This PDF is a selection from an out-of-print volume from the National Bureau of Economic Research

Volume Title: Business Cycles, Indicators and Forecasting

Volume Author/Editor: James H. Stock and Mark W. Watson, editors

Volume Publisher: University of Chicago Press

Volume ISBN: 0-226-77488-0

Volume URL: http://www.nber.org/books/stoc93-1

Conference Date: May 3-4, 1991

Publication Date: January 1993

Chapter Title: Estimating Event Probabilities from Macroeconometric Models Using Stochastic Simulation

Chapter Author: Ray C. Fair

Chapter URL: http://www.nber.org/chapters/c7191

Chapter pages in book: (p. 157 - 178)

3 Estimating Event Probabilities from Macroeconometric Models Using Stochastic Simulation

Ray C. Fair

Government policymakers and business planners are interested in knowing the probabilities of various economic events happening. In 1989 and 1990, for example, there was interest in the probability that a recession would occur in the near future. Model builders who make forecasts typically do not answer probability questions directly. They typically present a "base" forecast and a few alternative "scenarios." If probabilities are assigned to the scenarios, they are subjective ones of the model builders.[1]

Probability questions can, however, be directly answered within the context of macroeconometric models by using stochastic simulation. The first part of this paper (secs. 3.1–3.2) explains how this can be done and gives some examples. An adantage of this procedure is that the probabilities estimated from the stochastic simulation are objective in the sense that they are based on the use of estimated distributions. They are consistent with the probability structure of the model.

Estimated probabilities can also be used in the evaluation of a model. Consider, for example, the event that, in a five-quarter period, there is negative real GNP growth in at least two of the quarters. For any historical five-quarter period, this event either did or did not happen. The actual value or outcome is thus either zero or one. Now, for any five-quarter period for which data exist, one can estimate from a model the probability of the event occurring. If this is done for a number of five-quarter periods, one has a series of probability esti-

Ray C. Fair is professor of economics at Yale University and a research associate of the National Bureau of Economic Research.

The author is indebted to Douglas Hamilton for stimulating his interest in the use of stochastic simulation to estimate probabilities in econometric models. He is also indebted to James Hamilton, James Stock, Mark Watson, and other conference participants for very helpful comments.

1. Within the context of their leading indicator approach, Stock and Watson (1989) do present, however, estimates of the probability that the economy will be in a recession six months hence.

mates that can be compared to the actual (zero or one) values. One can thus evaluate how good the model is at predicting various events. An example of this type of evaluation is presented in the second part of this paper (sec. 3.3).

3.1 The Procedure

3.1.1 The Model

The model considered in this paper can be dynamic, nonlinear, and simultaneous and can have autoregressive errors of any order. Write the model as

$$(1) \qquad f_i(y_t, x_t, \alpha_i) = u_{it}, \quad i = 1, \ldots, n, t = 1, \ldots, T,$$

where y_t is an n-dimensional vector of endogenous variables, x_t is a vector of predetermined variables (both exogenous and lagged endogenous), α_i is a vector of unknown coefficients, and u_{it} is an error term. It is assumed that the first m equations are stochastic, with the remaining $u_{it}(i = m + 1, \ldots, n)$ identically zero for all t.

Each equation in (1) is assumed to have been transformed to eliminate any autoregressive properties of its error term. If the error term in the untransformed version, say, v_{it} in equation i, follows an rth-order autoregressive process,

$$v_{it} = \rho_{1i}v_{it-1} + \ldots + \rho_{ri}v_{it-r} + u_{it},$$

where u_{it} is i.i.d., then equation i is assumed to have been transformed into one with u_{it} on the right-hand side. The autoregressive coefficients $\rho_{1i}, \ldots, \rho_{ri}$ are incorporated into the α_i coefficient vector, and the additional lagged values that are involved in the transformation are incorporated into the x_t vector. This transformation makes the equation nonlinear in coefficients if it were not otherwise, but this adds no further complications to the model because it is already allowed to be nonlinear. It does result in the "loss" of the first r observations, but this has no effect on the asymptotic properties of the estimators. u_{it} in (1) can thus be assumed to be i.i.d. even though the original error term may follow an autoregressive process.

Let u_t be the m-dimensional vector $(u_{1t}, \ldots, u_{mt})'$. For the stochastic simulations below, it is assumed that u_t is distributed as multivariate normal $N(0, \Sigma)$, where Σ is $m \times m$. Although the normality assumption is commonly made, the general procedure discussed in this paper does not depend on it. If another distributional assumption were used, this would simply change the way in which the error terms were drawn for the stochastic simulations.

It is assumed that consistent estimates of α_i, denoted $\hat{\alpha}_i$, are available for all i. Given these estimates, consistent estimates of u_{it}, denoted $\hat{u}_{it}$, can be computed as $f_i(y_t, x_t, \hat{\alpha}_i)$. The covariance matrix Σ can then be estimated as $\hat{\Sigma} = (1/T)\hat{U}\hat{U}'$, where $\hat{U}$ is the $m \times T$ matrix of values of $\hat{u}_{it}$.

Let α be the k-dimensional vector $(\alpha_1', \ldots, \alpha_m')'$, where k is the total number of unrestricted coefficients in the model, including any autoregressive coefficients of the original error terms, and let $\hat{\alpha}$ denote the estimate of α. It is also assumed that an estimate of the covariance matrix of $\hat{\alpha}$, denoted $\hat{V}$, is available, where $\hat{V}$ is $k \times k$.

3.1.2 Estimating Standard Errors of Forecasts

It will be useful to consider first the use of stochastic simulation to estimate standard errors of forecasts. A forecast from a model is subject to four main sources of uncertainty—uncertainty from the structural error terms, from the coefficient estimates, from the exogenous-variable forecasts, and from the possible misspecification of the model. Stochastic simulation can easily handle the first three sources, but accounting for possible misspecification is much harder. A method is presented in Fair (1980) that uses stochastic simulation to estimate the degree of misspecification of a model and to adjust the standard errors for the misspecification. This method does not, however, carry over in any straightforward way to the estimation of probabilities, and, in this paper, only the first three sources of uncertainty are considered. The probability estimates are thus based on the assumption that the model is correctly specified.

Given $\hat{\Sigma}$ and $\hat{V}$, the uncertainty from the error terms and coefficient estimates can be estimated. Consider first drawing error terms. Let u_t^* denote a particular draw of the m error terms for period t from the $N(0, \hat{\Sigma})$ distribution. Given u_t^*, $\hat{\alpha}$, and x_t, one can solve the model for period t using a method like the Gauss-Seidel technique. This is merely a deterministic simulation for the given values of the error terms, coefficients, and predetermined variables. Call this simulation a "trial." Another trial can be made by drawing a new set of values of u_t^* and solving again. This can be done as many times as desired. From each trial, one obtains a prediction of each endogenous variable. Let y_{it}^j denote the value on the jth trial of endogenous variable i for period t. For J trials, the stochastic simulation estimate of the expected value of variable i for period t, denoted $\tilde{\mu}_{it}$, is

$$\tilde{\mu}_{it} = (1/J)\sum_{j=1}^{J} y_{it}^j. \quad (2)$$

The stochastic simulation estimate of the variance of the forecast error, denoted $\tilde{\sigma}_{it}^2$, is

$$\tilde{\sigma}_{it}^2 = (1/J)\sum_{j=1}^{J} (y_{it}^j - \tilde{\mu}_{it})^2. \quad (3)$$

If the forecast horizon is more than one period, then each trial is a dynamic simulation over the horizon, with predicted values computed for each endogenous variable for each period. Any lagged endogenous variables in the x_t vector are updated as the simulation proceeds. If, for example, the horizon is eight quarters, then eight vectors u_t^* are drawn ($t = 1, \ldots, 8$), the simulation is over the eight quarters, and eight means and variances are computed for each endogenous variable using formulas (2) and (3).

