This PDF is a selection from an out-of-print volume from the National Bureau of Economic Research

Volume Title: International and Interarea Comparisons of Income, Output, and Prices

Volume Author/Editor: Alan Heston and Robert E. Lipsey, editors

Volume Publisher: University of Chicago Press

Volume ISBN: 0-226-33110-5

Volume URL: http://www.nber.org/books/hest99-1

Publication Date: January 1999

Chapter Title: International Comparisons Using Spanning Trees

Chapter Author: Robert J. Hill

Chapter URL: http://www.nber.org/chapters/c8386

Chapter pages in book: (p. 109 - 120)

2 International Comparisons Using Spanning Trees

Robert J. Hill

A large number of index number methods have been proposed for making multilateral comparisons across countries. A distinction can be drawn between methods that compare all countries in a comparison simultaneously and those that compare countries by simply linking together bilateral comparisons. Methods of the former type are surveyed in Hill (1997). Here, I focus on methods of the latter type. This procedure of linking bilateral comparisons is often referred to as *chaining*. Chaining has a long history. In fact, it dates all the way back to Marshall (1887). However, historically, interest in chaining has focused primarily on time-series comparisons. This is because of the natural chronological ordering of time-series data. In particular, chronological chaining (i.e., linking together bilateral comparisons between adjacent time periods) has been widely advocated for measuring inflation.

Nevertheless, some work has been done on chaining across countries. Two notable references are Kravis, Heston, and Summers (1982) and Szulc (1996). Kravis, Heston, and Summers focus on chaining in the context of multilateral comparisons, while Szulc focuses on bilateral comparisons. More specifically, Szulc argues that, under certain conditions, a chained bilateral comparison is preferable to a direct bilateral comparison.

The analysis in this paper is framed using spanning trees. This is because spanning trees provide the underlying structure for any method of chaining. In fact, in a comparison between K countries, K^{K-2} different spanning trees are defined, each of which generates different results. This paper argues that the preferred method of linking should be the one that minimizes the sensitivity of the results to the choice of index number formula. Two methods are then proposed on the basis of this criterion. The minimum spanning tree (MST) method

Robert J. Hill is a senior lecturer in economics at the University of New South Wales in Sydney, Australia.

selects the spanning tree that is least sensitive to the choice of index number formula, while the shortest path (SP) method selects the path between two countries that is least sensitive to the choice of index number formula. The MST method develops the work of Kravis, Heston, and Summers, while the SP method builds on Szulc. These methods are illustrated using OECD data.

2.1 Spanning Trees

A spanning tree links vertices (in this case countries) in such a way that there is exactly one path between any pair of vertices. An edge connecting two vertices in a spanning tree denotes a bilateral index number comparison between those two countries. The comparison could be made using any bilateral formula, such as Paasche, Laspeyres, Fisher, or Törnqvist. Multilateral indexes are obtained by linking the bilateral indexes as specified by the spanning tree. However, it matters whether the bilateral formula satisfies the country reversal test. A purchasing power parity (PPP) index P_{jk} between countries j and k with j as the base satisfies the country reversal test if $P_{jk} = 1/P_{kj}$. Fisher and Törnqvist satisfy this test, while Paasche and Laspeyres do not. If the bilateral index that is used violates the country reversal test, then the edges in the spanning tree must have directional arrows to indicate the base country in each bilateral comparison. Directional arrows are not necessary if the bilateral formula satisfies the country reversal test.

