NBER TECHNICAL PAPER SERIES

FLEXIBLE FUNCTIONAL FORMS AND GLOBAL CURVATURE CONDITIONS

W. E. Diewert

T. J. Wales

Technical Working Paper No. 40

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
September 1984

The research reported here is part of the NBER's research program in Productivity. Any opinions expressed are those of the authors and not those of the National Bureau of Economic Research.

NBER Technical Working Paper #40
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ABSTRACT

Empirically estimated flexible functional forms frequently fail to satisfy the appropriate theoretical curvature conditions. Lau and Gallant and Golub have worked out methods for imposing the appropriate curvature conditions locally, but those local techniques frequently fail to yield satisfactory results. We develop two methods for imposing curvature conditions globally in the context of cost function estimation. The first method adopts Lau's technique to a generalization of a functional form first proposed by McFadden. Using this Generalized McFadden functional form, it turns out that imposing the appropriate curvature conditions at one data point imposes the conditions globally. The second method adopts a technique used by McFadden and Barnett, which is based on the fact that a non-negative sum of concave functions will be concave. Our various suggested techniques are illustrated using the U.S. Manufacturing data utilized by Berndt and Khaled.

W.E. Diewert
Department of Economics
University of British Columbia
Vancouver, Canada V6T1Y2

T.J. Wales
Department of Economics
University of British Columbia
Vancouver, Canada V6T1Y2

Flexible Functional Forms and Global Curvature Conditions

by W.E. Diewert and T.J. Wales*

1. Introduction

One of the most vexing problems applied economists have encountered in estimating flexible functional forms[1] in the production or consumer context is that the theoretical curvature conditions (concavity, convexity or quasiconvexity[2]) that are implied by economic theory are frequently not satisfied by the estimated cost, profit or indirect utility function. This problem has not gone unnoticed. Wales [1977], Christensen and Caves [1980] and Barrett and Lee [1984] have compared various flexible functional forms with respect to their regions in a parameter space where curvature conditions are satisfied. Lau [1978b] and Gallant and Golub]1984] have developed numerical methods for imposing curvature conditions locally, but these methods do not always yield satisfactory results in practice.[3] Moreover, for some purposes,[4] it is necessary to have estimated functional forms that satisfy globally the curvature conditions imposed by economic theory.

The primary purpose of this paper is to suggest and empirically test methods for imposing curvature conditions globally in the context of cost function estimation. We suggest two methods. The first method is explained in section 4 and is an adaptation of the techniques developed by McFadden [1978] and Lau [1978b]. The second method modifies the results of Barrett [1983] and is explained in section 5. We use the U.S. Manufacturing data utilized by Berndt and Khaled [1979] in order to test out empirically our new functional forms.

We also estimate translog and Generalized Leontief cost functions in order to compare our new functional forms with traditional flexible functional forms for cost functions. When comparing the various functional forms, we place some emphasis on obtaining stable estimates for various elasticities of demand since recent research[5] has indicated that flexible forms do not always generate empirically credible elasticity estimates.

In our empirical work, we maintain the hypothesis that the cost function is linearly homogeneous in prices. We also maintain the cross equation symmetry restrictions that occur in the various models. Thus we use economic theory to "well order" the economic universe to the greatest possible extent a priori. However, we do want our functional forms to be flexible not only with respect to input prices, but also with respect to output and time (or any exogenous indicator of "technical progress" such as a constant dollar stock of research and development expenditures). Thus we want to allow for arbitrary returns to scale and technical progress (to the second order) in our cost functions. In our four functional forms for cost functions, we allow second order flexibility in input prices, output and time under the maintained hypothesis of linear homogeneity in prices. Our functional forms can then be further restricted but yet still flexible under the additional maintained hypotheses of: (i) constant returns to scale in production, (ii) no technical progress so that the cost function does not depend on time and (iii) the conjunction of (i) and (ii). In section 2, we develop the necessary algebra for the translog functional form[6] while section 3 deals with the Generalized Leontief case.

2. The Translog Cost Function

Suppose that the technology of a firm can be represented by the production function f^{t*} in period t where $y = f^{t*}(x_1,x_2,\ldots,x_N)$, $x \equiv (x_1,x_2,\ldots,x_N)^T$ is the vector of inputs utilized and y is the maximal output that can be produced using this input vector in period t. Given a positive vector of input prices, $p \equiv (p_1,p_2,\ldots,p_N)^T \gg 0_N$, the period t cost function C* dual to the production function f^{t*} may be defined as follows:

$$(1) \qquad C^*(p,y,t) \equiv \min_x \{p^T x: \; f^{t*}(x) \geq y, \; x \geq 0_N\}.$$

C* will satisfy various regularity conditions depending on what assumptions we place on the production function f^{t*} (see Shephard [1953] or Diewert [1971] [1974] for examples of regularity conditions) but for our purposes, the most important conditions are: C* is a linearly homogeneous and concave function in the input prices p. Let $p^* \gg 0_N$, $y^* > 0$ and $t^* > 0$ and let C* be twice continuously differentiable with respect to its N+2 arguments at (p^*,y^*,t^*). Then the linear homogeneity property of C* in p and Euler's Theorem on homogeneous functions implies the following N+3 restrictions on the first and second derivatives of C*:

$$(2) \qquad p^{*T}\nabla_p C^*(p^*,y^*,t^*) = C^*(p^*,y^*,t^*),$$

$$(3) \qquad p^{*T}\nabla^2_{pp} C^*(p^*,y^*,t^*) = 0_N^T,$$

$$(4) \qquad p^{*T}\nabla_{py} C^*(p^*,y^*,t^*) = \nabla_y C^*(p^*,y^*,t^*) \equiv \partial C^*(p^*,y^*,t^*)/\partial y, \text{ and}$$

(5) $$p^{*T}\nabla^2_{pt}C^*(p^*,y^*,t^*) = \nabla_t C^*(p^*,y^*,t^*) \equiv \partial C^*(p^*,Yy^*,t^*)/\partial t$$

where $\nabla_p C^*$ denotes the column vector of the first order partial derivatives of C^* with respect to the components of p and $\nabla^2_{pp}C^*$ denotes the N by N matrix of second order partial derivatives of C^* with respect to the components of p.

The twice continuous differentiability assumption on C^* and Young's Theorem in calculus implies the following $(N+2)(N+1)/2$ symmetry restrictions on the second order derivatives of C^*:

(6) $$\nabla^2 C^*(p^*,y^*,t^*) = [\nabla^2 C^*(p^*,y^*,t^*)]^T$$

where $\nabla^2 C^*$ denotes the N+2 by N+2 matrix of second order partial derivatives of C^* with respect to all of its N+2 arguments, $(p_1,\ldots,p_N,y,t)$.

Finally, the concavity in prices property implies

(7) $\nabla^2_{pp}C^*(p^*,y^*,t^*)$ is a negative semidefinite matrix.

Diewert [1974;113] defined a flexible functional form for a cost function as one which could provide a second order differential approximation[7] to an arbitrary twice continuously differentiable cost function that satisfies the linear homogeneity in prices property. Thus a twice continuously differentiable cost function at the point (p^*,y^*,t^*), $C(p,y,t)$ say, is flexible if and only if it contains enough free parameters so that the following $1 + (N+2) + (N+2)^2$ equations can be satisfied:

(8) $$C(p^*,y^*,t^*) = C^*(p^*,y^*,t^*)$$

$$\nabla C(p^*,y^*,t^*) = \nabla C^*(p^*,y^*,t^*) \text{ and}$$

$$\nabla^2 C(p^*,y^*,t^*) = \nabla^2 C^*(p^*,y^*,t^*)$$

Hence the level, all N+2 first derivatives and all $(N+2)^2$ second order partial derivatives of C and C* coincide at (p^*,y^*,t^*).

If we impose linear homogeneity in prices on our candidate function for flexibility C, then C will also satisfy the $N + 3 + (N + 2)(N + 1)/2$ restrictions (2) - (6). Hence in order to be flexible, C must contain at least $1 + (N + 2) + (N + 2)^2 - [N + 3 + (N + 2)(N + 1)/2] = N(N + 1)/2 + 2N + 3$ free parameters.

Now we are ready to define the translog cost function C:

$$(9) \quad \ell n\, C(p,y,t) \equiv a_0 + \Sigma_{i=1}^N a_i \ell n\, p_i + a_y \ell n\, y + a_t t + (1/2)\, \Sigma_{i=1}^N \Sigma_{j=1}^N a_{ij} \ell n\, p_i \ell n\, p_j + \Sigma_{i=1}^N a_{iy} \ell n\, p_i \ell n\, y + \Sigma_{i=1}^N a_{it} t \ell n\, p_i + (1/2)\, a_{yy} \ell n\, y \ell n\, y + a_{yt} t \ell n\, y + (1/2)\, a_{tt} t^2;\ a_{ij} = a_{ji} \text{ for all } i,j.$$

Necessary and sufficient conditions insuring that C is linearly homogeneous in input prices are:

$$(10) \quad \Sigma_{i=1}^N a_i = 1;\ \Sigma_{j=1}^N a_{ij} = 0,\ i = 1,\ldots,N;\ \Sigma_{i=1}^N a_{iy} = 0;\ \Sigma_{i=1}^N a_{it} = 0.$$ [8]

C defined by (9) and (10) has $N(N + 1)/2 + 2N + 3$ free parameters, just enough to be a flexible functional form in the class of linearly homogeneous in p cost functions. In fact, we have:

<u>Theorem 1</u> (Woodland [1976;73], Khaled [1978;15]): The translog cost function C defined by (9) and (10) is a flexible (homogeneous in p) functional form.