Consider now drawing coefficients. Let α^* denote a particular draw of the coefficient vector α. Under the assumption that the asymptotic distribution of $\hat{\alpha}$ is multivariate normal with covariance matrix V, α^* can be drawn from the $N(\hat{\alpha}, \hat{V})$ distribution. (Again, the normality assumption is not necessary. Some other distribution could be assumed for $\hat{\alpha}$ and the draws made from it.) Each trial now consists of drawing both error terms and coefficients. If the forecast horizon is more than one period, only one coefficient draw should be done for the entire horizon. This is consistent with the assumption on which the estimation of a model is based, namely, that the coefficients do not change over time.

Accounting for exogenous-variable uncertainty is less straightforward than accounting for uncertainty from the error terms and coefficient estimates. Exogenous variables are by their nature exogenous, and no probability structure has been assumed for them. One might think that exogenous variables should always just be taken to be fixed, but, when comparing forecast-error variances across models, it is important to try to put each model on an equal footing regarding the exogenous variables. Otherwise, the model that takes more important and hard-to-forecast variables as exogenous has an unfair advantage. Therefore, some assumption about exogenous-variable uncertainty has to be made when comparing models.

One approach is to try to estimate variances of the exogenous-variable forecasts from past predictions that model builders and others have made of the exogenous variables. Given these estimates and a distributional assumption, one could then draw exogenous-variable values for each trial. Each trial would then consist of draws of the error terms, coefficients, and exogenous variables. An alternative approach is to estimate autoregressive or vector autoregressive equations for the exogenous variables and add them to the model. One would then have a model with no exogenous variables, and error terms and coefficients could be drawn from the expanded model. Either of these approaches is a way of trying to incorporate exogenous-variable uncertainty into the stochastic simulation estimates of the forecast-error variances.

3.1.3 Estimating Event Probabilities

Estimating event probabilities is straightforward once the stochastic simulation has been set up and the event defined. Consider an eight-quarter prediction period and the event that, within this period, there are two consecutive quarters of negative real GNP growth. Assume that 1,000 trials are taken. For each trial, one can record whether or not this event occurred. If it occurred, say, 150 times out of the 1,000 trials, its estimated probability would be 15 percent. It should be clear that as many events can be considered as desired. Almost no extra work is needed to estimate probabilities beyond what is needed to estimate means and variances, and there is wide latitude in the choice of events. The extra work is simply keeping track of how often each event occurs in the solution for each trial.

3.2 Estimated Probabilities for Three Events

3.2.1 The Model

Estimated probabilities for three events are presented in this section using the model in Fair (1984). There are two contractionary events and one inflationary event.

The model consists of thirty stochastic equations and ninety-eight identities. There are 179 estimated coefficients. The estimation period used for the present results is 1954:I–1989:IV (144 observations). The model is estimated by two-stage least squares with account taken when necessary of the autoregressive properties of the error terms. Ten of the equations are estimated under the assumption of a first-order autoregressive process of the error term, and two of the equations are estimated under the assumption of a third-order process. The autoregressive coefficients are included in the 179 coefficients. The 30×30 covariance matrix of the structural error terms was estimated as $(1/T)\hat{U}\hat{U}'$, where $\hat{U}$ is the $30 \times T$ matrix of estimated residuals (as noted above, T is 144). the 179×179 covariance matrix of the estimated coefficients was estimated using the formula in Fair (1984, 216–17). This matrix is *not* block diagonal even though the correlation of the error terms across equations is not taken into account in the estimation of each equation by two-stage least squares. The correlation affects the covariance matrix, so the matrix is not block diagonal.

There are eighty-two exogenous variables in the model, not counting the constant term, the time trend, and a few dummy variables. For the present results, exogenous-variable uncertainty was handled as follows. Each of the eighty-two exogenous variables was regressed on a constant, time, and its first four lagged values (over the same 1954:I–1989:IV estimation period).[2] The estimator was ordinary least squares. The 82×82 covariance matrix of the error terms was estimated as $(1/T)\hat{E}\hat{E}'$, where $\hat{E}$ is the $82 \times T$ matrix of estimated residuals from the exogenous-variable equations. Denote this estimated matrix as $\hat{S}$.

The eighty-two equations were then added to the model, leaving the expanded model with no exogenous variables except the constant term, the time trend, and a few dummy variables. The expanded model was restricted in two ways. First, the error terms in the thirty structural equations were assumed to be uncorrelated with the error terms in the eighty-two exogenous-variable

2. Many of the exogenous-variable equations were estimated in logs. Logs were not used for tax rates and for variables that were sometimes negative or very close to zero.

equations. The 112 × 112 estimated covariance matrix of all the error terms is thus block diagonal, with one block $\hat{\Sigma}$ and one block $\hat{S}$. This treatment is consistent with one of the assumptions on which the structural equations were estimated, namely, that the exogenous variables are uncorrelated with the structural error terms. Second, the coefficient estimates in the exogenous-variable equations were taken to be fixed in the stochastic simulations. In other words, only coefficients for the thirty structural equations were drawn. This lessens somewhat the uncertainty assumed for the exogenous variables, but it will be seen that the uncertainty from the coefficient estimates is small relative to the uncertainty from the error terms.

The key exogenous variables in the model are government fiscal policy variables, exports, and the price of imports. Monetary policy is endogenous—Fed behavior is explained by an interest rate reaction function, the interest rate reaction function being one of the thirty structural equations.

3.2.2 The Events

From about the beginning of 1989, there was concern that the economy might enter a recession in the near future, a recession generally being considered to be two consecutive quarters of negative real growth. It is thus of interest to examine this period. For the present results, the prediction period was taken to be the five quarters 1990:I–1991:I. Given this period, the following three events were considered:

A. At least two consecutive quarters of negative real GNP growth.
B. At least two quarters of negative real GNP growth.
C. At least two quarters in which inflation (percentage change in the GNP deflator) exceeded 7 percent at an annual rate.

Event A is a recession as generally defined. Event B allows the two or more quarters of negative growth not to be consecutive. Event C is a case in which people would probably start to worry about inflation picking up.

3.2.3 The Stochastic Simulations

Three stochastic simulations were performed, each based on 1,000 trials. For simulations 1 and 2, the exogenous-variable equations were *not* added to the model, and the exogenous-variable values were taken to be the actual values. For simulation 1, only error terms were drawn; for simulation 2, both error terms and coefficients were drawn.[3]

3. After the empirical work for this paper was finished, Gregory Chow suggested to me that one may not want to draw coefficients when estimating probabilities. Although coefficient estimates are uncertain, the true coefficients are fixed. In the real world, the reason that economic events are stochastic is because of the stochastic shocks (error terms), not because the coefficients are stochastic. (This is assuming, of course, that the true coefficients are fixed, which is the assumption on which the estimation of the model is based.) As a practical matter, it makes little difference whether or not one draws coefficients because, as will be seen below, most of the uncertainty is from the error terms, not the coefficient estimates. In future work, however, Chow's argument suggests that coefficients should not be drawn when estimating probabilities.

