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ON THE RELATIONSHIP BETWEEN DETERMINATE
AND MSV SOLUTIONS IN LINEAR RE MODELS

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ABSTRACT

This paper considers the possibility that, in linear rational expectations (RE) models, all determinate (uniquely non-explosive) solutions coincide with the minimum state variable (MSV) solution, which is unique by construction. In univariate specifications of the form $y(t) = AE(t)y(t+1) + Cy(t-1) + u(t)$ that result holds: if a RE solution is unique and non-explosive, then it is the same as the MSV solution. Also, this result holds for multivariate versions if the A and C matrices commute and a certain regularity condition holds. More generally, however, there are models of this form that possess unique non-explosive solutions that differ from their MSV solutions. Examples are provided and a strategy for easily constructing others is outlined.

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1. Introduction

Much recent research in economics has emphasized the concept of determinacy of rational expectation solutions—i.e., the property of a solution being the only non-explosive solution. It is well known that in linear rational expectations (RE) models a necessary and sufficient condition for determinacy is that the number of eigenvalues of the system's matrix pencil that exceed 1.0 in modulus equals the number of non-predetermined endogenous variables.[1] In various prominent cases, this condition does not obtain so there is no unique non-explosive solution.

Some researchers[2] have focused attention on the minimum state variable (MSV) solution, defined and promoted in McCallum (1983, 1999), which is by construction unique but possibly explosive, and exists if the model has any real (non-imaginary) solution.[3] It is obviously the case that some MSV solutions are not determinate, but it is unclear whether there are models in which a determinate solution exists but is not the MSV solution. That possibility has been hinted at by McCallum (1983, 1998) and Uhlig (1999), but examples have not been examined. There are some reasons, perhaps, to suspect that it might be true that all unique stable solutions are MSV solutions. Such a situation is easily seen to prevail in univariate models of the form $y_t = AE_t y_{t+1} + Cy_{t-1} + u_t$ and, as is shown below, also holds for multivariate versions if the A and C matrices commute and a regularity condition due to Binder and Pesaran (1995) obtains. Furthermore, there are recent results by Gauthier (2002, 2003) and Desgranges and Gauthier (2003) showing, among other things, that the same result holds in univariate

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[1] See Blanchard and Kahn (1980), Binder and Pesaran (1995), King and Watson (1998), among others.
[2] For example, Bullard and Mitra (2002), Barro (1989), Faust and Svensson (2001), and Leitemo (2003).
[3] It is important to note that the term "minimum state variable" is here being used in the manner of McCallum (1983, 1999) or Evans (1986), rather than that of Evans and Honkapohja (2001) or Gauthier (2003), which permits more than one MSV solution. See Section 2 below.

perfect foresight models with additional lagged terms and/or expected future values.

It transpires, nevertheless, that there can exist unique stable solutions that differ from the MSV solution. This is demonstrated below, in Section 4. In addition, the paper discusses (in Section 5 and elsewhere) various aspects of the two types of solutions and their implied criteria for selection of a RE solution. Section 2 outlines the specification to be utilized and provides preliminary results, while Section 3 mentions conditions under which unique stable solutions will invariably be MSV solutions.

2. Preliminaries

Because our main result consists of a counterexample, it will not be necessary to utilize a framework with full generality. Instead, it will be convenient to consider the specification treated by McCallum (1983, pp. 164-166). With y_t denoting a m×1 vector of endogenous variables, the system is

(1) $y_t = A \, E_t y_{t+1} + C \, y_{t-1} + u_t,$

where $u_t = R \, u_{t-1} + \varepsilon_t$, with R a stable m×m matrix and ε_t a white noise vector.[4] Also, it is assumed that A is nonsingular. That is a strong assumption, which renders the formulation (1) highly inconvenient from a practical perspective, but is acceptable for the purposes at hand. Furthermore, for other purposes the implied case can provide a useful precursor for more general analyses, as is illustrated in McCallum (2003).

