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6 Contingent Claims Valuation of Corporate Liabilities: Theory and Empirical Tests

E. Philip Jones, Scott P. Mason, and Eric Rosenfeld

6.1 Introduction

A fundamental issue in the study of capital structure is how securities issued by firms are valued in the financial markets. Typical corporate capital structures contain many individual securities, which in themselves are complicated by numerous covenants and indenture provisions. In addition, the valuation of any individual security must consider complex interactions among different claims. The corporate liability pricing model derived in Black and Scholes (1973) and Merton (1974) represents a theoretical breakthrough on this problem, with potentially significant practical applications. The critical insight of their model is that every security is a contingent claim on the value of the underlying firm. Hence these securities can be priced via an arbitrage logic which is independent of the equilibrium structure of risk and return. Every security must obey a general equation which depends only on riskless interest rates, the market value of the entire firm, and its volatility. The model distinguishes among securities via boundary conditions which correspond to covenants and indenture provisions. Since all of these data are directly observable or can be readily estimated, the model can be used to predict actual market prices.

Although this model has been the premier theory of how value is allocated among claimants on firms for almost a decade, its empirical validity remains an open question. Ingersoll has tested the model's ability to predict prices for dual purpose funds (1976) and to predict call policies

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for convertible bonds (1977). But we know of no test of the model in its presumably most important application, namely, the valuation of debt and equity in typical corporate capital structures. In addition to being of academic interest, such a test has significant practical implications. If it can be established that the model predicts actual market prices, then the model can be used to price new and untraded claims, to infer firm values from prices of traded claims like equity, and to price covenants separately.

In this paper evidence is presented on how well a model which makes the usual assumptions in the literature does in predicting market prices for claims in standard capital structures. The goal is to examine the predictive power of this prototypical model. The results suggest that the usual assumption list requires modification before it can serve as a basis for valuing corporate claims.

The usual assumptions made in the contingent claims valuation literature, for example, Ingersoll (1976, 1977), are as follows:

(A.1) Perfect markets: The capital markets are perfect with no transactions costs, no taxes, and equal access to information for all investors.

(A.2) Continuous trading.

(A.3) Itô dynamics: The value of the firm, V, satisfies the stochastic differential equation.

$$dV = (\alpha V - C)dt + \sigma V dz$$

where total cash outflow per unit time C is locally certain. α and σ^2 are the instantaneous expected rate of return and variance of return on the underlying assets.

(A.4) Constant σ^2.

(A.5) Nonstochastic term structure: The instantaneous interest rate $r(t)$ is a known function of time.

(A.6) Shareholder wealth maximization: Management acts to maximize shareholder wealth.

(A.7) Perfect bankruptcy protection: Firms cannot file for protection from creditors except when they are unable to make required cash payments. In this case perfect priority rules govern the distribution of assets to claimants.

(A.8) Perfect antidilution protection: No new securities (other than additional common equity shares) can be issued until all existing nonequity claims are extinguished. Deals between equity and subsets of other claimants are prohibited.

(A.9) Perfect liquidity: Firms can sell assets as necessary to make cash payouts, with no loss in total value.

Translating this set of assumptions into an explicit model for valuing claims in a typical capital structure is considerably more difficult than suggested by previous examples in the literature. A common capital structure consists of equity and multiple issues of callable nonconvertible sinking fund coupon debt. This differs from the standard example of a single issue of nonconvertible debt, due to Merton (1974), because of both the sinking fund and multiple issue features. One effect of sinking funds is to reduce the effective maturity of debt. Another effect, due to the option to retire at market or par (with or without an option to double the sinking fund payment), is to make debt more like equity. Multiple issues of debt introduce interactions among issues of debt, so that maximizing the value of equity need not be equivalent to minimizing the value of a given issue of debt, as in the single debt issue case. One accomplishment of this paper is to translate the usual assumption list into a model for realistic capital structures.

The plan of the paper is as follows. Section 6.2 presents a theoretical analysis of the valuation problem for a firm with equity and multiple issues of callable nonconvertible sinking fund coupon debt, based on the usual assumption list. Section 6.3 describes the empirical methodology, including numerical analysis techniques, sample data, and testing procedure. Section 6.4 presents an analysis of the results, and Section 6.5 gives a conclusion.

6.2 Theory

The theoretical basis of the corporate liability pricing model is developed in Black and Scholes (1973) and Merton (1974). They use an arbitrage argument to show that corporate liabilities which are functions of the value of the firm and time obey a partial differential equation which depends on the known schedule of interest rates $r \equiv r(t)$ and the variance rate of firm value σ^2, as well as on payouts and indentures on claims, but does not depend on expected returns on assets and liabilities of the firm. Nor does it depend on any equilibrium structure of risk and return. Readers are referred to these papers for a derivation of the basic partial differential equation.

A starting point for the analysis of realistic capital structures is the standard example of contingent claims valuation as applied to nonconvertible corporate bonds, namely the formulation in Merton (1974) of a callable coupon bond with no sinking fund. Merton shows that the equity $E(V, t)$ is a firm with one issue of such debt obeys the following partial differential equation and boundary conditions.

$$(1a) \qquad 0 = \tfrac{1}{2}\sigma^2 V^2 E_{VV} + (rV - cP - d)E_V + E_t - rE + d$$

$$E(0, t) = 0$$

$$E(V, t^*) = \max(0, V - P)$$

$$E(\bar{V}, t) = \bar{V} - k(t)P$$

$$E_V(\bar{V}, t) = 1\,,$$

where $P \equiv P(t)$ is the outstanding bond principal at time t, c is the coupon rate per unit principal, $k(t)$ is the call price schedule per unit principal, $d \equiv d(V, t)$ is the known dividend policy, and t^* is the maturity date of the bond. The upper free boundary, $\bar{V}(t)$, corresponds to the optimal call barrier at or above which the firm will call the bonds. Similarly, Merton shows that the valuation problem for the debt issue $D(V, t)$ can be formulated as follows:

$$
\text{(1b)} \qquad \begin{aligned}
&0 = \tfrac{1}{2}\sigma^2 V^2 D_{VV} + (rV - cP - d)D_V + D_t - rD + cP \\
&D(0, t) = 0 \\
&D(V, t^*) = \min(V, P) \\
&D(\bar{V}, t) = k(t)P \\
&D_V(\bar{V}, t) = 0 .
\end{aligned}
$$

The plan for section 6.2 is as follows. Section 6.2.1 generalizes the analysis of callable nonconvertible coupon bonds to allow for sinking funds, with and without noncumulative) options to double the sinking fund payment. Sinking funds are important because they dramatically decrease the effective maturity of bonds and because the option to sink at market or par makes bonds more like equity than otherwise. Section 6.2.2 then generalizes the analysis to deal with multiple issues of callable nonconvertible sinking fund coupon bonds. The ultimate contingent claims formulation of this valuation problem will bear only a generic resemblance to equations (1a) and (1b).