Three examples of spanning trees defined on the set of five vertices are depicted in figure 2.1. In general, K^{K-2} different spanning trees are defined on the set of K vertices. The *star* spanning tree depicted in figure 2.1*a* and the *string* spanning tree in figure 2.1*b* have both been widely used to measure inflation. An important issue in the inflation-measurement literature has been the debate over the relative merits of a fixed-base-price index as opposed to a chronologically chained price index. Ultimately, this is just a debate over two alternative spanning trees. By a *fixed-base-price index* is meant a price index constructed using the star spanning tree with the base time period placed at the center of the star. Conversely, by a *chronologically chained price index* is meant a price index constructed using the string spanning tree with the time periods linked chronologically.[1]

In the international comparison literature, Kravis, Heston, and Summers (1982) suggest using a variant on the star spanning tree. Using cluster analysis techniques, the set of countries can be divided into more homogeneous subsets. Various criteria are considered for measuring similarity across countries to enable cluster formation. These criteria range from geographic propinquity and price correlation coefficients to Paasche-Laspeyres spreads. Then star spanning trees defined over these clusters are linked to form a spanning tree defined

1. The measurement of inflation using spanning trees is discussed in greater detail in Hill (1999a).

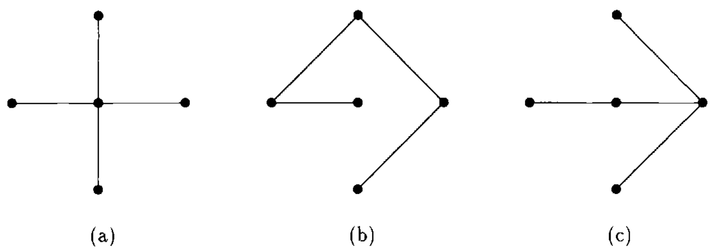

Fig. 2.1 Examples of spanning trees

over the whole set. However, the problem with this approach is that it imposes arbitrary constraints on the spanning tree. In particular, the center country for each cluster and the link countries between clusters are both chosen arbitrarily.

The minimum spanning tree (MST) method developed in this paper extends the pioneering work of Kravis, Heston, and Summers. However, rather than using Paasche-Laspeyres spreads to form clusters of countries along the lines of Kravis, Heston, and Summers and then constructing a spanning tree indirectly by linking together star spanning trees defined over these clusters, the MST method obtains a spanning tree directly by feeding the Paasche-Laspeyres spreads into Kruskal's minimum spanning tree algorithm without imposing any arbitrary restrictions. It is argued later that this spanning tree minimizes the sensitivity of the results of a multilateral comparison to the choice of bilateral index number formula.

Szulc instead focuses on bilateral comparisons and establishes conditions under which a chained comparison is preferable to a direct comparison. His criterion selects the path between two countries that most closely resembles successive linear combinations of their expenditure vectors. Sometimes the selected path is a direct comparison, while at other times it links the two countries indirectly via one or more other countries in the set. The shortest path (SP) method proposed here approaches the same problem from a different perspective. My criterion uses the shortest path algorithm to find the path between two countries with the smallest chained Paasche-Laspeyres spread. Again, it is argued later that such a path minimizes the sensitivity of the results of a bilateral comparison to the choice of bilateral index number formula. If the shortest path is calculated between one specific country and all other countries in the set, then the union of these shortest paths constitutes a spanning tree. Each country has its own shortest path spanning tree.

2.2 Notation and Definitions

The set of countries is indexed by $k = 1, \ldots, K$. It is assumed that each country supplies price and quantity data (p_{ki}, q_{ki}), defined over the same set of goods and services, indexed by $i = 1, \ldots, N$. Let P_{jk} and Q_{jk} denote, respectively, a bilateral purchasing power parity (PPP) and quantity index between countries j and k. Three important bilateral formulas are Paasche, Laspeyres, and Fisher. These indexes are defined as follows:

$$(1)\qquad \text{Paasche:}\quad P^P_{jk} = \frac{\sum_{i=1}^{N} p_{ki}q_{ki}}{\sum_{i=1}^{N} p_{ji}q_{ki}},\quad Q^P_{jk} = \frac{\sum_{i=1}^{N} p_{ki}q_{ki}}{\sum_{i=1}^{N} p_{ki}q_{ji}},$$