In this setting, the MSV solution will be of the form

(2) $y_t = \Omega \, y_{t-1} + \Gamma \, u_t.$

Accordingly, $E_t y_{t+1} = \Omega(\Omega y_{t-1} + \Gamma u_t) + \Gamma R u_t$ and straightforward undetermined-coefficient reasoning yields the requirement that the solution for Ω satisfies

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[4] In (1), constant terms are suppressed for notational simplicity while A and C are of dimension m×m.

(3) $A\Omega^2 - \Omega + C = 0,$

where all of the matrices are of order m×m. There are other implications, of course, but

the occurrence of multiple solutions arises entirely because of the nonlinear nature of (3);

for a given Ω, Γ is determined uniquely. In this setting, the MSV concept requires that

$\Omega = 0$ if $C = 0$, since otherwise the solution would in that case include extraneous

variables, and the MSV solution is defined as the one whose expression for Ω

continuously approaches 0 as C approaches a zero matrix.

      With A invertible, the matrix quadratic equation (3) can, as is well known, be

expressed in a first-order manner as

(4) $$\begin{bmatrix} \Omega \\ \Omega^2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A^{-1}C & A^{-1} \end{bmatrix} \begin{bmatrix} I \\ \Omega \end{bmatrix}.$$

Let M denote the 2m×2m matrix M in (4) and assume, without significant loss of

generality, that it is diagonalizable. Then it follows that $M = P^{-1}\Lambda P$, with Λ a diagonal

matrix with the eigenvalues of M on its diagonal. Then we can premultiply (4) by P,

where $P^{-1} = H$ is the matrix of (right) eigenvectors of M, to obtain

(5) $$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} \Omega \\ \Omega^2 \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} I \\ \Omega \end{bmatrix},$$

where the P_{ij} are submatrices of P and where the diagonals of Λ_1 and Λ_2 contain the

eigenvalues of M.

      To obtain the MSV solution, McCallum (1983) arranges the eigenvalues (and

associated eigenvectors) so that Λ_1 includes those that approach 0 as C approaches 0.[5]

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[5] The identification of this grouping is based on the continuity of eigenvalues with respect to the elements of the underlying matrix (M, in this case). We let C approach a zero matrix by replacing C with Cα in all relevant expressions and letting the real scalar α vary continuously from 1.0 to 0.

Then the MSV expression for Ω is implied by the second row of (5) to be[6]

(6) $\Omega = - P_{22}^{-1} P_{21}.$

Further, since PM = ΛP, we have (from the lower left submatrix) that $-P_{22}A^{-1}C = \Lambda_2 P_{21}$

so if the inverse of Λ_2 exists, (6) gives the solution

(7) $\Omega = P_{22}^{-1}\Lambda_2^{-1} P_{22}A^{-1}C,$

for which Ω approaches 0 as C approaches 0. For this conclusion, it needs to be true that

Λ_2^{-1} exists in the limit. But the eigenvalues of M are obtained from $\det[M - \lambda I] = 0$, and

using a result on the determinant of a partitioned matrix,[7] we have that

(8) $\det[M - \lambda I] = \det\begin{bmatrix} -\lambda I & I \\ -A^{-1}C & A^{-1} - \lambda I \end{bmatrix}$

$= \det[A^{-1} - \lambda I]\,\det[-\lambda I + I(A^{-1} - \lambda I)^{-1}A^{-1}C].$

From the latter we see that for any arrangement of the eigenvalues, half of them will

(continuously) approach zero and the other half will approach the eigenvalues of A^{-1} as C

approaches 0. Thus with the MSV definition of Λ_1, it is implied that the eigenvalues of

Λ_2 approach those of A^{-1}, which are all non-zero.

At this point it should be emphasized that expression (7) gives different solutions

for different groupings of eigenvalues into Λ_1 and Λ_2. Since M is $2m \times 2m$, there are

$(2m)!/(m!)^2$ different groupings, each of which provides a solution given by (6). There is

only one for which (7) is well defined in the limit as C approaches 0, however, since (8)

implies that all others feature Λ_2 matrices that are not invertible when C = 0.

