IRREVERSIBLE INVESTMENT, CAPACITY
CHOICE, AND THE VALUE OF THE FIRM

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ABSTRACT

A model of capacity choice and utilization is developed consistent with value maximization when investment is irreversible and future demand is uncertain. Investment requires the full value of a marginal unit of capacity to be at least as large as its full cost. The former includes the value of the firm's option not to utilize the unit, and the latter includes the opportunity cost of exercising the investment option. We show that for moderate amounts of uncertainty, the firm's optimal capacity is much smaller than it would be if investment were reversible, and a large fraction of the firm's value is due to the possibility of future growth. We also characterize the behavior of capacity and capacity utilization, and discuss implications for the measurement of marginal cost and Tobin's q.

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1. Introduction.

When investment is irreversible and future demand conditions are uncertain, a firm's investment expenditure involves the exercising, or "killing," of an option (the option to productively invest). One gives up the option of waiting for new information (about evolving demand and cost conditions), and using that information to re-evaluate the desirability and/or timing of the expenditure. This lost option value must be included as part of the cost of the investment. As a result, the standard investment rule "Invest when the marginal value of a unit of capital is at least as large as the purchase and installation cost of the unit" is not valid. Instead the marginal value of the unit must _exceed_ the purchase and installation cost by an amount equal to the value of keeping the firm's option to invest alive -- an opportunity cost of investing.

This aspect of investment has been explored in an emerging literature, and most notably in the recent paper by McDonald and Siegel (1986). They show that with even moderate levels of uncertainty, the value of this opportunity cost can be large, and an investment rule that ignores it will be grossly in error. Their calculations, and those in the related papers by Brennan and Schwartz (1985) and Majd and Pindyck (1985), show that in many cases projects should be undertaken only when their present value is at least double their direct cost.[1]

The existing literature has been concerned with investment decisions involving discrete projects, e.g. whether to build a factory. This paper examines the implications of irreversibility for capacity choice, e.g. how large a factory to build. In particular I focus on the _marginal_ investment decision. This provides a simple and intuitively appealing solution to the

optimal capacity problem, as well as insight into the sources and evolution of the firm's value. In addition, it clarifies issues related to the measurement of long-run marginal cost, and the interpretation and measurement of Tobin's q.

A firm's capacity choice is optimal when the value of the marginal unit of capacity is just equal to the total cost of that unit. This total cost includes the purchase and installation cost, plus the opportunity cost of exercising the option to buy the unit. An analysis of capacity choice therefore involves two steps. First, the value of a marginal unit of capacity must be determined, given that the firm already has capacity K. To do this we must account for the fact that if demand unexpectedly falls, the unit of capacity may be unutilized. Second, the value of the option to invest in the marginal unit must be determined (it will depend in part on the value of the marginal unit itself), together with the decision rule for exercising the option. In essence, this decision rule is the solution to the optimal capacity problem.

Because a marginal unit of capacity need not be utilized, it is worth more when demand fluctuates stochastically. This might suggest that the firm should hold more capacity when future demand is uncertain, but in fact the opposite is true. The reason is that uncertainty also increases the opportunity cost of exercising the option to invest in a marginal unit. Although the value of the marginal unit increases, this opportunity cost increases even more, so the net effect is to reduce the firm's optimal capacity.[2] Indeed, for many product markets the volatility of demand is such that firms should hold far less capacity than standard investment models would suggest.

This model of capacity choice also has implications for the valuation of firms. The value of a firm has two components: the value of installed capacity, and the value of the firm's options to install more capacity in the future. Solutions of the model based on plausible parameter values suggest that for typical firms, "growth options" should account for more that half (and in some cases much more than half) of market value.