By Shephard's [1953;11] Lemma, the firm's system of cost minimizing input demand functions, $x(p,y,t) \equiv [x_1(p,y,t),\ldots,x_N(p,y,t)]^T$, may be obtained by differentiating the cost function with respect to input prices; i.e., we have

(11) $x(p,y,t) = \nabla_p C(p,y,t)$.

Define the ith input share function by $s_i(p,y,t) \equiv p_i x_i(p,y,t)/C(p,y,t)$ for $i = 1,2,\ldots,N$. If we differentiate (9) with respect to $\ell n\ p_i$ and apply (11), we obtain the following system of input share equations:

(12) $s_i(p,y,t) = a_i + \Sigma_{j=1}^N a_{ij} \ell n\ p_j + a_{iy} \ell n\ y + a_{it} t;\ i = 1,\ldots,N.$

We now consider various interesting special cases of the general translog cost function defined by (9) and (10).

In order to make C defined by (9) and (10) linearly homogeneous in output y (so that the dual production function exhibits constant returns to scale), we need only impose the following N+2 additional linear restrictions:

(13) $a_y = 1\ ;\ a_{iy} = 0,\ i = 1,2,\ldots,N-1$ [9] $;\ a_{yy} = 0\ ;\ a_{yt} = 0.$

It can be shown that C defined by (9), (10) and (13) is a flexible functional form for arbitrary twice continuously differentiable cost functions C* that are separately linearly homogeneous in p and y.

In order to make C defined by (9) and (10) independent of time t (so that the dual production function does not exhibit any technical progress or regress), we need only impose the following additional N+2 linear restrictions:

(14) $a_t = 0\ ;\ a_{it} = 0,\ i = 1,2,\ldots,N-1$ [10] $;\ a_{yt} = 0\ ;\ a_{tt} = 0.$

Woodland [1976;25] indicated that C defined by (9), (10) and (14) is a flexible functional form for arbitrary cost functions C* that are homogeneous in p and independent of t.

Finally, if we wish to test the hypothesis that the technology is subject to constant returns to scale and subject to no technical progress, we need to test the validity of the following 2N+3 linear restrictions:

(15) $a_y = 1$; $a_t = 0$; $a_{iy} = 0, i = 1,\ldots,N-1$; $a_{it} = 0, i = 1,\ldots,N-1$;

$a_{yt} = 0$; $a_{tt} = 0$.

In our empirical work, we compute the following elasticities involving the estimated cost function C defined by (9) and (10):

(16) $\varepsilon_t^t \equiv \partial \ell n\ C(p^t, y^t,t)/\partial t, \quad t = 1,2,\ldots,T$ in all cases,

$\varepsilon_y^t \equiv \partial \ell n\ C(p^t,y^t,t)/\partial \ell n\ y,$

$\varepsilon_{it}^t \equiv \partial \ell n\ x_i(p^t,y^t,t)/\partial t, \quad i = 1,2,\ldots,N,$

$\varepsilon_{iy}^t \equiv \partial \ell n\ x_i(p^t,y^t,t)/\partial \ell n\ y, \quad i = 1,2,\ldots,N,$ and

$\varepsilon_{ij}^t \equiv \partial \ell n\ x_i(p^t,y^t,t)/\partial \ell n\ p_j, \quad i,j = 1,2,\ldots,N,$

where $x_i(p^t,y^t,t) \equiv \partial C(p^t,y^t,t)/\partial p_i$ for $i = 1,2,\ldots,N$ is our estimated ith input demand function. ε_t^t is the period t percentage change in total cost due to technical progress (if $\varepsilon_t^t > 0$, there is technical regress); ε_y^t is the period t percentage change in total cost due to a one percent change in output (if $\varepsilon_y^t < 1$, then the dual production function exhibits locally increasing returns to scale in period t[11]); if $\varepsilon_{it}^t < 0$, then technical change is biased towards input i in period t, i.e., it is input i saving; if $\varepsilon_{iy}^t < 1$, then input i is an inferior input in period i while if $\varepsilon_{iy}^t > 1$, then input i is normal, and finally ε_{ij}^t is the ordinary elasticity of demand for input i with respect to the price of input j (if the concavity property (7) is satisfied, $\varepsilon_{ii}^t < 0$ for all i and t).

3. The Generalized Leontief Cost Function

Consider the following functional form for a cost function:

$$(17)\quad C(p,y,t) \equiv \Sigma_{i=1}^{N} \Sigma_{j=1}^{N} b_{ij}p_i^{1/2}p_j^{1/2}y + \Sigma_{i=1}^{N} b_ip_i + \Sigma_{i=1}^{N} b_{it}p_ity + b_t(\Sigma_{i=1}^{N} p_i)t + b_{yy}(\Sigma_{i=1}^{N} p_i)y^2 + b_{tt}(\Sigma_{i=1}^{N} p_i)t^2y$$

with $b_{ij} = b_{ji}, i,j = 1,2,\ldots,N$. Note that C defined by (17) is linearly homogeneous in input prices p and it has $N(N + 1)/2 + 2N + 3$ independent b parameters, just the right number to be a flexible functional form. The first set of $N(N + 1)/2$ independent terms on the right hand side of (17) correspond to the Generalized Leontief cost function for a constant returns to scale technology with no technological progress (see Diewert [1971;497]). The ith input demand function corresponding to (17) can be obtained by differentiating C with respect to p_i (recall Shephard's Lemma, (11)):

$$(18)\quad x_i(p,y,t) \equiv \Sigma_{i=1}^{N} b_{ij}p_i^{-1/2}p_j^{1/2}y + b_i + b_{it}ty + b_tt + b_{yy}y^2 + b_{tt}t^2y, \quad i = 1,2,\ldots,N.$$

We now address the issue of the flexibility of the Generalized Leontief Cost function defined by (17). We assume that $N \geq 3$ throughout this paper.

Theorem 2: C defined by (17) is a flexible (homogeneous in p) cost function.

Proof: Let $p^* \gg 0_N, y^* > 0$ and $t^* > 0$ be given. Suppose C^* is twice continuously differentiable at (p^*,y^*,t^*) and is linearly homogeneous in p. We need to show that C defined by (17) can satisfy the $1 + (N + 2) + (N + 2)^2$ equations listed in (8). In what follows, it is understood that the functions

C and C* are both evaluated at (p^*,y^*,t^*). Step 1. Use the equations $\partial^2 C/\partial p_i \partial p_j = \partial^2 C^*/\partial p_i \partial p_j$ for $i \neq j$ to solve for the b_{ij} for $i \neq j$. Step 2. Determine b_{tt} by solving $\partial^2 C/\partial t^2 = \partial^2 C^*/\partial y^2$. Step 3. Determine b_{tt} by solving $\partial^2 C/\partial t^2 = \partial^2 C^*/\partial t^2$. Step 4. Consider the following system of N + 1 simultaneous equations involving b_t and the N b_{it}: $\partial^2 C/\partial y \partial t = \partial^2 C^*/\partial y \partial t$ and $\partial^2 C/\partial p_i \partial t = \partial^2 C^*/\partial p_i \partial t$, $i = 1,\ldots,N$. The coefficient matrix involving b_t and the b_{it} is nonsingular if $y^* > 0$ and $p^* >> 0_N$, so these equations determine b_t and the b_{it}. Step 5. Use the N equations $\nabla^2_{py} C = \nabla^2_{py} C^*$ to solve for b_{ii}, $i = 1,\ldots,N$. Step 6. Use the N equations $\nabla_p C = \nabla_p C^*$ to solve for the b_i, $i = 1,\ldots,N$. This determines all of the parameters in (17) uniquely. By construction, most of the equations in (8) are now satisfied. The remaining equations in (8) turn out to be satisfied because both C and C* satisfy the restrictions (2) - (6). Q.E.D.

It can readily be seen that C defined by (17) will be linearly homogeneous in y if and only if the following N + 2 additional linear restrictions on the σ parameters are satisfied.

(19) $\quad b_i = 0, i = 1,\ldots,N \; ; \; b_t = 0 \; ; \; b_{yy} = 0.$

Theorem 3: C defined by (17) and (19) is a flexible homogeneous in p and y cost function.