Let us now consider the particular solution given by (6) when the eigenvalues are

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[6] This row can be written as $(P_{21} + P_{22}\Omega)\Omega = \Lambda_2(P_{21} + P_{22}\Omega)$.
[7] See Johnston (1972, p. 95).

instead arranged so that Λ_1 includes those that are smallest (in modulus). Recall that H

denotes the eigenvectors of M so that PH = I. The latter implies that $P_{21}H_{11} + P_{22}H_{21} = 0$

and from the upper left-hand submatrix of MH = H Λ we have that $H_{21} = H_{11}\Lambda_1$.

Therefore $\Omega = -P_{22}^{-1}P_{21} = P_{22}^{-1}P_{22}H_{21}H_{11}^{-1} = H_{21}H_{11}^{-1} = H_{11}\Lambda_1 H_{11}^{-1}$. But the latter has

the same eigenvalues as Λ_1, which under present assumptions are the m smallest

eigenvalues of M. If there is a unique stable solution, it will feature an Ω whose eigen-

values are the m smallest eigenvalues of M. Thus, if there is a unique stable RE solution,

it will be given by expression (7) with Λ_1 including the smallest eigenvalues of M.

Is it likely that the unique stable solution and the MSV solution will coincide, if

the former exists? Clearly, if the entries in C are all small, so that C is close to a zero

matrix, they will coincide since the MSV solution for Ω will have near-zero

eigenvalues—and these will then tend to be the smallest of M's eigenvalues, which are

those that appear in Λ_1 for the unique stable solution. Thus there is a distinct tendency

for unique stable and MSV solutions to coincide. Indeed, they must coincide unless the

set of eigenvalues, which includes only the m smallest, changes in composition as α goes

from 1 to 0. For if it does not, then (7) will apply to the unique stable solution in the

limit, making it correspond to the MSV solution.

3. Special Cases

Let us now briefly consider the special cases mentioned above in which it is true

that the unique stable and MSV solutions coincide. The simplest example is that in

which m = 1, i.e., the model (1) is univariate. Write the quadratic (3) for that case as

$a\phi^2 - \phi + c = 0$, where we use ϕ in place of Ω. In this case we have roots for ϕ equal to

$\Lambda_1 = (1 - \sqrt{1 - 4ac})/2a$ and $\Lambda_2 = (1 + \sqrt{1 - 4ac})/2a$. The first of these approaches 0 as c

approaches 0, the second approaching a^{-1}, so it gives the MSV solution. Furthermore, the

first has the smaller modulus since $+\sqrt{1 - 4ac}$ has the same sign as 1. Consequently, the

stated coincidence obtains quite generally.

Somewhat less familiar is a result that obtains when A and C in (1) commute. If

two matrices commute (and are diagonalizable) then they can be diagonalized by the

same matrix. That implies that the same matrix, say T, diagonalizes both A and C and

also sums and products of those two matrices. Therefore it follows, as shown by Binder

and Pesaran (1995, p. 158), that the terms in (1) can be diagonalized to yield

(9) $\Lambda_A \Lambda_\Omega^2 - \Lambda_A + \Lambda_C = 0$,

where each Λ matrix includes the eigenvalues of the designated matrix on its diagonal

and zeros elsewhere. But (9) implies m distinct scalar equations of the quadratic form

considered in the previous paragraph. If the model has a unique stable solution, these m

equations must have m roots (eigenvalues of M) with modulus less than 1.0 and m with

modulus greater than 1.0. The MSV solution will assign one root—the one for which λ_Ω

approaches 0 as λ_C approaches 0—from each of these quadratic equations to the matrix

Λ_1, so it will coincide with the unique stable solution if and only if each of the quadratics

has one root greater (and one smaller) than 1.0 in modulus. Binder and Pesaran (1995, p.