This paper, like others cited above, stresses the options that firms have to productively invest. These options are important assets of firms. Firms hold them even if they are price-takers in product and input markets, and they can account for a good fraction of their market value. What gives firms these options? It may be a patent on a particular production technology, or ownership of land or natural resource reserves. But more generally, a firm's managerial resources and expertise, reputation, market position, and possibly scale, all of which may have been built up over time, enable it to productively undertake investments that individuals or other firms cannot undertake.[3]

The next section lays out a simple model of capacity choice with irreversible investment, based on the assumption that firms maximize their market value. It differs from previous models of this type, e.g. Brennan and Schwartz (1985) and McDonald and Siegel (1986), in that the focus is on the marginal investment decision, rather than the decision to invest in a discrete project. As the model is developed, a numerical example is used to show how the marginal value of capital, the opportunity cost of investing, and the firm's optimal capacity depend on current demand and uncetainty over future demand. Remaining sections of the paper use the model to study the value of the firm, the behavior of capacity and capacity utilization over

time, and implications for the measurement of marginal cost and Tobin's q.

2. A Model of Capacity Choice.

Consider the investment decisions of a firm that might have monopoly power, and faces the following demand function:[4]

$$P = \theta(t) - \gamma Q \qquad (1)$$

(The firm might be a price-taker, in which case $\gamma = 0$.) Here $\theta(t)$ evolves over time according to the following stochastic process:

$$d\theta = \alpha\theta dt + \sigma\theta dz \qquad (2)$$

where dz is the increment of a Weiner process, i.e. $dz = \varepsilon(t)(dt)^{1/2}$, with $\varepsilon(t)$ a serially uncorrelated and normally distributed random variable. Eqn. (2) says that the current value of θ (and thus the current demand function) is known to the firm, but _future_ values of θ are unknown, and are lognormally distributed with a variance that grows with the time horizon. Thus even though information arrives over time (the firm observes θ changing), future demand is always uncertain, as is the case in most real-world markets.[5]

Denote by $\rho_{\theta m}$ the correlation of θ with the market portfolio. Now suppose some asset or portfolio of assets exists with a stochastic return perfectly correlated with θ, so that if x is the price of this asset, it evolves according to:

$$dx = \mu x dt + \sigma x dz$$

By the CAPM, the expected return on this asset must be $\mu = r + \phi\rho_{\theta m}\sigma$, where ϕ is the market price of risk. We will assume that α, the expected percentage rate of change of θ, is _less_ than μ. (It will become clear later that if this were not the case, no firm in the industry would ever install

any capacity. No matter what the current level of θ, firms would always be better off waiting and simply holding the option to install capacity in the future.) Denote the difference between μ and α by δ, i.e. $\delta = \mu - \alpha$.

The firm's problem is to determine, initially and over time, its optimal investment spending on new production capacity. I assume that the firm starts with no capacity, so that at $t = 0$ it decides how much initial capacity to put in place. Later it may or may not add more capacity, depending on how demand evolves.

For simplicity I assume that new capacity can be installed instantly, and capital in place does not depreciate.[4] Another assumption -- and an important one -- is that investment is _irreversible_. That is, although capital in place can be sold by one firm to another, its scrap value is small because it has no alternative use than that originally intended for it. Thus a factory built to produce widgets can only be used to produce widgets, so if the demand for widgets falls, its market value will fall.

The fact that investment is irreversible implies that there is an opportunity cost associated with adding capacity -- adding capacity today forecloses the possibility of adding it instead at some point in the future (or never at all). Put another way, the firm currently has options to install capacity at various points in the future (options that can be exercised at the cost of purchasing the capital), and by installing capacity now, it closes those options. The optimal rule is to invest until the value of a marginal unit of capital is equal to its total cost -- the purchase and installation cost, plus the value of the option on the unit.[7]

A few more details are needed to finish specifying the model: (i) each unit of capital can be bought at a fixed price k per unit; (ii) each unit

of capital in place provides the capacity to produce one unit of output per time period; and (iii) the firm has an operating cost $C(Q)$ = c_1Q + $(1/2)c_2Q^2$. In general c_1 and/or c_2 can be zero, but if Y = 0 (so the firm is a price-taker), we require $c_2 > 0$ to bound the firm's size.