Proof: If C* is linearly homogeneous in y, so that $C^*(p,y,t) = yC^*(p,1,t)$ for $y > 0$, then C* satisfies the following N + 2 restrictions in addition to the restrictions (2) - (6):

(20) $\nabla^2_{py}C^*(p^*,y^*,t^*)y^* = \nabla_p C^*(p^*,y^*,t^*)$,

(21) $\nabla^2_{yy}C^*(p^*,y^*,t^*) \equiv \partial^2 C^*(p^*,y^*,t^*)/\partial y^2 = 0$, and

(22) $\nabla^2_{ty}C^*(p^*,y^*,t^*)y^* = \nabla_t C^*(p^*,y^*,t^*) \equiv \partial C^*(p^*,y^*,t^*)/\partial t$.

Now follow Step 1 in the proof of Theorem 1 to determine the b_{ij} for $i \neq j$. Step 2. Use $\nabla^2_{tt}C = \nabla^2_{tt}C^*$ to determine b_{tt}. Step 3. Use the N equations $\nabla^2_{pt}C = \nabla^2_{pt}C^*$ to determine b_{it} for $i = 1,\ldots,N$. Step 4. Use the N equations $\nabla^2_{p}C = \nabla_p C^*$ to determine b_{ii} for $i = 1,\ldots,N$. This determines all of the parameters in (17) subject to the restrictions (19) uniquely. The equations in (8) that are not satisfied by construction turn out to be satisfied because C and C* satisfy the restrictions (2) - (6) and (20) - (22). Q.E.D.

It can be verified that C defined by (17) will not depend on time t if and only if the following N + 2 linear restrictions on the b parameters are satisfied:

(23) $b_{it} = 0,\ i = 1,\ldots,N\ ;\ b_t = 0\ ;\ b_{tt} = 0$.

Theorem 4: C defined by (17) and (23) is a flexible homogeneous in p cost function in the class of cost functions that do not depend on time.

Proof: If C*(p,y,t) does not depend on time, then C* satisfies the following N+2 restrictions in addition to the restrictions (2) - (6):

(24) $\nabla^2_{pt}C^*(p^*,y^*,t^*) = 0_N$; (25) $\nabla^2_{tt}C^*(p^*,y^*,t^*) = 0$; (26) $\nabla^2_{ty}C^*(p^*,y^*t^*) = 0$.

Step 1. Use the equations $\partial^2 C/\partial p_i \partial p_j = \partial^2 C^*/\partial p_i \partial p_j$ to determine the parameters b_{ij} for $i \neq j$. Step 2. Use $\nabla^2_{yy}C = \nabla^2_{yy}C^*$ to determine b_{yy}. Step 3. Use the N equations $\nabla^2_{py}C = \nabla^2_{py}C^*$ to determine b_i for $i = 1,\ldots,N$. Step 4. Use the N equations $\nabla_p C = \nabla_p C^*$ to determine b_i for $i = 1,\ldots,N$. This determines all of parameters in (17) subject to the restrictions (23) uniquely., The equations in (8) that are not satisfied by construction turn out to be satisfied because both C and C* satisfy the restrictions (2) - (6) and (24) - (26). Q.E.D.

Theorems 2-4 are analogous to some theorems stated by Woodland [1976; 25 and 74].[12]

Finally, it may be verified that the Generalized Leontief cost function defined by (17) will be linearly homogeneous in y and independent of time if and only if the following 2N+3 linear restrictions are satisfied:

$$(25) \quad b_{it} = 0, 1 = 1,\ldots,N\ ;\ b_i = 0, i = 1,\ldots,N\ ;\ b_t = 0\ ;\ b_{yy} = 0\ ;\ b_{tt} = 0.$$

Diewert [1971; 506] showed that the cost function defined by (17) and (25) is flexible in the class of homogeneous in p, homogeneous in y and independent of t cost functions.

4. A Generalized McFadden Cost Function

Consider the following functional form for a cost function:

$$(26) \quad C^1(p,y,t) \equiv g^1(p)y + \Sigma_{i=1}^N b_{ii}p_i y + \Sigma_{i=1}^N b_i p_i + \Sigma_{i=1}^N b_{it}p_i ty$$
$$+ b_t(\Sigma_{i=1}^N p_i)t + b_{yy}(\Sigma_{i=1}^N p_i)y^2 + b_{tt}(\Sigma_{i=1}^N p_i)t^2 y$$

where the function g^1 is defined by

$$(27) \quad g^1(p) \equiv (1/2)\ p_1^{-1}\ \Sigma_{i=2}^N\ \Sigma_{j=2}^N\ c_{ij}p_i p_j \quad \text{where } c_{ij} = c_{ji} \quad \text{for } 2 \le i,j \le N.$$

There are $N(N-1)/2$ independent c_{ij} parameters and $3N + 3$ additional b parameters or $N(N+1)/2 + 2N + 3$ parameters in all for C^1. Note that the terms involving b's in (26) also occur in the Generalized Leontief form C defined by (17): the only difference is that the terms $\Sigma_{i=1}^N \Sigma_{j=1,i\neq j}^N b_{ij} p_i^{1/2} p_j^{1/2} y$ in (17) are replaced by the terms involved in the definition of $g^1(p)$. All of the terms involving b's in (26) are linear in input prices and hence they will not appear in the Hessian matrix of C with respect to p; in fact, we have $\nabla_{pp}^2 C^1(p,y,t) = \nabla_{pp}^2 g^1(p)y$. Note also that input 1 plays an asymmetric role in the definitions of g^1 (and hence C^1); this is why we have indexed C^1 and g^1 with a superscript 1. Finally, note that C^1 is linearly homogeneous in p.

Input demand functions for C^1, $x_i^1(p,y,t)$ say, may be obtained via Shephard's Lemma (11):

$$(28)\quad x_i^1(p,y,t) = y\partial g^1(p)/\partial p_i + b_{ii}y + b_i + b_{it}ty + b_t t + b_{yy}y^2 + b_{tt}t^2 y; i = 1,\ldots,N$$

where

$$(29)\quad \partial g^1(p)/\partial p_1 = -(1/2)\, p_1^{-2}\Sigma_{i=2}^N \Sigma_{j=2}^N c_{ij}p_i p_j \quad \text{and}$$

$$\partial g^1(p)/\partial p_i = \Sigma_{j=2}^N c_{ij} p_1^{-1} p_j \quad \text{for } i = 2,3,\ldots,N.$$

The matrix of second order partial derivatives of C^1 with respect to input prices is

$$(30)\quad \nabla_{pp}^2 C^1(p,y,t) = \nabla_{pp}^2 g^1(p) = \begin{bmatrix} p_1^{-3}\tilde{p}^T\tilde{C}\tilde{p}, & -p_1^{-2}\tilde{p}^T\tilde{C} \\ -p_1^{-2}\tilde{C}\tilde{p}, & p_1^{-1}\tilde{C} \end{bmatrix}$$

where $\tilde{p}^T \equiv [p_2, p_3, \ldots, p_N]$ and $\tilde{C}$ is the N-1 by N-1 matrix of c_{ij}'s.

Using (30), it can be verified that $\nabla_{pp}^2 C^1(p,y,t)$ is negative semidefinite for all $p \gg 0_N, y > 0, t > 0$ (recall (7)) if and only if $\tilde{C}$ is negative semidefinite. Thus if our estimated $\tilde{C}^1$ matrix turns out to be negative semidefinite, then C^1 will be globally concave.[13]

The functional form for a unit cost function defined by (27) and the terms $\Sigma_{i=1}^N b_{ii} p_i y$ is a (modest) generalization of a functional form due to McFadden [1978; 279][14]; hence we call the cost function defined by (26) the Generalized McFadden cost function:

Theorem 5: C^1 defined by (26) and (27) is a flexible (homogeneous in p) cost function.

Proof: Step 1. Use the equations $\partial^2 C(p^*,y^*,t^*)/\partial p_i \partial p_j = c_{ij} p_1^{*-1} y^* = \partial^2 C^*(p^*,y^*,t^*)/\partial p_i \partial p_j$ for $2 \leq i,j \leq N$ to solve for the c_{ij}. Steps 2 to 6 are exactly the same as in Theorem 2. The remaining equations in (8) not satisfied by construction turn out to be satisfied because both C^1 and C^* satisfy (2) - (6). Q.E.D.

It turns out that the same restrictions (19) that imposed the constant returns to scale property on the Generalized Leontief cost function will also do the job for the Generalized McFadden cost function. We also have the following counterpart to Theorem 3.

Theorem 6: C^1 defined by (26), (27) and (19) is a flexible homogeneous in p and y cost function.

Proof: Repeat step 1 of Theorem 5 to determine the c_{ij}. Then follow steps 2-4 in Theorem 3 to determine the b's. Q.E.D.

Theorem 7: C^1 defined by (26), (27) and (23) is a flexible homogeneous in p cost function in the class of cost functions that do not depend on time.

Proof: Repeat step 1 in Theorem 5 to determine the c_{ij}. Then follow steps 2-4 in Theorem 4 to determine the b's.