For simulation 3, the eighty-two exogenous-variable equations were added to the model in the manner discussed above. It is important to note that, in order to make this simulation comparable to the other two, the estimated residuals in the exogenous-variable equations were added to the equations and taken to be fixed across all the trials. The draws of the error terms for the exogenous-variable equations were then added to the fixed residuals. Adding the residuals to the exogenous-variable equations means that, when the expanded model is solved deterministically (by setting the error terms in the structural equations equal to zero), the solution is the same as when the nonexpanded model is solved using the actual values of the exogenous variables. This treatment of the exogenous-variable equations for simulation 3 means that the base paths of the exogenous variables are the actual paths (just as for simulations 1 and 2). The base paths, for example, are *not* the paths that would be predicted by the exogenous-variable equations if they were solved by setting their error terms equal to zero.

All three simulations are thus based on knowledge of the exogenous-variable values for the period 1990:I–1991:I. The simulations are, however, outside the estimation period since the estimation period ended in 1989:IV. Therefore, the simulations are predictions that could have been made as of the end of 1989:IV had all the exogenous-variable values for the next five quarters been known.

The same draws of the structural error terms were used for all three simulations, and the same draws of the coefficients were used for simulations 2 and 3. This means that the differences across the three simulations are not due to simulation error. There were no cases in which the model failed to solve for the three sets of 1,000 trials.

3.2.4 The Mean Forecasts and Their Standard Errors

It will be useful to present the mean forecasts and the standard errors of the forecasts before presenting the probabilities. The results for the percentage change in real GNP (denoted g) and the percentage change in the GNP deflator (denoted p) are presented in table 3.1. Two of the main features of the results in table 3.1, which are almost always true for stochastic simulations of macroeconometric models, are that the estimated forecast means are close to the predicted values from the deterministic simulation and that drawing coefficients has a small effect on the forecast standard errors. The first result means that the bias in the predicted values from the deterministic simulation, which arises from the nonlinearity of the model, is small. The second result means that the effect of coefficient uncertainty on the forecast standard errors is small—most of the effects come from the structural error terms and the exogenous variables.[4]

4. Another common result in this area is that the estimates are not sensitive to the use of more robust measures of central tendency and dispersion than the mean and variance. Forecast means and variances do not necessarily exist, but this does not appear to be a problem in practice. For

Table 3.1 Forecast Means and Standard Errors

	1990						1990				
	I	II	III	IV	1991:I		I	II	III	IV	1991:I
Actual[a]	1.73	.40	1.43	−1.59	−2.56	Actual[b]	4.87	4.72	3.86	2.56	5.20
	Forecast Means[a]						Forecast Means[b]				
det.	3.64	1.00	1.37	.98	−.19	det.	4.57	1.31	3.41	3.76	3.38
u	3.58	1.06	1.37	1.02	−.14	*u*	4.52	1.24	3.46	3.82	3.45
u, c	3.27	.82	1.19	.87	−.23	*u, c*	4.41	1.43	3.37	3.93	3.40
u, c, e	3.32	.91	1.43	1.02	−.11	*u, c, e*	4.35	1.60	3.27	3.82	3.39
	Forecast Standard Errors						Forecast Standard Errors				
u	1.84	2.03	2.07	2.01	2.18	*u*	1.69	1.75	1.63	1.62	1.65
u, c	1.94	2.13	2.23	2.14	2.24	*u, c*	1.74	1.81	1.69	1.65	1.68
u, c, e	2.84	3.23	3.37	3.24	3.50	*u, c, e*	2.26	2.33	2.36	2.32	2.45

[a]Percentage change in real GNP (g).

[b]Percentage change in the GNP deflator (p).

Note: All percentage changes are at annual rates. det. = deterministic simulation (error terms in the structural equations set to zero and the model solved once); u = structural error terms drawn; c = coefficients drawn; e = exogenous-variable equations added to the model as discussed in the text.

The actual values for g show that the growth rate was positive but very small in 1990:II and negative in 1990:IV and 1991:I. The forecast means for g are generally larger than the actual values for the five quarters. For 1990:IV, the means are about 1.0, compared to the actual value of −1.59, and, for 1991:I, the means are about −0.2, compared to the actual value of −2.56. Regarding the inflation predictions, 1990:II was underpredicted by about 3 percentage points, 1990:IV was overpredicted by about 1 percentage point, and 1991:I was underpredicted by about 2 percentage points. The predictions for the other two quarters are very close.

The exogenous variables add substantially to the forecast standard errors (compare the u, c rows to the u, c, e rows). It may be that the current treatment of exogenous-variable uncertainty has overestimated this uncertainty. When a model builder makes an actual ex ante forecast based on guesses of the future values of the exogenous variables, it may be that the average errors of the exogenous-variable guesses are less than those implied by adding the exogenous-variable equations to the model. In other words, one may know more in practice about the exogenous variables, particularly government pol-

more discussion of this, see Fair (1984, chap. 7). The use of more robust measures in the present case led to very similar results to those reported above.

icy variables, than is implied by the equations. The true forecast standard errors may thus lie somewhere between the *u*, *c* and the *u*, *c*, *e* cases above, and the probability estimates reported below may lie somewhere between the two cases.

Given that the predicted values of *g* are only around 1 percentage point for three of the five quarters and negative for another, and given that the standard errors are generally above 2 percentage points, it seems likely that a fairly large fraction of the trials will have two or more quarters of negative growth. The model is close to predicting negative growth for two or more quarters already, so, given the size of the standard errors, it would not be surprising that a fairly large probability of at least two quarters of negative growth was estimated.

3.2.5 The Estimated Probabilities

The probability estimates are shown in table 3.2. These estimates indicate that the probability of a recession or near recession occurring in the period 1990:I–1990:IV was fairly high according to the model. With the exogenous-variable equations added to the model, the estimated probability is greater than half for event B (two or more quarters of negative growth). The estimated probability of inflation being greater than 7 percent for two or more quarters (event C) is very small—less than 5 percent even with the exogenous-variable equations included.

Two other simulations were run to examine the sensitivity of the results to the exogenous-variable equations. For simulation 4, the error terms in the exogenous-variable equations were assumed to be uncorrelated with each other: $\hat{S}$ was taken to be diagonal. The three estimated probabilities in this case were .397, .529, and .077. Only the last estimate is changed much, where it is now slightly higher. Not accounting for the correlation of the exogenous-variable error terms appears to increase somewhat the variance of the inflation forecasts.

For simulation 5, the exogenous-variable equations were taken to be first-order autoregressive rather than fourth order. This had only a small effect on the results. The three estimated probabilities were .416, .538, and .037. It

Table 3.2 Probability Estimates for the Three Events

	Event		
Simulation	A	B	C
u	.275	.426	.002
u, *c*	.321	.483	.006
u, *c*, *e*	.393	.522	.049

Note: See the note to table 3.1 for the *u*, *c*, and *e* notation.

appears that little is gained in decreasing the estimated uncertainty from the exogenous variables by going from first to fourth order.[5]

Although the probability estimates for events A and B are fairly high, they are perhaps not as high as one might hope given that events A and B actually happened. The use of probability estimates to evaluate models will now be discussed.