157) describe a regularity condition that guarantees such a configuration, but it seems to

be of limited interest in the present context.

4. Examples

We now turn to a numerical specification that provides a counterexample to the

conjecture that all unique stable solutions to models of form (1) are also MSV solutions.

It is given by equation (1) with the following A and C matrices:

$$(10) \quad A = \begin{bmatrix} -1.5 & 1.2 \\ 0.5 & -1.3 \end{bmatrix} \qquad C = \begin{bmatrix} 1.2 & 0.5 \\ 0.5 & 1.6 \end{bmatrix}$$
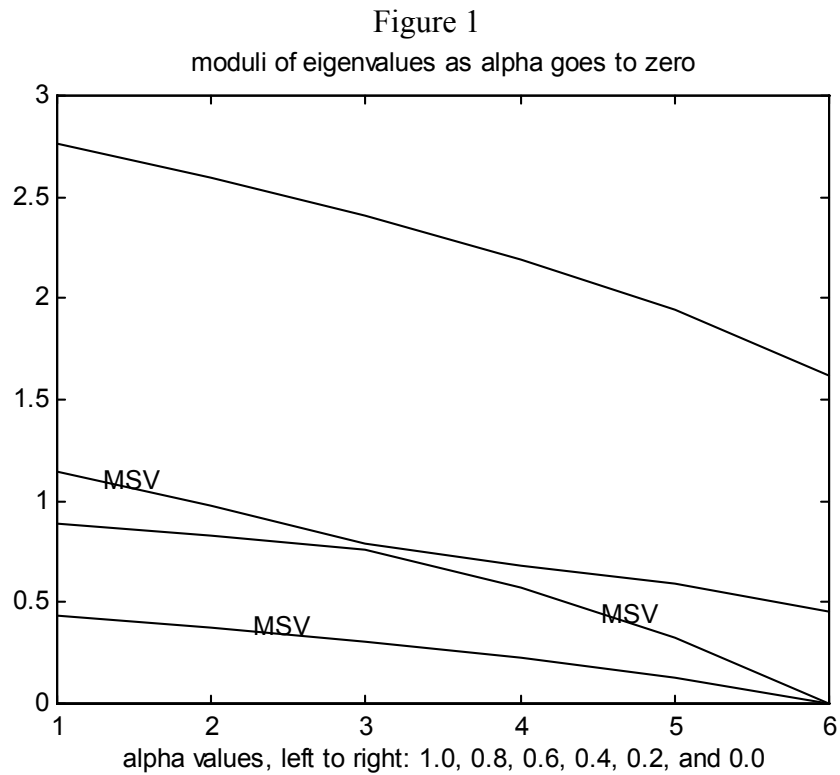
For simplicity, we take R to contain only zeros; that assumption does not affect the

solutions for Ω. The magnitudes relevant for our issues of concern are the eigenvalues of

M in the problem as just specified, i.e., with $\alpha = 1$, and for other values of α on the

interval [0, 1]. In Table 1, the eigenvalues are reported for α equal to 1.0, 0.8, 0.6, 0.4,

0.2, and 0.0. They are reported in order of decreasing modulus in each case. The

Table 1
Eigenvalues of M for various values of α

$\alpha = 1.0$	$\alpha = 0.8$	$\alpha = 0.6$	$\alpha = 0.4$	$\alpha = 0.2$	$\alpha = 0.0$
−2.7022	−2.5402	−2.3615	−2.1593	−1.9211	−1.6156
1.0887	0.9267	−0.7961	−0.7108	−0.6066	−0.4585
−0.9365	−0.8702	0.7479	0.5456	0.3070	0.0000
0.4759	0.4096	0.3357	0.2505	0.1466	0.0000

results for the actual problem at hand are given in the first column. It is readily seen to be

one in which there is a unique stable solution, since there are two eigenvalues with

modulus greater than 1.0. Thus −0.9365 and 0.4795 are the diagonal elements of the Λ_1