For purposes of comparison, note that if future demand were certain $(\sigma$ = 0), and if $\alpha \geq 0$, the firm's optimal initial capital stock would be $K^*(\theta)$ = $(\theta$ - c_1 - $rk)/(2Y+c_2)$, i.e. the firm should add capacity only if $\theta(K)$ > $(2Y+c_2)K$ + c_1 + rk. We will see that with moderate amounts of uncertainty over future demand, the optimal capital stock is considerably smaller than this.

The Value of a Marginal Unit of Capacity.

To solve the firm's investment problem we first determine the value of an incremental unit of capacity. That is, given that the firm already has capacity K, we want to find the value to the firm of an incremental unit, which we denote by $\Delta V(K)$. (Note that $\Delta V(K)$ is a function of θ as well as K.) This is just the present value of the expected flow of incremental profits from the marginal unit. Because the unit does not have to be utilized, future incremental profits are a nonlinear function of θ, which is stochastic. In particular, given a current capacity K, the incremental profit at any future time t resulting from a marginal unit of capacity is:

$$\Delta\pi_t(K) = \max [0, (\theta_t - (2Y+c_2)K - c_1)] \qquad (3)$$

Thus $\Delta V(K)$ can be written as:

$$\Delta V(K) = \int_0^\infty\int_0^\infty \Delta\pi_t(K;\theta)f(\theta,t)d\theta e^{-\mu t}dt \qquad (4)$$

where $f(\theta,t)$ is the density function for θ at time t, μ is the risk-adjusted discount rate, and $\Delta\pi_t(K;\theta)$ is given by eqn. (3). It is difficult, however,

to evaluate (4) directly. In addition, the rate μ might not be known.

Instead we obtain $\Delta V(K)$ by solving the following equivalent problem: What is the value of a factory that produces 1 unit of output per period, with operating cost $(2\gamma+c_2)K + c_1$, where the output is sold in a perfectly competitive market at a price θ_t, and where the factory can be shut down (temporarily and costlessly) if the price θ_t falls below the operating cost?[8] It is shown in the Appendix that the solution to this problem is:

$$\Delta V(K) = \begin{cases} b_1\theta^{\beta_1} & ; \quad \theta \leq (2\gamma+c_2)K + c_1 \\ b_2\theta^{\beta_2} + \theta/\delta - [(2\gamma+c_2)K + c_1]/r & ; \quad \theta \geq (2\gamma+c_2)K + c_1 \end{cases} \quad (5)$$

where:

$$\beta_1 = -\frac{(r-\delta-\sigma^2/2)}{\sigma^2} + \frac{1}{\sigma^2}[(r-\delta-\sigma^2/2)^2 + 2r\sigma^2]^{1/2} \qquad > 1$$

$$\beta_2 = -\frac{(r-\delta-\sigma^2/2)}{\sigma^2} - \frac{1}{\sigma^2}[(r-\delta-\sigma^2/2)^2 + 2r\sigma^2]^{1/2} \qquad < 0$$

$$b_1 = \frac{r - \beta_2(r-\delta)}{r\delta(\beta_1 - \beta_2)} [(2\gamma+c_2)K + c_1]^{1-\beta_1} \qquad > 0$$

$$b_2 = \frac{r - \beta_1(r-\delta)}{r\delta(\beta_1 - \beta_2)} [(2\gamma+c_2)K + c_1]^{1-\beta_2} \qquad > 0$$

A numerical example is useful to illustrate the characteristics of $\Delta V(K)$, as well as other aspects of the model. For this purpose I choose $r = \delta = .05$, $k = 10$, $c_1 = 0$, and either $\gamma = .5$ and $c_2 = 0$, or equivalently $\gamma = 0$ and $c_2 = 1$.[9] I vary θ or K, and consider values of σ in the range of 0 to .4.[10] For purposes of comparison, let $\Delta V_0(K)$ denote $\Delta V(K)$ for $\sigma = 0$, so $\Delta V_0(K) = \theta/\delta - [(2\gamma+c_2)K + c_1]/r$ for $\theta \geq (2\gamma+c_2)K + c_1$, and 0 otherwise. For our numerical example, $\Delta V_0(K) = 20(\theta-K)$ for $\theta \geq K$, and 0 otherwise.