3.3 Using Probability Estimates to Evaluate Models

As noted above, it is possible for a given event to compute a series of probability estimates and compare these estimates to the actual outcomes. Consider event A above, the event of at least two consecutive quarters of negative values of g in a five-quarter period. Let A_t denote this event for the five-quarter period that begins with quarter t, and let P_t denote a model's estimate of the probability of A_t occurring. Let R_t denote the actual outcome of A_t—one if A_t occurred, and zero otherwise. As Diebold and Rudebusch (1989) point out, two common measures of the accuracy of probabilities are the quadratic probability score (QPS),

$$\text{(4)} \qquad \text{QPS} = (1/T)\sum_{t=1}^{T} 2(P_t - R_t)^2,$$

and the log probability score (LPS),

$$\text{(5)} \qquad \text{LPS} = -(1/T)\sum_{t=1}^{T} [(1 - R_t)\log(1 - P_t) + R_t \log P_t],$$

where T is the total number of observations. It is also possible simply to compute the mean of P_t (say, $\bar{P}$) and the mean of R_t (say $\bar{R}$) and compare the two means. QPS ranges from zero to two, with zero being perfect accuracy, and LPS ranges from zero to infinity, with zero being perfect accuracy. Larger errors are penalized more under LPS than under QPS.

For the empirical work in this section, events A_t and B_t were analyzed for t ranging from 1954:I through 1990:I (145 observations). A_t is the event of at least two consecutive quarters of negative real GNP growth for the five-quarter period beginning with quarter t, and B_t is the event of at least two quarters of negative real GNP growth (not necessarily consecutive) for the five-quarter period beginning with quarter t.

Since t ranges over 145 observations, there are 145 A_t events and 145 B_t events. Estimating the probabilities of these events required 145 stochastic

5. Note that estimating, say, a fourth-order autoregressive equation for an exogenous variable with a constant term and time trend included is equivalent to estimating the equation with only a constant term and time trend included under the assumption of a fourth-order autoregressive process for the error term. The equations are simply accounting for the autoregressive properties of the error term once the mean and deterministic trend have been removed. The present results thus show that little is gained in going from a first-order autoregressive process for the error term to a fourth-order process.

simulations. Each stochastic simulation was for a five-quarter period. The beginning quarter for the first simulation was 1954:I, the beginning quarter for the second simulation was 1954:II, and so on through the beginning quarter for the 145th simulation, which was 1990:I. Two sets of 145 stochastic simulations were in fact made. For the first set, the exogenous-variable values were taken to be the actual values—the exogenous-variable equations were not used, and no draws of exogenous-variable errors were made. The model used for this set will be called model (u, c).

For the second set, the exogenous-variable equations were added to the model, and error terms were drawn for these equations. As was done for the results in section 3.2, the error terms in the exogenous-variable equations were assumed to be uncorrelated with the error terms in the structural equations, and no coefficients were drawn for the exogenous-variable equations. Unlike in section 3.2, however, the estimated residuals were not added to the exogenous-variable equations. The base values of the error terms in these equations were assumed to be zero, just as is always done for the structural equations. This means that the model's prediction of the five-quarter period is based only on information available prior to the period. The model used for this set of stochastic simulations will be called model (u, c, e). As noted in section 3.2, the use of the exogenous-variable equations may overestimate exogenous-variable uncertainty, so it is not clear that the structural model should be judged by model (u, c, e) rather than by model (u, c). The truth probably lies somewhere in between.

The number of trials for each stochastic simulation was 100. This means that each set of 145 stochastic simulations required solving the model over a five-quarter period 14,500 times. In some cases, the model failed to solve for the particular draws, and, in these cases, the trial was simply discarded. This means that some of the probability estimates are based on slightly fewer than 100 trials. Most of the failures occurred early in the sample period.

A simple autoregressive model for real GNP was also estimated and stochastically simulated. The model consisted of regressing the log of real GNP on a constant, time, and the first four lagged values of log real GNP. The estimation period was 1954:I–1989:IV, the same as for the structural model, and 145 stochastic simulations were made. In this case, 1,000 trials were made for each simulation. This model will be called model AR.

From this work, one has three sets of values of P_t $(t = 1, \ldots, 145)$ for each of the two events, one set for each model. One also has the values of R_t for each event. Given the values of R_t, a fourth model can be considered, which is the model in which P_t is taken to be equal to $\bar{R}$ for each observation, where $\bar{R}$ is the mean of R_t over the 145 observations. This is simply a model in which the estimated probability of the event is constant and equal to the frequency with which the event happened historically. This model will be called model CONSTANT. The results are shown in table 3.3.

Both the structural model and model AR overestimate $\bar{R}$. (Remember that model CONSTANT is constructed so that $\bar{P} = \bar{R}$.) Model AR has somewhat

Table 3.3 Measures of Probability Accuracy

	Event A				Event B		
	$\bar{P}$	QPS	LPS		$\bar{P}$	QPS	LPS
Actual ($\bar{P} = \bar{R}$)	.138			Actual ($\bar{P} = \bar{R}$)	.297		
Model (u, c)	.285	.192	.315	Model (u, c,)	.394	.249	.383
Model (u, c, e)	.336	.268	.416	Model (u, c, e)	.445	.322	.481
Model AR	.238	.239	.401	Model AR	.341	.361	.544
Model CONSTANT	.138	.238	.401	Model CONSTANT	.297	.417	.608

less bias than the structural model. Regarding QPS and LPS, model (u, c) is always the best. For event A, model (u, c, e) is the worst, but the results for it, model AR, and model CONSTANT are all fairly close. Model (u, c, e) is noticeably better than model AR and model CONSTANT for event B.

Table 3.4 presents the 145 values of R_t for each event and the 145 values of P_t for each event and each model except model CONSTANT. (P_t for model CONSTANT is simply .138 for all t for event A and .297 for all t for event B.) Figures 3.1–3.3 plot the values of R_t and P_t for event B for models (u, c), (u, c, e), and AR, respectively.

One knows from the QPS and LPS results above that models (u, c) and (u, c, e) do better than model AR, and the three figures provide a helpful way of seeing this. The probability estimates for model AR never get above .64, whereas they are close to 1.0 for models (u, c) and (u, c, e) around a number of the actual occurrences of event B. Remember that model (u, c, e) is based on predicted values of the exogenous variables, and even this version does a reasonable job of having high estimated probabilities when event B occurs and low estimated probabilities when event B does not occur. One of the main times during which the structural model gets penalized in terms of the QPS and LPS criteria is the second half of the 1960s, where the estimated probabilities were fairly high for a number of quarters before event B actually happened.

Note from the figures that the occurrence of event B for the period beginning in 1990:I was not well predicted relative to earlier occurrences. The recession of 1990:IV–1991:I was not an easy one to predict.[6]

6. Note that the values of P_t in the last row of table 3.4 for models (u, c) and (u, c, e)—.350 and .280 for event A and .510 and .340 for event B—are not the same as those presented in table 3.2—.321 and .393 for event A and .483 and .522 for event B—even though the five-quarter period is the same. For model (u, c), the differences are due to the use of 1,000 trials for the results in table 3.2 compared to 100 trials for the results in table 3.4. For model (u, c, e), the differences are further due to the use of predicted values as the base values for the exogenous variables in table 3.4 rather than the actual values in table 3.2. For model (u, c, e), the probability estimates are considerably lower when the predicted values of the exogenous variables are used. The exogenous-variable equations for some of the government spending variables failed to predict the slowdown in the growth rate of these variables that occurred, and this is one of the reasons for the lower probability estimates for model (u, c, e) in table 3.4.