matrix if the latter is defined as relevant for seeking a determinate solution, i.e., as

including the smallest (modulus) eigenvalues. But what is the MSV solution for the

model? Since eigenvalues are continuous functions of the model parameters, it is clear

that the second-listed eigenvalue in the first two columns is the "same" as the third

eigenvalue in the remaining columns.[8] Thus the composition of Λ_1 relevant for the MSV

solution includes 1.0887 and 0.4759. The MSV solution differs from the unique stable

solution; indeed, the MSV solution is dynamically explosive. A plot of the moduli of the

four eigenvalues against α is shown in Figure 1. There the diagram does not accurately

Figure 1

moduli of eigenvalues as alpha goes to zero

alpha values, left to right: 1.0, 0.8, 0.6, 0.4, 0.2, and 0.0

show the crossing of eigenvalue that occurs at a value of α slightly above 0.6 but,as one

eigenvalue is positive and the other negative, the numbers in Table 1 make the situation

clear. (The two curves pertaining to the MSV solution are so labeled in the figure.)

Reflection indicates that there is a simple way of generating an example with an

eigenvalue crossing that keeps a determinate solution from being the MSV solution.

Consider the m = 2 case, and suppose that the two rows of (1) represent separate

univariate models. One of these can be specified so as to imply an explosive univariate

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[8] Here "same" is used in the following sense: for each specified eigenvalue, its value is a continuous
function of each of the elements of the M matrix.

solution (one in which both moduli exceed 1.0) and the other to imply multiple stable

solutions (both are less than 1.0). A pair of such models does not constitute a non-

degenerate bivariate model, and will not permit RE solutions with some software.[9] But

by simply adding very small non-zero values for one or more of the off-diagonal

elements of A or C, a valid bivariate model of form (1) can be obtained. Yet with very

small values for these off-diagonal elements, the eigenvalues for this bivariate model will

be approximately the same as for the two univariate models taken together. Accordingly,

they will include two stable and two explosive eigenvalues. The bivariate system will

therefore be determinate; it will have one stable solution. The MSV solution will,

however, involve one eigenvalue from each of the univariate models and will therefore

differ from the unique stable solution.[10]

An example of this type is provided by the univariate models defined by a_{11} =

-0.4, $c_{11} = 1.5$ and $a_{22} = -1.5$, $c_{22} = 0.2$ with zeros elsewhere. The first has two explosive

roots (-3.5549 and 1.0549) and the second has two stable roots (-0.8277 and 0.1611).

Table 2
Eigenvalues of M for various values of α

$\alpha = 1.0$	$\alpha = 0.8$	$\alpha = 0.6$	$\alpha = 0.4$	$\alpha = 0.2$	$\alpha = 0.0$
-3.5563	-3.3873	-3.2038	-3.0012	-2.7719	-2.5011
1.0551	0.8862	-0.7703	-0.7387	-0.7044	-0.6666
-0.8275	-0.7998	0.7027	0.5001	0.2707	0.0000
0.1610	0.1332	0.1038	0.0721	0.0378	0.0000

[9] A necessary rank condition is not satisfied. See, e.g., King and Watson (1998) or McCallum (1998).
[10] Note that if both univariate systems have unique stable solutions, the MSV and unique stable bivariate solutions will coincide. If both are explosive, both solution criteria will indicate an explosive solution. If both have multiple solutions, so will the bivariate model—but one solution will be selected by the MSV criterion.

To create a non-degenerate bivariate model we change a_{12}, a_{21}, c_{12}, and c_{21} from 0.0 to the values 0.01, 0.02, 0.02, and 0.01, respectively. Then the resulting eigenvalues for various values of α are as reported in Table 2. As in the example of Table 1, there is a unique stable solution for the model (i.e., with $\alpha = 1$) but it differs from the MSV solution.