Figure 1 shows $\Delta V(K)$ as a function of θ for $K = 1$ and $\sigma = 0$, .2, and .4. Observe that $\Delta V(K)$ looks like the value of a call option -- indeed it is the sum of an infinite number of European call options (see Footnote 8).

As with a call option, $\Delta V(K)$ is increasing with σ, and for $\sigma > 0$, $\Delta V(K) > \Delta V_o(K)$ because the firm need not utilize its capacity. As $\theta \rightarrow \infty$, $\Delta V(K) \rightarrow \Delta V_o(K)$; for θ very large relative to K, this unit of capacity will almost surely be continuously utilized over a long period of time.

Figure 2 shows $\Delta V(K)$ as a function of K for $\theta = 2$, and $\sigma = 0$, .1, .2, and .4. Because demand evolves stochastically, a marginal unit of capacity has some positive value no matter how large is the existing capital stock; there is always some chance that it will be utilized over any finite period of time. The greater is σ, the more slowly $\Delta V(K)$ declines with K. Also, the smaller is K, the more likely it is that the marginal unit will be utilized, and so the smaller is $\Delta V(K) - \Delta V_o(K)$. When $K = 0$, $\Delta V(0) = \Delta V_o(K)$; with $c_1 = 0$, the _first_ marginal unit will always be utilized.

The fact that $\Delta V(K)$ is larger when $\sigma > 0$ might suggest that the firm should hold a larger amount of capacity, but just the opposite is true. As shown below, the firm's opportunity cost of exercising its option to invest in the marginal unit also becomes larger, and by an even greater amount.

The Decision to Invest in the Marginal Unit.

Having valued the marginal unit of capacity, we now value the firm's option to invest in this unit, and the optimal decision rule for exercising the option. This is analogous to a perpetual call option with exercise price k, on a stock that pays a proportional dividend at rate δ and has a current price $\Delta V(K)$. In the Appendix it is shown that its value, $\Delta F(K)$, is:

$$\Delta F(K) = \begin{cases} a\theta^{\beta_1} & ; \quad \theta \leq \theta^*(K) \\ \Delta V(K) - k & ; \quad \theta \geq \theta^*(K) \end{cases} \tag{6}$$

where: $\quad a = \dfrac{\beta_2 b_2}{\beta_1}(\theta^*)^{(\beta_2-\beta_1)} + \dfrac{1}{\delta\beta_1}(\theta^*)^{(1-\beta_1)} > 0$,

β_1, β_2, and b_2 are given under eqn. (5) above, and $\theta^*(K)$ is the critical value of θ at or above which it is optimal to purchase the marginal unit of capacity, i.e. the firm should purchase the unit if $\theta \geq \theta^*(K)$. This critical value $\theta^*(K)$ is in turn the solution to :

$$\frac{b_2(\beta_1 - \beta_2)}{\beta_1}(\theta^*)^{\beta_2} + \frac{(\beta_1 - 1)}{\delta\beta_1}\theta^* - \frac{(2\gamma + c_2)K + c_1}{r} - k = 0 \qquad (7)$$

Eqn. (7) can be solved numerically for θ^*, and eqn. (6) can then be used to calculate $\Delta F(K)$.

Recall our assumption that $\delta > 0$. The reader can verify that as $\delta \to 0$, $\theta^*(K) \to \infty$. Unless $\delta > 0$, the opportunity cost of investing in a unit of capacity always exceeds the benefit, and the firm will never install capacity.[11] Thus if firms in an industry are investing optimally and some positive amount of investment is taking place, we should observe $\delta > 0$.