6.2.1 Sinking Funds

Most issues of corporate debt specify the mandatory retirement of bonds via periodic sinking fund payments. Typically the firm is required to retire a specified fraction of the initial bonds each period. Generally the firm has the option to redeem these bonds through either of two mechanisms: (1) it can purchase the necessary bonds in the market and deliver them to the trustee or (2) it can choose the necessary bonds by lot and retire them by paying the standard principal amount to their owners. Often the firm also has the option to double the number of bonds retired each period if it wishes. Hence the firm faces the following choices each period: (1) Should the bonds be called? (2) Assuming the bonds are not called, should the mandatory number of bonds be sunk at market or par? (3) Assuming the bonds are not called, should the sinking fund payment be doubled? (If this option exists.)

First, we consider the contingent claims formulation of this problem where the firm has no option to double the sinking fund payment. Next the option to double is introduced.

Sinking Funds with No Option to Double. Suppose that the firm decides not to call its debt and has no option to double. Then it must decide whether to sink bonds at market or par. Since the only difference is in the cash payout involved, and since higher firm value implies higher equity value, management will choose whichever costs less. For any given $r(t)$, if the firm value is relatively low, then debt will trade below par and the firm will choose to sink at market. And, for some $r(t)$, if firm value is relatively high, then debt will trade above par and the firm will choose to sink at par.

Consider the stylized case of a continuous sinking fund. Let s be the rate at which bonds are sunk, and let $P(t) = P(0) - st$ be the remaining principal assuming the bonds have not been called. Then $\gamma(t) \equiv s / P(t)$ is the fractional rate at which bonds are sunk. If debt trades below par, then total sinking fund payments are $\gamma D(V, t)$ where $\gamma \equiv \gamma(t)$. If debt trades above par, then total sinking fund payments are $\gamma P \equiv s$. Hence a general expression for total sinking fund payments is γ min (D, P). Thus the contingent claims formulation of the valuation problem for equity in the presence of a single issue of callable nonconvertible sinking fund coupon debt, with no option to double, is

(2a)
$$0 = \tfrac{1}{2}\sigma^2 V^2 E_{VV} + [rV - \gamma \min(D, P) - cP - d]E_V + E_t - rE + d$$
$$E(0, t) = 0$$
$$E(V, t^*) = \max(0, V - P)$$
$$E(\bar{V}, t) = \bar{V} - kP$$
$$E_V(\bar{V}, t) = 1 .$$

Similarly, from (1b), the contingent claims formulation of the valuation problem for debt in this capital structure is

(2b)
$$0 = \tfrac{1}{2}\sigma^2 V^2 D_{VV} + [rV - \gamma \min(D, P) - cP - d]D_V + D_t - rD + \gamma \min(D, P) + cP$$
$$D(0, t) = 0$$
$$D(V, t^*) = \min(V, P)$$
$$D(\bar{V}, t) = kP$$
$$D_V(\bar{V}, t) = 0 .$$

In summary, the valuation problem for a capital structure with equity and a single issue of callable nonconvertible sinking fund coupon debt, with no option to double, divides into three regions of firm value as a function of time. One region is defined by the fact that debt trades below par. This region corresponds at the maturity of the debt issue to firm values where bankruptcy occurs. A "par barrier" separates this region from the one above. The region above lies between the par barrier sand the call barrier, so that debt trades between par and the call price. Since the call barrier converges to par at the maturity of the debt issue, this region converges to a point. The third region lies above the call barrier. It corresponds at the maturity of the debt issue to firm values where bankruptcy does not occur.

Sinking Funds with an Option to Double. Most sinking funds give the firm an option to double the sinking fund payments. This section deals with noncumulative options to double, where the right to double is unaffected by past doubling decisions. There also exist cumulative options to double, where the right to double is affected by past decisions. Given the option to double the sinking fund payment, the actual principal that will be outstanding at any future date is unknown. Hence the values of equity and debt can no longer be written as functions of firm value and time alone. However, the following theorem says that these values can be written as functions of firm value, current principal, and time:

THEOREM 1: Assume that the optimal retirement rate, $\dot{P}(V, P, t)$, for bonds can be expressed as a deterministic function of firm value, current principal, and time. Then equity and debt and functions $E(V, P, t)$ and $D(V, P, t)$ that obey the following partial differential equations:

$$(3a)\quad 0 = \tfrac{1}{2}\sigma^2 V^2 E_{VV} + [rV - \gamma^* \min(D, P) - cP - d]E_V - \gamma^* P E_P + E_t - rE + d$$

$$(3b)\quad 0 = \tfrac{1}{2}\sigma^2 V^2 D_{VV} + [rV - \gamma^* \min(D, P) - cP - d]D_V - \gamma^* P D_P + D_t - rD + \gamma^* \min(D, P) + cP$$

where $\gamma^*(V, P, t) \equiv -\dot{P}/P$.

Proof: Apply Itô's lemma to $E(V, P, t)$ and $D(V, P, t)$, noting that P is locally certain. Substitute this into the standard arbitrage proof given in Merton (1974). Q.E.D.

Theorem 1 provides a valuation logic once the optimal policy with respect to doubling the sinking fund payment has been determined. Consider the decision whether to double the current sinking fund payment, assuming that management acts optimally thereafter. Suppose that the sinking fund payment is not doubled, so that the fraction of bonds retired is $\gamma dt = s dt/P$. Let V and P be firm value and current principal before the sinking fund payment. Hence the value of equity after the sinking fund payment is $E[V - \min(D, P)\gamma dt, (1 - \gamma dt)P, t]$. Suppose alternatively that the sinking fund payment is doubled. By analogy the value of equity after the sinking fund payment is $E[V - 2\min(D, P)\gamma dt, (1 - 2\gamma dt)P, t]$. The difference between the two equity values is thus [min. (D, P), $E_V + PE_P]\gamma dt$. If the bracketed expression is positive, the firm should not double the sinking fund payment; otherwise it should.