$$(2)\qquad \text{Laspeyres:}\quad P^L_{jk} = \frac{\sum_{i=1}^{N} p_{ki}q_{ji}}{\sum_{i=1}^{N} p_{ji}q_{ji}},\quad Q^L_{jk} = \frac{\sum_{i=1}^{N} p_{ji}q_{ki}}{\sum_{i=1}^{N} p_{ji}q_{ji}},$$

$$(3)\qquad \text{Fisher:}\quad P^F_{jk} = (P^P_{jk}P^L_{jk})^{1/2},\quad Q^F_{jk} = (Q^P_{jk}Q^L_{jk})^{1/2}.$$

The Paasche-Laspeyres spread (PLS) index between countries j and k is defined as follows:

$$(4)\qquad \text{PLS}_{jk} = \log\left[\frac{\max(P^P_{jk},\ P^L_{jk})}{\min(P^P_{jk},\ P^L_{jk})}\right] = \log\left[\frac{\max(Q^P_{jk},\ Q^L_{jk})}{\min(Q^P_{jk},\ Q^L_{jk})}\right].$$

The PLS index has the following properties:

PROPERTY 1: $\text{PLS}_{jj} = 0$.

PROPERTY 2: $\text{PLS}_{jk} = \text{PLS}_{kj}$.

PROPERTY 3: $\text{PLS}_{jk} \geq 0$.

In a bilateral context, the spread between corresponding Paasche and Laspeyres indexes may be interpreted as a measure of the sensitivity of the results to the choice of index number formula. This is because, as Paasche and Laspeyres converge, so do all other bilateral index number formulas.[2] In the limit, if the price data satisfy the conditions for Hicks's (1946) composite commodity theorem, then all price index formulas give the same answer.[3] Similarly, in the limit, if the quantity data satisfy the conditions for Leontief's (1936) aggregation theorem, then all quantity index formulas give the same answer.[4] Under both scenarios, $\text{PLS}_{jk} = 0$. This paper develops a framework for generalizing this idea to both chained bilateral and multilateral comparisons.

2. In fact, if preferences are homothetic, then Paasche and Laspeyres provide lower and upper bounds on the true underlying cost-of-living index.

3. The price data of countries j and k satisfy the conditions for Hicks's composite commodity theorem if $p_{ji} = \lambda p_{ki}$, $\forall\ i = 1, \ldots, N$, where λ denotes an arbitrary positive scalar.

4. The quantity data of countries j and k satisfy the conditions for Leontief's aggregation theorem if $q_{ji} = \lambda q_{ki}$, $\forall\ i = 1, \ldots, N$, where λ denotes an arbitrary positive scalar.

2.3 The Shortest Path (SP) Method

The SP method selects the path between two countries with the smallest summed PLS index. For example, suppose that there are three countries, A, B, and C, and that we wish to find the shortest path between A and B. If $PLS_{AB} \leq PLS_{AC} + PLS_{CB}$, then the shortest path is a direct comparison between A and B. Otherwise, the shortest path is via country C. The shortest path is the path between two countries with the smallest chained Paasche-Laspeyres spread. Hence, it minimizes the sensitivity of the results to the choice of bilateral index number formula.

The shortest path can be calculated between one specific country and all other countries in the set. The union of these $K - 1$ shortest paths is a spanning tree. This spanning tree for each country is easily calculated using the shortest path spanning tree algorithm run on Mathematica (see Skiena 1990). Figures 2.2 and 2.3 depict the shortest path spanning trees, respectively, of the United States and Turkey calculated over the set of twenty-four OECD countries for 198 goods and services headings in 1990. In figure 2.2, for only seven of twenty-three countries is a direct comparison the shortest path to the United States. For example, the shortest path between the United States and Sweden

