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# VOLATILITY COMOVEMENT:
# A MULTIFREQUENCY APPROACH

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## ABSTRACT

We implement a multifrequency volatility decomposition of three exchange rates and show that components with similar durations are strongly correlated across series. This motivates a bivariate extension of the Markov-Switching Multifractal (MSM) introduced in Calvet and Fisher (2001, 2004). Bivariate MSM is a stochastic volatility model with a closed-form likelihood. Estimation can proceed by ML for state spaces of moderate size, and by simulated likelihood via a particle filter in high-dimensional cases. We estimate the model and confirm its main assumptions in likelihood ratio tests. Bivariate MSM compares favorably to a standard multivariate GARCH both in- and out-of-sample. We extend the model to multivariate settings with a potentially large number of assets by proposing a parsimonious multifrequency factor structure.

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## 1. Introduction

A growing body of research investigates the comovement of volatility in financial series. The motivation underlying this effort is well-known. Joint movements in volatility influence the distribution of portfolio returns, and thus play an important role in risk management, portfolio selection, and derivative pricing. Comovements in volatility also help our understanding of financial markets, and shed light on issues such as contagion and the transmission of shocks through the financial system.[1] This motivation is particularly strong in the exchange rate literature, where first moments of currency returns relate weakly to fundamentals at medium frequencies, and movements in volatility can be large and persistent (e.g., Meese and Rogoff, 1983; Rogoff, 1999; Sarno and Taylor, 2002; Clarida *et al.,* 2003).[2]

Multivariate GARCH, pioneered by Kraft and Engle (1982) and Bollerslev, Engle and Wooldridge (1988), is perhaps the most commonly used class of models. A natural extension of GARCH, these processes assume that a vector transform of the covariance matrix can be written as a linear combination of its lagged values and the return innovations. Andersen, Bollerslev and Lange (1999) show that these models perform well relative to competing alternatives. General formulations are, however, hampered by difficulties. The dimensionality of the parameter space grows quickly with the number of assets, and positive-definiteness of the covariance-matrix is not easily guaranteed. This has led to simplified versions including the constant-conditional correlation GARCH specification introduced by Bollerslev (1990).[3] Multivariate stochastic volatility processes share these difficulties, although this has not prevented the growth of an impressive literature, including work by Harvey, Ruiz, and Shephard (1994), Andersen, Benzoni and Lund (2002), and Johannes, Polson and Stroud (2002).

We propose a new approach based on a recent advance in univariate time series, the Markov Switching Multifractal (MSM) of Calvet and Fisher (2001, 2004). This earlier research uses Markov-switching to develop the first weakly convergent sequence of discrete filters for time-stationary multifractal diffusions. In MSM, total volatility is the multiplicative product of a large number of random components that are independent and statistically identical except for heterogeneity in their durations. The construction is parsimonious and delivers volatility persistence, substantial outliers, and a decomposition of volatility into frequency-specific components. MSM compares favorably to earlier specifications both in- and out-of-sample. Univariate multifractal forecasts slightly im-

[1] See for instance Engle, Ito and Lin (1990), or Edwards and Susmel (2003) and the references therein.

[2] See Lyons (1995, 2001) for stronger evidence at high frequency.

[3] Researchers have also considered weaker restrictions (Engle and Kroner, 1995; Engle and Mezrich, 1996; Engle 2002), factor structures (e.g., Engle, 1987; Diebold and Nerlove, 1989; Engle, Ng and Rotshchild, 1990), and estimation methods other than maximum likelihood (Ledoit, Santa-Clara and Wolf, 2003).

prove on GARCH(1,1) at daily and weekly intervals, and provide considerable gains in accuracy at horizons of 10 to 50 days.

This paper investigates comovement in MSM volatility components across exchange rates. We consider three series, the German Mark, the British Pound and the Japanese Yen, all versus the US Dollar over the period 1973-2003. Our results show that components from different series with similar frequencies tend to move together. In contrast, components with very different frequencies display less correlation, both within and across series. We then attempt to relate MSM volatility components to macroeconomic indicators, and find no robust pattern using variables such as GDP, inflation, money supply, interest rates, and stock market volatility. On the other hand, oil and gold prices both correlate positively with currency volatility over the past three decades, consistent with the view that these commodities act as proxies for global economic and political risk. An MSM volatility decomposition reveals that these correlations exist only at low frequencies. This result encourages the econometrician to be cautious about the out-of-sample behavior of these apparent regularities.

Our findings motivate the construction of a bivariate model of currency volatility. This specification, called bivariate MSM, offers several advantages. It is relatively parsimonious, as the number of parameters is independent of $\bar{k}$. There is no issue of positive semi-definiteness. The likelihood function can be written in closed-form and ML estimation can be implemented for state spaces of reasonable size.

To accommodate larger state spaces, we develop a particle filter that permits convenient inference and forecasting using simulations. The good performance of the method is checked in Monte Carlo experiments. The algorithm broadens the range of computationally tractable models to include cases where the number of volatility components is very large, and to cases where the state variables are drawn from continuous rather than binomial distributions. This innovation thus opens econometric research on multifractal processes to a much wider range of specifications in both the univariate and multivariate cases.

We estimate the bivariate model by maximum likelihood and verify that the goodness of fit increases with the number of frequency-specific volatility components. Likelihood ratio tests confirm that the main assumptions of the model are empirically valid. Bivariate MSM compares favorably to constant correlation GARCH (CC-GARCH) in-sample. Out-of-sample, integral transforms and the Cramer-von Mises statistic indicate that, in contrast to CC-GARCH, bivariate MSM captures well the conditional distribution of a variety of currency portfolios. MSM also provides reasonable measures of value at risk, while CC-GARCH tends to underestimate the riskiness of a currency position.

We conclude the paper by examining generalizations of the model to a larger number of assets. We show that bivariate MSM extends easily to larger economies, but can become complicated in its general formulation. We thus propose a factor model of multifrequency stochastic volatility, specified by a number of volatility parameters that grows linearly in the number of assets. Estimation can be conducted by maximizing the closed-form likelihood or by implementing the particle filter.

The rest of the paper is organized as follows. Section 2 reviews univariate MSM and relates the volatility components to other financial and macroeconomic indicators. Section 3 introduces the bivariate model. Section 4 develops inference methods, including likelihood estimation and the particle filter. Empirical results are reported in Section 5. Extensions to many assets are discussed in Section 6. Unless stated otherwise, all proofs are given in the Appendix.

## 2. Comovement of Univariate Volatility Components

### 2.1. The Univariate Stochastic Volatility Model

We begin by reviewing the Markov-Switching Multifractal (MSM), a discrete-time Markov process with multi-frequency stochastic volatility. Consider an economic series $p_t$ defined in discrete time on the regular grid $t = 0, 1, 2, ..., \infty$. In applications, $p_t$ is the log-price of a financial asset or exchange rate. We consider an economy with $\overline{k}$ volatility components $M_{1,t}, M_{2,t}, ..., M_{\overline{k},t}$, which decay at heterogeneous frequencies $\gamma_1, .., \gamma_{\overline{k}}$. The notation MSM($\overline{k}$) refers to versions of the model with $\overline{k}$ frequencies.

The innovations $x_t \equiv p_t - p_{t-1}$ are specified as

$$x_t = (M_{1,t}M_{2,t}...M_{\overline{k},t})^{1/2}\varepsilon_t, \tag{2.1}$$

where the random variables $\varepsilon_t$ are IID standard Gaussians $\mathcal{N}(0, \sigma^2)$. The random *multipliers* or *volatility components* $M_{k,t}$ are persistent, non-negative and satisfy $\mathbb{E}(M_{k,t}) = 1$. We assume for simplicity that the multipliers $M_{1,t}, M_{2,t}...M_{\overline{k},t}$ at a given time $t$ are statistically independent. The parameter $\sigma$ then equals the unconditional standard deviation of the innovation $x_t$.

Equation (2.1) defines a return process with stochastic volatility $\sigma_t = \sigma(M_{1,t}M_{2,t}...M_{\overline{k},t})^{1/2}$. We conveniently stack the period $t$ volatility components into the $1 \times \overline{k}$ row vector

$$M_t = (M_{1,t}, M_{2,t}, ..., M_{\overline{k},t}).$$

The vector $M_t$ is first-order Markov and is called the *volatility state.* The econometrician observes the returns $x_t$ but not the vector $M_t$ itself. MSM is thus a hidden Markov chain model of volatility. The latent state $M_t$ is inferred recursively by Bayesian updating, and estimation is possible by maximum likelihood. MSM is thus a tractable high-dimensional version of the regime-switching models advocated by Hamilton (1989).

The volatility components $M_{k,t}$ follow processes that are identical except for time scale. Assume that the state vector has been constructed up to date $t-1$. For each

$k \in \{1, .., \bar{k}\}$, the next period multiplier $M_{k,t}$ is either drawn from a fixed distribution $M$ with probability $\gamma_k$, or otherwise remains equal to its current value: $M_{k,t} = M_{k,t-1}$. The dynamics of $M_{k,t}$ can be summarized as:

$$\begin{array}{ll} M_{k,t} \text{ drawn from distribution } M & \text{with probability } \gamma_k \\ M_{k,t} = M_{k,t-1} & \text{with probability } 1 - \gamma_k. \end{array}$$

The switching events and new draws from $M$ are assumed to be independent across $k$ and $t$. The volatility components $M_{k,t}$ thus differ in their transition probabilities $\gamma_k$ but not in their marginal distribution $M$.

The transition probabilities are specified as $\gamma_k = 1 - (1 - \gamma_{\bar{k}})^{\left(b^{k-\bar{k}}\right)}$, where $\gamma_{\bar{k}} \in (0, 1)$ and $b \in (1, \infty)$. This specification is introduced in Calvet and Fisher (2001) in connection with the discretization of a Poisson arrival process. The pair $(\gamma_{\bar{k}}, b)$ thus provides a numerically convenient specification for the transition probabilities.

The multifractal construction imposes only minimal restrictions on the marginal distribution of the multipliers: $M \geq 0$ and $\mathbb{E}\, M = 1$. While flexible parametric or even nonparametric specifications of $M$ can be used, this paper focuses on the parsimonious setup in which $M$ is drawn from a binomial random variable taking values $m_0 \in [1, 2]$ or $2 - m_0 \in [0, 1]$ with equal probability. The full parameter vector is then

$$\psi \equiv (m_0, \sigma, b, \gamma_{\bar{k}}) \in \mathbb{R}^4_+,$$

where $m_0$ characterizes the distribution of the multipliers, $\sigma$ is the unconditional standard deviation of returns, and $b$ and $\gamma_{\bar{k}}$ define the set of switching probabilities.

The multiplicative structure (2.1) is appealing to model the outliers and volatility persistence exhibited by financial time series. Changes in low level multipliers lead to discrete shifts in volatility that can be maintained over long periods of time. Such risks are important, for example, to a market maker pricing long-lived options. In addition, variations in high frequency multipliers help capture extreme tail events in short-run returns. This has obvious implications for pricing shorter-lived options or for calculating Value-at-Risk.

In exchange rate series, the estimated duration of the most persistent component, $1/\gamma_1$, is typically of the same order as the length of the data. The process thus generates volatility cycles with periods proportional to the sample size, a property also apparent in the sample paths of long memory processes. Fractionally integrated processes generate such patterns by assuming that an innovation linearly affects future periods at a hyperbolically declining weight. As a result, fractional integration tends to produce smooth volatility processes. By contrast, our approach generates long cycles with a switching mechanism that also gives abrupt volatility changes.[4] The combination of

[4]Long memory is often defined by a hyperbolic decline in the autocovariance function as the lag

long-memory behavior with sudden volatility movements in MSM has a natural appeal for financial econometrics.

The continuous-time version of MSM can be conveniently constructed and lies outside the class of Itô diffusions when $\bar{k} \to \infty$. The sample paths are continuous but exhibit a high degree of heterogeneity in local behavior, which is characterized by a continuum of local Hölder exponents in any finite time interval. Calvet and Fisher (2001, 2002) fully develop the continuous-time limit.

### 2.2. Comovement of Exchange Rate Volatility

The empirical analysis investigates daily returns on the Deutsche Mark (DM), Japanese Yen (JA) and British Pound (UK), all against the US Dollar. The returns are imputed from noon daily prices reported by the Federal Reserve Bank of New York.[5] The series begin on 1 June 1973, shortly after the demise of the Bretton Woods fixed exchange rate system. The Deutsche Mark is replaced by the Euro at the beginning of 1999. Each series ends on 30 October 2003 and contains 7,635 observations.

For each currency, we estimate MSM by maximum likelihood on the entire sample and report the results in Table 1. The columns correspond to the number of frequencies $\bar{k}$ varying from 1 to 8. The first column is thus a standard Markov-switching model with only two possible levels of volatility. As $\bar{k}$ increases, the number of states increases at the rate $2^{\bar{k}}$. The multiplier value $m_0$ tends to decline with $\bar{k}$. With a larger number of frequencies, less variability is required from each individual component to match the volatility fluctuations of the data. Estimates of $\sigma$ fluctuate across $\bar{k}$ with no apparent trend. When $\bar{k} = 1$, the parameter $\hat{\gamma}_{\bar{k}}$ indicates that the single multiplier has a duration of a few weeks. As $\bar{k}$ increases, the switching probability of the highest frequency multiplier increases until a switch occurs almost every day for large $\bar{k}$. At the same time, the growth rate $\hat{b}$ decreases steadily with $\bar{k}$. In the DM series with $\bar{k} = 8$, a switch in the lowest frequency multiplier occurs approximately once every eight years, or about one fourth the sample size. Thus, as $\bar{k}$ increases, frequencies tend to span a wider range while becoming more tightly spaced.

We use the ML estimates to compute, for each currency, the smoothed state probabilities (Kim, 1993) and the expectation of the multipliers conditional on the entire sample: $\hat{M}_{k,t} = \mathbb{E}(M_{k,t}|x_1, ..., x_T)$. The correlations of the smoothed components $\hat{M}_{k,t}$ are reported in Table 2. In the first panel, we see that different components of the DM exchange rate are moderately correlated, and correlation decreases in the distance

goes to infinity. As shown in Calvet and Fisher (2004), MSM mimics the hyperbolic auocorrelation in the size of the returns exhibited by many financial series (e.g., Ding, Granger and Engle, 1993). The multifractal model thus illustrates the difficulty of distinguishing between long memory and structural change in finite samples, as in Hidalgo and Robinson (1996) Diebold and Inoue (2001).

[5] More specifically, the data consist of buying rates for wire transfers at 12:00 PM Eastern time.

between frequencies.[6] We report only DM results for space constraints, but obtain similar results with the UK and JA series. The second and third panels of Table 2 show inferred comovement of the DM components with JA and UK. Correlation between the smoothed beliefs $\hat{M}^{\alpha}_{k,t}$ and $\hat{M}^{\beta}_{k',t}$ of two currencies tends to be high when $k$ and $k'$ are close, and is low otherwise. This suggests that the volatility components of two exchange rates are correlated only if their frequencies are similar.

The interpretation is slightly complicated, however, by the fact that each currency may have a different set of volatility frequencies. To address this issue, we now introduce a simple bivariate model in which currencies are statistically independent but have identical frequency parameters $b$ and $\gamma_{\bar{k}}$. The log-likelihood of the two series is then

$$L(x^{\alpha}_t; m^{\alpha}_0, \sigma_{\alpha}, b, \gamma_{\bar{k}}) + L(x^{\beta}_t; m^{\beta}_0, \sigma_{\beta}, b, \gamma_{\bar{k}}), \tag{2.2}$$

where $L$ denotes the log-likelihood of univariate MSM. This specification, called the *combined univariate*, is an important building block of the bivariate model introduced in the next section.