Since min (D, P) is less than the call price kP, doubling the sinking fund payment is a cheap way of calling a fraction of the bonds. Hence there will be a "doubling barrier" $\bar{\bar{V}}(P, t)$ which lies below the call barrier $\bar{V}(P, t)$. The firm will double the sinking fund payment above the doubling barrier but not below it. The firm is indifferent between doubling and not doubling right at the barrier; hence the expression we just derived vanishes at the barrier. Using this logic in (3a), the contingent claims formulation of the valuation problem for equity in the presence of a single issue of callable nonconvertible sinking fund coupon debt, with a noncumulative option to double, is as follows:

(4a)
$$0 = \tfrac{1}{2}\sigma^2 V^2 E_{VV} + [rV - \gamma\min(D, P) - cP - d]E_V - \gamma PE_P + E_t - rE + d,\ 0 \le V \le \bar{\bar{V}}(P, t)$$

$$0 = \tfrac{1}{2}\sigma^2 V^2 E_{VV} + [rV - 2\gamma\min(D, P) - cP - d]E_V - 2\gamma PE_P + E_t - rE + \mathrm{d},\ \bar{\bar{V}}(P, t) \le V \le \bar{V}(p, t)$$

$$E(0, P, t) = 0$$

$$E(V, 0, t) = V$$

$$E(V, P, t^*) = \max(0, V - P)$$

$$\min[D(\bar{\bar{V}}, P, t), P]E_V(\bar{\bar{V}}, P, t) + PE_P(\bar{\bar{V}}, P, t) = 0$$

$$E(\bar{V}, P, t) = \bar{V} - kP$$

$$E_V(\bar{V}, P, t) = 1\,.$$

ontingent claims formulation of the valuation problem for debt in this capital structure is

(4b)
$$0 = \tfrac{1}{2}\sigma^2 V^2 D_{VV} + [rV - \gamma\min(D, P) - cP - d]D_V - \gamma PD_P + D_t - rD + \gamma\min(D, P) + cP \quad \text{for } 0 < V < \bar{\bar{V}}(P, t)$$

$$0 = \tfrac{1}{2}\sigma^2 V^2 D_{VV} + [rV - 2\gamma\min(D, P) - cP - d]D_V - 2\gamma PD_P + D_t - rD + 2\gamma\min(D, P) + cP \quad \text{for } \bar{\bar{V}}(P, t) \le V \le \bar{V}(P, t)$$

$$D(0, P, t) = 0$$

$$D(V, 0, t) = 0$$

$$D(V, P, t^*) = \min(V, P)$$

$$\min[D(\bar{\bar{V}}, P, t), P]\,[D_V(\bar{\bar{V}}, P, t) - 1] + PD_P(\bar{\bar{V}}, P, t) = 0$$

$$D(\bar{V}, P, t) = kP$$

$$D_V(\bar{V}, P, t) = 0\,.$$

Actual sinking fund indentures cause claims to be nonhomogeneous functions of firm value and current principal. The reason is that the fractional rate at which bonds are retired (γ or 2γ where $\gamma = s/P$) grows as current principal declines. However, there is a reasonable approximation to actual sinking fund indentures that simplifies the analysis and leads to additional insights. Namely, assume that the fractional rate at which bonds must be sunk is γ, a constant, or 2γ if the sinking fund payment is doubled. In effect this assumes that the current decision whether to double the sinking fund payment does not affect permitted future fractional rates at which bonds are sunk.

This assumption plus the assumption that dividends are proportional to firm value reduce the dimensionality of the equations in (4a) and (4b). Consider standardized values for firm value ($x \equiv V/P$), equity ($f \equiv E/P$), and debt ($g \equiv D/P$); and define the proportional dividend rate as $\delta \equiv d/V$. Substituting these into (4a) and using the new assumptions, the following standardized formulation

$$
\begin{aligned}
\text{(5a)}\qquad & 0 = \tfrac{1}{2}\sigma^2 x^2 f_{xx} + [(r+\gamma-\delta)x - \gamma\min(g, 1) - c] f_x \\
& \qquad + f_t - (r+\gamma)f + \delta x,\ 0 \le x \le \bar{\bar{x}}\ (t) \\
& 0 = \tfrac{1}{2}\sigma^2 x^2 f_{xx} + [(r+2\gamma-\delta)x - 2\gamma - c] f_x \\
& \qquad + f_t - (r+2\gamma)f + \delta x,\ \bar{\bar{x}}\ (t) \le \bar{x} \le x(t) \\
& f(0, t) = 0 \\
& f(x, t^*) = \max\,(0, x-1) \\
& (1-\bar{\bar{x}}) f_x(\bar{\bar{x}}, \mathrm{t}) + f(\bar{\bar{x}}, t) = 0 \\
& f(\bar{x}, t) = \bar{x} - k \\
& f_x(\bar{x}, t) = 1\,.
\end{aligned}
$$

Note that this implies a doubling barrier which lies between the par barrier and the call barrier, so that the firm is always sinking at par if it doubles the sinking fund payment. To see that this is so, reconsider the expression derived before, namely, min $(D, P)E_V + PE_P$. Suppose that the debt is trading below par, so that this expression is $DE_V + PE_P = (V + E)\ E_V - PE_P$. Under the new assumptions, equity is a homogeneous function of firm value and current principal. Hence by Euler's condition $E = VE_V + PE_P$. Substituting this into the expression gives $E(1 - E_V) \geq 0$, which says that the sinking fund payment should not be doubled.

Similarly, using (4b), debt is proportional to a standardized solution, $g(x, t)$, where

(5b)
$$0 = \tfrac{1}{2}\sigma^2 x^2 g_{xx} + [(r+\gamma-\delta)x - \gamma\min(g,1) - c]g_x + g_t - (r+\gamma)g + \gamma\min(\gamma\min(g,1) + c,\ 0 \le x \le \bar{\bar{x}}(t)$$

$$0 = \tfrac{1}{2} + \sigma^2 x^g{}_{xx} + [(r+2\gamma-\delta)x - 2\gamma - c]g_x + g_t - (r+2\gamma)g + 2\gamma + c,\ \bar{\bar{x}}(t) \le x \le \bar{x}(t)$$

$$g(0,t) = 0$$

$$g(x,t^*) = \min(x,1)$$

$$(1-\bar{\bar{x}})g_x(\bar{\bar{x}},t) + g(\bar{\bar{x}},t) - 1 = 0$$

$$g(\bar{x},t) = k$$

$$g_x(\bar{x},t) = 0.$$

In summary, the valuation problem for a capital structure with equity and a single issue of callable nonconvertible sinking fund coupon debt, with a noncumulative option to double, divides into four regions of firm value as a function of time. One region is defined by the fact that debt trades below par. In this region bonds are sunk at market and sinking fund payments are not doubled. This region corresponds at the maturity of the debt issue to firm values where bankruptcy occurs. A second region lies between the par barrier and the doubling barrier. In this region bonds are sunk at par and sinking fund payments are not doubled. A third region lies between the doubling barrier and the call barrier. In this region bonds are sunk at par and sinking fund payments are doubled. Since the call barrier converges to par at the maturity of the debt issue, both the second and third regions coverage to a point. The fourth region lies above the call barrier. It corresponds at the maturity of the debt issue to firm values where bankruptcy does not occur. For some given $r(t)$, $k(t)$, and c, it is possible that debt will always trade below par. Thus bonds are always sunk at market and the sinking fund payment is never doubled. In these cases there is only one region, since the par barrier, doubling barrier and the call barrier do not exist.