Theorem 8: C^1 defined by (26), (27), (19) and (23) is a flexible homogeneous in p and y cost function in the class of cost functions that do not depend on time.

Proof: Repeat step 1 in Theorem 5 to determine the c_{ij}. Then use the N equations $\nabla_p C^1(p^*,y^*,t^*) = \nabla_p C^*(p^*,y^*,t^*)$ to determine b_{ii} for $i = 1, \ldots, N$. The remaining equations in (8) turn out to be satisfied because both C^1 and C^* satisfy the restrictions (2)-(6), (20)-(22) and (24)-(26). Q.E.D.

If C, the matrix of c_{ij}'s, is negative definite, then a locally valid explicit dual production function to the cost function C^1 defined by (26), (27), (19) and (23) may be calculated as follows. Suppose $x^* = x^1(p^*,y^*,t^*)$ where $x^1(p,y,t)$ is defined by (28) and $x^* \gg 0_N$, $p^* \gg 0_N$, $y^* > 0$. Then for (x,p,y) in a neighborhood of (x^*,p^*,y^*), y will be the maximal output producible by the input vector x only if p is such that the following equations are satisfied (where $x^T \equiv (x_1, x_2, \ldots, x_N) \equiv (x_1, \tilde{x}^T)$ and $\tilde{b} \equiv (b_{22}, b_{33}, \ldots, b_{NN})^T$):

$$(31) \qquad x_1 = (b_1 - (1/2)\, p_1^{2}\, \tilde{p}^T \tilde{C} \tilde{p})y \;;\; \tilde{x} = (\tilde{b} + p_1^{-1} \tilde{C} \tilde{p})y.$$

Equations (31) are simply equations (28) rewritten using the notation below (30). Equations (28) are simply the equations which result when Shephard's Lemma is applied to the C^1 defined by (26) and (27), (19) and (23). Now define the vector of relative input prices by $\tilde{q} \equiv p_1^{-1}\tilde{p}$ and eliminate $\tilde{q}$ from

(31). Since $\tilde{C}$ is assumed to be negative definite for purposes of this exercise, $\tilde{C}^{-1}$ exists and we obtain the following quadratic equation in y^{-1}:

$$(32) \qquad y^{-1}x_1 = b_1 - (1/2)\ (y^{-1}\tilde{x} - \tilde{b})^T\ \tilde{C}^{-1}(y^{-1}\tilde{x} - \tilde{b}).$$

For $(x_1, \tilde{x})$ close to $(x_1^*, \tilde{x}^*)$, we may solve (32) for the right $y = f(x)$ and thus we have our local representation of the production function:[15]

Up to this point, we have established that the Generalized McFadden cost function defined by (26) is equivalent to the Generalized Leontief cost function defined by (17) in terms of its flexibility, ease of estimation, and hypothesis testing capabilities. However, if the estimated GM cost function does not satisfy the concavity restrictions (7), then we may readily impose these restrictions globally.

In order to impose the concavity restrictions (7) on the functional form defined by (26) and (27), we use the following technique due to Wiley, Schmidt and Bramble [1973; 318]: we reparameterize the matrix $\tilde{C} \equiv [c_{ij}]$ by replacing it by minus the product of a lower triangular matrix of dimension N-1 by N-1, A say, times its transpose, A^T; i.e.,

$$(33) \qquad \tilde{C} = -AA^T;\ A = [a_{ij}],\ i,\ j = 1, \ldots, N-1\ ;\ a_{ij} = 0 \text{ for } i < j.$$

Lau [1978b; 427][16] shows that every positive semidefinite matrix C (equal to $-\tilde{C}$ say) has the following representation:

(34) $\qquad C = BDB^T$ where $B = [b_{ij}]$, $i, j = 1, \ldots, N-1$; $b_{ij} = 0$ for $i < j$; and $b_{ii} = 1$, $i = 1, \ldots, N-1$, and D is a non-negative diagonal matrix.

Using Lau's theorem, we may show that any negative semidefinite $\tilde{C}$ has the representation given by (33). This follows from the following theorem.

Theorem 9: Let C be an N-1 by N-1 symmetric matrix. Then C is positive semidefinite if and only if $C = AA^T$ for some lower triangular matrix A (i.e., A satisfies the restrictions in (33)).

Proof: Let C be positive semidefinite. Then a B and D exist which satisfy the restrictions in (34) by Lau's theorem. Let $D^{1/2}$ be a diagonal matrix which has the non-negative square roots of the corresponding diagonal elements of D running down its main diagonal. Then

$$C = BDB^T = BD^{1/2}D^{1/2}B^T = (BD^{1/2})(BD^{1/2})^T = AA^T$$

where $A \equiv BD^{1/2}$ satisfies the restrictions in (33).

On the other hand, let $C = AA^T$. Then for any vector z, $z^TCz = z^TAA^Tz = (A^Tz)^T(A^Tz) \geq 0$. Hence C is positive semidefinite. Q.E.D.

Thus the Wiley, Schmidt and Bramble technique for imposing negative semidefiniteness on a matrix turns out to be perfectly general (and equivalent to Lau's technique).

Recall that input 1 played an asymmetric role in the definition of g^1, (27), and hence in the definition of C^1 as well. In fact, we could single out any input to play the asymmetric role. For example, define for $k = 1, \ldots, N$:

$$(35) \qquad g^k(p) \equiv (1/2)\, p_k^{-1} \Sigma_{i=1,i\neq k}^N \Sigma_{j=1,j\neq k}^N c_{ij}p_ip_j, \qquad c_{ij} = c_{ji}.$$

We may then define $C^k(p,y,t)$ as in (26) except g^1 is replaced by the above

g^k. The resulting C^k has all of the flexibility properties that C^1 had. Moreover, for each of the functional forms C^k, it is very easy to impose globally the concavity property (7) that well behaved cost functions must satisfy. Thus it would seem that the Generalized McFadden family of functional forms would be ideal for applied work. Unfortunately, our empirical work did not completely justify our high a priori hopes for this functional form: we found that our estimated elasticities changed considerably in some cases as we changed our "numeraire" good. Thus inspired by the work of Barnett [1983], we were led to consider yet another family of cost functions.

5. A Generalized Barnett Cost Function

Consider the cost function defined by (26) except in this section, define the function g^1 by:

$$(36)\qquad g^1(p) \equiv \Sigma_{i=2}^N \Sigma_{j=2,i\neq j}^N b_{ij}\, p_i^{1/2} p_j^{1/2} - \Sigma_{i=2}^N \Sigma_{j=2,i\neq j}^N d_{ij} p_1^2 p_i^{-1/2} p_j^{-1/2}$$

$$- \Sigma_{i=2}^N d_i p_1^2 p_i^{-1} \ ; \ b_{ij} = b_{ji} \geq 0 \ ; \ d_{ij} = d_{ji} \geq 0; \ b_{ij} d_{ij} = 0 \text{ for all } i,j.$$

The function g^1 depends on p_1 in an asymmetric way but is linearly homogeneous in input prices p. It has $(N-1)(N-2)/2$ non-negative b_{ij} parameters, $(N-1)(N-2)/2$ non-negative d_{ij} parameters and $N-1$ unrestricted d_i parameters. However, the equality restrictions $b_{ij}d_{ij} = 0$ set $(N-1)(N-2)/2$ of the parameters equal to zero, so there are only $N(N-1)/2$ independent parameters in g^1. We call the cost function defined by (26) and (36) the Modified Barnett cost function, since it is a straightforward modification of Barnett's [1983; 21] Miniflex Laurent functional form from the consumer context (where "income" acts as a numeraire good) to the producer context where we have chosen p_1 to play the role of the numeraire good. The input demand functions which correspond to this new functional form are linear in the unknown parameters (use (28) where g^1 is defined by (36)).

Theorem 10: C^1 defined by (26) and (36) is a flexible (homogeneous in p) cost function:

Proof: Step 1. Consider the following equations for $2 \leq i < j \leq N$:

$$(37) \qquad \partial^2 C^1/\partial p_i \partial p_j = \tfrac{1}{2} b_{ij}(p_i^* p_j^*)^{-1/2} y^* - \tfrac{1}{2} d_{ij}(p_i^* p_j^*)^{-3/2} y^* = \partial^2 C^*(p^*, y^*, t^*)/\partial p_i \partial p_j .$$

If $\partial^2 C^*/\partial p_i \partial p_j \geq 0$ set $d_{ij} = 0$, and solve for $b_{ij} \geq 0$. If $\partial^2 C^*/\partial p_1 \partial p_j < 0$, set $b_{ij} = 0$ and solve for $d_{ij} > 0$. Now use the equations $\partial^2 C^1(p^*, y^*, t^*)/\partial p_i \partial p_j = \partial^2 C^*(p^*, y^*, t^*)/\partial p_1 \partial p_i$ for i = 2, 3, ..., N to solve for the d_i, i = 2, 3, ..., N. Steps 2 to 6 now proceed in the same manner as in Theorem 2. Q.E.D.