In Table 3, we report results for the combined univariate model. Panel A shows ML estimates for the (DM,JA) series. For low values of $\overline{k}$, some parameter estimates differ noticeably from the unrestricted univariate results in Table 1, but generally the frequency restrictions do not appear problematic. To confirm this, for $\overline{k} = 8$ we compare the likelihood of $-13063.11$ with the sum of unrestricted likelihoods, i.e. $-6885.90 - 6174.96 = -13060.86$. Under the combined univariate, the difference of 2.25 is asymptotically distributed as a chi-squared with two degrees of freedom. This difference is not significant at any conventional level, confirming that the frequency restrictions are reasonable. The second part of Panel A repeats this likelihood ratio (LR) test for each frequency and currency combination, reporting $p$-values. Except for very low values of $\overline{k}$, the frequency restrictions are not rejected. Panel B then shows correlations between smoothed volatility components for the (DM,JA) series under the combined univariate model. With frequencies now identical across currencies, we expect results to be stronger than in Table 2, and this is confirmed. Results for other currency pairs are similar. We thus find that (1) restricting frequencies to be identical across currencies is reasonable, and (2) components of similar frequencies tend to move together while components with very different frequencies do not. These conclusions are useful in developing a bivariate MSM specification in Section 3.

### 2.3. Currency Volatility and Macroeconomic Indicators

We now investigate whether currency volatility comovement relates to other macroeconomic and financial variables. Earlier research leads us to be relatively pessimistic. For

[6] Note, however, that because the econometrician does not directly observe changes in multipliers, correlation in smoothed beliefs can be consistent with independence between $M_{k,t}$ and $M_{k',t}$, $k \neq k'$.

instance, the first moments of exchange rates are weakly linked to fundamentals (e.g., Meese and Rogoff, 1983; Andersen and Bollerslev, 1998*a*; Rogoff, 1999; Sarno and Taylor, 2002). Variances are also difficult to explain, at least in stock market data (e.g., Schwert, 1989). We examine whether the new multifrequency decomposition confirms these negative results.

We first consider IMF monthly data for 1973-2000, including monetary aggregates (M1, M2 and M3), short and long interest rates, producer price index, consumer price index, wages and the growth rate of industrial production. We compute the correlation between monthly volatility and the macro variables of each country, their difference and the absolute value of their difference. We use several measures of volatility, such as the absolute value of the monthly return, the realized monthly volatility, and MSM volatility components. In unreported work using a variety of lag structures, we find no robust link between currency volatility and these variables. These results are consistent with the findings of Andersen and Bollerslev (1998), who show that macro announcements induce volatility shocks that are of comparable magnitude to daily volatility. It is thus not surprising that little impact is found at the monthly frequency.

Economic theory suggests that exchange rates might be more strongly linked to equity markets, since both classes of instruments incorporate forward-looking information about rates of return, national economic conditions and corporate profits.[7] In Table 4, we investigate the comovement of each currency with volatility in domestic and US stock markets. Daily returns are imputed from the US value-weighted CRSP index, the German CDAX Composite Price Index, the UK FT-Actuaries All Share Index and the Japanese Nikkei 225 Stock Average. The sum of squared daily returns measures realized monthly stock volatility, and is compared to the currency return, absolute return, realized volatility (RV) and MSM frequency-specific components. The reported correlation is positive for the Deutsche Mark and the Yen, and weakly negative for the Pound. We thus find no robust link between currency and stock volatility.

Oil prices are often viewed as proxies for global economic and political uncertainty (e.g., Hamilton, 2003). As seen in Table 4, the dollar price of oil correlates positively with the RV of DM and UK,[8] and the MSM decomposition further reveals that this is primarily a low frequency phenomenon. The results become more intriguing for JA. While the raw oil price shows little correlation with the RV of the Yen, it is again strongly correlated with low-frequency MSM components. The MSM decomposition thus finds evidence of a regularity that direct analysis of realized volatility would not uncover. Similar results are obtained with gold, further suggesting that currency volatility and certain commodity prices may be linked at low frequencies through an unidentified

[7] See Sarno and Taylor (2002) for a recent review of the economics of exchange rates.

[8] We use the domestic first purchase price of crude oil expressed in dollars per barrel provided by Global Insight/DRI.

global risk factor. As in Stock and Watson (2003), we view these findings with caution since it is unclear whether oil and gold prices will continue to be effective proxies for global risk in the future.

Our results thus deepen the puzzle regarding the link between exchange rates and fundamentals. Volatility components are strongly correlated across currencies but only weakly related to other macroeconomic and financial variables. This motivates the development of a multivariate multifrequency model of exchange rates.

## 3. A Bivariate Multifrequency Model

### 3.1. The Stochastic Volatility Specification

We consider two financial series $\alpha$ and $\beta$ defined on the regular grid $t = 0, 1, 2, ..., \infty$. Their log-returns[9] $x_t^\alpha$ and $x_t^\beta$ in period $t$ are stacked into the column vector

$$x_t = \begin{bmatrix} x_t^\alpha \\ x_t^\beta \end{bmatrix} \in \mathbb{R}^2.$$

As in univariate MSM, volatility is stochastic and hit by shocks of heterogeneous frequencies indexed by $k \in \{1, ..., \bar{k}\}$. For every frequency $k$, the currencies have volatility components $M_{k,t}^\alpha$ and $M_{k,t}^\beta$. Consider

$$M_{k,t} = \begin{bmatrix} M_{k,t}^\alpha \\ M_{k,t}^\beta \end{bmatrix} \in \mathbb{R}_+^2.$$

The period-$t$ volatility column vectors $M_{k,t}$ are stacked into the $2 \times \bar{k}$ matrix

$$M_t = (M_{1,t}; M_{2,t}; ...; M_{\bar{k},t}).$$

Each row of the matrix contains the volatility components of a particular currency, while each column corresponds to a particular frequency. As in univariate MSM, we assume that $M_{1,t}, M_{2,t}...M_{\bar{k},t}$ at a given time $t$ are statistically independent. The main task is to choose appropriate dynamics for each vector $M_{k,t}$.

Economic intuition suggests that volatility arrivals are correlated but not necessarily simultaneous across currency markets. For this reason, we allow arrivals across series to be characterized by a correlation coefficient $\lambda$. Assume that the volatility vector $M_{k,s}$ has been constructed up to date $t-1$. In period $t$, each series $c \in \{\alpha, \beta\}$ is hit by an arrival with probability $\gamma_k$. Let $1_{k,t}^c$ denote the random variable equal to 1 if there is an arrival on $M_{k,t}^c$, and equal to 0 otherwise. The arrival vector $1_{k,t} = (1_{k,t}^\alpha, 1_{k,t}^\beta)$ is specified to be IID, and its unconditional distribution is defined by three conditions.

[9] If $X_t^\alpha$ denote the value of the exchange rate at date $t$, the log-return is $x_t^\alpha = \ln(X_t^\alpha / X_{t-1}^\alpha)$.

First, the arrival vector is symmetrically distributed: $(1^{\beta}_{k,t}, 1^{\alpha}_{k,t}) \overset{d}{=} (1^{\alpha}_{k,t}, 1^{\beta}_{k,t})$. Second, the switching probability of a series is equal to an exogenous constant:

$$\mathbb{P}(1^{\alpha}_{k,t} = 1) = \gamma_k.$$

Third, there exists $\lambda \in [0,1]$ such that

$$\mathbb{P}(1^{\beta}_{k,t} = 1 | 1^{\alpha}_{k,t} = 1) = (1-\lambda)\gamma_k + \lambda.$$

As shown in the Appendix, these three conditions define a unique distribution for $1_{k,t}$. Arrivals are independent if $\lambda = 0$ and simultaneous if $\lambda = 1$. We easily check that $\lambda$ is the unconditional correlation between $1^{\alpha}_{k,t}$ and $1^{\beta}_{k,t}$.

Given the realization of the arrival vector $1_{k,t}$, the construction of the volatility components $M_{k,t}$ is based on a bivariate distribution $M = (M^{\alpha}, M^{\beta}) \in \mathbb{R}^2_+$. We briefly postpone the choice of $M$, and assume for now that it is defined by two parameters $m^{\alpha}_0$ and $m^{\beta}_0$. If arrivals hit both series ($1^{\alpha}_{k,t} = 1^{\beta}_{k,t} = 1$), the state vector $M_{k,t}$ is drawn from $M$. If only series $c \in \{\alpha, \beta\}$ receives an arrival, the new component $M^{c}_{k,t}$ is sampled from the marginal $M^c$ of the bivariate distribution $M$. Finally, $M_{k,t} = M_{k,t-1}$ if there is no arrival ($1^{\alpha}_{k,t} = 1^{\beta}_{k,t} = 0$). The construction thus implies that switching vectors and draws from $M$ are independent across $k$ and $t$. In vector notation, it can be summarized as:

$$M_{k,t} \overset{d}{=} M_{k,t-1} + 1_{k,t} * (M - M_{k,t-1}),$$

where $*$ denote element by element multiplication. The volatility components $M_{k,t}$ differ in their transition probabilities $\gamma_k$, but not in their marginal distribution $M$ or arrival correlation $\lambda$. These features greatly contribute to the parsimony of the model.

The volatility vectors $M_{k,t}$ are persistent, non-negative and satisfy $\mathbb{E}(M_{k,t}) = \mathbf{1}$, where $\mathbf{1} = (1,1)'$. Consistent with previous notation, let $g(M_t)$ denote the $2 \times 1$ vector $M_{1,t} * M_{2,t} * ... * M_{\bar{k},t}$. The return vector $x_t$ is specified as

$$x_t = [g(M_t)]^{1/2} * \varepsilon_t, \tag{3.1}$$

where the column vectors $\varepsilon_t \in \mathbb{R}^2$ are IID Gaussian $\mathcal{N}(0, \Sigma)$. The covariance matrix $\Sigma$ can be written as

$$\Sigma = \begin{bmatrix} \sigma^2_{\alpha} & \rho_{\varepsilon}\sigma_{\alpha}\sigma_{\beta} \\ \rho_{\varepsilon}\sigma_{\alpha}\sigma_{\beta} & \sigma^2_{\beta} \end{bmatrix}.$$

The construction thus permits correlation in volatility across series through the bivariate distribution $M$, and correlation in returns through $\rho_{\varepsilon}$. As in the univariate case, the transition probabilities $(\gamma_1, \gamma_2, ..., \gamma_{\bar{k}})$ are defined as

$$\gamma_k = 1 - (1 - \gamma_{\bar{k}})^{\left(b^{k-\bar{k}}\right)}, \tag{3.2}$$

where $\gamma_{\bar{k}} \in (0,1)$ and $b \in (1,\infty)$. This completes the specification of *bivariate MSM.*

We observe that under bivariate MSM, returns satisfy

$$\begin{aligned} x_t^\alpha &= (M_{1,t}^\alpha M_{2,t}^\alpha ... M_{\bar{k},t}^\alpha)^{1/2} \varepsilon_t^\alpha, \\ x_t^\beta &= (M_{1,t}^\beta M_{2,t}^\beta ... M_{\bar{k},t}^\beta)^{1/2} \varepsilon_t^\beta. \end{aligned}$$

Their univariate dynamics thus coincide with univariate MSM. In particular, the parameter $\sigma_c$ is again the unconditional standard deviation of each univariate series $c \in \{\alpha, \beta\}$. Bivariate MSM thus requires eight parameters

$$\psi \equiv (\sigma_\alpha, \sigma_\beta, m_0^\alpha, m_0^\beta, b, \gamma_{\bar{k}}, \rho_\varepsilon, \lambda),$$

where $\sigma_\alpha$ and $\sigma_\beta$ are the unconditional standard deviations of the return series, $m_0^\alpha$ and $m_0^\beta$ determine the distribution of volatility components, $\gamma_{\bar{k}}$ their transition probabilities, $\rho_\varepsilon$ the correlation of the Gaussian innovations, and $\lambda$ the correlation of arrivals across series.

The bivariate construction imposes few restrictions on the distribution of vector $M$. For the empirical applications in the remainder of this paper, we investigate a simple specification, which assumes that each $M_{kt}$ is drawn from a bivariate binomial distribution $M = (M^\alpha, M^\beta)'$. The first element $M^\alpha$ takes values $m_0^\alpha \in [1,2]$ and $m_1^\alpha = 2 - m_0^\alpha \in [0,1]$ with equal probability. Similarly, $M^\beta$ is either $m_0^\beta \in [1,2]$ or $m_1^\beta = 2 - m_0^\beta$. The random vector $M$ can thus take four possible values, whose probabilities are parameterized by the matrix $(p_{i,j}) = (\mathbb{P}\{M = (m_i^\alpha, m_j^\beta)\})$. The conditions $\mathbb{P}(M^\alpha = m_0^\alpha) = 1/2$ and $\mathbb{P}(M^\beta = m_0^\beta) = 1/2$ impose that

$$\begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} = \begin{bmatrix} \frac{1+\rho_m^*}{4} & \frac{1-\rho_m^*}{4} \\ \frac{1-\rho_m^*}{4} & \frac{1+\rho_m^*}{4} \end{bmatrix}$$

for some $\rho_m^* \in [-1,1]$. We easily check that $\rho_m^*$ is the correlation between components $M^\alpha$ and $M^\beta$ under the distribution $M$. The empirical section focuses on the specification $\rho_m^* = 1$, which in unreported work is never rejected on the currency data. Because distributions with $\rho_m^* < 1$ may be useful for series with weaker correlation in volatility, we report in the next subsection all theoretical results for arbitrary values of $\rho_m^*$.

### 3.2. Properties

The dynamics of $M_{k,t}$ are determined by the switching vector $1_{k,t}$ and the bivariate distribution $M$. We show in the Appendix that $M_{k,t}$ has a unique ergodic distribution $\bar{\Pi}_k$. Let $\bar{\Pi}_k^{HH}$, $\bar{\Pi}_k^{HL}$, $\bar{\Pi}_k^{LH}$ and $\bar{\Pi}_k^{LL}$ respectively denote the probabilities of states $(m_0^\alpha, m_0^\beta)$, $(m_0^\alpha, m_1^\beta)$, $(m_1^\alpha, m_0^\beta)$, and $(m_1^\alpha, m_1^\beta)$. The symmetry of the construction implies that $\bar{\Pi}_k^{HH} = \bar{\Pi}_k^{LL}$ and $\bar{\Pi}_k^{HL} = \bar{\Pi}_k^{LH}$. When $\rho_m^* > 0$, the multipliers are more likely to be either both high or both low: $\bar{\Pi}_k^{HH} = \bar{\Pi}_k^{LL} > 1/4 > \bar{\Pi}_k^{HL} = \bar{\Pi}_k^{LH}$.

Since different components are statistically independent, the ergodic distribution of the volatility state $M_t = (M_{1,t}; ...; M_{\bar{k},t})$ is the product measure $\bar{\Pi} = \bar{\Pi}_1 \otimes ... \otimes \bar{\Pi}_{\bar{k}}$. Under this distribution, the cross-currency correlation of volatility components is given by

$$Corr(M_{kt}^{\alpha}, M_{kt}^{\beta}) = \rho_m^* \frac{(1-\lambda)\gamma_k + \lambda}{2 - [(1-\lambda)\gamma_k + \lambda]} \leq \rho_m^*.$$

This coefficient is large when the transition probability $\gamma_k$ or the correlation of switching events $\lambda$ are high, i.e. when arrivals tend to happen at the same time. The upper bound $\rho_m^*$ is reached when either $\lambda = 1$ or $\gamma_k \to 1$.