Table 3.4 **Estimated Probabilities from the Three Models**

	Event A				Event B			
Beg. Quar.	Act.	Model (u, c,)	Model (u, c, e)	Model AR	Act.	Model (u, c,)	Model (u, c, e)	Model AR
1954:I	1.0	.462	.407	.275	1.0	.667	.531	.408
1954:II	.0	.439	.254	.246	.0	.585	.476	.350
1954:III	.0	.200	.188	.171	.0	.345	.266	.264
1954:IV	.0	.215	.170	.122	.0	.557	.318	.180
1955:I	.0	.314	.292	.132	.0	.407	.326	.194
1955:II	.0	.229	.355	.160	.0	.554	.516	.234
1955:III	.0	.337	.394	.190	.0	.584	.564	.276
1955:IV	.0	.363	.474	.200	.0	.758	.684	.290
1956:I	.0	.454	.552	.202	.0	.619	.750	.296
1956:II	.0	.330	.474	.225	.0	.546	.680	.332
1956:III	.0	.418	.474	.195	.0	.622	.680	.294
1956:IV	.0	.280	.408	.178	1.0	.730	.653	.261
1957:I	1.0	.585	.495	.158	1.0	.702	.657	.233
1957:II	1.0	.565	.465	.166	1.0	.765	.636	.241
1957:III	1.0	.442	.434	.183	1.0	.651	.566	.274
1957:IV	1.0	.402	.455	.163	1.0	.644	.646	.241
1958:I	.0	.223	.380	.205	.0	.362	.522	.286
1958:II	.0	.051	.088	.199	.0	.127	.176	.267
1958:III	.0	.063	.068	.082	.0	.125	.136	.132
1958:IV	.0	.156	.110	.085	.0	.200	.154	.116
1959:I	.0	.097	.081	.103	.0	.172	.152	.146
1959:II	.0	.085	.112	.128	1.0	.340	.337	.188
1959:III	.0	.258	.253	.126	1.0	.526	.444	.191
1959:IV	.0	.289	.333	.161	1.0	.495	.455	.230
1960:I	.0	.287	.454	.133	1.0	.468	.546	.214
1960:II	.0	.333	.340	.116	1.0	.548	.515	.169
1960:III	.0	.138	.214	.153	.0	.298	.337	.223
1960:IV	.0	.079	.234	.151	.0	.213	.362	.226
1961:I	.0	.032	.082	.130	.0	.053	.112	.192
1961:II	.0	.021	.051	.088	.0	.031	.071	.135
1961:III	.0	.010	.074	.096	.0	.030	.096	.131
1961:IV	.0	.020	.071	.104	.0	.061	.122	.146
1962:I	.0	.061	.111	.118	.0	.101	.152	.166
1962:II	.0	.091	.131	.136	.0	.131	.192	.195
1962:III	.0	.106	.206	.152	.0	.223	.278	.223
1962:IV	.0	.104	.206	.151	.0	.219	.268	.224
1963:I	.0	.053	.165	.170	.0	.095	.216	.243
1963:II	.0	.040	.122	.132	.0	.070	.184	.204
1963:III	.0	.060	.121	.139	.0	.120	.182	.203
1963:IV	.0	.160	.162	.158	.0	.220	.242	.235
1964:I	.0	.210	.180	.180	.0	.300	.220	.263
1964:II	.0	.160	.232	.172	.0	.230	.333	.263
1964:III	.0	.121	.232	.210	.0	.222	.364	.307
1964:IV	.0	.121	.337	.221	.0	.182	.480	.331
1965:I	.0	.071	.354	.225	.0	.131	.485	.333
1965:II	.0	.081	.480	.210	.0	.212	.653	.309

(continued)

Table 3.4 (continued)

	Event A				Event B			
Beg. Quar.	Act.	Model $(u, c,)$	Model (u, c, e)	Model AR	Act.	Model $(u, c,)$	Model (u, c, e)	Model AR
1965:III	.0	.192	.430	.255	.0	.273	.570	.355
1965:IV	.0	.232	.390	.280	.0	.323	.530	.398
1966:I	.0	.276	.500	.315	.0	.357	.640	.428
1966:II	.0	.327	.600	.370	.0	.541	.780	.491
1966:III	.0	.388	.680	.424	.0	.592	.860	.576
1966:IV	.0	.434	.700	.379	.0	.707	.880	.534
1967:I	.0	.500	.670	.380	.0	.630	.850	.530
1967:II	.0	.404	.530	.377	.0	.566	.690	.532
1967:III	.0	.112	.410	.365	.0	.276	.580	.518
1967:IV	.0	.273	.490	.362	.0	.525	.680	.496
1968:I	.0	.354	.400	.393	.0	.455	.590	.540
1968:II	.0	.470	.390	.392	.0	.600	.670	.536
1968:III	.0	.616	.622	.410	.0	.808	.776	.538
1968:IV	.0	.790	.610	.448	1.0	.940	.830	.584
1969:I	1.0	.700	.590	.459	1.0	.830	.750	.627
1969:II	1.0	.850	.910	.408	1.0	.970	.970	.552
1969:III	1.0	.850	.770	.437	1.0	.950	.900	.588
1969:IV	1.0	.780	.690	.418	1.0	.930	.900	.587
1970:I	1.0	.680	.680	.424	1.0	.820	.830	.584
1970:II	.0	.530	.660	.344	1.0	.680	.800	.509
1970:III	.0	.550	.660	.303	1.0	.710	.790	.440
1970:IV	.0	.350	.350	.278	1.0	.600	.490	.401
1971:I	.0	.080	.110	.326	1.0	.110	.200	.459
1971:II	.0	.090	.220	.245	1.0	.230	.360	.383
1971:III	.0	.090	.130	.304	.0	.230	.210	.428
1971:IV	.0	.100	.190	.293	.0	.190	.230	.452
1972:I	.0	.020	.120	.273	.0	.020	.180	.405
1972:II	.0	.020	.180	.247	.0	.020	.260	.371
1972:III	.0	.020	.320	.307	.0	.030	.470	.415
1972:IV	.0	.110	.350	.350	.0	.130	.500	.496
1973:I	.0	.220	.300	.352	1.0	.260	.410	.488
1973:II	.0	.370	.560	.408	1.0	.470	.660	.537
1973:III	.0	.540	.620	.479	1.0	.790	.810	.624
1973:IV	1.0	.720	.710	.451	1.0	.920	.890	.632
1974:I	1.0	.830	.790	.385	1.0	.940	.950	.542
1974:II	1.0	.760	.660	.405	1.0	.900	.840	.562
1974:III	1.0	.870	.630	.366	1.0	.950	.830	.529
1974:IV	1.0	.750	.740	.371	1.0	.800	.860	.514
1975:I	.0	.430	.510	.300	.0	.480	.670	.446
1975:II	.0	.020	.070	.286	.0	.090	.120	.390
1975:III	.0	.050	.000	.148	.0	.070	.000	.252
1975:IV	.0	.090	.050	.159	.0	.130	.060	.227
1976:I	.0	.100	.080	.191	.0	.150	.080	.270
1976:II	.0	.040	.080	.211	.0	.050	.080	.296
1976:III	.0	.040	.090	.242	.0	.050	.100	.344
1976:IV	.0	.030	.100	.236	.0	.140	.120	.360

Table 3.4 (continued)