We may use the Modified Barnett cost function to test for constant returns to scale and no technical progress, i.e., we have the following counterparts to Theorems 6, 7 and 8:

Theorem 11: C^1 defined by (26), (36) and (19) is a flexible homogeneous in p and y cost function.

Theorem 12: C^1 defined by (26), (36) and (23) is a flexible homogeneous in p cost function in the class of cost functions that do not depend on time.

Theorem 13: C^1 defined by (26), (36), (19) and (23) is a flexible homogeneous in p and y cost function in the class of cost functions that do not depend on time.

The next Theorem explains why we have imposed non-negativity restrictions on the b_{ij}'s and d_{ij}'s which occur in (36).

Theorem 14: If the d_i parameters which occur in (36) are non-negative then the Modified Barnett cost function defined by (26) and (36) is globally concave in input prices.

Proof: It can be verified that the Hessian matrices of the functions $p_i^{1/2}p_j^{1/2}$, $-p_i^{-1/2}p_j^{-1/2}p_1^2$ and $-p_i^{-1}p_1^2$ are negative semidefinite. Hence these functions are concave, and since a non-negative sum of concave functions is concave, $g^1(p)$ defined by (36) is a globally concave function over $p \gg 0_N$. Since $C^1(p,y,t)$ equals $g^1(p)y$ plus functions which are linear in p (and hence concave), $C^1(p,y,t)$ is globally concave in p. Q.E.D

Theorems 13 and 14 are modifications of Theorems A.2 and A.3 in Barnett [1983, 21-22] to the producer context. The basic idea for generating globally concave cost functions by taking non-negative sums of concave functions may be found in Diewert [1971] and McFadden [1978].

Theorem 14 is a nice result in the sense that the addition of the non-negativity constraints $d_i \geq 0$ makes the Modified Barnett cost function defined by (26) and (36) globally concave in input prices. Unfortunately, these additional non-negativity restrictions destroy the flexibility properties of g^1; i.e., Theorems 10-13 are no longer true if we restrict the d_i to be non-negative. If the $d_i \geq 0$, then it must be the case that input 1 is an Allen-Uzawa[17] substitute for every other input; i.e., input 1 is not allowed to be complementary with any other input. If the b_{ij}, d_{ij} and d_i parameters in (36) are restricted to be non-negative then inputs i and j are complementary only if $d_{ij} > 0$. But there are no d_{ij} coefficients involving good 1, so good 1 cannot be complementary with any other input.[18]

The above difficulty with the Modified Barnett functional form leads us to define the following <u>Generalized Barnett cost function</u>: define C^1 by (26) and $g^1(p)$ by:

$$(38) \qquad g^1(p) \equiv \Sigma_{i=1}^N \Sigma_{j=1,i\neq j}^N b_{ij} p_i^{1/2} p_j^{1/2} - \Sigma_{i=2}^N \Sigma_{j=2,i\neq j}^N d_{ij} p_1^2 p_i^{-1/2} p_j^{-1/2}$$
$$- \Sigma_{i=2}^N \Sigma_{j=2,i\neq j}^N e_{ij} p_1^{-1/2} p_i^{-1/2} p_j^2$$

where $b_{ij} = b_{ji} \geq 0$, $d_{ij} = d_{ji} \geq 0$ and $e_{ij} \geq 0$ for all i, j.

There are $N(N-1)/2$ independent non-negative b_{ij}'s, $(N-1)(N-2)/2$ independent non-negative d_{ij}'s and $(N-2)^2$ independent non-negative e_{ij}'s (note that the matrix of e_{ij}'s is <u>not</u> taken to be symmetric). If we allowed the N-1 coefficients e_{1j}, j=2, ..., N to be unrestricted in sign, then the g^1 defined by (38) would contain the g^1 defined by (36) as a special case. However, we do not allow the e_{ij} to be negative since this would destroy the global concavity of g^1 as defined by (38).[19]

It can be seen that the Generalized Barnett cost function is linearly homogeneous in input prices. As usual, input demand functions for the C^1 defined by (26) and (38) may be obtained via Shephard's Lemma; see equations (28) where g^1 is defined by (38). The resulting demand functions are linear in the unknown b_{ij}, d_{ij} and e_{ij} parameters As usual, linear homogeneity in y can be imposed as C^1 by imposing (19). C^1 can be made independent of time by imposing (23). Finally, C^1 can be made homogeneous in y and independent of time by imposing (19) and (23).

The major advantage of the Generalized Barnett Cost function is its global concavity in input prices. Some disadvantages associated with it are: (i) a priori, it requires many more parameters than other flexible functional forms, (ii) it requires the imposition of a large number of inequality constraints on the coefficients, and (iii) even with the large number of parameters, we cannot prove that the cost function defined by (26) and (38) is flexible. However, we can prove a limited flexibility result for the Generalized Barnett functional form. We first require a definition.

Suppose that we are given two twice continuously differentiable (at $p^* \gg 0_N$) linearly homogeneous functions, $c(p)$ and $c^*(p)$. Suppose c is such that the following $1 + N + (N-1)(N-2)/2$ equations are satisfied:

(39) (i) $c(p^*) = c^*(p^*)$, (ii) $\nabla c(p^*) = \nabla c^*(p^*)$ and

(iii) $\partial^2 c(p^*)/\partial p_i \partial p_j = \partial^2 c^*(p^*)/\partial p_i \partial p_j$ for all i, j expect $i \neq j$, $i \neq k$, $j \neq k$.

Then we say that c is quasiflexible relative to the numeraire good k. If we look at the system of N^2 equations $\nabla^2 c(p^*) = \nabla^2 c^*(p^*)$, then if c is quasiflexible relative to good k, we have equality of the second order partial derivatives of c and c* except possibly along the main diagonal of the matrix equation $\nabla^2 c(p^*) = \nabla^2 c^*(p^*)$ and along the k^{th} row (and column).[20]

<u>Theorem 15</u>: $c(p) \equiv g^1(p) + \Sigma_{i=1}^N b_{ii} p_i$ where g^1 is defined by (38) is quasi-flexible for <u>any</u> choice of the numeraire good k.

<u>Proof</u>: Let the numeraire good k=1. Consider the following system of equations for $2 \leq i < j \leq N$:

(40) $$\partial^2 c(p^*)/\partial p_i \partial p_j = \partial^2 c^*(p^*)/\partial p_i \partial p_j \equiv c^*_{ij}.$$

If $c^*_{ij} \geq 0$, set $d_{ij} = e_{ij} = e_{ji} = 0$ and solve for $b_{ij} \geq 0$. If $c^*_{ij} < 0$, set $b_{ij} = e_{ij} = e_{ji} = 0$ and solve for $d_{ij} > 0$. This solves equations (39)(iii) when k = 1. Now use equations (39)(ii) to solve for the b_{ii}. Equations (39)(i) will be satisfied since $c(p^*) = p^{*T} \nabla c(p^*) = p^{*T} \nabla c^*(p^*) = c^*(p^*)$ using the homogeneity properties of c and c*.

Now let the numeraire good be an arbitrary $k \geq 2$. Set $e_{ij} = 0$ for all i and j except when j = k. Set $d_{kj} = b_{jk} = 0$ for $j = 2, 3, \ldots, N$, $j \neq k$.[21] Now consider the system of equations (40) for $2 \leq i < j \leq N$, $i \neq k$, $j \neq k$. If $c^*_{ij} \geq 0$, set $d_{ij} = e_{ij} = e_{ji} = 0$ and solve for $b_{ij} \geq_{ij} 0$. If $c^* < 0$, set $b_{ij} = e_{ij} = e_{ji} = 0$ and solve for $d_{ij} > 0$. Now solve the N-2 equations $\partial^2 c(p^*)/\partial p_1 \partial p_j = c^*_{1j}$, $j = 2, 3, \ldots, N$, $j \neq k$ for non-negative b_{1j} or e_{jk} for $j = 2, 3, \ldots, N$, $j \neq k$. Thus equations (39)(iii) are satisfied by construction. Equations (39)(i) and (ii) may now be satisfied by choosing the b_{ii} appropriately. Q.E.D

Thus the Generalized Barnett cost function defined by (26) and (38) appears to be "reasonably" flexible but we cannot prove that it is completely flexible.

6. Empirical Results

In this section we report on results obtained from estimating our functional forms using data utilized by Berndt and Khaled [1979]. The data contain information for the period 1947-71 on output of U.S. Manufacturing industries together with information on prices and quantities for four inputs: capital (K), labor (L), energy (E) and materials (M). Although the functional forms that we estimate have been discussed above we present them here for the reader's convenience. For the translog (TL) we have the system consisting of of the logarithm of the cost function (9) together with the following N-1 share equations for each time period t:

$$(41) \quad s_i(p,y,t) = a_i + \Sigma_{j=1}^{N} a_{ij}\ell n\, p_j + a_{ij}\ell n\, y + a_{it}t + u_i, \quad i = 1, \ldots, N-1 \,;$$

where the a_{ij} satisfy the restrictions (10), and u_i is a disturbance term for the i^{th} share equation, i=1,2,...,N-1 and u_N is a disturbance term for (9).