The return series have unconditional means equal to zero: $\mathbb{E}\, x_t = 0$. By the Cauchy-Schwarz inequality, their correlation satisfies

$$Corr(x_t^{\alpha}; x_t^{\beta}) = \rho_{\varepsilon} \prod_{k=1}^{\bar{k}} \mathbb{E}[(M_{k,t}^{\alpha} M_{k,t}^{\beta})^{1/2}] \leq \rho_{\varepsilon}. \tag{3.3}$$

The upper bound $\rho_{\varepsilon}$ is attained when the multipliers of both series are perfectly correlated. On the other hand when $\rho_{\varepsilon} < 1$, uncorrelated changes in volatility represent additional sources of noise that reduce the correlation of asset returns.

The econometrician does not observe the volatility state, but only the set of past returns $X_t \equiv \{x_s\}_{s=1}^{t}$. Returns are unpredictable with this information set: $\mathbb{E}_{t-1} x_t = 0$, and the model is thus consistent with some standard forms of market efficiency.[10] Comovement is quantified by the conditional covariance $Cov_t(x_{t+n}^{\alpha}; x_{t+n}^{\beta}) = \rho_{\varepsilon}\sigma_{\alpha}\sigma_{\beta} \prod_{k=1}^{\bar{k}} \mathbb{E}_t[(M_{k,t+n}^{\alpha} M_{k,t+n}^{\beta})^{1/2}]$, and the conditional correlation

$$Corr_t(x_{t+n}^{\alpha}; x_{t+n}^{\beta}) = \rho_{\varepsilon} \prod_{k=1}^{\bar{k}} \frac{\mathbb{E}_t[(M_{k,t+n}^{\alpha} M_{k,t+n}^{\beta})^{1/2}]}{[(\mathbb{E}_t M_{k,t+n}^{\alpha})(\mathbb{E}_t M_{k,t+n}^{\beta})]^{1/2}} \leq \rho_{\varepsilon}. \tag{3.4}$$

These quantities fluctuate through time with the multipliers. Thus while the construction assumes constant correlation coefficients $\rho_{\varepsilon}$ and $\rho_m$, the conditional correlation of returns is time-varying. We easily check that it is large when the volatility components of the currencies are high.

Comovement in volatility can be similarly investigated. We observe that when $\rho_m > 0$ and $\bar{k}$ is large, the conditional correlation of absolute returns satisfies

$$Corr_t(|x_{t+n}^{\alpha}|; |x_{t+n}^{\beta}|) \sim C_{\varepsilon} \prod_{k=1}^{\bar{k}} \frac{\mathbb{E}_t[(M_{k,t+n}^{\alpha} M_{k,t+n}^{\beta})^{1/2}]}{[(\mathbb{E}_t M_{k,t+n}^{\alpha})(\mathbb{E}_t M_{k,t+n}^{\beta})]^{1/2}}, \tag{3.5}$$

where $C_{\varepsilon} = \mathbb{E}(\left|\varepsilon_1^{\alpha}\varepsilon_1^{\beta}\right|)$. Consistent with previous intuition, correlation between absolute returns is high in periods of high volatility.

[10] See Campbell, Lo and MacKinlay (ch2, 1997) for a discussion.

## 4. Inference

We now develop inference methods for bivariate MSM. Analytical Bayesian updating and a closed-form likelihood are practical when the number of volatility components is not too large. Alternative computational methods are developed for high-dimensional state spaces.

### 4.1. Closed-Form Likelihood

Since each frequency vector $M_{kt}$ is drawn from a bivariate binomial, the volatility state $M_t = (M_{1,t}; M_{2,t}; ...; M_{\overline{k},t})$ takes $d = 4^{\overline{k}}$ possible values $m^1, ..., m^d \in \mathbb{R}_+^{\overline{k}}$. The dynamics of $M_t$ are characterized by the $d \times d$ transition matrix $A = (a_{i,j})_{1 \leq i,j \leq d}$ with components $a_{ij} = \mathbb{P}(\, M_{t+1} = m^j \big| \, M_t = m^i)$.

The econometrician does not observe the volatility state, but only the set of past returns $X_t \equiv \{x_s\}_{s=1}^t$. The corresponding probabilities

$$\Pi_t^j = \mathbb{P}\left(M_t = m^j \,|X_t\right)$$

are computed recursively by Bayesian updating. Let $\Pi_t = \left(\Pi_t^1, .., \Pi_t^d\right) \in \mathbb{R}_+^d$. In the next period, state $M_{t+1}$ is drawn and the econometrician observes the return vector $x_{t+1}$. Conditional on the new volatility state, the joint return $x_{t+1}$ has bivariate Gaussian density $f_{x_{t+1}}\left(x_{t+1}|M_{t+1} = m^i\right)$. By Bayes' rule, the updated probability is

$$\Pi_{t+1} = \frac{f\left(x_{t+1}\right) * \Pi_t A}{\left[f\left(x_{t+1}\right) * \Pi_t A\right] \mathbf{1'}}, \tag{4.1}$$

where $\mathbf{1} = (1, ., 1) \in \mathbb{R}^d$ and $f\left(x\right)$ is the vector of conditional densities $(f_{x_{t+1}}(x_{t+1}|M_{t+1} = m^i))_i$. The belief $\Pi_{t+1}$ is thus a function of the observation $x_{t+1}$ and the prior probability $\Pi_t$. In empirical applications, the initial vector $\Pi_0$ is chosen equal to the ergodic distribution $\bar{\Pi}$ of the Markov chain.

We easily check that the log-likelihood function has the closed-form expression:

$$\ln L\left(x_1, ..., x_T; \psi\right) = \sum_{t=1}^{T} \ln[f(x_t) \cdot (\Pi_{t-1} A)].$$

For fixed $\bar{k}$, the maximum likelihood estimator (ML) is consistent and asymptotically efficient as $T \to \infty$. Analytical multistep forecasting can proceed using updated beliefs and the transition matrix $A$ as in Calvet and Fisher (2001).

### 4.2. Particle Filter

The transition matrix $A$ contains $4^{\overline{k}} \times 4^{\overline{k}}$ elements, increasing exponentially in the number of frequencies. When $\overline{k} = 8$, the transition matrix thus has cardinality $2^{32} \approx$

$4 \times 10^9$, and is computationally expensive to use. Following the recent literature on Markov chains,[11] we propose a simulation-based inference methodology. Specifically, we introduce a particle filter, a recursive algorithm that generates independent draws $M_t^{(1)}, \ldots, M_t^{(B)}$ from the conditional distribution $\Pi_t$.

We begin at $t = 0$ by drawing $M_0^{(1)}, \ldots, M_0^{(B)}$ from the ergodic distribution $\bar{\Pi}$. For any $t \geq 0$, assume that $\{M_t^{(b)}\}_{b=1}^B$ have been independently sampled from $\Pi_t$. Given a new return $x_{t+1}$, an approximation to Bayes' rule gives draws $\{M_{t+1}^{(b)}\}_{b=1}^B$ from the new belief $\Pi_{t+1}$. More specifically, we rewrite the updating formula (4.1) as

$$\Pi_{t+1}^j \propto f_{x_{t+1}}\left(x_{t+1}|M_{t+1} = m^j\right) \sum_{i=1}^d \mathbb{P}\left(M_{t+1} = m^j|M_t = m^i\right) \Pi_t^i.$$

The vectors $M_t^{(1)}, \ldots, M_t^{(B)}$ are independent draws from $\Pi_t$. This suggests the Monte Carlo approximation:

$$\Pi_{t+1}^j \propto f_{x_{t+1}}\left(x_{t+1}|M_{t+1} = m^j\right) \frac{1}{B} \sum_{b=1}^B \mathbb{P}\left(M_{t+1} = m^j|M_t = M_t^{(b)}\right).$$

As shown in the Appendix, we complete the approximation by simulating each $M_t^{(b)}$ one-step forward and reweighting using an importance sampler:

1. Simulate the Markov chain one-step ahead to obtain $\hat{M}_{t+1}^{(1)}$ given $M_t^{(1)}$. Repeat $B$ times to generate $B$ draws $\hat{M}_{t+1}^{(1)}, \ldots, \hat{M}_{t+1}^{(B)}$. This preliminary step only uses information available at date $t$, and must therefore be adjusted to account for the new return.

2. Draw a random number $q$ from 1 to $B$ with probability

$$\mathbb{P}(q = b) \equiv \frac{f_{x_{t+1}}(x_{t+1}|M_{t+1} = \hat{M}_{t+1}^{(b)})}{\sum_{b'=1}^B f_{x_{t+1}}(x_{t+1}|M_{t+1} = \hat{M}_{t+1}^{(b')})}.$$

   The vector $M_{t+1}^{(1)} = \hat{M}_{t+1}^{(q)}$ is a draw from $\Pi_{t+1}$. Repeat $B$ times to obtain $B$ draws $M_{t+1}^{(1)}, \ldots, M_{t+1}^{(B)}$.

This recursive procedure provides a discrete approximation to Bayesian updating that is computationally convenient in large state spaces.

[11] See for instance Chib, Nardari and Shephard (2002), Jacquier, Polson and Rossi (1994), and Pitt and Shephard (1999).

### 4.3. Simulated Likelihood

We can use the particle filter to compute the likelihood function. Each one-step ahead density satisfies $f(x_t|X_{t-1}) = \sum_{i=1}^{d} f(x_t|M_t = m^i)\, \mathbb{P}(M_t = m^i|X_{t-1})$. Given simulated draws $\hat{M}_t^{(b)}$ from $M_t \mid X_{t-1}$, the Monte Carlo estimate of the conditional density is thus

$$\widehat{f}(x_t|X_{t-1}) \equiv \frac{1}{B}\sum_{b=1}^{B} f(x_t|M_t = \hat{M}_t^{(b)}),$$

and the log-likelihood is approximated by $\sum_{t=1}^{T} \ln \hat{f}(x_t|X_{t-1})$. We can use these calculations to carry out simulated likelihood estimation. In practice, an arbitrarily close approximation can be achieved by increasing $B$. Larger state spaces require more draws to achieve the same degree of precision.

Table 5 presents a Monte Carlo assessment of this method. We focus on the univariate specification with $\bar{k} = 8$ components. Using the particle filter, we generate 500 approximations of the log-likelihood of the univariate DM series at the optimized ML estimates from Table 1. Each calculation uses independent sets of Monte Carlo draws. We then compare the mean, standard deviation, and quantiles of the estimates with the exact value obtained in Table 1 by analytical Bayesian updating. All particle filter evaluations use $B = 10,000$ random draws. The particle filter estimate of the log-likelihood has a relatively small standard deviation and the average across simulations, $-6887.3$, is close to the true value of $-6885.9$. The quantiles are tightly clustered as well. The table also shows particle filter estimates of the forecast variance, which are accurate and approximately unbiased. These results confirm that the particle filter produces reasonable estimates of the likelihood and moments of the series for problems of reasonable size.

The particle filter extends the range of computationally feasible multifractal specifications. In previous work with univariate processes, Calvet and Fisher (2004) report an approximate computational upper bound of ten binomial state variables, or $2^{10}$ states. While this gives good results in the univariate case, multivariate models require a correspondingly larger number of state variables. We will thus show in the empirical section that the particle filter produces good results in a bivariate model with $\bar{k} = 8$ components, or $2^{16}$ states. The particle filter also permits implementation of specifications where the state vector $M$ is drawn from a continuous distribution. Since earlier research (Calvet and Fisher, 2002) suggest that exchange rates are best fit by lognormal multipliers, it will be interesting in future work to revisit the lognormal specification and compare its performance to the binomial.

### 4.4. Two-Step Estimation

Two-step estimation offers additional computational benefits, permitting the econometrician to decompose inference into a sequence of lower-dimensional optimization problems. In the bivariate multifractal, each series $c \in \{\alpha, \beta\}$ follows a univariate MSM with parameters $m_0^c, \sigma_c, b$ and $\gamma_{\bar{k}}$. This implies that we can estimate six of the eight parameters using the likelihood and smaller state space of the univariate model. Additionally, univariate estimation gives good precision in finite samples (Calvet and Fisher, 2004). This motivates us to develop the two-step method described below. The Appendix shows that this procedure is a special case of GMM, implying consistency and asymptotic normality of the estimator.

In the first stage, we obtain the parameters $(m_0^\alpha, m_0^\beta, \sigma_\alpha, \sigma_\beta, b, \gamma_{\bar{k}})$ by optimizing the sum of the two univariate log-likelihoods, as in (2.2). Intuitively, this gives consistent estimates for all parameters since the gradient of this sum with respect to the true parameters is zero. Because this objective function coincides with the likelihood of the combined univariate, the first step has already been completed in Section 2.

The second stage gives estimates for the remaining two parameters, $(\rho_\varepsilon, \lambda)$, which are unique to the bivariate model. When the state space is not too large, $(\bar{k} \leq 5)$, computation of the analytical bivariate likelihood is practical. We therefore maximize the exact bivariate MSM probability density conditional on the first-stage estimates. For higher-dimensional specifications, $(\bar{k} = 6, 7, 8)$, computation of the analytical bivariate likelihood is difficult. We therefore use the particle filter to optimize the simulated likelihood as described in Section 4.3.[12] In this paper, the two-step procedure aids empirical implementation of bivariate specifications with state spaces as large as $2^{16}$.

## 5. Empirical Results

### 5.1. Bivariate MSM Estimates

We report in Table 6 full analytical ML estimates of bivariate MSM for $\bar{k} \leq 5$ and exchange rate pairs (DM,JA), (DM,UK) and (JA,UK). As in univariate MSM, $\widehat{m}_0$ declines with $\bar{k}$, while the standard deviations $\widehat{\sigma}_\alpha$ and $\widehat{\sigma}_\beta$ are variable but display no apparent trend. The correlation between Gaussian innovations $\widehat{\rho}_\varepsilon$ is positive and roughly constant across $\bar{k}$. The arrival correlation $\widehat{\lambda}$ is also large and approximately invariant to the number of volatility components. Both parameters new to the bivariate model seem precisely estimated. Finally, the estimated $\widehat{\lambda}$ is highest when $\widehat{\rho}_\varepsilon$ is highest, and lowest

[12] One could of course more generally match to any relevant moments in the second stage. Our view is that simulated likelihood is an excellent choice for intermediate problems because it potentially entails a small loss in efficiency. For very large problems, including many assets as discussed in Section 6, it would be natural to consider matching moments such as (3.3) and (3.5). This could potentially further reduce computational requirements.

when $\widehat{\rho}_\varepsilon$ is lowest. We infer that volatility is most correlated when returns are most correlated, which is intuitive.

The likelihood functions sharply increase with the number of frequencies. For instance with (DM,JA), the log-likelihood increases by more than 800 when $\bar{k}$ goes from 1 to 5. Since the models are non-nested and specified by the same number of parameters, this is a very substantial increase of fit in-sample. We also compare the goodness of fit to the independent case in Table 3, and find that for (DM,JA) with $\bar{k} = 5$, the gain in likelihood is over 1300 points. Results are similar for other currencies, demonstrating that the bivariate model improves over independent univariate models.

In Table 7, we reestimate bivariate MSM with the two-step procedure of subsection 4.4. For $\bar{k} \leq 5$, the second stage uses the analytical bivariate likelihood, and for $\bar{k} = 6, 7, 8$ the particle filter is implemented. Comparing the results with $\bar{k} \leq 5$ to full MLE, we observe that the parameter estimates are comparable, and two step estimation appears to work well. For $\bar{k} = 6, 7, 8$, the results appear consistent with the univariate MLE estimates of Table 1 as well as the estimates of the lower dimensional bivariate models. The particle filter is thus effective in extending the range of tractable models.