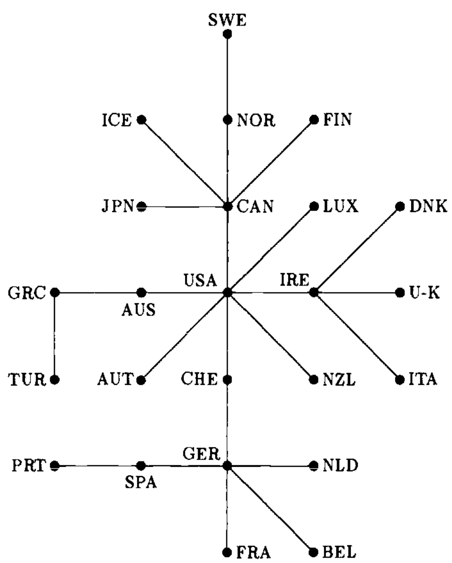

Fig. 2.2 Shortest path spanning tree for the United States
Note: The country codes are as follows: GER = Germany, FRA = France, ITA = Italy, NLD = Netherlands, BEL = Belgium, LUX = Luxembourg, U-K = United Kingdom, IRE = Ireland, DNK = Denmark, GRC = Greece, SPA = Spain, PRT = Portugal, AUT = Austria, CHE = Switzerland, FIN = Finland, ICE = Iceland, NOR = Norway, SWE = Sweden, TUR = Turkey, AUS = Australia, NZL = New Zealand, JPN = Japan, CAN = Canada, USA = United States.

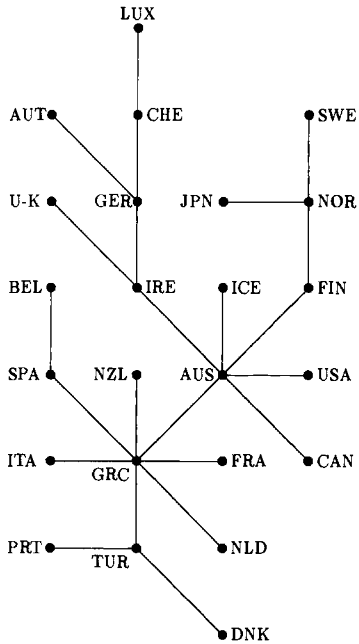

Fig. 2.3 Shortest path spanning tree for Turkey
Note: See note to fig. 2.2.

is via Canada and Norway. Similarly, in figure 2.3, for only three countries is a direct comparison the shortest path to Turkey. Hence, the results obtained here support the conclusion of Szulc (1996) that chaining is often desirable even in bilateral comparisons.

Table 2.1 compares the direct Paasche-Laspeyres spreads with the shortest path chained Paasche-Laspeyres spreads obtained from figures 2.2 and 2.3 for the United States and Turkey. Chaining along the shortest path reduces the ratio of Laspeyres to Paasche by up to 10 percent. The biggest reduction in this ratio is obtained in a comparison between Turkey and Luxembourg. Table 2.2 compares the direct Fisher PPPs with the shortest path chained Fisher PPPs for the United States and Turkey. Again, the biggest change is obtained in a comparison between Turkey and Luxembourg. This time the change is about 11 percent. It is worth noting that Turkey appears relatively poorer compared with most other OECD countries if it is compared directly using Fisher indexes as opposed to along the shortest path. By contrast, no systematic pattern emerges for the United States. For eight countries, it appears relatively richer when compared by chaining Fisher indexes along the shortest path, but, conversely, for eight other countries, it appears relatively richer when compared directly. For the other seven countries, it makes no difference since the shortest path is a direct comparison.