Denoting $u = (u_1, \ldots, u_N)^T$, assume that u has a multivariate normal distribution with $E(u) = 0$, $E(uu)^T = \Omega$ and that Ω is constant over time. These assumptions about the disturbances are maintained for all of our functional forms.

For our Generalized McFadden (GM) form we have the following system of N equations in each period:

$$(42) \quad x_i y^{-1} = \Sigma_{j=1,j\neq k}^{N} c_{ij} p_k^{-1} p_j + b_{ii} + b_i y^{-1} + b_{it}t + b_t t y^{-1} + b_{yy} y + b_{tt} t^2 + u_i$$

$$\text{for } i = 1, \ldots, N \text{ and } i \neq k;$$

$$x_k y^{-1} = -(1/2)\, p_k^{-2} \Sigma_{i=1,i\neq k}^{N} c_{ij} p_i p_j + b_{kk} + b_k y^{-1} + b_{kt} t + b_t t y^{-1} + b_{yy} y + b_{tt} t^2 + u_k$$

with $c_{ij} = c_{ji}$ for $1 \leq i, j \leq N$, $i \neq k$, $j \neq k$, and where k is the numeraire input. The dependent variables are input levels divided by output rather than input expenditure levels since this makes the assumption of homoskedasticity of the disturbances more plausible. The cost equation is not estimated since it contains no additional information.

For our generalized Barnett (GB) form we have the following system of N equations for each period:

$$\begin{aligned}
(43)\quad x_i y^{-1} = {} & \Sigma_{j=1,j\neq i}^{N} b_{ij} p_i^{-1/2} p_j^{1/2} + \Sigma_{j=1,j\neq k,j\neq i}^{N} d_{ij} p_i^{-3/2} p_j^{-1/2} p_k^2 \\
& + (1/2)\, \Sigma_{j=1,j\neq k,j\neq i}^{N} e_{ij} p_i^{-3/2} p_j^2 p_k^{-1/2} - 2\, \Sigma_{j=1,j\neq k,j\neq i}^{N} e_{ji} p_i p_j^{-1/2} p_k^{-1/2} \\
& + b_{ii} + b_i y^{-1} + b_{it} t + b_t t y^{-1} + b_{yy} y + b_{tt} t^2 + u_i;\ i=1, \ldots, N,\ i\neq k; \\
x_k y^{-1} = {} & \Sigma_{j=1,j\neq k}^{N} b_{kj} p_k^{-1/2} p_j^{1/2} - 2\, \Sigma_{i=1,i\neq k}^{N} \Sigma_{j=1,j\neq k}^{N} d_{ij} p_i^{-1/2} p_j^{-1/2} p_k \\
& + (1/2)\, \Sigma_{i=1,i\neq k}^{N} \Sigma_{j=1,j\neq k}^{N} e_{ij} p_i^{-1/2} p_j^2 p_k^{-3/2} \\
& + b_{kk} + b_k y^{-1} + b_{kt} t + b_t t y^{-1} + b_{yy} y + b_{tt} t^2 + u_k
\end{aligned}$$

with $b_{ij} = b_{ji} \geq 0$, $d_{ij} = d_{ji} \geq 0$ and $e_{ij} \geq 0$ for all i, j and again k is the numeraire input. Finally, the Generalized Leontief form is obtained from (43) by setting $d_{ij} = e_{ij} = 0$ for all i and j, and by leaving the b_{ij} unconstrained.

All systems were estimated by the method of nonlinear maximum likelihood.[22] For the GB model the non-negativity constraints were imposed by maximizing the likelihood with respect to new parameters defined as the squares of the original ones.[23] For the GM model the global concavity condition was imposed using the method due to Wiley, Schmidt and Bramble [1973; 318] discussed above. Neither procedure presented any convergence problems during the estimations.

Table 1 contains summary statistics for our estimated functional forms. In all models the imposition of constant returns to scale or of no technological change involves six restrictions. As is clear from the table all models reject the hypothesis of constant returns to scale or of no technological change at the 5 percent level of significance since the critical Chi-square value is 12.6.[24] Although not recorded in the table the joint hypothesis of constant returns and no technological change is also clearly rejected. Hence in the analysis below we present results only for the full models. It is interesting to note that the choice of numeraire inputs has very little effect on the log likelihood values and consequently on the test statistics for the GB model. On the other hand, the results for the GM and GMC (Constrained Generalized McFadden) models are more sensitive. Finally, it may be noted that imposition of concavity in the GM model causes only a slight reduction in the value of the likelihood, regardless of the choice of numeraire.

In Table 2, we present price elasticities and R^2 values. The elasticities are evaluated at the first sample point, while in Table 3 we present them evaluated at the final sample point in order to give some idea about the variation over the period. Table 2 reinforces our finding from Table 1 that using different inputs as the numeraire in the GB model does not lead to very different results. Exactly the same pattern of substitutability and complementarity prevails regardless of the choice of numeraire, and the values of the elasticities are very similar. Indeed the differences between elasticities resulting from the choice of numeraire input in the GB model appear to be of the same order of magnitude as the differences between elasticities in the traditional GL and TL models. On the other hand, for the GM and GMC models, some of the elasticities vary considerably depending on which input is used as the numeraire. In the GM model only 8 of the 16 elasticities have the same sign for all four numeraires, while in the GMC model only ten have the same sign. A comparison of elasticities between the GM and GMC models suggests roughly comparable results in terms of signs and magnitudes. Of course in the GM model the own price elasticities are constrained to be negative; however in the GM model, the own elasticities for labor and materials are only slightly positive.

The R^2 values for the GB model are comparable to those for the GL model and do not vary much with the choice of numeraire.[25] Indeed changing the numeraire changes the R^2 value by three percentage points or fewer in all instances. For the GM and GMC models, the R^2 values are the same order of magnitude as for the other models, but are more sensitive to the choice of numeraire, particularly for the materials equation. In general, all R^2 values appear to be satisfactory, particularly in view of the fact that the dependent variables are input-output ratios rather than input levels.

Table 3 contains estimates of price elasticities evaluated at the final sample point. Since the same general conclusions may be drawn from Table 3 and from Table 2 there is no need to discuss them further. In general, a comparison of the first and last sample point reveals that the elasticities varied the least for the TL and GL models, and the most for the GM and GMC models.

In Table 4, we summarize our findings on the effects of technological change on input use and total cost for the first and last sample points. Since the results for the GM and GMC models are virtually identical we provide only those for the GM model. The table entries are $\partial \ell n\ x_i(p,y,t)/\partial t$ for inputs $i =$ K,L,E,M and $\partial \ell n\ C(p,y,t)/\partial t$. Clearly disembodied technological change has been negligible in the period, with the effect on total cost being essentially zero as estimated by all functional forms.[26] The effect on individual inputs varies, depending on the functional form and numeraire input, with no clear pattern emerging. In any event the effects are small.

In Table 5 we summarize our findings on scale effects. Once again the GMC and GM results are virtually identical and we present only those for the GM model. The table entries are $\partial \ell n\ x_i(p,y,t)/\partial \ell n\ y$ for each input $i =$ K,L,E,M and $\partial \ell n\ C(p,y,t)/\partial \ell n\ y$. The inverse of the latter is often used as a measure of returns to scale.[27] All functional forms yield plausible estimates of returns to scale except possibly for the TL model with a final year cost elasticity of .62, implying very high economies of scale. Once again the choice of numeraire in the GB model affects the estimates much less than it does in the GM model. For the individual input elasticities, the estimates are fairly stable across functional forms for labor and materials but not for capital and energy. It is worth noting that labor and materials account for roughly 90 percent of total costs (with labor about 30 percent and materials about 60 percent) and hence their behavior has a dominant effect on the elasticity of total cost with respect to output.[28]

In all cases the labor elasticity is below, and the materials elasticity above, the total cost elasticity. The materials elasticity is generally just slightly less than unity indicating, not surprisingly, few economies in the major cost component. The capital and energy estimates vary from positive to negative depending on functional form, and it is not clear what conclusion can be reached about possible economies for these inputs. However, the fact that they each account for about only 5 percent of total cost suggests that they will likely not contribute substantially to overall economies.

In the discussions above we have found that the GM (and GMC) model results are more sensitive to the choice of numeraire than are those for the GB model. Inspection of the GM input demand equations (42) suggests a possible explanation. For each non-numeraire equation, all the price terms on the right hand side have the numeraire price as a divisor, while for the numeraire good they have the square of the numeraire prices as a divisor. Therefore unless there is a high correlation between all the price series, there may be a substantial difference in results due to varying the numeraire input. On the other hand for the GB model this problem does not arise. For each non-numeraire input there is a set of terms that does not involve the numeraire price, a set in which it multiplies other prices in squared form, and a set in which it divides other prices in square root form. For the numeraire good there is a set of terms in which it multiplies prices, and one in which it divides prices, raised to a power. Thus varying the numeraire may simply lead to different parameters being set to zero without any substantial change in the results. This is consistent with our findings.