Unfortunately, incorporating the option to double the sinking fund payment in a capital structure with numerous debt issues dramatically increases the dimensionality of the valuation problem. Therefore the option to double is ignored in the numerical approximations. The results in this section imply that this leads to underpricing of equity and the overpricing of debt.

6.2.2 Multiple Debt Issues

This section generalizes the analysis to allow for multiple debt issues. This feature of debt is important because it introduces interactions among bonds that are not present in the standard example of one debt issue. For expositional simplicity, this section considers the case of two issues of callable nonconvertible sinking fund coupon debt (with no options to double).

The value of any remaining claims in a capital structure initially composed of equity and two issues of callable nonconvertible sinking fund coupon debt, with no options to double, will depend on whether either debt issue has been redeemed via a call decision, as well as on firm value and time. In effect the capital structure of the firm can be in any one of four states, which is indexed by the variable θ. If there are n debt issues then there are 2^n such states. $\theta = 0$ in the state where both issues of debt have been previously called. The valuation problem in this state is trivial; equity value equals firm value. $\theta = 1$ is the state where bond 1 is alive but bond 2 has been called. $\theta = 2$ is the state where bond 2 is alive but bond 1 has been called. Finally, $\theta = 3$ is the state where neither bond has been called.

With this notation the values of claims can be written as functions of the current capital structure state as well as firm value and time. Letting $E(V, \theta, t)$ $D(V, \theta, t)$, and D' (V, θ, t) be the values of equity and the two debt issues, they obey the following system of partial differential equations in any relevant capital structure state:

$$\text{(6a)} \quad 0 = \tfrac{1}{2}\sigma^2 V^2 E_{VV} + (rV - \pi - \pi' - d)E_V + E_t - rE + d; \quad \theta = 1, 2, 3$$

$$\text{(6b)} \quad 0 = \tfrac{1}{2}\sigma^2 V^2 D_{VV} + (rV - \pi - \pi' - d)D_V + D_t - rD + \pi; \quad \theta = 1, 3$$

$$\text{(6c)} \quad 0 = \tfrac{1}{2}\sigma^2 V^2 D'_{VV} + (rV - \pi - \pi' - d)D'_V + D'_t - rD' + \pi'; \quad \theta = 2, 3.$$

π and π' are simply total cash payouts to the two debt issues. Taking account of whether bonds have been called and whether it makes sense to sink at market or par,

$$\pi(V, 1, t) = \pi(V, 3, t) \equiv \gamma \min(D, P) + cP$$

$$\pi(V, 2, t) \equiv 0$$

$$\pi'(V, 2, t) = \pi'(V, 3, t) \equiv \gamma' \min(D', P') + c'P'$$

$$\pi'(V, 1, t) = 0.$$

Note how current values of debt issues enter into valuation equations for other claims. Hence equations (6a)–(6c) must generally be solved simultaneously. It is always possible to eliminate one relevant equation, since the claims sum to firm value.

Boundary conditions are needed to relate the solutions to (6a), (6b), and (6c) for different capital structure states to each other and to complete the contingent claims formulation of the general valuation problem. For each relevant security in each state a lower boundary condition, a terminal boundary condition, and an upper (free) boundary condition must be specified. The lower boundary condition in every case is trivial; limited liability translates zero firm value into zero value for every claim: $E(0, \theta, t) = D(0, \theta, t) = D'(0, \theta, t) = 0$.

Each state has a unique terminal boundary. Let t^* be the maturity of debt issue D and let $t^{*\prime}$ be the maturity of debt issue D'. Without loss of generality $t^* \leq t^{*\prime}$. First suppose that the firm is in capital structure state $\theta = 1$, where the second debt issue has been called. Then the terminal boundary coincides with the maturity of the first debt issue. The terminal boundary condition in this case is standard for a capital structure with a single issue of callable nonconvertible coupon debt:

$$E(V, 1, t^*) = \max[0, V - P(t^*)]$$

$$D(V, 1, t^*) = \min[V, P(t^*)].$$

Next suppose that the firm is in capital structure state $\theta = 2$, where the first debt issue has been called. Then the terminal boundary coincides with the maturity of the second debt issue. Again the terminal boundary condition is standard:

$$E(V, 2, t^*) = \max[0, V - P'(t^{*\prime})]$$

$$D'(V, 2, t^{*\prime}) = \min[V, P'(t^{*\prime})].$$

Finally, suppose that the firm is in capital structure state $\theta = 3$, where neither debt issue has been called. Then the terminal boundary coincides with the earlier maturity date, since the firm must transit to a new capital structure state on this date. In the example the first debt issue matures at t^*. Since the debt is callable, the only relevant region has to do with firm values which are insufficient to cover the remaining principal on the first debt issue, so that the firm is bankrupt. Since firm value is insufficient to meet principal payments on the first debt issue alone, equity is worthless in this region: $E(V, 3, t^*) = 0$. The value of the two debt issues in this region depends on seniority. If the first issue is senior, then

$$D(V, 3, t^*) = V$$

$$D'(V, 3, t^*) = 0.$$

If the second issue is senior, then

$$D(V, 3, t^*) = \max[0, V - P'(t^*)]$$

$$D'(V, 3, t^*) = \min[V, P'(t^*)].$$

Finally, if neither issue is senior, then both get pro rata shares:

$$D(V, 3, t^*) = VP(t^*)/[P(t^*) + P'(t^*)]$$
$$D'(V, 3, t^*) = VP'(t^*)/[P(t^*) + P'(t^*)].$$

It remains to specify upper free boundary conditions corresponding to optimal call decisions in each of the capital structure states. First suppose that the firm is in capital structure states $\theta = 1$, where the second debt issue has been called. The upper free boundary conditions in this case are standard for a capital structure with a single issue of callable conconvertible coupon debt:

$$E[\bar{V}(1, t), 1, t] = \bar{V}(1, t) - k(t)P(t)$$
$$E_V[\bar{V}(1, t), 1, t] = 1.$$

Similarly, if the firm is in capital structure state $\theta = 2$, where the first debt issue has been called, then the conditions are

$$E[\bar{V}(2, t), 2, t] = \bar{V}(2, t) - k'(t)P'(t)$$
$$E_V[\bar{V}(2, t), 2, t] = 1.$$