We compare bivariate MSM with the constant correlation GARCH (CC-GARCH) of Bollerslev (1990), which is a standard benchmark in the multivariate volatility literature. Returns are specified as:

$$x_t^\alpha = \sqrt{h_t^\alpha}\varepsilon_t^\alpha, \quad x_t^\beta = \sqrt{h_t^\beta}\varepsilon_t^\beta,$$

where $\varepsilon_t^\alpha$ and $\varepsilon_t^\beta$ are two standard normals with correlation $\rho_\varepsilon$. The conditional variances $h_t^\alpha$ and $h_t^\beta$ satisfy the recursions $h_{t+1}^c = \omega_c + a_c(\varepsilon_t^c)^2 + b_c h_t^c$ and for each $c \in \{\alpha, \beta\}$. CC-GARCH is thus specified by 7 parameters as compared to 8 with multivariate MSM.

We report in Table 8 an in-sample comparison of bivariate MSM with $\bar{k} = 5$ components against CC-GARCH. It is immediately clear that MSM gives much higher likelihoods although it has only one additional parameter. For all three pairs of exchange rates, the difference in likelihood is over 1000 points. The same results hold whether comparing full MLE results from the two models, or the likelihoods obtained under two-step estimation. To account for the difference in the number of parameters, we compute the BIC statistic for each model. We then test the significance of the difference using the original method suggested by Vuong (1989), and the HAC-adjusted version developed in Calvet and Fisher (2004). In all cases, the $p$-value from the test that CC-GARCH has a superior BIC statistic to multivariate MSM is substantially less than 1%. The in-sample evidence thus strongly favors multivariate MSM.

### 5.2. Integral Transforms

We now use probability integral transforms to analyze out-of-sample properties, as in Diebold, Gunther and Tay (1998) and Elerian, Chib and Shephard (2001). In all remaining empirical work, we use the MSM specification with $\bar{k} = 5$ components. We first estimate MSM and CC-GARCH on the 1973-1989 subsample. The out-of-sample evaluations use the 3473 observations from 1990 to 2003. Let $y_{t,n} \equiv \sum_{i=1}^{n} x_{t+i}$ denote the forward-looking $n$-period return at time $t$. Either MSM or CC-GARCH can be used to define a conditional forecast distribution

$$F_{t,n}(y) \equiv \mathbb{P}(y_{t,n} \leq y | x_1, .., x_t).$$

Under correct specification, the random variables $U_{t,n} = F_{t,n}(y_{t,n})$ are uniformly distributed on $[0, 1]$, and if $n = 1$ they are also independent.

In Figure 1, we compare histograms of selected integral transforms $\{U_{t,n}\}$ for the two models. Histograms are shown for $n = 1$ and $n = 5$ days using as portfolios DM, JA, an equal-weighted position in the two currencies, and a hedge portfolio with weights $(1, -1)$.[13] We see that MSM provides approximately uniform histograms. In contrast, CC-GARCH generates tent-shaped plots with a large concentration of values around 0 and 1. These feature are symptomatic of tails that are too thin in the estimated CC-GARCH process. Similar results are obtained with other currencies.

The Cramer-von Mises (CVM) criterion confirms these graphical results. Let $T^*$ denote the number of out-of-sample periods, and $\hat{F}_U$ the empirical distribution of the transforms $U_{t,1}$. As $T^* \to \infty$, the Cramer-von Mises criterion $T^* \int_0^1 [y - \hat{F}_U(y)]^2 dy$ weakly converges to a weighted series of independent $\chi^2$ random variables:

$$T^* \int_0^1 [y - \hat{F}_U(y)]^2 dx \Rightarrow \sum_{j=1}^{\infty} \left( \frac{z_j}{j\pi} \right)^2$$

where the $\{z_j\}$ are IID $\mathcal{N}(0, 1)$.[14] We report in Table 8 the CVM statistics for all currencies and portfolios. At the 1% level, we reject MSM in only 2 out of 12 cases, while CC-GARCH is rejected 10 out 12 cases.[15] The CVM statistics thus confirm that CC-GARCH provides inaccurate conditional density forecasts, while MSM is broadly consistent with exchange rate data.

[13] The random variables $U_t$ are constructed as follows. In every period, we use the particle filter to draw $B$ values $y_{t,n}^{(1)}, ..., y_{t,n}^{(B)}$ from the conditional distribution of $y_{t,n}$ given $x_1, ..., x_t$. We then approximate $F_{t,n}(y)$ by the empirical c.d.f. $\hat{F}_{t,n}(y) = \frac{1}{B} \sum_{b=1}^{B} 1\{y_{t,n}^{(b)} \leq y\}$. Sensitivity tests indicate that $B = 10,000$ draws are more than sufficient to provide a good approximation.

[14] See Shorack and Wellner (1986) for further details.

[15] We do not adjust the critical values for estimation error. Earlier work (e.g., Thompson, 2000) suggests that such adjustments would only have small effects.

### 5.3. Value at Risk

The tail properties of financial series are of direct interest for risk management and financial regulation. Value at risk (VaR) is a particularly widespread method that summarizes the expected maximum loss over a target horizon within a given confidence interval. Given a confidence level $p$, we define the value at risk of a portfolio to be the $1-p^{th}$ quantile of the conditional return distribution:

$$VaR_{t,n}(p) \equiv F_{t,n}^{-1}(1-p).$$

Thus with probability $p$ we expect to lose no more than $VaR_{t,n}(p)$ over the next $n$ days.

The accuracy of a model for value at risk is most easily verified by recording the failure rate, i.e. the number of times VaR is exceeded in a given sample (e.g., Kupiec, 1995; Jorion, 1997).[16] Table 9 reports the failure rates of MSM and CC-GARCH for portfolios held for $n = 1$ and 5 days and confidence levels of 1%, 5% and 10%. As described in the Appendix, we forecast for each bivariate process the value at risk of individual currencies, equal-weighted portfolios, and hedge portfolios.

The results in Table 9 lead to two conclusions: MSM is more conservative and more accurate than CC-GARCH. MSM is more conservative because it tends to fail less than CC-GARCH. For example, when the 1-day predicted failure rate is 1%, actual portfolio losses exceed the MSM VaR forecast more than 1% of the time for 3 out of 12 portfolios. Actual losses exceed the 1% CC-GARCH quantile more than 1% of the time in 11 out of 12 portfolios. Of course, an excessively conservative model does not necessarily lead to superior risk management. Statistical tests suggest that MSM is not overly conservative. For each portfolio and VaR quantile we test the null hypothesis that the empirical failure rate is equal to the expected failure rate. For the MSM model, the failure rates are statistically different from the 1% prediction for only 1 out of 12 portfolios. The CC-GARCH failure rates are statistically different from 1% in 11 out of 12 portfolios. MSM thus provides more accurate quantile forecasts than CC-GARCH.

### 5.4. Specification Tests

Our comparisons of bivariate MSM with CC-GARCH have shown that the new model does well. It is now natural to investigate whether, in an absolute sense, the restrictions imposed by our model are supported by the data. We weaken one assumption at a time, and assess improvement in fit by likelihood ratio (LR) tests. When a restriction applies equally to the univariate and bivariate models, we choose to test on the univariate series. This allows us to distinguish between misspecifications originating in univariate MSM and those unique to the bivariate approach.

[16] The failure rate is thus the proportion of days in the out-of-sample data in which $x_{t+1}+\cdots+x_{t+n} < VaR_{t,n}(p)$.

Heterogeneity in volatility persistence is made parsimonious by the frequency parameterization (3.2). We focus on univariate models with $\bar{k} = 8$ components. For each currency, we consider the *restricted* univariate MLE estimates in Table 1 and denote by $L_r$ the corresponding likelihood. In contrast, we call *unrestricted* model $k \in \{1, .., \bar{k}\}$ the extension in which frequency parameter $\gamma_k$ is free and all other frequencies satisfy (3.2). We estimate the $k^{\text{th}}$ unrestricted model and denote by $L_u(k)$ the corresponding likelihood. Under the restricted model, $2[L_u(k) - L_r]$ converges to $\chi^2(1)$ as $T \to \infty$. This methodology generates eight LR statistics for each of the three currencies. For space constraints, we report in the text only the salient features of the analysis. For DM, none of the tests provides evidence against the MSM frequency restrictions at the 1% level. One statistic ($k = 6$) is significant for JA, and two tests ($k = 6, 7$) are significant for UK. Evidence against the frequency specification is thus limited to three of the twenty-four tests.

We similarly assess on univariate series whether volatility components have identical distributions across frequencies. Unrestricted specification $k$ permits component $M_{k,t}$ to have its own distribution parameter $m_0(k)$. Results are mixed. For DM, only two of the eight tests ($k = 1, 6$) are significant at the 1% level. For JA, the first five tests suggest a value of $m_0(k)$ larger than for the other components. Similarly for the UK series, LR tests of the first 4 components suggest stronger shocks at low than at high frequency variation, while higher frequencies ($k = 6, 7$) suggest less variability. Overall, the DM data seems to match the MSM model exceptionally well, while JA and UK appear to prefer stronger low-frequency variation. These results are consistent with Calvet and Fisher (2002), who show that DM best matches the moment-scaling restrictions implied by the multifractal model. We also note that the current analysis only considers binomial MSM. Multivariate distributions $M$ such as the lognormal may better accommodate the strong low-frequency variations in UK and JA over the last three decades.

We finally test the restrictions imposed by bivariate MSM on volatility comovement. For each currency pair, the restricted model is given by the full MLE estimates with $\bar{k} = 5$ in Table 6. Unrestricted model $k$ permits that component $k$ may have its own unique arrival correlation $\lambda_k$. We report no rejections at the 1% level for JA-UK, one significant test ($k = 5$) for the DM-UK pair, and two significant statistics ($k = 2, 5$) for the DM-JA pair. Overall, MSM incorporates empirically reasonable restrictions that permit parsimonious specification of bivariate multifrequency volatility.

## 6. Extension to Many Assets

### 6.1. Multivariate MSM

Bivariate MSM can be readily extended to economies with an arbitrary number $N$ of financial prices. The construction assumes a volatility component $M_{k,t}^n \in \mathbb{R}_+$ for each frequency $k \in \{1, ..., \bar{k}\}$ and asset $n \in \{1, ..., N\}$. As in the bivariate case, components $M_{k,t}^n$ and $M_{k',t}^{n'}$ are statistically independent if $k \neq k'$, but can be correlated if $n \neq n'$. The dynamics of volatility components are determined by a multivariate distribution $M$ on $\mathbb{R}_+^N$, and an arrival vector $1_{k,t} \in \{0,1\}^N$ for each frequency $k \in \{1, .., \bar{k}\}$. The component of each asset switches with unconditional probability $\gamma_k$, and arrivals across assets are characterized by correlation coefficients $\lambda_{n,n'}$ :

$$\mathbb{E}1_{k,t} = \gamma_k \mathbf{1}, \qquad Corr(1_{k,t}) = (\lambda_{n,n'})_{1 \leq n,n' \leq N}.$$

The state vector is constructed recursively. At time $t$, we draw the independent arrival vector $(1_{k,t})_{k=1,...,\bar{k}}$, and sample the new components $M_{k,t}$ from the corresponding marginal distribution of $M$.

The volatility state is fully specified by $N \times \bar{k}$ matrix $M_t = (M_{k,t}^n)_{n,k}$. The econometrician again has beliefs $\Pi_t$ over the state space that can be updated using Bayes' rule. When the distribution of $M$ is discrete, the likelihood function is available in closed-form. For large state spaces, estimation can be carried out using a particle filter. We refer the reader to the Appendix for further details.

While natural, this approach requires the specification and estimation of the multivariate distribution $M$ and the arrival correlation matrix $(\lambda_{n,n'})_{1 \leq n,n' \leq N}$. In a general formulation, the number of parameter therefore grows at least as fast as a quadratic function of $N$. Like other specifications such as multivariate GARCH, the model is computationally expensive for a large number of assets. We now propose an overlapping class of models that is based on the same principles as multivariate MSM and remains tractable with many assets.

### 6.2. Factor MSM

A factor model for stochastic volatility gives a parsimonious multifrequency specification. Consider $L$ volatility factors $F_t^\ell = (F_{k,t}^\ell)_{1 \leq k \leq \bar{k}} \in \mathbb{R}_+^{\bar{k}}$, which can jointly affect all currencies. Each vector $F_t^\ell$ contains $\bar{k}$ frequency-specific components and follows a specific univariate MSM process with parameters $(b, \gamma_{\bar{k}}, m_0^\ell)$. The volatility of each currency $n$ is also affected by an idiosyncratic shock $E_t^n$, which is specified by parameters $(b, \gamma_{\bar{k}}, m_0^{L+n})$. Draws of the factors $F_{k,t}^\ell$ and idiosyncratic shocks $E_{k,t}^n$ are independent, but the timing of arrivals may be correlated. Factors and idiosyncratic components thus follow univariate MSM with identical frequencies.

For every currency $n \in \{1, ..., N\}$, the volatility component $M_{k,t}^n$ is the weighted product of the factors and idiosyncratic shock of same frequency:

$$M_{k,t}^n = C_n \left(F_{k,t}^1\right)^{w_1^n} ... (F_{k,t}^L)^{w_L^n} \quad (E_{k,t}^n)^{w_{L+1}^n}.$$

The weights are non-negative and add up to one. The constant $C_n$ is chosen to guarantee that $\mathbb{E}M_{kt}^n = 1$, and is thus not a free parameter.[17] In logarithms, we obtain the more familiar expression:

$$\ln M_{kt}^n = \ln C_n + \sum_{\ell=1}^{L} w_\ell^n \ln F_{kt}^\ell + w_{L+1}^n \ln E_{kt}^n$$

Returns are then defined as previously as $x_t = (M_{1t} * ... * M_{\bar{k}t})^{1/2} * \varepsilon_t$, where $\varepsilon_t$ is a centered multivariate Gaussian noise: $\varepsilon_t \sim N(0, \Sigma)$. We show in the Appendix that as with multivariate GARCH, the estimation of $\Sigma$ can be carried out directly from sample autocorrelations of the series.

Two special cases of this setup are of particular interest. First, when arrivals for all factors and idiosyncratic components are simultaneous, factor MSM is a special case of the multivariate MSM in the previous subsection. New draws of $M_{kt}^n$ are then independent of all past multipliers, and the factor model generates univariate series that are consistent with univariate MSM. Further, when the distribution of factors and idiosyncratic shocks is lognormal, the resulting multipliers $M_{kt}^n$ are lognormal as well. This is convenient as we know that lognormal multipliers fit well the moment-scaling properties of financial series, including exchange rates (Calvet and Fisher, 2002). Stochastic volatility is now fully specified by: (1) the frequency parameters $b$ and $\gamma_{\bar{k}}$; (2) the distribution parameters of factors and idiosyncratic shocks $(m_0^1, .., m_0^{L+N})$; and (3) the factor loadings $w^n = (w_1^n, .., w_L^n)$ of each asset. The model is thus defined by $N(L+1) + L + 2$ volatility parameters.

The second interesting special case is when arrivals of factors and idiosyncratic shocks are independent. It is easy to verify that this specification has the same number of parameters as when arrivals are simultaneous. Further, this choice permits that at time $t$ some but not all factors and idiosyncratic components may change. The univariate volatility components $M_{kt}^n$ then takes a new value without requiring a completely independent draw from $M$. Thus, the implied univariate volatility dynamics are smoother than standard MSM, but can generate the same thick tails and long-memory volatility persistence. These specifications are thus both practical to implement and deserving of further empirical investigation.