7. Conclusions

The primary purpose of this paper is to propose and test empirically methods for imposing curvature conditions globally in the context of cost function estimation. We suggest two methods; the first builds on the work of McFadden [1978] and Lau [1978b], while the second is developed from the work of Barnett [1983]. We estimate these models using data utilized by Berndt and Khaled [1979] and find that they yield results that are generally comparable, in terms of price, output and technological change effects, to those given by traditional flexible forms such as the Translog and Generalized Leontief.

One drawback of our new forms is that they require one input to be singled out as the numeraire input, and hence there are potentially as many sets of estimated parameters as there are inputs. Our empirical results suggest that this is a more serious problem for the GM model than it is for the GB model. Indeed the latter appears to be sufficiently flexible so as to yield results that do not vary much with the choice of numeraire input, at least with the data set used here. However, the GM model should work well if the variability in the data p,y,t is "reasonably" small so that any quadratic approximation should yield "roughly" equivalent results.

Our techniques can readily be adapted to profit function estimation when there is only one fixed factor: simply reinterpret our cost function $C(p,y,t)$ as the negative of a profit function, $-\pi(p,y,t)$, where $\pi(p,y,t) \equiv \max_z \{p \cdot z: z \varepsilon S^t(y)\}$. The variable y is now interpreted as the non-negative amount of a fixed factor used during the period, p is a positive vector of variable input and output prices, z is a net output vector (inputs are indexed with a negative sign) and $S^t(y)$ is a period t technology set that depends on y. Our old input vector x is now replaced with -z. With these changes, all of our analysis carries through (except for the translog case).[29]

Footnotes

*Our thanks to E.R. Berndt for his valuable comments.

1. The two most commonly used flexible functional forms are the Generalized Leontief introduced by Diewert [1971] and the translog introduced by Christensen, Jorgenson and Lau [1971; 1973] and Sargan [1971]. One concept of flexibility was defined by Diewert [1974; 113] and the equivalence of various definitions of flexibility was demonstrated in Barnett [1983; 19-20].

2. For definitions and alternative characterizations of these curvature concepts, see Diewert, Avriel and Zang [1977].

3. When Jorgenson and Fraumeni [1981] applied their version of Lau's [1978] method for imposing curvature conditions in their 36 industry translog study of U.S. industries, they ended up setting 204 out of 360 second order parameters equal to zero. Alt [1982] and Berndt [1984] explain the various methods for imposing curvature conditions either globally or over a region.

4. For example, functional forms for production and utility functions used in applied general equilibrium models should satisfy curvature conditions globally. For an excellent review of the applied general equilibrium modelling approach, see Shoven and Whalley [1984].

5. See the discussion in White [1980] and Elbadawi, Gallant and Souza [1983].

6. Much of this analysis is in Woodland [1976] and Khaled [1978].

7. The term "differential approximation" is due to Lau [1974; 184]. On the equivalence of differential approximations to other concepts of second order approximations, see Barnett [1983; 19-21].

8. In view of the symmetry conditions, $a_{ij} = a_{ji}$ for all i, j, any one of these N constraints is redundant.

9. We also need $a_{Ny} = 0$ but this is implied by $a_{iy} = 0$ for i=1, ..., N-1 and $\sum_{i=1}^{N} a_{iy} = 0$ which is part of (10).

10. $a_{Nt} = 0$ is implied by (10) and (14).

11. ε_y^t is Ohta's [1974] dual rate of returns to scale.

12. Instead of using our terms $\Sigma_i b_i p_i$ and $b_{yy}(\Sigma_i p_i)y^2$ in (17), Woodland used the terms $\Sigma_i b_i p_i y^2$ and $b_{yy}(\Sigma_i p_i)$. Our parameterization has the advantage that the demand equations (18) have constant terms b_i on the right hand sides.

13. This very important observation is due to Lau [1974; 196].

14. McFadden restricted the coefficients of $\tilde{C}$ in a complex way so that good 1 was substitutable with every other good. Our functional form (26) also bears a superficial resemblance to the normalized quadratic profit function derived by Lau [1978a; 194]; however, Lau's function form is dual to a quadratic production function which loses its flexibility in the class of linearly homogeneous production functions.

15. Note that the production function is linearly homogeneous as it should be in this case.

16. Lau's Theorem is an advance over the usual representation theorem derived in good matrix analysis texts (e.g., see Strang [1976; 245]) which states that $PCP^T = BDB^T$ where C, B and D are subject to the restrictions in (34) and P is a permutation matrix; i.e, Lau's Theorem allows us to set P = I, an identity matrix. Lau calls the decomposition of C, given by (34), the Cholesky decomposition of C where Strang [1976; 241] calls the decomposition of C defined by $C = AA^T$, where A satisfies the restrictions in (33), the Cholesky decomposition of C.

17. Inputs i and j are Allen [1938; 504] - Uzawa [1962] substitutes (complements) if and only if $\partial x_i(p,y,t)/\partial p_j = \partial^2 C(p,y,t)/\partial p_i dp_j > 0$ (<0).

18. Thus the Modified Barnett functional form defined by (26) and (36) with the $d_i > 0$ is similar to the functional form defined by McFadden [1978; 279-280]. In fact, we may obtain exact counterparts to Theorems 10-14 if in definition (36), we replace $-d_{ij}p_1^2p_i^{-1/2}p_j^{-1/2}$ by $-d_{ij}p_1^{-1}(p_i+p_j)^2$. In both cases, if $d_{ij} > 0$, then inputs i and j are complements but inputs 1 and i and 1 and j are substitutes (when $d_i > 0$ for i = 2, ..., N). Thus there is a connection between Barnett's [1983; 21-22] Theorems A.2 and A.3 and McFadden's [1978; 279] Lemma 4.

19. We can prove that C^1 defined by (26) and (38) is globally concave in input prices using the same technique as used in Theorem 14.

20. Recall that the linear homogeneity of c and c* implies that $\nabla^2 c(p^*)p^* = \nabla^2 c^*(p^*)p^* = 0_N$.

21. We do not have to set these $b_{kj} = 0$; we could try to solve the equations $\partial^2 c(p^*)/\partial p_k dp_j = c^*_{kj}$ for $j = 1, \ldots, N$, $j \neq k$ for the b_{kj} after all of the other parameters have been determined, but the resulting b_{kj} need not be non-negative.

22. In the estimations, we used an algorithm due to Fletcher [1972].

23. It is interesting that although the GB model potentially contains more nonzero parameters (30) than the GL, GL or GM models (21), imposition of the non-negativity constraints in practice never resulted in more nonzero parameters than in the other models. For three of the numeraire choices there were 21 nonzero parameters in the GB model and for the other there were 20.

24. We ignore here the inference problems associated with the GB and GMC models. For the GB model, our Chi-square tests are conditional on the non-negativity constraints for the zero parameters being binding, and for three of the GMC models, conditional on at least one of the determinantal inequalities (that principal minors alternate in sign) being binding. As far as we know, the (unconditional) sampling theory involved in maximum likelihood estimation subject to inequality restrictions has not been developed.

25. The R^2 values for the TL that appear in Table 2 have been obtained by using the parameter estimates from the TL share equations to calculate predicted input-output ratios. This may explain why they differ somewhat from those for the other forms.

26. Berndt and Khaled [1979] reach essentially the same conclusion

with their estimates of the "dual rate of total cost diminution", $-\partial \ln C(p,t,y)/\partial t$, ranging from .00719 to -.0003 for various models.

27. See, for example, Berndt and Khaled [1979] or Ohta [1974].

28. In 1947 the actual cost shares for K,L,E and M were .051, .247, .043 and .659, while in 1971 they were .047, .289, .045 and .619. Of course the share-weighted sum of the input elasticities is equal to the cost elasticity.

29. Minus profits will be negative and we cannot take the log of a negative number.

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Table 1

Summary Statistics

Functional Form	Numeraire Input	Log L	Test Statistic for CRTS	Test Statistic for No Technical Change	Local Concavity Violations
GB	K	523.4	64.3	29.7	0
	L	524.1	66.0	31.5	0
	E	523.6	64.9	30.5	0
	M	523.5	64.2	37.7	0
GM	K	518.6	58.1	27.7	25
	L	521.7	46.1	15.8	25
	E	526.7	67.7	32.3	25
	M	521.5	57.2	17.3	0
GMC	K	517.1	55.9	25.3	0
	L	520.9	47.2	18.7	0
	E	526.3	67.8	44.3	0
	M	521.5	57.2	19.5	0
GL	-	522.9	59.4	17.0	0
TL	-	508.0	44.6	21.0	6

Notes:

1. GMC is the GM model constrained to be globally concave.

2. Log L is the log likelihood for the full model allowing for technical change and returns to scale.

3. The test statistic for CRTS is minus twice the difference between the log likelihoods in the full model and that with CRTS imposed. The test statistic for the no technical change is defined analagously.