Finally, suppose that the firm is in capital structure state $\theta = 3$, where both debt issues are alive. The upper free boundary in this state corresponds to the barrier where the firm calls one of the bond issues and thus transits to another state. Since management chooses the bond to call so as to maximize shareholder wealth,

$$E[\bar{V}(3, t), 3, t] = \max\{E[\bar{V}(3, t) - k(t)P(t), 2, t], \\ E[\bar{V}(3, t) - k'(t)P'(t), 1, t]\}.$$

Similarly the "high contact" optimization condition is

$$E_V[\bar{V}(3, t), 3, t] = \partial \max E[\bar{V}(3, t) - k(t)P(t), 2, t], \\ E[\bar{V}(3, t) - k'(t)P'(t), 1, t]/\partial V$$

Suppose that it is optimal to call the first debt issue at $\bar{V}(3, t)$, then the values of the debt issues on this barrier are

$$D[\bar{V}(3, t), 3, t] = k(t)P(t)$$
$$D'[\bar{V}(3, t), 3, t] = D'[\bar{V}(3, t) - k(t)P(t), 2, t].$$

Conversely suppose that it is optimal to call the second debt issue, then

$$D[V(3, t), 3, t] = D[V(3, t) - k'(t)P'(t), 1, t]$$
$$D'[V(3, t), 3, t] = k'(t)P'(t).$$

In summary, the valuation problem for capital structures containing equity and two issues of callable nonconvertible sinking fund coupon debt corresponds to the simultaneous solution of a system of partial differential equations. Appropriate combinatorial application of these principles leads directly to a formulation of the valuation problem for capital structures containing equity and n issues of callable nonconvertible sinking fund coupon debt. This approach is necessitated by the fundamental problem of determining the optimal call policy governing the n callable bonds. This formulation identifies that policy which maximizes the value of the equity.

It is important to understand the dimensionality of the n issue case. First note that there are 2^n possible capital structure states, including the trivial state of an all-equity firm. Furthermore, there are a number of securities to be value in each state. One way to calculate the number of different solutions to partial differential equations required in the n issue case is as follows. There are $\binom{n}{n} = 1$ capital structure states corresponding to 0 bonds having been called. In this one state there are $n + 1$ securities outstanding for a total of $n + 1$ solutions. There are $\binom{n}{n-1} = n$ capital structure states corresponding to one bond's having been called. In each of these n states there are n securities outstanding. Continuing in this way, we find that there are $\Sigma_{j=0}^{n-1} \binom{n}{n-j} (n + 1 - j)$ solutions in all. Hence one high priority line of research in terms of applying contingent claims valuation to realistic capital structures is the derivation of rational theorems which rule out various capital structure states (e.g., which show that certain kinds of bonds are always called first.

6.3 Data and Methodology

Data were collected for 15 firms on a monthly basis from January 1975 to January 1981. The firms were chosen based on a number of criteria at the beginning of 1975:

1. Simple capital structures (i.e., one class of stock, no convertible bonds, small number of debt issues, no preferred stock).
2. Small proportion of private debt to total capital.
3. Small proportion of short-term notes payable or capitalized leases to total capital.
4. All publicly traded debt is rated.

Based on these criteria the following firms were selected:

1. Allied Chemical
2. Anheuser Busch
3. Brown Group
4. Bucyrus Erie
5. Champion Spark Plug
6. Cities Service
7. CPC
8. MGM
9. Proctor & Gamble
10. Pullman
11. Raytheon
12. Republic Steel
13. Seagram
14. Sunbeam
15. Upjohn

The contingent claims valuation model requires three kinds of data in order to solve for prices of individual claims as functions of total firm value: (1) indenture data, (2) variance rate data, and (3) interest rate data. The bond indentures define the boundary conditions which constitute the economic description of various claims. For example, the following data were collected for each bond for each firm: principal, coupon rate, call price schedule, call protection period, sinking fund payments, and options to sink at market or par. The bond convenant data were collected from Moody's *Bond Guide*, except that sinking fund payments were collected from the monthly Standard and Poor's *Bond Guide*. For purposes of testing the model, actual bond prices were also collected from the latter sources.

The following procedure was used to estimate a variance rate for each firm in the sample, as of each January from 1977 through 1981. For each of the trailing 24 months, we calculated the percentage return on the total of all claims, including any payouts, that were outstanding at the beginning of the month. (To estimate the market value of nontraded debt, we assumed that the ratio of market value to book value was the same as for traded debt.) The sample variance of this percentage return gives an estimate of the variance rate for the firm as a whole. Table 6.1 summarizes the estimates.

The standard assumption in contingent claims analysis is that the future course of interest rates, $r(t)$, is known. Specifically, it is often assumed that the instantaneous rate of interest is constant through time (i.e., a flat term structure). The assumption of a flat term structure results in a fundamental problem for the empirical test of the contingent claims

Table 6.1 **Estimates of Standard Deviation of Returns to Firm (Annualized) January 1977–January 1981**

	1977	1978	1979	1980	1981
1. Allied Chemical	.193	.204	.184	.196	.185
2. Anheuser Busch	.225	.228	.217	.245	.255
3. Brown Group	.200	.152	.151	.157	.192
4. Bucyrus Erie	.301	.268	.211	.231	.268
5. Champion Spark Plug	.257	.178	.215	.227	.220
6. Cities Service	.160	.149	.129	.169	.327
7. CTC	.191	.176	.143	.131	.173
8. MGM	.190	.155	.258	.303	.420
9. Proctor & Gamble	.150	.146	.165	.149	.156
10. Pullman	.330	.236	.308	.348	N.A.
11. Raytheon	.278	.182	.227	.280	.291
12. Republic Steel	.168	.170	.141	.158	.173
13. Seagram	.268	.234	.171	.216	.380
14. Sunbeam	.258	.204	.240	.287	.288
15. Upjohn	.320	.207	.215	.233	.216

model. If a flat term structure is assumed then the model will misprice riskless bonds. Therefore the test of whether contingent claims analysis can price risky bonds is systematically flawed. This problem is handled by the assumption that the future course of the one-year rate of interest will be consistent with the one-year forward interest rates implied by the current term structure. This procedure will result in the correct pricing of riskless bonds. The following procedure was used to estimate implied one-year forward interest rates for 25 years, as of each January from 1977 through 1982. First, identify all par government bonds as of that date. These data were gathered from the *Wall Street Journal*. There are usually fewer than 25 such bonds. Therefore linear interpolation was used to complete a 25-year yield curve for par government bonds. Then this yield curve was solved for implied one-year forward rates. Hence the implied forward rates pertain to a par term structure.