## 7. Conclusion

This paper uses the Markov-Switching Multifractal (MSM) of Calvet and Fisher (2001, 2004) to implement a univariate frequency decomposition of volatility in several exchange rate series. We find that the estimated components are generally difficult to

[17] We thus have $C_n = 1/\mathbb{E}[\left(F_{k,t}^1\right)^{w_1^n}]...\mathbb{E}[(F_{k,t}^L)^{w_L^n}] \quad \mathbb{E}[(E_{k,t}^n)^{w_{L+1}^n}]$. This computation is particularly easy when the marginal distribution of the shocks are multinomial or lognormal.

relate to standard macroeconomic variables. Low frequency volatility components from all currencies covary positively with oil and gold prices, suggesting that these commodities may act as proxies for global economic risk. Relative to previous measures of volatility, the component decomposition increases the strength and cross-sectional robustness of our results. At the same time, our analysis encourages the econometrician to be cautious since the source of covariation is limited to low frequencies.

We identify strong patterns in volatility comovement between currencies. Across exchange-rate pairs, volatility components tend to have high correlation when their durations are similar and low correlations otherwise. This motivates our development of bivariate MSM, a multifrequency model of comovement in stochastic volatility and covariation in financial prices. The model permits a parsimonious specification of bivariate shocks with heterogeneous durations, capturing the economic intuition that shared fundamentals may have different innovation frequencies. We show that Bayesian updating and the likelihood function are always available in closed-form, but are practical to implement only when the state space is of moderate size. We therefore develop a particle filter suitable for larger state spaces, and show its good performance in inference and forecasting. We estimate bivariate MSM on three exchange rate pairs, and show that it performs well in- and out-of-sample relative to a standard benchmark model, CC-GARCH. We also use likelihood ratio tests to confirm some of the principle restrictions of the model. We conduct inference and forecasting for good performing pure regime-switching models with $2^{16}$ states and only eight parameters.

The methods developed in this paper open a number of new directions for future research. First, MSM offers an economically appealing and computationally tractable alternative to previous multivariate GARCH and stochastic volatility models. Comparisons of the approaches in different applications can and should be developed. Additionally, the particle filter methodology opens new frontiers for conducting estimation and inference in MSM processes with very large state spaces. The particle filter also permits examination of processes where the volatility component distribution takes values on a continuous support. Earlier work suggests that lognormal distributions might be particularly appealing. This specification becomes computationally accessible with the particle filter, and can be compared to the binomial specification used in this paper and earlier research. Finally, we propose in this paper a multifrequency factor structure appropriate for multivariate settings with potentially large numbers of assets. This appears promising for future empirical research.

## 8. Appendix

### 8.1. Switching Vector

The probability of a simultaneous switch is $\mathbb{P}(1^{\alpha}_{k,t} = 1^{\beta}_{k,t} = 1) = \mathbb{P}(1^{\alpha}_{k,t} = 1)\mathbb{P}(1^{\beta}_{k,t} = 1|1^{\alpha}_{k,t} = 1)$. Overall, the vector $1_{k,t}$ has joint distribution

| | Arrival on $\beta$ | No arrival on $\beta$ |
|---|---|---|
| Arrival on $\alpha$ | $\gamma_k[(1-\lambda)\gamma_k + \lambda]$ | $\gamma_k(1-\lambda)(1-\gamma_k)$ |
| No arrival on $\alpha$ | $\gamma_k(1-\gamma_k)(1-\lambda)$ | $(1-\gamma_k)[1-\gamma_k(1-\lambda)]$ |

### 8.2. Ergodic Distribution

The bivariate process $(M^{\alpha}_{k,t}, M^{\beta}_{k,t})$ can take values $s^1 = (H,H)$, $s^2 = (H,L)$, $s^3 = (L,H)$ and $s^4 = (L,L)$. The transition matrix is $T = (t_{ij})$, where $t_{ij} = \mathbb{P}(s_{t+1} = s^j | s_t = s^i)$. It satisfies

$$T = \begin{bmatrix} p_k & 1-\frac{\gamma_k}{2}-p_k & 1-\frac{\gamma_k}{2}-p_k & \gamma_k - 1 + p_k \\ 1-\frac{\gamma_k}{2}-q_k & q_k & \gamma_k - 1 + q_k & 1-\frac{\gamma_k}{2}-q_k \\ 1-\frac{\gamma_k}{2}-q_k & \gamma_k - 1 + q_k & q_k & 1-\frac{\gamma_k}{2}-q_k \\ \gamma_k - 1 + p_k & 1-\frac{\gamma_k}{2}-p_k & 1-\frac{\gamma_k}{2}-p_k & p_k \end{bmatrix},$$

where

$$\begin{aligned} p_k &= 1-\gamma_k + \gamma_k[(1-\lambda)\gamma_k + \lambda]\frac{1+\rho^*_m}{4}, \\ q_k &= 1-\gamma_k + \gamma_k[(1-\lambda)\gamma_k + \lambda]\frac{1-\rho^*_m}{4}. \end{aligned}$$

Simple manipulation implies that the characteristic polynomial of $T$ is

$$P_T(x) = (1-x)(1-\gamma_k - x)^2[2(p_k + q_k + \gamma_k) - 3 - x].$$

We easily check that $|2(p_k + q_k + \gamma_k) - 3| < 1$ and infer that $T$ has a unique ergodic distribution $\bar{\Pi}_k = (\bar{\Pi}^{HH}_k, \bar{\Pi}^{HL}_k, \bar{\Pi}^{LH}_k, \bar{\Pi}^{LL}_k)$. The symmetry of the transition matrix implies that $\bar{\Pi}^{HH}_k = \bar{\Pi}^{LL}_k$ and $\bar{\Pi}^{HL}_k = \bar{\Pi}^{LH}_k$. We easily check that $\bar{\Pi}^{HH}_k = \frac{1}{4}\frac{2-2q_k-\gamma_k}{2-(p_k+q_k)-\gamma_k}$ or equivalently

$$\bar{\Pi}^{HH}_k = \frac{1}{4}\,\frac{1-(1-\rho^*_m)[(1-\lambda)\gamma_k + \lambda]/2}{1-[(1-\lambda)\gamma_k + \lambda]/2},$$

and finally note that $\bar{\Pi}^{HL}_k = 1/2 - \bar{\Pi}^{HH}_k$.

## 8.3. Particle Filter

As discussed in the main text, the vectors $\hat{M}_{t+1}^{(1)}$, ..., $\hat{M}_{t+1}^{(B)}$ are independent draws from the probability distribution $h(m) \equiv \mathbb{P}(M_{t+1} = m|X_t)$. Consider a continuous function $Y$ defined on $\mathbb{R}_+^{\bar{k}}$ and taking values on the real line. The conditional expectation $\mathbb{E}[Y(M_{t+1})|X_{t+1}] = \sum_{j=1}^d \mathbb{P}(M_{t+1} = m^j|X_{t+1})\, Y(m^j)$ is conveniently rewritten as

$$\mathbb{E}[Y(M_{t+1})|X_{t+1}] = \sum_{j=1}^{d} h(m^j) \frac{\mathbb{P}(M_{t+1} = m^j|X_{t+1})}{h(m^j)} Y(m^j)\,.$$

The Monte Carlo approximation to this integral is

$$\mathbb{E}[Y(M_{t+1})|X_{t+1}] \approx \frac{1}{B} \sum_{b=1}^{B} \frac{\mathbb{P}(M_{t+1} = \hat{M}_{t+1}^{(b)}|X_{t+1})}{h(\hat{M}_{t+1}^{(b)})} Y(\hat{M}_{t+1}^{(b)}).$$

Bayes' rule implies

$$\frac{\mathbb{P}(M_{t+1} = \hat{M}_{t+1}^{(b)}|X_{t+1})}{B\, h(\hat{M}_{t+1}^{(b)})} = \frac{f_{x_{t+1}}(x_{t+1}|M_{t+1} = \hat{M}_{t+1}^{(b)})}{B\, f_{x_{t+1}}\,(x_{t+1}|X_t)}.$$

Since $f_{x_{t+1}}\,(x_{t+1}|X_t) \approx \frac{1}{B}\sum_{b'=1}^{B} f_{x_{t+1}}(x_{t+1}|\hat{M}_{t+1}^{(b')})$, we infer that

$$\frac{\mathbb{P}(M_{t+1} = \hat{M}_{t+1}^{(b)}|X_{t+1})}{B\, h(\hat{M}_{t+1}^{(b)})} \approx \frac{f_{x_{t+1}}(x_{t+1}|M_{t+1} = \hat{M}_{t+1}^{(b)})}{\sum_{b'=1}^{B} f_{x_{t+1}}(x_{t+1}|M_{t+1} = \hat{M}_{t+1}^{(b')})}.$$

The right-hand side defines a probability $\mu_b$ for every $b \in \{1, .., B\}$. We infer that the random variable $Y(M_{t+1})$ has conditional expectation $\mathbb{E}[Y(M_{t+1})|X_{t+1}] \approx \sum_{b=1}^{B} \mu_b\, Y(\hat{M}_{t+1}^{(b)})$. Since this result is valid for any function $Y$, we conclude that $\Pi_{t+1}$ can be approximated with a discrete distribution taking on the value $\hat{M}_{t+1}^{(b)}$ with probability $\mu_b$.

## 8.4. Two-Step Estimation

We partition the parameter vector into $\psi \equiv (\psi_1', \psi_2')'$, with $\psi_1 = (\sigma_\alpha, \sigma_\beta, m_0^\alpha, m_0^\beta, b, \gamma_{\bar{k}})'$ and $\psi_2 = (\rho_\varepsilon, \lambda)'$. In the first step, we compute the vector $\hat{\psi}_1$ that maximizes the combined univariate likelihood $L(x_t^\alpha; m_0^\alpha, \sigma_\alpha, b, \gamma_{\bar{k}}) + L(x_t^\beta; m_0^\beta, \sigma_\beta, b, \gamma_{\bar{k}})$. Note that $\hat{\psi}_1$ is a GMM estimator based on the moment conditions $\partial L(x_t^\alpha)/\partial\psi_1 + \partial L(x_t^\beta)/\partial\psi_1$. Under correct specification the expectation of each derivative is zero, which implies consistency and asymptotic normality of $\hat{\psi}_1$. In the second step, we estimate $\psi_2$ by maximizing the simulated bivariate likelihood $L(x_t^\alpha, x_t^\beta; \hat{\psi}_1, \psi_2)$ given the first stage estimate $\hat{\psi}_1$. The simulated likelihood is computed using the particle filter with $B = 10,000$ draws.

Standard errors for the two-step estimates are obtained by restating the algorithm as a GMM estimator based on the moment conditions $T^{-1}\sum_{t=1}^{T} g_t(\widehat{\psi}) = 0$, where

$g_t(\psi)$ is the column vector with components $\partial[\ln f(x_t^\alpha|X_{t-1}^\alpha) + \ln f(x_t^\beta|X_{t-1}^\beta)]/\partial\psi_1$ and $\partial \ln f(x_t^\alpha, x_t^\beta|X_{t-1}^\alpha, X_{t-1}^\beta)/\partial\psi_2$. Standard GMM arguments imply asymptotic normality

$$\sqrt{T}(\widehat{\psi} - \psi_0) \xrightarrow{d} \mathcal{N}[0, H^{-1}V\left(H'\right)^{-1}]$$

with $H = -\mathbb{E}\partial g_t(\psi_0)/\partial\psi'$ and $V = \text{Var}\left[T^{-1/2}\sum g_t(\psi_0)\right]$. To estimate $V$, we approximate $g_t$ by taking finite difference derivatives of the objective function. Then we estimate $V$ using the formula of Newey and West (1987) with 10 lags. When calculating finite difference derivatives using the particle filter, we use $15,000$ simulations. We estimate $H$ by calculating the sample variance of the first derivatives:

$$\widehat{H} = \begin{pmatrix} \widehat{H}_{1,1} + \widehat{H}_{1,2} & 0 \\ \widehat{H}_{2,1} & \widehat{H}_{2,2} \end{pmatrix},$$

where $\widehat{H}_{1,1}$ and $\widehat{H}_{1,2}$ are the $6 \times 6$ matrices

$$\begin{aligned} \widehat{H}_{1,1} &= T^{-1}\sum \frac{\partial \ln f(x_t^\alpha|X_{t-1}^\alpha)}{\partial\psi_1}\frac{\partial \ln f(x_t^\alpha|X_{t-1}^\alpha)}{\partial\psi_1'} \approx -\mathbb{E}\left[\frac{\partial^2 \ln f(x_t^\alpha|X_{t-1}^\alpha)}{\partial\psi_1\partial\psi_1'}\right] \\ \widehat{H}_{1,1} &= T^{-1}\sum \frac{\partial \ln f(x_t^\beta|X_{t-1}^\beta)}{\partial\psi_1}\frac{\partial \ln f(x_t^\beta|X_{t-1}^\beta)}{\partial\psi_1'} \approx -\mathbb{E}\left[\frac{\partial^2 \ln f(x_t^\beta|X_{t-1}^\beta)}{\partial\psi_1\partial\psi_1'}\right]. \end{aligned}$$

Similarly, $(\widehat{H}_{21}, \widehat{H}_{22})$ are the bottom two rows of the $8 \times 8$ matrix

$$T^{-1}\sum \frac{\partial \ln f(x_t^\alpha, x_t^\beta|X_{t-1}^\alpha, X_{t-1}^\beta)}{\partial\psi}\frac{\partial \ln f(x_t^\alpha, x_t^\beta|X_{t-1}^\alpha, X_{t-1}^\beta)}{\partial\psi'}.$$

The matrix $\widehat{H}$ is consistent since its elements are second derivatives of the univariate or bivariate likelihoods.

### 8.5. VaR Forecasts

We use the particle filter to calculate the VaR implied by MSM. The algorithm in section 3.4 is used to generate volatility draws $M_t^{(1)}$, ..., $M_t^{(B)}$ from the distribution $\Pi_t$. For each draw $M_t^{(b)}$, we simulate the bivariate series forward $n$ days to obtain $B$ draws from the cumulative return on the portfolio. We then estimate $VaR_{t,n}(p)$ as the $1 - p^{th}$ empirical quantile.

For 1 day forecasts CC-GARCH provides a closed form expression for value at risk, namely $VaR_{t,1}(p) = -Q_{1-p}\,\sigma_{t|t-1}$, where $Q_{1-p}$ is the $(1-p)^{th}$ quantile of a standard normal variable and $\sigma_{t|t-1}$ is the standard deviation implied by CC-GARCH. The 5-day CC-GARCH forecasts are calculated by simulation. In all cases we use $B = 10000$ simulated draws.

### 8.6. Inference in the Multivariate Model

For either multivariate MSM or factor MSM, we seek to estimate the covariance matrix $\Sigma$ and the vector of volatility parameters $\psi$. One possibility is to choose a tight specification for $\Sigma$ and use the particle filter to optimize the simulated likelihood over all parameters. For example, our bivariate estimates show that currency pairs with strong volatility comovement also have high correlation in innovations. This suggests using the same factor structure that controls volatility to parsimoniously specify $\Sigma$.