4. Concavity violations are sample points at which the estimated cost function is not concave. The sample size is 25.

5. The log likelihood for the TL model is not comparable with the others since it involves cost shares rather than input-output ratios as the dependent variables.

Table 2

Input Price Elasticities and R^2 Values

Numeraire	GB K	GB L	GB E	GB M	GL	TL
KK	-.26	-.33	-.21	-.47	-.24	-.34
KL	.22	.23	.18	.20	.36	.48
KE	-.04	-.03	-.02	-.02	-.08	-.09
KM	.08	.13	.05	.28	-.04	-.05
LK	.05	.05	.04	.04	.08	.11
LL	-.31	-.28	-.25	-.35	-.33	-.20
LE	.11	.07	.07	.06	.12	.08
LM	.16	.16	.14	.25	.13	.01
EK	-.05	-.03	-.03	-.03	-.11	-.12
EL	.64	.41	.42	.37	.74	.44
EE	-.70	-.57	-.60	-.50	-.73	-.62
EM	.10	.19	.21	.16	.10	.30
MK	.00	.01	.00	.02	.01	-.01
ML	.06	.06	.05	.09	.05	.01
ME	.01	.01	.01	.01	.01	.02
MM	-.07	-.08	-.07	-.13	-.05	-.02
R^2 K	.84	.87	.86	.84	.86	.68
L	.97	.96	.96	.96	.97	.96
E	.86	.83	.83	.86	.83	.77
M	.70	.71	.70	.71	.70	.82

Table 2 Continued

Input Price Elasticities and R^2 Values

Numeraire	GM K	GM L	GM E	GM M	GMC K	GMC L	GMC E
KK	-.08	-.43	-.42	-.38	-.15	-.48	-.44
KL	.35	.94	.42	.27	.09	.72	.41
KE	.00	-.08	.09	-.09	-.02	-.11	.09
KM	-.28	-.43	-.09	.19	.07	-.13	-.06
LK	.08	.20	.09	.06	.02	.16	.09
LL	.02	-.08	-.11	-.21	-.08	-.29	-.11
LE	.12	.13	.04	.10	.10	.12	.04
LM	-.22	-.26	-.03	.06	-.04	.02	-.02
EK	.00	-.11	.11	-.12	-.02	-.14	.10
EL	.67	.76	.26	.55	.58	.68	.24
EE	-.92	-.70	-.61	-.71	-.84	-.71	-.61
EM	.24	.04	.24	.27	.29	.17	.27
MK	-.02	-.04	-.01	.02	.01	-.01	-.01
ML	-.08	-.10	-.01	.02	-.02	.01	-.01
ME	.02	.00	.02	.02	.02	.01	.02
MM	.09	.13	.00	-.06	-.01	-.01	-.01
R^2 K	.84	.78	.86	.83	.80	.76	.86
L	.97	.97	.96	.97	.96	.97	.96
E	.92	.85	.86	.85	.91	.85	.87
M	.53	.74	.67	.75	.64	.73	.67

Notes to Table 2

1. KL means the elasticity of demand for capital with respect to the price of labor. Other entries are defined analagously.

2. The R^2 values are calculated for each equation separately as $1 - \text{var}(\varepsilon)/\text{var}(z)$ where $\text{var}(\varepsilon)$ is the variance of the residuals and $\text{var}(z)$ is the variance of the relevant dependent variable.

3. Price elasticities are evaluated at the first sample point.

4. The GM model with M as numeraire satisfies the global concavity conditions, hence the GM and GMC estimates are the same in this case.

Table 3

Input Price Elasticities

Numeraire	GB K	GB L	GB E	GB M	GL	TL
KK	-.49	-.42	-.52	-.27	-.36	-.26
KL	.58	.46	.60	.10	.48	.56
KE	-.14	-.10	-.15	-.03	-.09	-.11
KM	.04	.06	.06	.20	-.04	-.19
LK	.10	.08	.10	.02	.08	.09
LL	-.46	-.51	-.51	-.54	-.37	-.24
LE	.18	.18	.20	.16	.14	.07
LM	.18	.25	.22	.37	.14	.07
EK	-.15	-.11	-.17	-.03	-.09	-.12
EL	1.20	1.23	1.31	1.07	.95	.48
EE	-1.14	-1.18	-1.25	-1.03	-.95	-.63
EM	.10	.05	.10	-.02	.09	.27
MK	.00	.01	.01	.02	.00	-.01
ML	.09	.12	.11	.18	.07	.04
ME	.01	.00	.01	.00	.01	.02
MM	-.10	-.13	-.11	-.20	-.07	-.04

Numeraire	GM K	GM L	GM E	GM M	GMC K	GMC L	GMC E
KK	-.76	-.16	-.26	-.26	-.69	-.18	-.28
KL	.61	.42	.61	.43	.90	.31	.60
KE	-.57	-.04	-.27	-.09	-.51	-.06	-.27
KM	.71	-.21	-.08	-.09	.30	-.06	-.05
LK	.10	.07	.10	.07	.15	.05	.10
LL	.08	.00	-.27	-.57	-.27	-.11	-.29
LE	.24	.07	.21	.15	.21	.06	.21
LM	-.41	-.13	-.04	.35	-.08	.01	-.02
EK	-.60	-.05	-.29	-.09	-.55	-.06	-.29
EL	1.54	.43	1.38	.98	1.33	.39	1.41
EE	-1.25	-.41	-1.39	-.74	-1.15	-.42	-1.40
EM	.31	.02	.31	-.14	.37	.10	.27
MK	.06	-.02	-.01	-.01	.03	-.01	.00
ML	-.21	-.07	-.02	.17	-.04	.00	-.01
ME	.02	.00	.02	-.01	.03	.01	.02
MM	.13	.08	.00	-.15	-.01	-.00	-.01

Notes:

1. See Table 2.

2. Elasticities are evaluated at the last sample point.

Total Cost .76 .76 .75 .79 .79 .93

Table 4

Effect of Technological Change on Inputs and Total Cost

Numeraire		GB K	GB L	GB E	GB M	GL	TL
Input	K	.04	.06	.06	.04	.04	.02
	L	.01	.01	.01	.01	.01	.00
	E	.05	.05	.01	.05	.04	.01
	M	.00	.00	.00	.00	.00	-.01
Total Cost		.01	.01	.01	.01	.01	-.01
Input	K	-.01	.01	.01	-.01	.01	.04
	L	.00	.00	.00	.00	.00	.01
	E	-.01	.01	-.06	-.01	.00	.03
	M	.00	.00	-.01	.00	.00	.01
Total Cost		.00	.00	.00	.00	.00	.01

Numeraire		GM K	GM L	GM E	GM M
Input	K	.05	.02	.06	.04
	L	.01	.00	.01	.01
	E	.06	.02	.05	.04
	M	.01	.00	.00	.00
Total Cost		.01	.00	.01	.01
Input	K	-.03	.02	.00	.01
	L	-.02	.01	-.01	.00
	E	-.04	.03	-.03	.00
	M	.00	.00	.00	.00
Total Cost		-.01	.00	.00	.00

Notes:

1. The first 5 rows for each form are for the first sample period and the last 5 are for the final sample period.

2. Table entries are $\partial \ell n\, x_i(p, y, t) / \partial t$ for input i and $\partial \ell n\, C(p, y, t) / \partial t$ for total cost.

Table 5

Output Elasticities of Input Demand and Total Cost

Numeraire		GB K	GB L	GB E	GB M	GL	TL
Input	K	-.01	-.06	-.06	-.04	.07	.21
	L	.61	.59	.57	.63	.67	.80
	E	.07	.14	.08	.30	.16	.28
	M	.93	.93	.93	.95	.94	1.08
Total Cost		.76	.76	.75	.79	.79	.93
Input	K	.54	.47	.51	.36	.42	-.20
	L	.72	.68	.69	.66	.70	.52
	E	.46	.41	.46	.31	.30	.00
	M	.96	.96	.96	.95	.95	.79
Total Cost		.84	.83	.83	.81	.82	.62

Numeraire		GM K	GM L	GM E	GM M
Input	K	-.25	.19	-.24	.01
	L	.58	.75	.56	.61
	E	-.40	.34	.03	.11
	M	.86	.95	.90	.96
Total Cost		.68	.83	.72	.79
Input	K	.94	.28	.53	.40
	L	.90	.69	.71	.66
	E	.99	.12	.56	.29
	M	.98	.93	.95	.96
Total Cost		.95	.79	.84	.81

Notes:

1. See note 1, Table 4.

2. Table entries are $\partial \ell n\ x_i(p, y, t) / \partial \ell n\ y$ for input i and $\partial \ell n\ C(p, y, t) / \partial \ell n\ y$ for total cost.