	Event A				Event B			
Beg. Quar.	Act.	Model (u, c,)	Model (u, c, e)	Model AR	Act.	Model (u, c,)	Model (u, c, e)	Model AR
1977:I	.0	.100	.090	.210	.0	.120	.090	.314
1977:II	.0	.110	.150	.225	.0	.130	.210	.321
1977:III	.0	.100	.230	.242	.0	.110	.330	.345
1977:IV	.0	.210	.300	.268	.0	.370	.440	.383
1978:I	.0	.110	.370	.332	.0	.190	.520	.464
1978:II	.0	.170	.360	.275	.0	.220	.570	.426
1978:III	.0	.390	.560	.287	.0	.450	.750	.396
1978:IV	.0	.400	.540	.359	1.0	.540	.760	.498
1979:I	.0	.580	.710	.371	1.0	.780	.850	.537
1979:II	.0	.460	.730	.384	1.0	.790	.890	.537
1979:III	.0	.740	.750	.368	1.0	.890	.920	.532
1979:IV	.0	.860	.780	.305	1.0	.950	.900	.448
1980:I	.0	.680	.660	.325	.0	.840	.820	.465
1980:II	.0	.740	.660	.283	1.0	.920	.830	.420
1980:III	.0	.470	.530	.386	.0	.710	.740	.515
1980:IV	.0	.200	.390	.220	1.0	.310	.460	.375
1981:I	1.0	.600	.500	.174	1.0	.750	.640	.258
1981:II	1.0	.950	.610	.213	1.0	.960	.810	.299
1981:III	1.0	.850	.660	.257	1.0	.990	.800	.371
1981:IV	1.0	.980	.830	.230	1.0	1.000	.900	.359
1982:I	.0	.860	.730	.244	1.0	.950	.800	.356
1982:II	.0	.700	.390	.224	.0	.780	.480	.327
1982:III	.0	.580	.220	.112	.0	.660	.330	.174
1982:IV	.0	.180	.080	.113	.0	.270	.080	.170
1983:I	.0	.000	.000	.096	.0	.000	.000	.155
1983:II	.0	.000	.000	.090	.0	.000	.000	.126
1983:III	.0	.000	.010	.091	.0	.000	.010	.127
1983:IV	.0	.000	.020	.112	.0	.010	.020	.153
1984:I	.0	.050	.050	.128	.0	.050	.050	.185
1984:II	.0	.100	.170	.148	.0	.140	.250	.203
1984:III	.0	.320	.250	.179	.0	.480	.300	.251
1984:IV	.0	.400	.260	.204	.0	.520	.410	.293
1985:I	.0	.120	.210	.197	.0	.190	.270	.284
1985:II	.0	.070	.210	.171	.0	.090	.240	.250
1985:III	.0	.030	.110	.185	.0	.060	.140	.262
1985:IV	.0	.040	.160	.183	.0	.080	.180	.263
1986:I	.0	.030	.110	.185	.0	.050	.120	.262
1986:II	.0	.010	.050	.186	.0	.020	.060	.267
1986:III	.0	.000	.000	.220	.0	.000	.000	.317
1986:IV	.0	.010	.030	.201	.0	.010	.050	.300
1987:I	.0	.000	.020	.163	.0	.000	.020	.237
1987:II	.0	.000	.020	.163	.0	.010	.020	.234
1987:III	.0	.010	.110	.181	.0	.010	.150	.254
1987:IV	.0	.070	.160	.185	.0	.070	.190	.263
1988:I	.0	.020	.150	.188	.0	.030	.260	.265
1988:II	.0	.010	.110	.203	.0	.030	.170	.285

(continued)

Table 3.4 (continued)

	Event A				Event B			
Beg. Quar.	Act.	Model (u, c,)	Model (u, c, e)	Model AR	Act.	Model (u, c,)	Model (u, c, e)	Model AR
1988:III	.0	.020	.170	.224	.0	.050	.230	.321
1988:IV	.0	.130	.290	.230	.0	.150	.400	.332
1989:I	.0	.120	.310	.223	.0	.160	.400	.323
1989:II	.0	.170	.300	.213	.0	.370	.470	.307
1989:III	.0	.270	.360	.229	.0	.450	.460	.329
1989:IV	.0	.350	.300	.227	.0	.470	.450	.328
1990:I	1.0	.350	.280	.226	1.0	.510	.340	.321

Note: "Beg. Quar." = beginning quarter. "Act." = actual.

As a final comment, the results in this section are all within sample except for the results for the last five quarters. Even model CONSTANT is within sample because it uses the sample mean over the entire period. In future work, it would be of interest to do rolling regressions and have all the simulations be outside sample. This is expensive because covariance matrices also have to be estimated each time, and it limits the number of observations for which P_t can be computed because observations are needed at the beginning for the initial estimation period. In future work, it would also be useful to do more than one hundred trials per stochastic simulation. There is still considerable stochastic-simulation error with only one hundred trials.

3.4 Conclusion

This paper shows that stochastic simulation can be used to answer probability questions about the economy. The procedure discussed here is flexible in allowing for different models, different assumptions about the underlying probability distributions, different assumptions about exogenous-variable uncertainty, and different events for which probabilities are estimated. The paper also shows that a series of probability estimates can be computed and that these estimates can then be used to evaluate a model's ability to predict the various events.

References

Diebold, Francis X., and Glenn D. Rudebusch. 1989. Scoring the leading indicators. *Journal of Business* 62:369–91.

Fair, Ray C. 1980. Estimating the expected predictive accuracy of econometric models. *International Economic Review* 21 (June): 355–78.

———. 1984. *Specification, estimation, and analysis of macroeconometric models.* Cambridge, Mass.: Harvard University Press.

