.

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Table 1. *Simulation results*

$\alpha = 0.8, \beta = 0.3, A = 20, \rho = 0.84, \theta = 0.25, v = 0.3$				
Tax rate $\tau$ (%)	Saving rate $s$ (%)	Fertility $N$	Output $Y$	Welfare
Case 1. <i>No externality</i> : $\delta = 0, \tau^* = 0$				
0*	20.0	1.273	18.11	12.3424
5	20.0	1.228	18.72	12.3392
10	20.0	1.181	19.40	12.3287
20	20.0	1.076	20.98	12.2779
Case 2. <i>With externality</i> : $\delta > 0$				
Case 2.1. $\delta = 0.05, \tau^* = 14.02\%$				
0	20.0	1.273	19.51	12.6262
10	20.0	1.181	21.11	12.6514
14*	20.0	1.141	21.86	12.6541
20	20.0	1.076	23.12	12.6471
First-best solution	24.00	1.141	21.86	12.6886
Case 2.2. $\delta = 0.10, \tau^* = 27.27\%$				
0	20.0	1.273	21.27	12.9415
20	20.0	1.076	25.86	13.0574
27*	20.0	0.994	28.14	13.0700
30	20.0	0.956	29.28	13.0678
First-best solution	28.00	0.990	28.24	13.2102
Case 2.3. $\delta = 0.20, \tau^* = 51.72\%$				
0	20.0	1.273	26.49	13.6905
45	20.0	0.736	56.89	14.3797
52*	20.0	0.613	70.40	14.4096
55	20.0	0.554	78.56	14.3998
First-best solution	36.00	0.618	69.72	15.0008

*Note:* \* indicates the optimal social security tax rate for each value of  $\delta$ .