In the general case, estimation can be conducted in two steps: (1) Estimate the covariance matrix of the Gaussian noises; (2) Use the particle filter to estimate the volatility parameters $\psi$ by simulated maximum likelihood. Step (2) is straightforward, and step (1) can be conducted as follows. For any two assets $n$ and $p$, we know that

$$\mathbb{E}[x_t^{(n)} x_t^{(p)}] = \Gamma_{n,p} \mathbb{E}[\varepsilon_t^{(n)} \varepsilon_t^{(p)}] \text{ and } \mathbb{E}|x_t^{(n)} x_t^{(p)}| = \Gamma_{n,p} \mathbb{E}|\varepsilon_t^{(n)} \varepsilon_t^{(p)}|,$$

where $\Gamma_{n,p} = \prod_{k=1}^{k} \mathbb{E}\{[M_{kt}^n M_{kt}^p]^{1/2}\}$. We infer

$$\frac{\sum_t x_t^{(n)} x_t^{(p)}}{\sum_t |x_t^{(n)} x_t^{(p)}|} \overset{a.s.}{\rightarrow} \frac{\mathbb{E}[\varepsilon_t^{(n)} \varepsilon_t^{(p)}]}{\mathbb{E}|\varepsilon_t^{(n)} \varepsilon_t^{(p)}|} = \varphi(\rho_{n,p}),$$

where $\rho_{n,p} = Corr[\varepsilon_t^{(n)}; \varepsilon_t^{(p)}]$ and $\varphi(\rho) \equiv \frac{\pi}{2} \frac{\rho}{\sqrt{1-\rho^2} + \rho \arcsin \rho}$. The function $\varphi$ is strictly increasing and maps $[-1, 1]$ onto $[-1, 1]$. A consistent estimator of the correlation coefficient is therefore

$$\hat{\rho}_{n,p} = \varphi^{-1} \left( \frac{\sum_t x_t^{(n)} x_t^{(p)}}{\sum_t |x_t^{(n)} x_t^{(p)}|} \right).$$

The variance of the Gaussians is consistently estimated by $\hat{\sigma}_n^2 = \frac{1}{T} \sum_t [x_t^{(n)}]^2$. The covariance matrix defined by $\hat{\sigma}_n^2$ and $\hat{\rho}_{n,p}$ may not be positive-definite. We thus apply the methodology of Ledoit, Santa-Clara and Wolf (2003) to obtain a positive semi-definite matrix $\hat{\Sigma}$.

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TABLE 1. – UNIVARIATE MLE

| | $\bar{k}=1$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| | | | | *Deutsche Mark* | | | | |
| $\hat{m}_0$ | 1.617 | 1.556 | 1.535 | 1.472 | 1.445 | 1.396 | 1.365 | 1.338 |
| | (0.019) | (0.015) | (0.012) | (0.012) | (0.013) | (0.012) | (0.011) | (0.011) |
| $\hat{\sigma}$ | 0.672 | 0.649 | 0.594 | 0.567 | 0.504 | 0.537 | 0.549 | 0.552 |
| | (0.012) | (0.017) | (0.013) | (0.015) | (0.016) | (0.027) | (0.020) | (0.021) |
| $\hat{\gamma}_{\bar{k}}$ | 0.074 | 0.086 | 0.841 | 0.779 | 0.812 | 0.909 | 0.979 | 0.998 |
| | (0.002) | (0.018) | (0.096) | (0.082) | (0.083) | (0.103) | (0.036) | (0.008) |
| $\hat{b}$ | - | 6.85 | 34.31 | 11.86 | 9.02 | 5.83 | 4.67 | 3.82 |
| | | ( 2.44) | ( 10.55) | ( 1.99) | ( 1.24) | ( 0.82) | ( 0.60) | ( 0.49) |
| ln $L$ | -7121.92 | -6975.92 | -6916.81 | -6900.06 | -6891.67 | -6888.91 | -6885.60 | -6885.90 |
| | | | | *Japanese Yen* | | | | |
| $\hat{m}_0$ | 1.783 | 1.774 | 1.688 | 1.644 | 1.579 | 1.567 | 1.559 | 1.508 |
| | (0.011) | (0.009) | (0.011) | (0.011) | (0.010) | (0.010) | (0.010) | (0.010) |
| $\hat{\sigma}$ | 0.632 | 0.537 | 0.568 | 0.473 | 0.473 | 0.634 | 0.514 | 0.508 |
| | (0.011) | (0.009) | (0.019) | (0.017) | (0.023) | (0.023) | (0.019) | (0.017) |
| $\hat{\gamma}_{\bar{k}}$ | 0.208 | 0.358 | 0.276 | 0.713 | 0.861 | 0.894 | 0.894 | 0.977 |
| | (0.022) | (0.038) | (0.048) | (0.082) | (0.053) | (0.060) | (0.058) | (0.030) |
| $\hat{b}$ | - | 147.47 | 11.76 | 15.73 | 9.13 | 8.22 | 7.60 | 5.88 |
| | | ( 59.61) | ( 2.02) | ( 2.67) | ( 1.18) | ( 0.99) | ( 0.87) | ( 0.74) |
| ln $L$ | -6776.19 | -6421.01 | -6279.02 | -6216.85 | -6196.55 | -6184.90 | -6181.29 | -6174.96 |
| | | | | *British Pound* | | | | |
| $\hat{m}_0$ | 1.708 | 1.666 | 1.640 | 1.612 | 1.574 | 1.529 | 1.498 | 1.457 |
| | (0.013) | (0.013) | (0.011) | (0.014) | (0.011) | (0.012) | (0.011) | (0.010) |
| $\hat{\sigma}$ | 0.606 | 0.580 | 0.523 | 0.516 | 0.431 | 0.455 | 0.385 | 0.380 |
| | (0.009) | (0.018) | (0.018) | (0.016) | (0.015) | (0.017) | (0.013) | (0.014) |
| $\hat{\gamma}_{\bar{k}}$ | 0.113 | 0.213 | 0.271 | 0.549 | 0.617 | 0.782 | 0.817 | 0.959 |
| | (0.016) | (0.036) | (0.065) | (0.086) | (0.074) | (0.078) | (0.083) | (0.001) |
| $\hat{b}$ | - | 18.69 | 13.92 | 14.39 | 11.59 | 8.49 | 6.83 | 5.33 |
| | | ( 4.84) | ( 2.68) | ( 2.67) | ( 1.84) | ( 1.16) | ( 0.87) | ( 0.04) |
| ln $L$ | -6220.55 | -5987.37 | -5882.60 | -5826.92 | -5792.97 | -5778.58 | -5771.92 | -5770.20 |

*Notes:* This table shows maximum likelihood estimation results for binomial MSM. Columns correspond to the number $\bar{k}$ of volatility components. Asymptotic standard errors are in parenthesis.

TABLE 2. – CORRELATION OF SMOOTHED UNIVARIATE VOLATILITY COMPONENT BELIEFS

| | DM1 | DM2 | DM3 | DM4 | DM5 | DM6 | DM7 | DM8 | $\lvert x_{\mathrm{DM}}\rvert$ | $x^2_{\mathrm{DM}}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| DM1 | 1.000 | 0.762 | 0.377 | 0.174 | 0.099 | 0.066 | 0.031 | 0.020 | 0.189 | 0.115 |
| DM2 | 0.762 | 1.000 | 0.600 | 0.328 | 0.151 | 0.093 | 0.043 | 0.028 | 0.255 | 0.174 |
| DM3 | 0.377 | 0.600 | 1.000 | 0.603 | 0.307 | 0.168 | 0.077 | 0.052 | 0.312 | 0.245 |
| DM4 | 0.174 | 0.328 | 0.603 | 1.000 | 0.738 | 0.432 | 0.201 | 0.137 | 0.374 | 0.295 |
| DM5 | 0.099 | 0.151 | 0.307 | 0.738 | 1.000 | 0.792 | 0.420 | 0.297 | 0.463 | 0.373 |
| DM6 | 0.066 | 0.093 | 0.168 | 0.432 | 0.792 | 1.000 | 0.770 | 0.610 | 0.667 | 0.539 |
| DM7 | 0.031 | 0.043 | 0.077 | 0.201 | 0.420 | 0.770 | 1.000 | 0.961 | 0.887 | 0.713 |
| DM8 | 0.020 | 0.028 | 0.052 | 0.137 | 0.297 | 0.610 | 0.961 | 1.000 | 0.894 | 0.716 |
| $\lvert x_{\mathrm{DM}}\rvert$ | 0.189 | 0.255 | 0.312 | 0.374 | 0.463 | 0.667 | 0.887 | 0.894 | 1.000 | 0.872 |
| $x^2_{\mathrm{DM}}$ | 0.115 | 0.174 | 0.245 | 0.295 | 0.373 | 0.539 | 0.713 | 0.716 | 0.872 | 1.000 |
| JA1 | 0.590 | 0.287 | 0.036 | 0.020 | 0.036 | 0.032 | 0.012 | 0.007 | 0.051 | 0.002 |
| JA2 | 0.611 | 0.302 | 0.048 | 0.023 | 0.038 | 0.034 | 0.013 | 0.008 | 0.056 | 0.006 |
| JA3 | 0.788 | 0.440 | 0.172 | 0.063 | 0.065 | 0.048 | 0.021 | 0.013 | 0.104 | 0.048 |
| JA4 | 0.368 | 0.185 | 0.162 | 0.064 | 0.030 | 0.036 | 0.020 | 0.013 | 0.073 | 0.062 |
| JA5 | 0.157 | 0.177 | 0.150 | 0.231 | 0.169 | 0.109 | 0.053 | 0.036 | 0.103 | 0.084 |
| JA6 | 0.058 | 0.062 | 0.127 | 0.279 | 0.349 | 0.284 | 0.155 | 0.111 | 0.192 | 0.174 |
| JA7 | 0.029 | 0.023 | 0.032 | 0.106 | 0.206 | 0.321 | 0.312 | 0.267 | 0.284 | 0.258 |
| JA8 | 0.012 | 0.008 | 0.011 | 0.043 | 0.095 | 0.209 | 0.339 | 0.353 | 0.333 | 0.303 |
| $\lvert x_{\mathrm{JA}}\rvert$ | 0.187 | 0.108 | 0.092 | 0.134 | 0.177 | 0.256 | 0.328 | 0.327 | 0.363 | 0.346 |
| $x^2_{\mathrm{JA}}$ | 0.091 | 0.048 | 0.065 | 0.101 | 0.142 | 0.209 | 0.261 | 0.256 | 0.297 | 0.344 |
| UK1 | 0.819 | 0.525 | 0.170 | 0.081 | 0.052 | 0.042 | 0.018 | 0.011 | 0.114 | 0.054 |
| UK2 | 0.730 | 0.558 | 0.246 | 0.165 | 0.094 | 0.062 | 0.028 | 0.018 | 0.134 | 0.073 |
| UK3 | 0.464 | 0.526 | 0.254 | 0.163 | 0.072 | 0.050 | 0.023 | 0.015 | 0.128 | 0.086 |
| UK4 | 0.251 | 0.505 | 0.308 | 0.195 | 0.075 | 0.049 | 0.021 | 0.014 | 0.143 | 0.111 |
| UK5 | 0.070 | 0.131 | 0.516 | 0.571 | 0.365 | 0.200 | 0.091 | 0.062 | 0.213 | 0.169 |
| UK6 | 0.149 | 0.162 | 0.239 | 0.440 | 0.536 | 0.423 | 0.228 | 0.162 | 0.291 | 0.242 |
| UK7 | 0.082 | 0.079 | 0.092 | 0.185 | 0.319 | 0.463 | 0.431 | 0.362 | 0.407 | 0.354 |
| UK8 | 0.030 | 0.030 | 0.035 | 0.074 | 0.145 | 0.301 | 0.473 | 0.488 | 0.471 | 0.409 |
| $\lvert x_{\mathrm{UK}}\rvert$ | 0.168 | 0.213 | 0.221 | 0.254 | 0.273 | 0.360 | 0.462 | 0.462 | 0.564 | 0.524 |
| $x^2_{\mathrm{UK}}$ | 0.081 | 0.135 | 0.178 | 0.213 | 0.234 | 0.305 | 0.390 | 0.391 | 0.508 | 0.571 |

*Notes:* This table shows correlations from a frequency decomposition of binomial MSM with $\bar{k} = 8$ components for the univariate DM series with itself, JA, and UK. For each series, the smoothed probabilities $\hat{M}_{k,t} = \mathbb{E}(M_{k,t}|x_1, ..., x_T)$ of different volatility states are calculated. For convenience, we denote these probabilities by DM1,...,DM8,JA1,...,JA8,UK1,...,UK8. The table then shows correlations of the DM decomposition with decompositions from all three series. Correlations are generally strongest near the diagonal where components have similar indices.

TABLE 3. – COMBINED UNIVARIATE RESULTS

A. MLE Estimation

| | $\bar{k}=1$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| | | | | *DM-JA Parameter Estimates* | | | | |
| $\hat{m}_0^{DM}$ | 1.643 | 1.618 | 1.515 | 1.474 | 1.445 | 1.405 | 1.397 | 1.367 |
| | (0.013) | (0.014) | (0.013) | (0.013) | (0.011) | (0.013) | (0.012) | (0.011) |
| $\hat{m}_0^{JA}$ | 1.775 | 1.757 | 1.687 | 1.638 | 1.578 | 1.565 | 1.522 | 1.488 |
| | (0.018) | (0.010) | (0.010) | (0.011) | (0.010) | (0.010) | (0.010) | (0.014) |
| $\hat{\sigma}_{DM}$ | 0.669 | 0.577 | 0.597 | 0.569 | 0.504 | 0.565 | 0.449 | 0.472 |
| | (0.010) | (0.011) | (0.015) | (0.018) | (0.015) | (0.021) | (0.018) | (0.018) |
| $\hat{\sigma}_{JA}$ | 0.613 | 0.544 | 0.565 | 0.487 | 0.476 | 0.632 | 0.384 | 0.532 |
| | (0.016) | (0.010) | (0.018) | (0.018) | (0.023) | (0.024) | (0.016) | (0.027) |
| $\hat{\gamma}_{\bar{k}}$ | 0.129 | 0.257 | 0.301 | 0.756 | 0.844 | 0.872 | 0.959 | 0.982 |
| | (0.013) | (0.026) | (0.068) | (0.057) | (0.055) | (0.054) | (0.036) | (0.022) |
| $\hat{b}$ | - | 69.57 | 11.97 | 13.21 | 9.14 | 7.16 | 6.16 | 4.93 |
| | | ( 21.57) | ( 2.04) | ( 1.61) | ( 0.95) | ( 0.65) | ( 0.57) | ( 0.45) |
| $\ln L$ | -13913.86 | -13424.24 | -13203.13 | -13119.07 | -13088.39 | -13077.02 | -13072.22 | -13063.11 |
| | | | | *LR Tests Against Unrestricted Univariate* | | | | |
| DM-JA | 0.000 | 0.000 | 0.026 | 0.341 | 0.922 | 0.201 | 0.069 | 0.326 |
| DM-UK | 0.369 | 0.049 | 0.021 | 0.192 | 0.062 | 0.062 | 0.065 | 0.334 |
| JA-UK | 0.060 | 0.003 | 0.837 | 0.601 | 0.132 | 0.487 | 0.869 | 0.929 |

B. Correlation of Smoothed Volatility Component Beliefs

| | DM1 | DM2 | DM3 | DM4 | DM5 | DM6 | DM7 | DM8 | $\|x_{\text{DM}}\|$ | $x^2_{\text{DM}}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| JA1 | 0.628 | 0.714 | 0.349 | 0.072 | 0.009 | 0.038 | 0.020 | 0.007 | 0.081 | 0.028 |
| JA2 | 0.690 | 0.774 | 0.405 | 0.135 | 0.016 | 0.048 | 0.027 | 0.011 | 0.101 | 0.043 |
| JA3 | 0.595 | 0.686 | 0.228 | 0.140 | 0.018 | 0.049 | 0.028 | 0.012 | 0.078 | 0.033 |
| JA4 | 0.306 | 0.234 | 0.147 | 0.114 | 0.036 | 0.022 | 0.027 | 0.013 | 0.065 | 0.065 |
| JA5 | -0.019 | -0.034 | 0.052 | 0.113 | 0.302 | 0.227 | 0.116 | 0.056 | 0.117 | 0.102 |
| JA6 | 0.028 | 0.023 | 0.040 | 0.084 | 0.255 | 0.352 | 0.258 | 0.145 | 0.206 | 0.186 |
| JA7 | 0.008 | 0.009 | 0.007 | 0.021 | 0.103 | 0.224 | 0.342 | 0.294 | 0.299 | 0.274 |
| JA8 | 0.004 | 0.004 | 0.002 | 0.009 | 0.048 | 0.123 | 0.287 | 0.353 | 0.333 | 0.304 |
| $\|x_{\text{JA}}\|$ | 0.177 | 0.191 | 0.093 | 0.088 | 0.128 | 0.193 | 0.301 | 0.326 | 0.363 | 0.346 |
| $x^2_{\text{JA}}$ | 0.087 | 0.094 | 0.039 | 0.066 | 0.100 | 0.156 | 0.243 | 0.254 | 0.297 | 0.344 |

*Notes:* Panel A shows maximum likelihood estimation results for the combined univariate model, which for two series $\alpha$ and $\beta$ has likelihood $L(x_t^\alpha; m_0^\alpha, \sigma_\alpha, b, \gamma_{\bar{k}}) + L(x_t^\beta; m_0^\beta, \sigma_\beta, b, \gamma_{\bar{k}})$. This corresponds to the likelihood of two statistically independent univariate MSM processes constrained to have the same frequency parameters $b$ and $\gamma_{\bar{k}}$ . Columns correspond to the number of frequencies $\bar{k}$ in the estimated model, and estimation results with asymptotic standard errors in parentheses are presented for the DM-JA currency pair only. The second part of Panel A shows $p$-values from a likelihood ratio test of the combined univariate against two unrestricted independent MSM processes. A low $p$-value indicates that the restrictions imposed by assuming the frequency parameters to be identical across currencies are rejected. Panel B then shows correlations from a frequency decomposition of the DM-JA combined univariate model with eight components. For each series, the smoothed probabilities $\hat{M}_{k,t} = \mathbb{E}(M_{k,t}|x_1, ..., x_T)$ of volatility states are calculated. For convenience, we denote these probabilities by DM1,...,DM8,JA1,...,JA8. Correlations are strongest near the diagonal where components have similar frequencies, and these results are strengthened relative to Table 2 where frequency restrictions are not enforced.