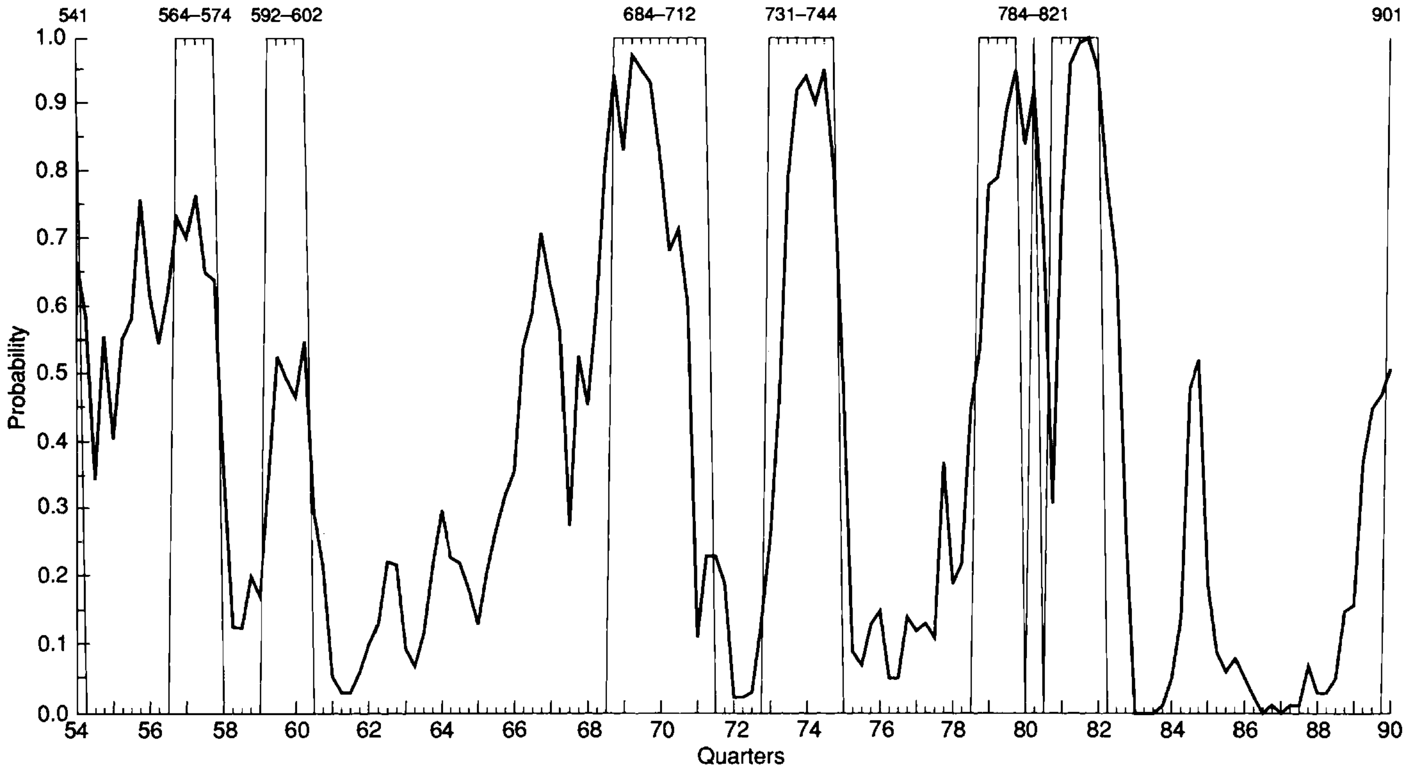

Fig. 3.1 Estimated probabilities for event B for model (*u*, *c*)

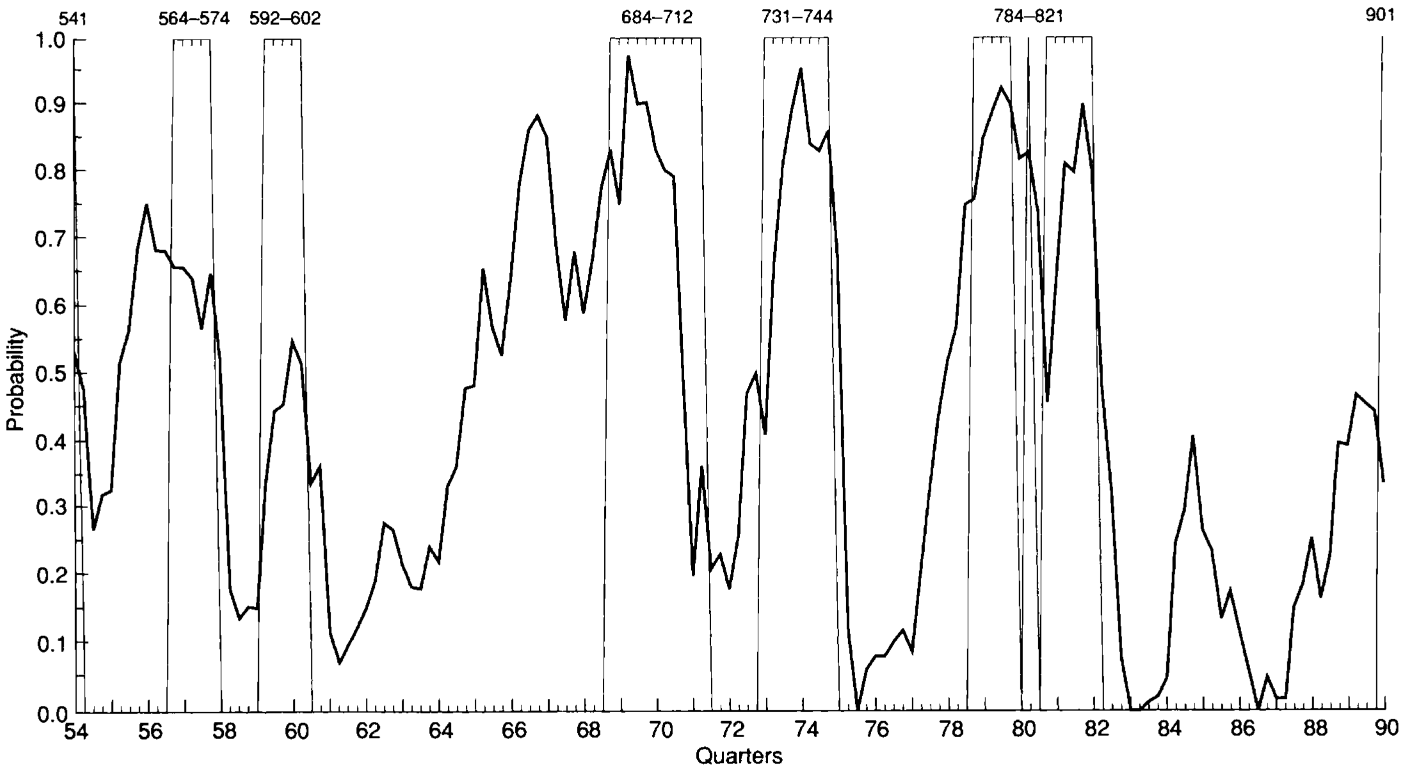

Fig. 3.2 Estimated probabilities for event B for model (*u*, *c*, *e*)