As with a call option on a dividend-paying stock, both $\Delta F(K)$ and the critical value $\theta^*(K)$ increase as σ increases. Figure 3 shows $\Delta F(K)$ as a function of θ for $K = 1$ and $\sigma = 0$, .2, and .4. When $\sigma = 0$, $\theta^* = 1.5$, i.e. the firm should increase capacity only if θ exceeds 1.5. For $\sigma = .2$ and .4, θ^* is 2.45 and 3.44 respectively. The opportunity cost of exercising the firm's option to invest in additional capacity is $\Delta F(K)$, which increases with σ, so a higher σ implies a higher critical value $\theta^*(K)$. Also, it is easily shown that $\theta^*(K)$ is monotonically increasing in K.

The Firm's Optimal Capacity.

The function $\theta^*(K)$ is the firm's optimal investment rule; if θ and K are such that $\theta > \theta^*(K)$, the firm should add capacity, increasing K until θ^* rises to θ. Equivalently we can substitute for $b_2(K)$ and rewrite eqn. (7) in terms of $K^*(\theta)$, the firm's optimal capacity:

$$\frac{r - \beta_1 (r - \delta)}{r \delta \beta_1} \theta^{\beta_2} [(2\gamma + c_2)K^* + c_1]^{1 - \beta_2} - \frac{[(2\gamma + c_2)K^* + c_1]}{r} + \frac{(\beta_1 - 1)}{\delta \beta_1} \theta - k = 0 \qquad (7')$$

Figure 4 shows $K^*(\theta)$ for $\sigma = 0$, .2 and .4. (For many industries .2 is a conservative value for σ -- see Footnote 10.) Observe that K^* is much smaller when future demand is uncertain. For $\sigma = .4$, θ must be more than three times as large as when $\sigma = 0$ before any capacity is installed.

Another way to see how uncertainty over future demand affects the firm's optimal capacity is by comparing $\Delta F(K)$, the value of the option to invest in a marginal unit, with $\Delta V(K) - k$, the net (of purchase cost) value of the unit. The optimal capacity $K^*(\theta)$ is the maximum K for which these two quantities are equal. Note from eqn. (6) that for $\theta \geq \theta^*$, or equivalently, $K \leq K^*$, exercising the option to invest maximizes its value, so that $\Delta F(K) = \Delta V(K) - k$, but for $K > K^*$, $\Delta F(K) > \Delta V(K) - k$, and the option to invest is worth more "alive" than "dead."

This is shown in Figure 5, which plots $\Delta F(K)$ and $\Delta V(K) - k$ as functions of K, for $\theta = 2$ and $\sigma = .2$. Recall that $\Delta V(K)$ is larger when future demand is uncertain. As the figure shows, if the opportunity cost of exercising the option to invest were ignored (i.e. the firm added capacity until $\Delta V(K) - k$ was zero), the firm's capacity would be about 2.3 units, as opposed to 1.5 units when $\sigma = 0$. But at these capacity levels the opportunity cost of investing in a marginal unit exceeds the net value of the unit, so the value of the firm is not maximized. The optimal capacity is only $K^* = .67$, the largest K for which $\Delta F(K) = \Delta V(K) - k$, and the solution to eqn. (7).

3. The Value of The Firm.

As noted above, $K^*(\theta)$ is the capacity level which maximizes the firm's

market value, net of cash outlays for the purchase of capital. This can be seen algebraically and from Figure 5 by noting that the value of the firm has two components, the value of installed capacity, and the value of the firm's options to install more capacity in the future. The firm's net value as a function of its capacity K is thus given by:

$$\text{Net Value} = \int_{0}^{K} \Delta V(v)\,dv + \int_{K}^{\infty} \Delta F(v)\,dv - kK \tag{8}$$

Differentiating with respect to K shows that this is maximized when K = K* such that ΔV(K*) - ΔF(K*) - k = 0.