TABLE 4. – CORRELATION OF EXCHANGE RATES WITH OTHER FINANCIAL PRICES

| | $x_t$ | $\|x_t\|$ | RV | MSM Volatility Component Beliefs $k=1$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | *Deutsche Mark* | | | | | |
| CRSP RV | -0.0849 | 0.1712 | 0.0904 | 0.0930 | 0.0360 | 0.0316 | 0.1964 | 0.1421 | 0.0678 | -0.0056 | -0.0479 |
| DAX RV | -0.1515 | 0.0916 | 0.1205 | 0.2087 | 0.1296 | 0.1086 | 0.1541 | 0.1041 | 0.0111 | -0.0363 | -0.0395 |
| Oil | 0.1142 | 0.0235 | 0.2138 | 0.5254 | 0.4859 | 0.2685 | 0.1649 | 0.0241 | -0.0178 | -0.1062 | -0.1137 |
| Gold | 0.0774 | 0.0652 | 0.1642 | 0.7378 | 0.4979 | 0.1334 | 0.0571 | -0.0104 | -0.0613 | -0.0755 | -0.0497 |
| | | | | | | *Japanese Yen* | | | | | |
| CRSP RV | -0.0969 | 0.1696 | 0.2097 | -0.1841 | -0.1924 | -0.2137 | -0.1439 | -0.0920 | 0.0011 | -0.0136 | 0.0221 |
| NIKKEI RV | -0.0280 | 0.1155 | 0.2291 | 0.3069 | 0.3013 | 0.2764 | -0.3071 | -0.0965 | -0.0526 | -0.0781 | -0.0400 |
| Oil | 0.0465 | 0.0429 | 0.0046 | 0.2803 | 0.2858 | 0.2838 | 0.1306 | -0.1608 | 0.0335 | -0.0422 | 0.0052 |
| Gold | -0.0135 | 0.1323 | 0.1444 | 0.5233 | 0.5306 | 0.5332 | 0.2071 | 0.0586 | 0.0417 | -0.0633 | -0.1053 |
| | | | | | | *British Pound* | | | | | |
| US CRSP RV | 0.0480 | 0.0590 | -0.0746 | -0.0299 | 0.0907 | 0.0496 | -0.0597 | 0.0291 | 0.0570 | -0.0342 | -0.0405 |
| FTSE RV | 0.0113 | 0.0390 | -0.0986 | -0.4642 | -0.2036 | -0.0865 | -0.1638 | 0.1450 | 0.0405 | -0.0329 | -0.0518 |
| Oil | -0.0900 | 0.0864 | 0.2583 | 0.5495 | 0.3972 | 0.2652 | 0.2846 | 0.2087 | 0.0766 | -0.0175 | -0.0455 |
| Gold | 0.0396 | 0.0837 | 0.1466 | 0.6963 | 0.5087 | 0.4084 | 0.2824 | -0.1019 | 0.0658 | 0.0083 | -0.0243 |

*Notes:* This table investigates for each country the comovement between exchange rates and four financial variables: the monthly realized volatility (RV) on the US and domestic stock market, the oil price (in USD/barrel) and the gold price (in USD/oz). Monthly realized volatilities are imputed as the sum of squared daily returns. Correlation between currency and equity RV is positive for DM and JA, but negative for UK. Oil and gold prices are positively correlated to exchange rate volatility for all countries, and the MSM decomposition reveals that this result is primarily a low-frequency phenomenon.

TABLE 5. – EVALUATION OF PARTICLE FILTER

| | $\ln L$ | $E_t \sum_{j=1}^n x_{t+j}^2$ $n=1$ | 5 | 10 | 20 | 50 |
|---|---|---|---|---|---|---|
| True value | -6885.9 | 0.432 | 2.194 | 4.442 | 8.991 | 22.66 |
| Simulation average | -6887.3 | 0.431 | 2.187 | 4.426 | 8.953 | 22.54 |
| Standard deviation | 1.851 | 0.012 | 0.064 | 0.142 | 0.325 | 0.983 |
| 1% quantile | -6892.1 | 0.405 | 2.036 | 4.076 | 8.103 | 19.79 |
| 25% quantile | -6888.4 | 0.423 | 2.147 | 4.338 | 8.747 | 21.97 |
| 50% quantile | -6887.3 | 0.431 | 2.191 | 4.435 | 8.957 | 22.66 |
| 75% quantile | -6886.2 | 0.439 | 2.231 | 4.525 | 9.179 | 23.23 |
| 99% quantile | -6883.3 | 0.458 | 2.330 | 4.739 | 9.633 | 24.42 |

*Notes:* This table compares values generated by the particle filter with their true values generated by exact Bayesian updating. $\ln L$ is the value of the log-likelihood function for the Deutsche Mark series with $\bar{k} = 8$ evaluated at the maximum likelihood estimates in Table 1. The forecasted variance of the series is denoted $E_t \sum_{j=1}^n x_{t+j}^2$. For each quantity, the table provides the true value along with the average, standard deviation, and quantiles over 500 particle filter approximations using independent sets of random draws. Each approximation uses $B = 10000$ random draws.

TABLE 6. – BIVARIATE MSM ESTIMATES
MAXIMUM LIKELIHOOD

| | $\bar{k}=1$ | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| | | | *DM and JA* | | |
| $\hat{m}_0^{DM}$ | 1.637 | 1.589 | 1.543 | 1.484 | 1.447 |
| | (0.011) | (0.013) | (0.013) | (0.013) | (0.011) |
| $\hat{m}_0^{JA}$ | 1.718 | 1.701 | 1.667 | 1.621 | 1.573 |
| | (0.011) | (0.009) | (0.010) | (0.010) | (0.010) |
| $\hat{\sigma}_{DM}$ | 0.679 | 0.621 | 0.575 | 0.559 | 0.524 |
| | (0.009) | (0.011) | (0.014) | (0.017) | (0.015) |
| $\hat{\sigma}_{JA}$ | 0.683 | 0.649 | 0.577 | 0.573 | 0.509 |
| | (0.011) | (0.014) | (0.017) | (0.018) | (0.024) |
| $\hat{\gamma}_{\bar{k}}$ | 0.122 | 0.217 | 0.732 | 0.828 | 0.905 |
| | (0.013) | (0.022) | (0.066) | (0.049) | (0.038) |
| $\hat{b}$ | - | 16.23 | 23.71 | 13.60 | 8.70 |
| | | ( 3.09) | ( 4.54) | ( 1.48) | ( 0.83) |
| $\hat{\rho}_\epsilon$ | 0.580 | 0.589 | 0.576 | 0.580 | 0.580 |
| | (0.008) | (0.009) | (0.010) | (0.009) | (0.009) |
| $\hat{\lambda}$ | 0.647 | 0.641 | 0.589 | 0.634 | 0.637 |
| | (0.041) | (0.039) | (0.056) | (0.048) | (0.049) |
| ln $L$ | -12519.99 | -12001.70 | -11797.05 | -11688.44 | -11655.80 |
| | | | *DM and UK* | | |
| $\hat{m}_0^{DM}$ | 1.651 | 1.570 | 1.522 | 1.492 | 1.484 |
| | (0.012) | (0.010) | (0.012) | (0.012) | (0.011) |
| $\hat{m}_0^{UK}$ | 1.731 | 1.656 | 1.624 | 1.588 | 1.564 |
| | (0.010) | (0.010) | (0.009) | (0.017) | (0.010) |
| $\hat{\sigma}_{DM}$ | 0.681 | 0.706 | 0.626 | 0.560 | 0.498 |
| | (0.009) | (0.014) | (0.011) | (0.031) | (0.012) |
| $\hat{\sigma}_{UK}$ | 0.629 | 0.658 | 0.573 | 0.506 | 0.458 |
| | (0.009) | (0.015) | (0.012) | (0.042) | (0.015) |
| $\hat{\gamma}_{\bar{k}}$ | 0.227 | 0.422 | 0.746 | 0.791 | 0.864 |
| | (0.021) | (0.052) | (0.057) | (0.067) | (0.040) |
| $\hat{b}$ | - | 13.29 | 15.24 | 11.71 | 10.83 |
| | | ( 2.28) | ( 2.26) | ( 1.68) | ( 1.35) |
| $\hat{\rho}_\epsilon$ | 0.707 | 0.714 | 0.707 | 0.708 | 0.710 |
| | (0.007) | (0.007) | (0.007) | (0.007) | (0.007) |
| $\hat{\lambda}$ | 0.837 | 0.852 | 0.833 | 0.822 | 0.827 |
| | (0.023) | (0.023) | (0.026) | (0.027) | (0.025) |
| ln $L$ | -10894.41 | -10513.18 | -10335.82 | -10270.90 | -10240.51 |
| | | | *JA and UK* | | |
| $\hat{m}_0^{JA}$ | 1.764 | 1.718 | 1.693 | 1.629 | 1.608 |
| | (0.014) | (0.008) | (0.009) | (0.010) | (0.010) |
| $\hat{m}_0^{UK}$ | 1.729 | 1.661 | 1.633 | 1.595 | 1.571 |
| | (0.005) | (0.012) | (0.012) | (0.011) | (0.010) |
| $\hat{\sigma}_{JA}$ | 0.655 | 0.619 | 0.531 | 0.489 | 0.709 |
| | (0.008) | (0.014) | (0.015) | (0.014) | (0.021) |
| $\hat{\sigma}_{UK}$ | 0.603 | 0.578 | 0.514 | 0.474 | 0.385 |
| | (0.006) | (0.012) | (0.018) | (0.011) | (0.009) |
| $\hat{\gamma}_{\bar{k}}$ | 0.219 | 0.304 | 0.449 | 0.748 | 0.791 |
| | (0.011) | (0.027) | (0.054) | (0.046) | (0.043) |
| $\hat{b}$ | - | 21.50 | 15.08 | 13.21 | 11.91 |
| | | ( 4.32) | ( 2.08) | ( 1.43) | ( 1.40) |
| $\hat{\rho}_\epsilon$ | 0.447 | 0.453 | 0.449 | 0.438 | 0.440 |
| | (0.007) | (0.004) | (0.011) | (0.012) | (0.011) |
| $\hat{\lambda}$ | 0.499 | 0.565 | 0.560 | 0.544 | 0.535 |
| | (0.048) | (0.047) | (0.054) | (0.056) | (0.059) |
| ln $L$ | -12247.45 | -11647.36 | -11404.09 | -11266.91 | -11211.52 |

*Notes:* This table shows maximum likelihood estimation results for bivariate MSM. Columns correspond to the number $\bar{k}$ of volatility components. Asymptotic standard errors are in parenthesis.