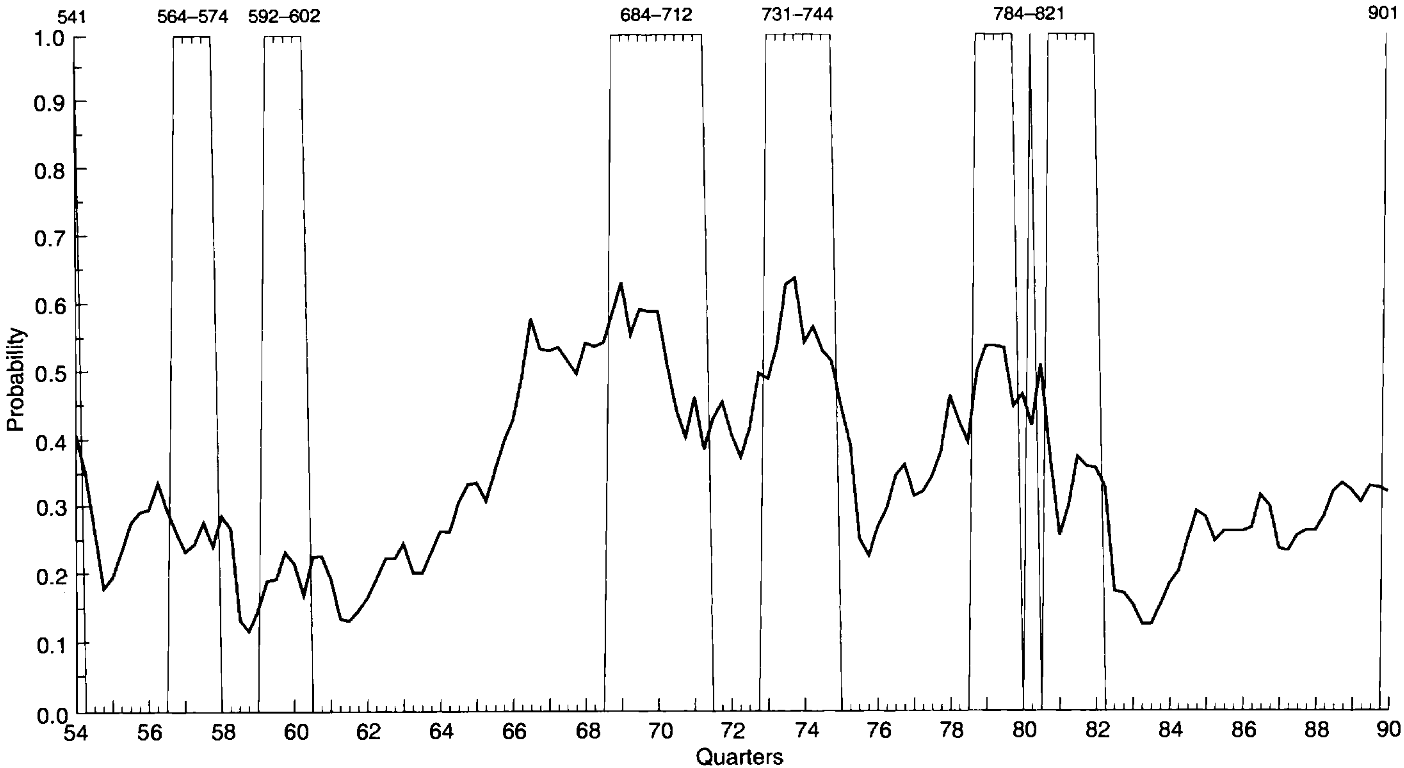

Fig. 3.3 Estimated probabilities for event B for model AR

Stock, James H., and Mark W. Watson. 1989. New indexes of coincident and leading economic indicators. Discussion Paper no. 178D. Harvard University, Kennedy School of Government, April.

Comment James D. Hamilton

If one has a fully specified econometric model of the economy, as Ray Fair does, and if one has no compunction about torturing a computer, as Fair apparently does not, then one need not be limited to reporting just the point forecast for GNP predicted by the model. By simulating the model, one can in principle calculate the probability distribution of any future economic event. This distribution can reflect both uncertainty about the future course of the economy and uncertainty about the true values of the structural parameters. Fair offers a nice illustration of this method using his model of the U.S. economy.

Although the calculated probabilities of future events provide some of the most interesting insights from Fair's analysis, I would like to begin with the simple point forecasts of real GNP growth, in order to compare Fair's predictions with those of other models. Table 3C.1 compares the results from Fair's simulations with two real-time forecasts. The first forecast is based on a vector autoregression maintained by Christopher Sims at the Federal Reserve Bank of Minneapolis, and the second is based on a survey of alternative forecasts compiled by Victor Zarnowitz for the National Bureau of Economic Research (for sources, see table 3C.1). Between 1954 and 1990, quarterly U.S. real GNP growth averaged 2.9 percent at an annual rate, with a standard deviation of 4 percent. The economy grew unusually slowly during 1990, with a recession beginning in the fourth quarter. Fair's model tracks this outcome fairly well, in contrast to the ex ante forecasts produced by many economists at the beginning of the year.

It is worth emphasizing that Fair's (u, c, e) simulations reported in his table 3.1 are not strictly comparable to these real-time ex ante forecasts. In Fair's simulations, the exogenous variables are not drawn from their conditional distribution based on information available at the beginning of 1990 but are instead drawn from a distribution based on the actual ex post values of these variables. Fair argues that a practical user of his model has better information about the future values of the exogenous variables than is captured by simple autoregressions. Even if one were uncomfortable with this argument, the parameters of his model were estimated without using the 1990 data, with the result that his simulations clearly offer evidence that his model contains an

James D. Hamilton is professor of economics at the University of California, San Diego, and research adviser to the Federal Reserve Bank of Richmond.

Table 3C.1 **Comparison of Forecasts of Real GNP Growth**

	1990				
	I	II	III	IV	1991:1
Actual	1.7	.4	1.4	−1.6	−2.6
Fair	3.3	.9	1.4	1.0	−.1
	(2.8)	(3.2)	(3.4)	(3.2)	(3.5)
Minnesota	2.7	2.9	3.1	3.1	3.1
	(2.5)	(3.3)	(3.0)	(3.1)	(4.5)
NBER	1.2	1.8	2.1	3.0	2.1

Sources: Minnesota: Christopher Sims, "Economic Forecasts from a Vector Autoregression" (Minneapolis: Federal Reserve Bank of Minneapolis, 30 December 1989 [release date]). NBER: Victor Zarnowitz, "Economic Outlook Survey" *NBER Reporter* (Spring 1990).

accurate description of the economy and of the exogenous variables that contributed to the recession of 1990–91.

Although the point estimates in Fair's table 3.1 do not incorporate ex ante uncertainty about the exogenous variables, the standard errors that he calculates are very similar to those that would be calculated from a simulation based solely on historically available information. It is instructive to note that the standard errors for his five-quarter-ahead (u, c, e) simulation for real GNP growth are close to the unconditional standard deviation. This suggests that, in the absence of better information about the values of the exogenous variables than contained in an autoregression, the model does not offer much improvement over a simple forecast that a year from now GNP growth will proceed at its historical average rate. Recall the familiar result that the mean squared error is equal to the variance plus the square of the bias:

$$E_Y[Y - E(Y)]^2 = E_X E_{Y|X}[Y - E(Y|X)]^2 + E_X[E(Y) - E(Y|X)]^2.$$

Let Y in this formula stand for GNP growth; then $E(Y)$ is the unconditional average growth of GNP, and $E_Y[Y - E(Y)]^2$ is the unconditional variance around this mean. Let X represent information on which a forecast of GNP might be based. Then $E_{Y|X}[Y - E(Y|X)]^2$ is the variance of this forecast, the magnitude that would be calculated from Fair's simulations in his equation (3), while $[E(Y) - E(Y|X)]$ is the difference between the forecast and the historical mean. For typical values of X, if the forecast variance $E_{Y|X}[Y - E(Y|X)]^2$ equals the unconditional variance $E_Y[Y - E(Y)]^2$, then the forecast $E(Y|X)$ should be equal to the unconditional mean $E(Y)$.

This suggests another role that simulation might play in model verification. If one finds that the model generates longer-run forecasts that differ significantly from the unconditional mean but that the standard deviations for these forecasts equal or exceed the unconditional standard deviation, then the forecasts might be improved by Bayesian shrinkage toward the unconditional mean. Confidence intervals should also be correspondingly tightened to reflect the value of the prior information, with the result that the distribution of forecast errors converges to the unconditional distribution as the forecast horizon grows larger.

A similar issue may apply to the calculations of probabilities of recessions. Fair's model appears to be helpful in predicting turning points, as measured by both the quadratic probability score and the log probability score. On average, however, the model errs in overpredicting recessions, with the bias most severe for the (u, c, e) simulation. This could result from either the mean GNP growth implied by the model being too low or the standard deviation being too high. Again, Bayesian adjustment so that the distribution of longer-run forecast errors converges to the unconditional distribution of GNP growth might prove helpful. The simpler expedient of shrinking the probabilities of turning points calculated by the model toward the unconditional probabilities would also be interesting to explore.

Overall, Fair has proposed a valuable tool for economic research and practical forecasting, and the results seem quite favorable for his model of the economy.