The method of Markov chains is used to approximate solutions to the problems posed in the previous section. Parkinson (1977), Mason (1978), and Cox et al. (1979) use Markov chains to approximate solutions to valuation problems similar to the ones considered in this paper. The method of finite differences has been used by Brennan and Schwartz (1976*a*, 1976*b*) to treat similar contingent claims equations. The methods of Markov chains and finite differences are very similar, as demonstrated in Brennan and Schwartz (1978) and Mason (1978). Readers are referred to these papers for background on numerical analysis techniques.

If all claims are publicly traded, then the value of the firm can be observed and prices for all claims, relative to the observed firm value, can be predicted. However, since all claims on the test firms are not publicly traded, an alternative approach had to be taken. Namely, the equity pricing function was used to estimate firm value. In other words, what firm value is consistent with the actual equity value? Then this estimated firm value was used to predict debt prices. Note that this procedure amplifies systematic errors in pricing the debt. For example, suppose that the model systematically underprices equity and overprices debt, as functions of firm value. Then this procedure will make two compounding errors. First, it will overestimate the value of the firm. Then it will overestimate debt as a function of firm value. Hence it will overestimate debt for both reasons. Counting each year from 1977 through 1981, and counting each bond existing in each year for each of the 15 firms, we solved numerically for prices of 163 bonds, as well as for equity values. The next section describes our results.

6.4 Empirical Results

Table 6.2 summarizes the empirical results for the 163 bonds in the sample. It reveals that the average percentage pricing error—defined as

Table 6.2 Percentage Pricing Error for Various Partitions of Sample

Partition	Number of Bonds	Percent of Sample	Error Mean	Error S.D.	Absolute Error Mean	Absolute Error S.D.	Difference of Means Test*
All	163	100.0	.0149	.0714	.0589	.0430	N.A.
High rated	133	81.6	.0023	.0637	.0518	.0373	5.072
Low rated	30	18.4	.0707	.0767	.0906	.0517	
Low variance†	40	24.5	.0096	.0611	.0527	.0323	1.597
High variance	43	26.4	.0339	.0746	.0660	.0485	
Short term	74	45.4	.0077	.0761	.0611	.0460	1.164
Long term	89	54.6	.0208	.0667	.0571	.0403	
Senior	147	90.2	.0083	.0701	.0570	.0417	3.678
Junior	16	9.8	.0751	.0523	.0763	.0506	
Low coupon	64	39.3	−.0228	.0669	.0589	.0391	5.937
High coupon	99	60.7	.0392	.0632	.0589	.0454	
Discount	143	87.7	.0075	.0703	.0569	.0420	3.700
Premium	20	12.3	.0679	.0547	.0731	.0475	
Low current yield	86	52.8	−.0065	.0684	.0557	.0403	4.238
High current yield	77	47.2	.0388	.0670	.0625	.0456	

*(Equal variances). The test assumes independent observations. As might be expected, our sample exhibits positive correlation among errors for the same bond in different years (as well as in other ways). This biases the test toward detecting a significant difference in means.

†Since firm variance rates are remarkably similar, we examined the two extreme quartiles.

predicted price minus actual price, divided by actual price—is less than 1½%. The standard deviation of the percentage pricing error is about 7%. The average absolute value of the percentage pricing error is about 6%. The accompanying histogram in figure 6.1 gives additional information on these errors.

Although there is almost no systematic bias in pricing errors for the sample as a whole, there might be systematic bias among subsets of bonds that simply cancel out in the entire sample. We tested for this by dividing the sample according to conventional classifications. For example, table 6.2 indicates that the model underprices bonds with high ratings ($\geq$ A rating) and overprices bonds with low ratings ($<$ A rating) and that this difference is statistically significant.

Statistical significance is measured by a difference of means test. This test assumes that the two underlying populations are normally distributed with the same variance. In addition it is assumed that the samples are made up of independent draws. To the extent that the samples are not made up of independent draws, the test is biased. It is likely that the samples studied in this section have positively correlated errors; thus the reports of statistical significance are biased upward.

Table 6.2 shows that the model underprices bonds on firms with low variance rates and overprices bonds on firms with high variance rates.

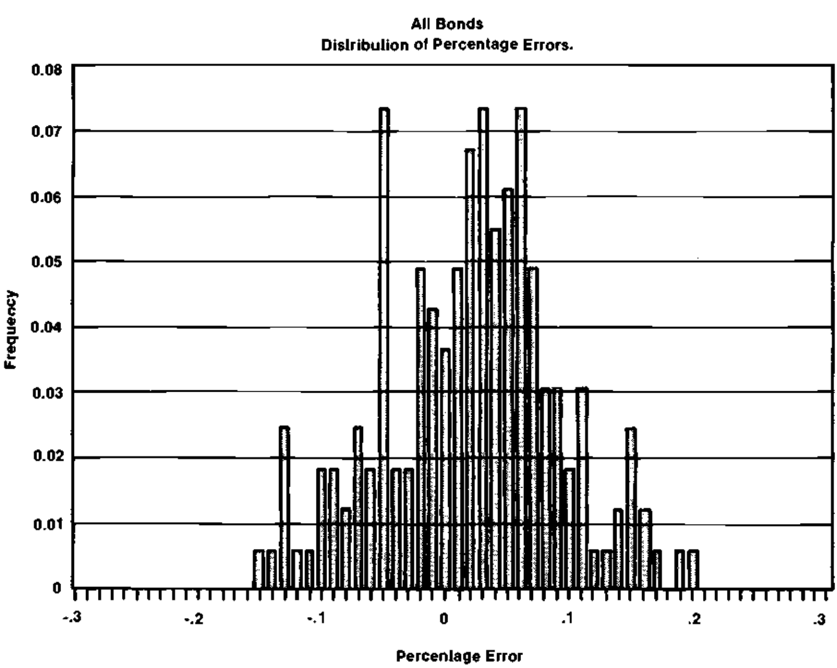

Fig. 6.1 All bonds.

The table indicates that the model underprices bonds with stated maturities less than 15 years and overprices bonds with stated maturities greater than 15 years. Of course, total variance equals the variance rate multiplied by time. Hence overpricing high variance and long maturity bonds may be two sides of the same coin. The table shows that the model prices senior bonds correctly on average, but overprices junior bonds. Finally, the table shows that the model underprices low coupon bonds (coupon rate $\leq 7\%$) and overprices high coupon bonds (coupon rate $> 7\%$).