The value of the firm's installed capacity, V(K*), is just the first integral in eqn. (8). In Figure 5 it is the area under the curve ΔV(K) - k from K = 0 to K*, plus the purchase cost kK*. The value of the firm's options to expand is the second integral, which in Figure 5 is the area under the curve ΔF(K) from K = K* to ∞. As the figure suggests, the value of the firm's growth options is a large portion of its total value.

The sensitivity of firm value and its components to uncertainty over future demand can be seen from Table 1, which shows K*, V(K*), F(K*), and total value for different values of σ and θ. When σ = 0, the value of the firm is only the value of its installed capacity. Whatever the value of θ, the firm is worth more the the more volatile is demand. A larger σ implies a larger value for each unit of installed capacity, and a much larger value for the firm's options to expand. Also, the larger is σ, the larger is the fraction of firm value attributable to its growth options. When σ = .2 or more, more than half of the firm's value is F(K*), the value of its growth options. Even when σ = .1, F(K*) accounts for more than half of total value when θ is 1 or less. (When demand is currently small, it is

the possibility of greater demand in the future that gives the firm much of its value.) And there is always a range of θ for which K* is zero, so that all of the firm's value is due to its growth options.

As mentioned earlier, σ = .2 or more should be typical for many industries. A testable implication of the model is that for firms in such industries, the fraction of market value attributable to the value of capital in place should not be much more than one half. A second implication is that this fraction should be smaller the greater is the volatility of market demand. I have not tried to test either of these implications (valuing capital in place is itself a difficult task). However, calculations reported by Kester (1984) are consistent with both of them. He estimated the value of capital in place for 15 firms in 5 industries by capitalizing a flow of anticipated earnings from this capital, and found it is half or less of market value in the majority of cases. Furthermore, this fraction is only about 1/5 to 1/3 in industries where demand is more volatile (electronics, computers), but more than 1/2 in industries with less volatile demand (tires and rubber, food processing).

4. The Dynamics of Capacity, Capacity Utilization, and Firm Value.

If the firm begins with no capacity, it initially observes θ and installs a starting capacity K*(θ). If θ then increases, it will expand capacity accordingly, and the value of the firm will rise. The value of its growth options will also rise, but will become a smaller fraction of total value (see Table 1). However if θ decreases, it will find itself holding more capacity than it would have chosen had the decrease been anticipated. The firm's value will fall, and depending on how much θ decreases, some of

its capacity may become unutilized.

Because capital does not depreciate in this model, the firm's capacity is non-decreasing, but will rise only periodically. The dynamics of capacity are characterized in Figure 6, which shows a sample path for $\theta(t)$, and the corresponding behavior of $K(t)$. (The duration of continuous upward movements in $K(t)$ is exaggerated.) The firm begins at t_0 by installing $K^*(\theta_0) \equiv K_0^*$. Then θ increases until it reaches a (temporary) maximum θ_1 at t_1, and K is increased accordingly to K_1^*. Here it remains fixed until t_2, when θ again reaches θ_1. Afterwards K is increased as θ increases, until t_3 when θ begins to decline from a new maximum, and K remains fixed at K_3^*.

Thus an implication of the model is that investment occurs only in spurts, when demand is rising, and only when it is rising above historic levels.[12] Firms usually increase capacity only periodically, and this is often attributed to the "lumpiness" of investment. But lumpiness is clearly not required for this behavior.

Let us now examine the firm's capacity utilization. Clearly during periods of expansion, all capacity will be utilized. When demand falls, however, some capacity may go unutilized, but only if it falls far enough.

If the firm had unlimited capacity it would maximize current profits by setting output at $Q^* = (\theta - c_1)/(2\gamma + c_2)$. However $K^*(\theta) < (\theta - c_1)/(2\gamma + c_2)$, and as shown in Section 2, can be much less even for moderate values of σ. Thus for θ in the range $\underline{\theta}(K) \equiv (2\gamma + c_2)K + c_1 \le \theta \le \theta^*(K)$, capacity will remain fixed but will be fully utilized. Capacity will go unutilized only when $\theta < \underline{\theta}(K)$. In Figure 6 this occurs during the intervals (t_a, t_b) and (t_c, t_d).