Table 2.1 **Laspeyres PPPs Divided by Paasche PPPs**

	Direct (D), United States	Chained (C), United States	D/C, United States	Direct, Turkey	Chained, Turkey	D/C, Turkey
Germany	1.055	1.050	1.005	1.270	1.167	1.088
France	1.092	1.075	1.016	1.192	1.157	1.030
Italy	1.098	1.084	1.014	1.163	1.152	1.010
Netherlands	1.081	1.066	1.014	1.225	1.169	1.049
Belgium	1.081	1.070	1.010	1.247	1.169	1.067
Luxembourg	1.051	1.051	1	1.312	1.189	1.104
United Kingdom	1.079	1.058	1.021	1.180	1.171	1.008
Ireland	1.047	1.047	1	1.177	1.159	1.016
Denmark	1.128	1.078	1.046	1.162	1.162	1
Greece	1.142	1.118	1.022	1.079	1.079	1
Spain	1.098	1.095	1.003	1.129	1.121	1.007
Portugal	1.189	1.143	1.040	1.094	1.094	1
Austria	1.051	1.051	1	1.216	1.179	1.031
Switzerland	1.049	1.049	1	1.237	1.168	1.059
Finland	1.074	1.065	1.009	1.168	1.166	1.002
Iceland	1.079	1.078	1.001	1.189	1.182	1.005
Norway	1.090	1.061	1.027	1.281	1.173	1.092
Sweden	1.097	1.079	1.017	1.215	1.192	1.020
Turkey	1.317	1.207	1.091	1	1	1
Australia	1.056	1.056	1	1.192	1.143	1.044
New Zealand	1.068	1.068	1	1.180	1.160	1.018
Japan	1.101	1.079	1.021	1.298	1.203	1.079
Canada	1.034	1.034	1	1.263	1.169	1.080
United States	1	1	1	1.317	1.207	1.091

Table 2.2 **Direct and Chained Fisher PPPs**

	Direct, United States = 1	Chained, United States = 1	Max(Direct, Chained)/Min(Direct, Chained), United States	Direct, Turkey = 1,000	Chained, Turkey = 1,000	Max(Direct, Chained)/Min(Direct, Chained), Turkey
Germany	2.050	2.107	1.028	1.373	1.430	1.042
France	6.585	6.605	1.003	4.490	4.662	1.038
Italy	1,438	1,475	1.026	923.0	1,002	1.086
Netherlands	2.126	2.176	1.024	1.490	1.501	1.007
Belgium	38.91	39.91	1.026	26.52	27.33	1.031
Luxembourg	40.02	40.02	1	25.00	27.87	1.115
United Kingdom	.6074	.6180	1.017	.4060	.4152	1.023
Ireland	.7100	.7100	1	.4340	.4769	1.099
Denmark	9.208	9.656	1.049	6.364	6.364	1
Greece	141.0	142.4	1.010	98.51	98.51	1
Spain	112.2	111.0	1.011	75.14	76.20	1.014
Portugal	104.0	103.5	1.005	70.88	70.88	1
Austria	14.28	14.28	1	9.126	9.667	1.059
Switzerland	2.190	2.190	1	1.478	1.486	1.005
Finland	6.489	6.213	1.044	4.343	4.407	1.015
Iceland	82.58	80.12	1.031	54.01	56.16	1.040
Norway	9.795	9.464	1.035	6.268	6.675	1.065
Sweden	9.246	9.020	1.025	6.199	6.361	1.026
Turkey	1,481	1,446	1.024	1,000	1,000	1
Australia	1.381	1.381	1	.9590	.956	1.003
New Zealand	1.587	1.587	1	1.122	1.118	1.004
Japan	197.2	190.2	1.037	136.0	131.5	1.034
Canada	1.274	1.274	1	.8850	.8944	1.011
United States	1	1	1	.6750	.6918	1.025

2.4 The Minimum Spanning Tree (MST) Method

The observation that the PLS index between two countries provides a measure of the sensitivity of the results of a bilateral comparison to the choice of index number formula can be generalized to spanning trees. A comparison between K countries has a $K \times K$ matrix of PLS indexes. The matrix is symmetrical (property 2) with zeros on the lead diagonal (property 1). Hence, the matrix has $K(K - 1)/2$ distinct PLS indexes. However, a spanning tree defined on K vertices has only $K - 1$ edges. Each edge has a corresponding PLS index. Therefore, each spanning tree uses only $K - 1$ of the possible $K(K - 1)/2$ PLS indexes. An overall measure of sensitivity to the choice of index number formula for each spanning tree can be obtained from the $K - 1$ PLS indexes of the bilateral comparisons contained within the spanning tree.