In summary, the model tends to underprice safe bonds and overprice risky bonds in a systematic way. This leads us to conclude that the usual assumptions in the contingent claims valuation literature are violated in some systematic way. Three assumptions are questioned in particular: (1) the assumption of zero personal taxes, (2) the assumption of a constant variance rate, and (3) the assumption of perfect antidilution protection. The plan is as follows. First, there is a discussion of what kinds of pricing errors would ensue from violation of each of these three assumptions. Then empirical evidence is presented from the sample that is designed to discriminate among pricing errors induced by violation of each of these assumptions.

6.4.1 Personal Tax Assumption

According to assumption (A.1), which is standard in the contingent claims valuation literature, there are no personal taxes. This implies that investors capitalize ordinary income and capital gains in the same way. However, conventional wisdom says that investors prefer capital gains to ordinary income for tax reasons. Furthermore, Ingersoll (1976) finds that inclusion of differential taxes on ordinary income and capital gains improves the ability of the contingent claims valuation model to predict prices for the income and capital shares of dual funds.

If differential taxes cause investors to capitalize ordinary income differently from capital gains, then failure to include this in the model could lead to overpricing bonds with higher current yields relative to bonds with lower current yields. (See Ingersoll [1976, p. 110] for a careful discussion of this issue.) First consider highly rated bonds. Recall that the interest rates in the model are derived from a term structure for par government bonds. Given the tax treatment of bonds trading in the secondary market, high-quality discount bonds should be underpriced relative to high-quality premium bonds. This is due to the fact that the IRS allows investors to amortize secondary market premiums against interest income while also allowing realized gains due to secondary market discounts to be taxed at capital gains rates.

Another dimension of any tax effect has to do with risk. Consider low-quality par bonds versus high-quality par bonds (e.g., new issue bonds on high-variance vs. low-variance firms). The expected capital loss on the low-quality bonds is larger in absolute terms than the expected capital loss on the high-quality bonds. Hence the low-quality bonds will have a higher coupon rate than the high-quality bonds. Since the higher taxes on the low-quality bond are ignored, any tax effect will cause low-quality to be overpriced relative to high-quality bonds. In particular, since government par bonds are perfectly safe, any tax effect will cause corporate par bonds to be overpriced in general. Similar considerations say that any tax effect will cause junior par bonds to be overpriced relative to senior par bonds. And similar considerations also suggest that any tax effect will cause longer maturity par bonds to be overpriced relative to shorter maturity par bonds.

6.4.2 Variance Rate Assumption

According to assumption (A.4), which is standard in the contingent claims valuation literature, the variance rate of firm value σ^2 is a constant. Empirical evidence for common equity suggests that its variance rate goes up as its level goes down. Of course, this is consistent with a constant variance rate for firm value—because of the possibility of leverage effects. However, it is also consistent with a nonconstant firm value variance rate.

Suppose that the variance rate of firm value is not a constant, but rather increases as firm value decreases. For example, the stochastic process for firm value might belong to the constant elasticity of variance class. And suppose that a constant variance rate is falsely assumed in estimating σ^2. What kinds of pricing errors would this include? These errors would be similar in type to those induced by an underestimate of a variance rate that is in fact constant. In other words, in either case the probability of financial distress is significantly underestimated.

Underestimating the variance will not matter much for high-quality bonds. But it will cause low-quality bonds to be overpriced by a significant amount. Hence underestimating the variance will cause corporate bonds to be overpriced in general and will cause low-quality bonds to be overpriced relative to high-quality bonds. Similar considerations suggest that underestimating the variance will cause junior bonds to be overpriced relative to senior bonds and longer maturity bonds to be overpriced relative to shorter maturity bonds.

6.4.3 Dilution Assumption

According to the perfect antidilution assumption in (A.8), which is standard in the contingent claims valuation literature, no new bonds can be issued until all old bonds have been extinguished. Furthermore, according to the perfect liquidity assumption in (A.9), firms can simply sell assets in order to make cash payouts. Hence in the model equity maximizes its value by funding all cash payouts through asset sales.

However, firms which call bonds normally have the option to fund the call by issuing new bonds with the same priority. Holding firm value constant, this allows management to dilute any remaining bonds, as compared to the model which allows for no dilution. On the other hand, the model causes firm value to go down when bonds are called, as compared to refunding with new bonds that keeps firm value constant. Now suppose equity can choose between refunding and asset liquidation to finance a call decision. The option to refund can have value to equity. Failure to include the option to refund in our model will cause equity to be underpriced and debt to be overpriced in general. Since the option to refund has value because of the possibility of diluting existing debt, junior debt will be overpriced relative to senior debt and longer maturity debt will be overpriced relative to shorter maturity debt. In other words, debt can be economically junior either because it is explicitly junior or because it has a relatively longer maturity than other debt.

6.4.4 Empirical Evidence on Violation of These Assumptions

The empirical evidence tends to confirm the existence of a tax effect, a variance effect, and a dilution effect. Table 6.2 gives evidence of a tax effect. It shows that the model underprices discount bonds relative to premium bonds. These results continue to obtain when examining only high-quality bonds, where variance rate effects and dilution effects are minimal. Table 6.2 gives further evidence of a tax effect. It shows that the model overprices bonds with above-average coupon yields relative to bonds with below-average coupon yields. (The median coupon yield in the sample is approximately 9%.) Again, the results continue to obtain when examining only high-quality bonds. Hence there is unambiguous evidence for the existence of a tax effect.

There is also empirical evidence for a variance effect. A naive test for the existence of a variance effect in whether bonds of firms with high estimated variance rates are overpriced relative to bonds of firms with low estimated variance rates, since risky bonds are more sensitive to underestimating variance than safe bonds. Table 6.2 showed that this is the case. However, this is a naive test, because a tax effect alone would cause risky bonds to be overpriced relative to safe bonds. This is because, everything else equal, risky bonds have higher expected capital losses than safe bonds, which is compensated for by higher current yield. To test for a variance effect independent of any tax effect, the sample is first split according to high versus low current yield. This is done to control for the tax effect. Then pricing errors are compared for bonds of high- versus low-variance firms within each subsample. Table 6.3 reports these re-

Table 6.3 Percentage Pricing Errors for Various Subpartitions of Sample

Subpartition	Number of Bonds	Percent of Sample	Error		Absolute Error		Difference of Means Test
			Mean	S.D.	Mean	S.D.	
Low current yield:							
Low variance	22	13.5	−.0058	.0592	.0480	.0352	.656
High variance	16	9.8	.0080	.0666	.0536	.0903	
High current yield:							
Low variance	18	11.0	.0284	.0580	.0585	.0272	.976
High variance	27	16.6	.0492	.0748	.0734	.0514	
High current yield and variance:							
Senior	20	12.3	.0297	.0691	.0623	.0422	2.459
Junior	7	4.3	.1050	.0612	.1050	.0612	

sults. It shows that bonds of high-variance firms continue to be overpriced relative to low-variance firms within each subsample, although the effect is more pronounced for bonds with high current yield. Furthermore, almost identical results hold when junior bonds are excluded from the sample, to check against the possibility that variance only proxies for a dilution effect. These results are interpreted as evidence for a variance effect in addition to a tax effect.