5. Discussion

In light of the last example, it is interesting to note that whenever one combines one explosive and one indeterminate univariate model and adds very small off-diagonal elements, the resulting MSV solution will be explosive; one of its Λ_1 eigenvalues will exceed 1.0 in modulus. There will, nevertheless, be a unique stable solution. Consider, then, the two approaches for designation of the relevant RE solution, one being to adopt only unique stable solutions and the other being to adopt the MSV solution. Clearly these approaches will lead to different predictions about the dynamic behavior of the bivariate model in the type of case being considered. This observation leads naturally to the question: Which outcome would actually prevail in such cases? One promising possibility is to determine, as in the work of Evans and Honkapohja (2001), which (if either) of the solutions is E-stable, a property that is closely related to the least-squares learnability of the solution. (For some relevant results, see McCallum (2003).)

Continuing in this vein, we note that the continuity properties of the MSV and smallest-eigenvalue solution concepts are very different. By construction, the MSV solution's Ω matrix will vary continuously with the model's parameters (i.e., elements of A and C). The smallest-eigenvalue criterion permits changes in the group of eigenvalues included in Λ_1, however, which are likely to result in major discontinuities in Ω and to involve changes in the existence or absence of a unique stable solution.

References

Barro, R.J. (1989) "Interest-Rate Targeting," Journal of Monetary Economics 23, 3-30.

Binder, M., and M. H. Pesaran (1995) "Multivariate Rational Expectations Models and Macroeconomic Modeling: A Review and Some New Results," Handbook of Applied Econometrics, eds. M. H. Pesaran and M. Wickens. Basil Blackwell.

Blanchard, O.J., and C.M. Kahn (1980) "The Solution of Linear Difference Models Under Rational Expectations," Econometrica 48, 1305-1311.

Bullard, J., and K. Mitra (2002) "Learning About Monetary Policy," Journal of Monetary Economics 49, 1105-1129.

Desgranges, G., and S. Gauthier (2003) "Uniqueness of Bubble-Free Solution in Linear Linear Rational Expectations Models," Macroeconomic Dynamics (forthcoming).

Evans, G.W. (1986) "Selection Criteria for Models with Non-Uniqueness," Journal of Monetary Economics 18, 147-157.

Evans, G.W., and S. Honkapohja (2001) Learning and Expectations in Macroeconomics. Princeton Univ. Press.

Faust, J., and L.E.O. Svensson (2001) "Transparency and Credibility: Monetary Policy Policy with Unobservable Goals," International Economic Review 42, 369-397.

Gauthier, S. (2002) "Determinacy and Stability Under Learning of Rational Expectations Equilibria," Journal of Economic Theory 102, 354-374.

_____ (2003) "Dynamic Equivalence Principle in Linear Rational Expectations Models," Macroeconomic Dynamics 7, 63-88.

Johnston, J. (1972) Econometric Methods, 2nd ed. McGraw-Hill.

King, R.G., and M.W. Watson (1998) "The Solution of Singular Linear Difference Systems Under Rational Expectations," International Economic Review 39, 1015-1026.

Leitemo, K. (2003) "Targeting Inflation by Constant-Interest Rate Forecasts," Journal of Money, Credit, and Banking 35, 609-626.

McCallum, B.T. (1983) "On Non-Uniqueness in Rational Expectations Models: An Attempt at Perspective," Journal of Monetary Economics 11, 139-168.

_____ (1998) "Solutions to Linear Rational Expectations Models: A Compact Exposition," Economics Letters 61, 143-147.

_____ (1999) "Role of the Minimal State Variable Criterion in Rational Expectations Models," <u>International Tax and Public Finance</u> 6, 621-639.

_____ (2003) "The Unique Minimum State Variable RE Solution is E-Stable in All Well Formulated Linear Models," NBER WP 9960.

Uhlig, H. (1999) "A Toolkit for Analyzing Nonlinear Dynamic Stochastic Models Easily," <u>Computational Methods for the Study of Dynamic Economies</u>, eds. R. Marimon and A. Scott. Oxford University Press.