The minimum spanning tree is the spanning tree defined on a set of vertices with the smallest sum of weights. In our context, the weight on each edge is its corresponding PLS index. Hence, the minimum spanning tree is the spanning tree with the smallest sum of PLS indexes. The minimum spanning tree is a natural extension of the shortest path to multilateral comparisons.

A number of equivalent algorithms exist in the graph theory literature for computing the minimum spanning tree of a graph. The minimum spanning tree for the OECD countries in figure 2.4 was computed using Kruskal's algorithm run on Mathematica (again, see Skiena 1990). It should be noted that the algorithm is very efficient. In a comparison of over 24^{22} spanning trees, it finds the optimal spanning tree almost instantly.

Kruskal's algorithm proceeds as follows. The algorithm begins by ranking the edges according to the size of their weights (PLS indexes).[5] Then the edge with the smallest weight is selected, subject to the constraint that it does not create a cycle. If selecting this edge creates a cycle, then the algorithm skips it and moves on to the edge with the next smallest weight. This procedure for selecting edges is repeated until it is no longer possible to select any more edges without creating a cycle, at which point the algorithm terminates. In practice, this implies that the algorithm always selects exactly $K - 1$ edges. The set of vertices and selected edges constitutes the minimum spanning tree. Hence, the minimum spanning tree is constructed from edges connecting pairs of countries with the smallest Paasche-Laspeyres spreads. A proof that Kruskal's algorithm finds the spanning tree with the smallest sum of weights can be found in Wilson (1985, 55).

The minimum spanning tree in figure 2.4 is compared to the star spanning trees with the United States and Turkey at the center, respectively, in table 2.3. The PPPs in table 2.3 are calculated by chaining Fisher PPPs across the respective spanning tree. Each set of PPPs is normalized so that the PPP for

5. The probability of encountering ties becomes negligible if the PLS indexes are calculated to a sufficiently large number of decimal places.

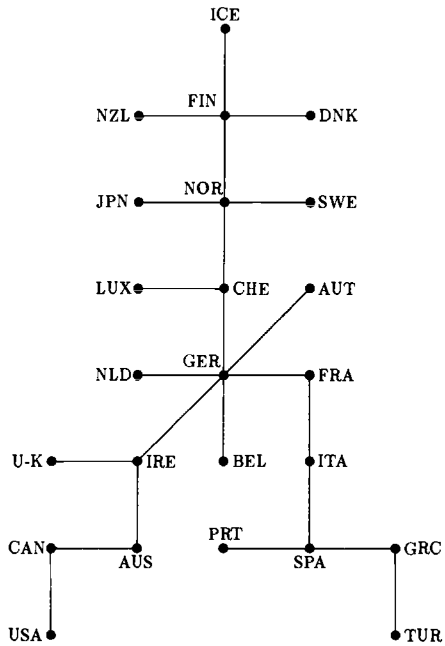

Fig. 2.4 Minimum spanning tree for the OECD
Note: See note to fig. 2.2.

the United States equals one. A clear pattern emerges from table 2.3. For eight of twenty-three countries, the MST PPP lies between the two star PPPs. However, for the remaining fifteen countries, the MST PPP is less than both star PPPs. This implies that the United States will tend to appear richer relative to the other OECD countries in 1990 if a comparison is made by chaining Fisher PPPs across either star spanning tree as opposed to the minimum spanning tree. (It is not clear whether this result generalizes to other years.)