Lastly, the question remains, Is there evidence for a dilution effect in addition to a tax effect and a variance effect? A naive test for the existence of a dilution effect is whether economically junior bonds are overpriced—that is, either bonds which are explicitly junior or bonds that are effectively junior because of their longer maturity—relative to economically senior bonds. Table 6.2 showed that this is the case; junior bonds are overpriced relative to senior bonds.

As before, this is a naive test, because either a tax effect or a variance effect alone would cause junior bonds to be overpriced relative to senior bonds. To get a more sophisticated test, the sample is first restricted to bonds with high current coupon yield issued by corporations with high variance rates, which tends to control for tax and variance effects. Table 6.3 shows the results. Although junior bonds continue to be overpriced relative to economically senior bonds, the effect is not strong. Hence there appears to be a dilution effect, but it is not as strong as the tax and variance effects.

6.5 Conclusion

In this paper a theoretical model is derived for valuing claims in realistic capital structures containing equity and multiple issues of callable nonconvertible sinking fund coupon debt, based on the usual assumptions in the contingent claims valuation literature. This model is tested on a number of bonds for 15 firms yearly from 1977 through 1982. The predicted prices are not systematically different from actual prices for the sample as a whole. However, predicted prices are systematically different from actual prices for various types of bonds in the sample. Evidence exists for a systematic tax effect and a systematic variance effect in the results. There is also evidence for a less significant dilution effect associated with the option to refund.

Establishing the empirical validity of contingent claims analysis as a corporate liability pricing model is a large and complex task. A number of theoretical and methodological problems must be addressed. For example, as demonstrated in this paper, sinking funds and optimal call policies for multiple bond capital structures warrant further theoretical study. It has also been demonstrated that the detailed consideration of the interaction of multiple bond convenants can significantly increase the dimensionality of the overall valuation problem. This underscores the need for research into more efficient numerical analysis methods.

We view this paper as an important first step in establishing the empirical validity of contingent claims analysis. Given the results of the paper, current research is under way, using an expanded database, where the problem formulation takes explicit account of personal taxes, the option to refund, the cost of financial distress, and changing variance rates. Once the results of this current research are known, a portfolio test will be conducted to determine if market inefficiencies can explain any of the discrepancies between the model prices and market prices.

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Comment Fischer Black

This is a costly model. It uses a lot of computer time. At the end of it all, the average error in pricing bonds is 6%. I am surprised that Jones et al. are able to create a model with such a large error. Surely an investment banker can price a new bond more accurately than that. I am disappointed, because I think that the best application of option theory is to risky bonds. I hope that the best models will be more accurate.

Part of the problem is that the authors handicap themselves. They don't allow themselves to use some of the information that an investment banker is able to use. For example, they don't use information about the current prices of comparable bonds.

Overall, I like the paper very much. I find it very stimulating. I think it is the most thought-provoking paper on the valuation of corporate bonds that I've seen. It is well written, too. There is math in it, but the math is sufficiently hidden that it doesn't get in the way of understanding what the paper is saying.

In trying to figure out how Jones et al. can be so far off in pricing the bonds, I began to think about the assumptions they make. They are very careful about certain assumptions, such as looking at the exact indenture provisions on the bonds. There are other assumptions, though, that one might take differently than they did. These assumptions might make a difference in the values they get.

For example, they assume that a firm goes along, makes the sinking fund and other payments on its outstanding bonds, and eventually pays off its bonds. The firm ends up with no debt. In fact, firms don't seem to do that. They go along for a while, paying off some of their existing debt. But then they decide to make some new investments, so they issue more debt. That affects the value of the debt that is already outstanding. Putting this feature into their model could make a significant difference in the values they get.

Another assumption that can be handled in different ways is the assumption that firms behave in the way the model thinks is optimal. In Ingersoll's study of convertible bonds, using methods like those in this paper, it appears that firms call their bonds too late. They don't call them at what seems to be the optimal time. Maybe the same thing applies to the firms in this paper. Maybe they are not behaving in a way that the model says is optimal. That may explain some of the differences here between value and price.

I think that if you ask corporate treasurers how they decide what to do, they will often give relatively unsophisticated answers. They will give you rules of thumb that incorporate factors we think ought to be incorpo-

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rated, but usually not in an elegant way. It's conceivable to me that if we are able to incorporate these rules of thumb in the model, we might get better values.

Another important issue is the way one estimates the volatility of the firm. With stock options, the volatility is perhaps the most important input to the option valuation model. With corporate bonds, the volatility may be less important, but it is important enough to make the difference between a correct valuation and an incorrect valuation in most cases. I believe that the procedure that is followed in this paper is essentially equivalent to taking the actual historical volatility of the firm as the estimated future volatility. That's going to give you incorrect volatility estimates.

Moreover, the errors in estimating volatility will be correlated across firms. There will be times when the volatility estimates are too low for most firms, and other times when they're too high for most firms. In the period covered by this paper I think the volatility estimates are too low, since volatilities increased over that period.

Errors in estimating volatility are especially important when the authors look at the pattern of errors across high- and low-volatility firms. Firms that seem to have high volatility will often be firms for which we have overestimated volatility, and firms that seem to have low volatility will often be firms for which we have underestimated volatility. I think it might be better if the authors used implied volatilities in place of historical volatilities. A firm's implied volatility is the volatility that gives the right equity value when used in the model.

There's another point I can't resist making, because it's related to a discovery Scott Mason reported in his dissertation. He found that there is some uncertainty about how bonds will be handled in case of bankruptcy. Suppose we are in a period where interest rates have risen. A firm with low coupon bonds outstanding gets into bankruptcy. For one reason or another, it has enough assets so that it could buy back at least one issue of its bonds. The bonds are not due for several years. Does the firm have to buy back the bonds at par, or can it buy them back at the present value of a riskless bond with that coupon and other provisions? In this paper, the authors assume that the bonds will be bought back at par, even when the present value of a riskless bond with similar terms is below par. This assumption is probably realistic, and probably won't make much difference in most cases anyway, but it will make some differences in the values. The authors mention allowing for changes in the firm's volatility as its value changes, and taking into account the fact that interest rates are stochastic. These assumptions will make a difference, too, but I don't think they will make as much of a difference as using implied volatilities instead of historical volatilities and taking account of future debt issues by the firm.