TABLE 7. – BIVARIATE MSM ESTIMATES
TWO-STEP

| | k=1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| | | | | DM and JA | | | | |
| $\widehat{m}_0^1$ | 1.643 | 1.618 | 1.515 | 1.474 | 1.445 | 1.405 | 1.397 | 1.367 |
| | (0.020) | (0.019) | (0.022) | (0.023) | (0.022) | (0.022) | (0.022) | (0.022) |
| $\widehat{m}_0^2$ | 1.775 | 1.757 | 1.687 | 1.638 | 1.578 | 1.565 | 1.522 | 1.488 |
| | (0.013) | (0.012) | (0.016) | (0.017) | (0.020) | (0.018) | (0.019) | (0.021) |
| $\widehat{\sigma}_1$ | 0.669 | 0.577 | 0.597 | 0.569 | 0.504 | 0.565 | 0.449 | 0.472 |
| | (0.014) | (0.011) | (0.019) | (0.020) | (0.021) | (0.018) | (0.027) | (0.035) |
| $\widehat{\sigma}_2$ | 0.613 | 0.544 | 0.565 | 0.487 | 0.476 | 0.632 | 0.384 | 0.532 |
| | (0.010) | (0.010) | (0.018) | (0.016) | (0.021) | (0.017) | (0.022) | (0.041) |
| $\widehat{\gamma_{\bar{k}}}$ | 0.129 | 0.257 | 0.301 | 0.756 | 0.844 | 0.872 | 0.959 | 0.982 |
| | (0.014) | (0.024) | (0.037) | (0.072) | (0.075) | (0.081) | (0.047) | (0.027) |
| $\widehat{b}$ | - | 69.57 | 11.97 | 13.21 | 9.14 | 7.16 | 6.16 | 4.93 |
| | | (21.80) | (2.20) | (2.11) | (1.32) | (1.29) | (0.86) | (0.56) |
| $\widehat{\rho}_\epsilon$ | 0.566 | 0.570 | 0.581 | 0.574 | 0.578 | 0.581 | 0.581 | 0.618 |
| | (0.013) | (0.014) | (0.016) | (0.017) | (0.017) | (0.049) | (0.009) | (0.010) |
| $\widehat{\lambda}$ | 0.587 | 0.544 | 0.646 | 0.585 | 0.624 | 0.633 | 0.659 | 0.633 |
| | (0.065) | (0.067) | (0.064) | (0.082) | (0.080) | (0.032) | (0.038) | (0.023) |
| | | | | DM and UK | | | | |
| $\widehat{m}_0^1$ | 1.626 | 1.565 | 1.519 | 1.473 | 1.452 | 1.406 | 1.401 | 1.370 |
| | (0.020) | (0.021) | (0.022) | (0.023) | (0.022) | (0.022) | (0.023) | (0.022) |
| $\widehat{m}_0^2$ | 1.697 | 1.657 | 1.641 | 1.602 | 1.573 | 1.521 | 1.492 | 1.454 |
| | (0.016) | (0.017) | (0.019) | (0.021) | (0.022) | (0.022) | (0.022) | (0.022) |
| $\widehat{\sigma}_1$ | 0.671 | 0.645 | 0.599 | 0.568 | 0.493 | 0.563 | 0.471 | 0.474 |
| | (0.013) | (0.017) | (0.018) | (0.021) | (0.018) | (0.018) | (0.028) | (0.033) |
| $\widehat{\sigma}_2$ | 0.605 | 0.588 | 0.515 | 0.468 | 0.422 | 0.457 | 0.391 | 0.385 |
| | (0.011) | (0.015) | (0.015) | (0.016) | (0.017) | (0.020) | (0.019) | (0.026) |
| $\widehat{\gamma_{\bar{k}}}$ | 0.090 | 0.151 | 0.388 | 0.683 | 0.672 | 0.798 | 0.844 | 0.969 |
| | (0.011) | (0.019) | (0.052) | (0.082) | (0.087) | (0.093) | (0.092) | (0.043) |
| $\widehat{b}$ | - | 12.33 | 15.25 | 11.97 | 10.09 | 6.97 | 6.23 | 5.02 |
| | | (3.38) | (3.02) | (2.07) | (1.68) | (1.48) | (0.88) | (0.65) |
| $\widehat{\rho}_\epsilon$ | 0.697 | 0.703 | 0.704 | 0.709 | 0.711 | 0.700 | 0.689 | 0.704 |
| | (0.010) | (0.011) | (0.012) | (0.013) | (0.012) | (0.011) | (0.012) | (0.010) |
| $\widehat{\lambda}$ | 0.814 | 0.818 | 0.826 | 0.802 | 0.820 | 0.790 | 0.844 | 0.800 |
| | (0.041) | (0.037) | (0.042) | (0.048) | (0.045) | (0.050) | (0.022) | (0.027) |
| | | | | JA and UK | | | | |
| $\widehat{m}_0^1$ | 1.776 | 1.762 | 1.688 | 1.645 | 1.631 | 1.568 | 1.558 | 1.507 |
| | (0.062) | (0.031) | (0.032) | (0.025) | (0.024) | (0.024) | (0.021) | (0.023) |
| $\widehat{m}_0^2$ | 1.728 | 1.680 | 1.640 | 1.607 | 1.575 | 1.525 | 1.499 | 1.458 |
| | (0.016) | (0.016) | (0.019) | (0.020) | (0.021) | (0.022) | (0.021) | (0.021) |
| $\widehat{\sigma}_1$ | 0.624 | 0.546 | 0.569 | 0.471 | 0.702 | 0.631 | 0.514 | 0.509 |
| | (0.011) | (0.011) | (0.017) | (0.014) | (0.028) | (0.030) | (0.025) | (0.031) |
| $\widehat{\sigma}_2$ | 0.606 | 0.561 | 0.522 | 0.508 | 0.432 | 0.457 | 0.384 | 0.379 |
| | (0.012) | (0.012) | (0.015) | (0.015) | (0.016) | (0.021) | (0.018) | (0.023) |
| $\widehat{\gamma_{\bar{k}}}$ | 0.161 | 0.303 | 0.275 | 0.635 | 0.697 | 0.847 | 0.864 | 0.970 |
| | (0.015) | (0.027) | (0.030) | (0.062) | (0.068) | (0.064) | (0.067) | (0.034) |
| $\widehat{b}$ | - | 43.46 | 12.73 | 14.55 | 13.60 | 8.22 | 7.24 | 5.60 |
| | | (10.62) | (2.27) | (2.14) | (2.08) | (1.04) | (0.91) | (0.70) |
| $\widehat{\rho}_\epsilon$ | 0.439 | 0.439 | 0.448 | 0.439 | 0.439 | 0.436 | 0.414 | 0.436 |
| | (0.017) | (0.018) | (0.019) | (0.021) | (0.021) | (0.010) | (0.018) | (0.017) |
| $\widehat{\lambda}$ | 0.494 | 0.519 | 0.570 | 0.549 | 0.524 | 0.575 | 0.625 | 0.561 |
| | (0.068) | (0.071) | (0.063) | (0.076) | (0.080) | (0.049) | (0.027) | (0.015) |

*Notes:* This table shows two-step estimates for bivariate MSM. Columns correspond to the number $\bar{k}$ of volatility components. First stage estimates are obtained by optimizing the combined univariate likelihood as in Table 3A. As described in the Appendix, this provides consistent estimates for the parameters $(m_0^\alpha, m_0^\beta, \sigma_\alpha, \sigma_\beta, b, \gamma_{\bar{k}})$. For $\bar{k} \leq 5$, the second stage optimizes the analytically calculated bivariate MSM likelihood conditional on the first stage estimates. For $\bar{k} = 6, 7, 8$, the optimization of the likelihood is numerically implemented using the particle filter approximation. Standard errors in parentheses are calculated by recasting the optimization in a GMM context, as described in the Appendix, and are HAC adjusted using Newey and West (1987).

TABLE 8. – In-Sample Model Comparison

| | No. of Parameters | $\ln L$ | BIC | BIC $p$-value vs. Multifractal: Vuong (1989) | BIC $p$-value vs. Multifractal: HAC Adj |
|---|---|---|---|---|---|
| **A. Full MLE Estimates** | | | | | |
| | | *DM and JA* | | | |
| Bivariate MSM | 8 | -11655.80 | 3.0626 | | |
| CC GARCH | 7 | -12825.63 | 3.3679 | $< 0.001$ | $< 0.001$ |
| | | *DM and UK* | | | |
| Bivariate MSM | 8 | -10240.51 | 2.6919 | | |
| CC GARCH | 7 | -11331.00 | 2.9764 | $< 0.001$ | $< 0.001$ |
| | | *JA and UK* | | | |
| Bivariate MSM | 8 | -11211.52 | 2.9462 | | |
| CC GARCH | 7 | -12550.49 | 3.2958 | $< 0.001$ | $< 0.001$ |
| **B. Two-Step Estimates** | | | | | |
| | | *DM and JA* | | | |
| Bivariate MSM | 8 | -11658.89 | 3.0634 | | |
| CC GARCH | 7 | -12830.98 | 3.3693 | $< 0.001$ | $< 0.001$ |
| | | *DM and UK* | | | |
| Bivariate MSM | 8 | -10262.05 | 2.6975 | | |
| CC GARCH | 7 | -11434.17 | 3.0034 | $< 0.001$ | $< 0.001$ |
| | | *JA and UK* | | | |
| Bivariate MSM | 8 | -11233.59 | 2.9521 | | |
| CC GARCH | 7 | -12559.72 | 3.2982 | $< 0.001$ | $< 0.001$ |

*Notes:* This table summarizes information about in-sample goodness of fit. The Bayesian Information Criterion is given by $BIC = T^{-1}(-2\ln L + NP \ln T)$. The last two columns give $p$-values from a test that the corresponding model dominates the multifractal model by the BIC criterion. The first value uses the Vuong (1989) methodology, and the second value adjusts the test for heteroskedasticity and autocorrelation as described in Calvet and Fisher (2004). A low $p$-value indicates that the CC GARCH model would be rejected in favor of the multifractal model. Panel A presents the results when both models have been estimated by Full MLE. Panel B presents results where both models are estimated by two-step procedures.

TABLE 9. – GOODNESS OF FIT
ONE-DAY FORECASTS

| | Bivariate MSM | | | CC GARCH | | |
|---|---|---|---|---|---|---|
| | *DM,JA* | *DM,UK* | *JA,UK* | *DM,JA* | *DM,UK* | *JA,UK* |
| Currency $\alpha$ | 0.22 | 0.16 | 0.54 | **1.07** | **1.07** | **1.70** |
| Currency $\beta$ | 0.47 | 0.51 | 0.29 | **1.70** | **2.57** | **2.56** |
| Equal-Weight | 0.67 | 0.05 | 0.23 | **2.48** | **1.06** | **2.51** |
| Hedge | **2.27** | 0.59 | **1.06** | 0.56 | **5.93** | 0.52 |

*Notes:* This table shows the Cramer-vonMises distance between a uniform distribution and the empirical distribution of the probability integral transform of the corresponding model forecast. The bivariate MSM specification uses $\bar{k} = 5$ components. Currency $\alpha$ and $\beta$ refer to the first currency and second currency in each pair. Equal-Weight is an equal weighted portfolio, and Hedge is a zero investment portfolio of $\alpha - \beta$. Under correct specification, the reported statistics are greater than 0.73 in about 1% of samples. A high value of the statistic thus indicates rejection of the corresponding model. Rejections at the 1% level are indicated by bold face.

TABLE 10. – FAILURE RATES
VALUE AT RISK FORECASTS

| | Bivariate MSM | | | CC GARCH | | |
|---|---|---|---|---|---|---|
| | 1% | 5% | 10% | 1% | 5% | 10% |
| | | | A. One Day Horizon | | | |
| | | | *DM and JA* | | | |
| Currency $\alpha$ | 0.69 | 4.35 | 9.10 | **1.81** | 5.13 | 9.01 |
| Currency $\beta$ | 0.95 | 4.81 | 9.56 | **2.30** | 5.38 | 9.10 |
| Equal-Weight | 0.86 | **3.92** | **8.32** | 1.30 | 4.66 | **8.21** |
| Hedge | 0.69 | 5.64 | **12.21** | **2.25** | **6.68** | **11.81** |
| | | | *DM and UK* | | | |
| Currency $\alpha$ | 0.92 | 4.92 | 10.14 | **1.81** | 5.13 | 9.01 |
| Currency $\beta$ | 0.72 | 5.27 | 10.68 | **1.44** | 4.61 | **8.29** |
| Equal-Weight | 1.07 | 4.69 | 10.28 | **1.87** | 5.18 | 8.98 |
| Hedge | **0.55** | 4.72 | 9.13 | 0.92 | **4.00** | **7.00** |
| | | | *JA and UK* | | | |
| Currency $\alpha$ | 1.01 | 5.04 | 9.88 | **2.30** | 5.38 | 9.10 |
| Currency $\beta$ | 0.60 | 4.41 | 9.70 | **1.44** | 4.61 | **8.29** |
| Equal-Weight | 0.84 | 4.55 | 8.78 | **1.64** | 4.69 | **8.03** |
| Hedge | 1.15 | 5.64 | **11.37** | **2.25** | **6.25** | 10.34 |
| | | | B. Five-Day Horizon | | | |
| | | | *DM and JA* | | | |
| Currency $\alpha$ | 0.78 | 4.21 | 9.57 | 1.61 | 5.48 | 10.72 |
| Currency $\beta$ | 1.07 | 5.30 | 10.55 | **2.31** | **7.06** | 11.62 |
| Equal-Weight | 0.72 | 4.44 | 8.50 | 1.64 | 5.25 | 9.14 |
| Hedge | 0.92 | 5.42 | 12.16 | **3.29** | **8.39** | **13.46** |
| | | | *DM and UK* | | | |
| Currency $\alpha$ | 0.95 | 5.13 | 10.35 | 1.64 | 5.51 | 10.72 |
| Currency $\beta$ | 0.75 | 5.28 | 11.01 | 0.98 | 4.79 | 9.77 |
| Equal-Weight | 0.84 | 5.07 | 10.93 | 1.27 | 5.94 | 10.61 |
| Hedge | 0.69 | 4.01 | 8.76 | 0.86 | 3.86 | 8.48 |
| | | | *JA and UK* | | | |
| Currency $\alpha$ | 1.21 | 5.53 | 10.75 | **2.36** | 6.86 | 11.70 |
| Currency $\beta$ | **0.46** | 4.67 | 9.02 | 1.53 | 4.93 | 9.57 |
| Equal-Weight | 0.84 | 4.12 | 9.02 | 1.53 | 4.93 | 9.57 |
| Hedge | 1.73 | 6.40 | 11.24 | 1.76 | 6.34 | 11.53 |

*Notes:* This table displays the frequency of returns that exceed the Value at Risk forecasted by the model. The bivariate MSM specification uses $\bar{k} = 5$ components. For quantile $p$% the number reported is the frequency of portfolio returns below quantile $p$ predicted by the model. If the VaR forecast is correct, the observed failure rate should be close to the prediction. Boldface numbers are statistically different from $\alpha$ at the 1% level. Panel A shows results for a one day horizon, while Panel B shows a five-day horizon. Currency $\alpha$ and $\beta$ refer to the first currency and second currency in each pair. Equal-Weight is an equal weighted portfolio, and Hedge is a zero investment portfolio of $\alpha - \beta$. Standard errors in Panel A are computed by $p * (1 - p)/3473$, where 3473 is the number of out-of-sample observations. Standard errors in Panel B are computed using Newey and West (1987).

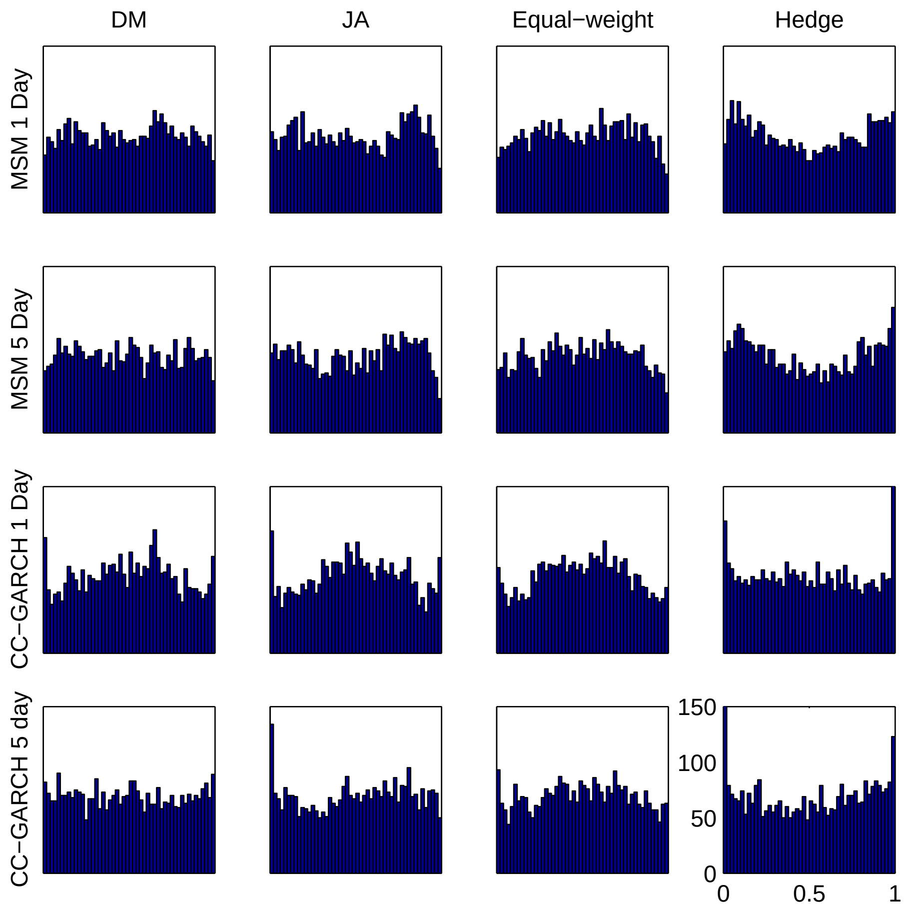


**Figure 1: Probability Integral Transforms.** These figures show histograms of the probability integral transforms $\{U_{t,n}\}$ for horizons (in rows) of $n = 1$ and $n = 5$ days and portfolios (in columns) of DM, JA, an equal-weighted portfolio of the two currencies, and a hedge portfolio with weights $(1, -1)$. The models considered are bivariate MSM with $\bar{k} = 5$ components and CC-GARCH. Under correct specification, the integral transforms are uniformly distributed.