Once selected, the minimum spanning tree can be used for subsequent multilateral comparisons.[6] This greatly simplifies international comparisons by dramatically reducing the number of countries that must be compared directly. By chaining across the minimum spanning tree, each country is compared only with its neighbors in the tree. Hence, these comparisons can be made over a more representative set of goods and services, especially since by construction the neighbors have relatively similar price and expenditure patterns. In contrast, most other multilateral PPP methods require all countries in a compari-

6. Admittedly, the minimum spanning tree is unlikely to be very robust. However, the minimum spanning tree at one point in time is likely to be only slightly suboptimal at a later date. Hence, it is not necessarily unreasonable to use the same spanning tree for a number of years. A sensitivity analysis of the MST method is provided in Hill (1999b).

Table 2.3 Star and Minimum Spanning Tree Chained Fisher PPPs

	Star with United States at Center	Star with Turkey at Center	Minimum Spanning Tree
Germany	2.050	2.034	2.037
France	6.585	6.652	6.384
Italy	1,438	1,367	1,372
Netherlands	2.126	2.207	2.104
Belgium	38.91	39.28	38.58
Luxembourg	40.02	37.03	39.70
United Kingdom	.607	.601	.591
Ireland	.710	.643	.679
Denmark	9.208	9.428	9.208
Greece	141.0	145.9	137.2
Spain	112.2	111.3	106.1
Portugal	104.0	105.0	99.03
Austria	14.28	13.52	13.77
Switzerland	2.190	2.190	2.117
Finland	6.489	6.434	6.244
Iceland	82.58	80.01	79.43
Norway	9.795	9.286	9.457
Sweden	9.246	9.184	9.013
Turkey	1,481	1,481	1,393
Australia	1.381	1.421	1.361
New Zealand	1.587	1.662	1.577
Japan	197.2	201.5	186.3
Canada	1.274	1.311	1.274
United States	1	1	1

son to supply price and expenditure data over the same basket of goods and services. This requirement creates difficulties since a staple good in one country may be rare or even unobtainable in another country. This is particularly a problem in comparisons between rich and poor countries.

2.5 Conclusion

Any method of chaining index numbers has an underlying spanning tree. However, a comparison between K countries has K^{K-2} possible spanning trees, each of which generates different results. This paper uses this insight to develop two new methods of making international comparisons. Both methods discriminate between spanning trees on the basis of the sensitivity of their resulting indexes to the choice of index number formula. The shortest path (SP) method minimizes sensitivity in bilateral comparisons, while the minimum spanning tree (MST) method minimizes sensitivity in multilateral comparisons. Both methods are illustrated using OECD data.

References

Hicks, J. R. 1946. *Value and capital.* 2d ed. Oxford: Clarendon.

Hill, R. J. 1997. A taxonomy of multilateral methods for making international comparisons of prices and quantities. *Review of Income and Wealth,* ser. 43, no. 1:49–69.

———. 1999a. Measuring inflation and growth using spanning trees. *International Economic Review,* forthcoming.

———. 1999b. Comparing price levels across countries using minimum-spanning trees. *Review of Economics and Statistics* 81, no. 1:135–42.

Kravis, I. K., A. Heston, and R. Summers. 1982. *World product and income: International comparisons of real gross product.* Baltimore: Johns Hopkins University Press.

Leontief, W. 1936. Composite commodities and the problem of index numbers. *Econometrica* 4:39–59.

Marshall, A. 1887. Remedies for fluctuations of general prices. *Contemporary Review* 51:355–75.

Skiena, S. 1990. *Implementing discrete mathematics: Combinatorics and graph theory with mathematica.* Redwood City, Calif.: Addison-Wesley.

Szulc, B. 1996. Criterion for adequate linking paths in chain indices. In *Improving the quality of price indices,* ed. L. Biggeri. Luxembourg: Eurostat.

Wilson, R. J. 1985. *Introduction to graph theory.* 3d ed. New York: Longman.