The irreversibility of investment induces firms to hold less capacity as a buffer against unanticipated drops in demand. As a result there will

be periods of low demands when capacity is fully utilized. A large drop in demand is required for capacity utilization to fall below 100%.

The value of the firm will move in the same direction as θ. Most of the time the firm's capacity K will be above $K^*(\theta)$ -- in Figure 6 exceptions are during the intervals (t_0, t_1) and (t_2, t_3) -- but given θ and K, the firm's value can always be computed from:

$$\text{Value} = \int_0^K \Delta V(v;\theta)dv + \int_K^\infty \Delta F(v;\theta)dv \qquad (9)$$

The share of the firm's value due to its growth options will also fluctuate with θ. For example as Table 1 shows, during periods when capacity is growing (so that $K = K^*(\theta)$), this share falls. It also falls when θ is falling and $K > K^*(\theta)$. Thus as firms in this model evolve over time, growth options tend to account for a smaller share of value.

5. The Measurement of Long-Run Marginal Cost.

The measurement of long-run marginal cost and its relationship to price can be important for industry analyses in general, and antitrust applications in particular. As shown below, when investment is irreversible, traditional measures will understate marginal cost and overstate the amount by which it differs from price, even in a competitive market. This problem is particularly severe when product markets are volatile.

Suppose $\sigma = 0$. Then $\Delta F(K) = 0$, and the firm sets K (and Q) so that:

$$\Delta V(K) = \theta/\delta - [(2\gamma+c_2)K + c_1]/r = k \qquad (10)$$

Note that $\Delta V(K)$, the value of a marginal unit of capacity, is net of (capitalized) marginal operating cost. Let us rewrite (10) as follows:

$$\theta/\delta - 2\gamma K/r = (c_1 + c_2 K)/r + k \qquad (10')$$

The left-hand side of (10') is capitalized marginal revenue (θ is capital-ized at a rate δ because it is growing at rate α; with $\sigma = 0$, $\delta = r - \alpha$). The right-hand side is full marginal cost, the capitalized operating cost, plus the purchase cost of a unit of capital. Eqn. (10') is the usual relation between marginal revenue and marginal cost when the former is increasing at a deterministic rate.

Now suppose $\sigma > 0$, and $K = K^*(\theta)$. Then from eqn. (5), the optimality condition $\Delta V(K) = k + \Delta F(K)$ can be written as:

$$\Delta V(K) = b_2\theta^{\beta_2} + \theta/\delta - [(2\gamma+c_2)K + c_1]/r = k + \Delta F(K) \qquad (11)$$

or:
$$\theta/\delta - 2\gamma K/r = -b_2\theta^{\beta_2} + (c_2 K + c_1)/r + k + \Delta F(K) \qquad (11')$$

Observe that two adjustments must be made to obtain full (capitalized) marginal cost, the RHS of (11'). The first term on the RHS of (11') is the value of the firm's option to let the marginal unit of capacity go unutil-ized, and must be subtracted from capitilized operating cost. The last term is the opportunity cost of exercising the option to invest. As we have seen in Section 2, the last term dominates the first, so that K must be smaller to satisfy (11'), and marginal cost as conventionally measured will under-state true marginal cost.

If the firm is a price-taker, $\gamma = 0$ and $P = \theta$. Price will equal marginal cost, if the latter is defined correctly as in (11'). Unfortunate-ly the first and last terms on the RHS of (11') are difficult to measure, particularly with aggregate data. But if one wishes to compare price with marginal cost, ignoring them can be misleading.[13]

6. Marginal q and Investment.

The q theory of investment says that firms have an incentive to invest

whenever marginal q -- the increase in market value resulting from a marginal unit of capital divided by the cost of that unit -- exceeds one. But models based on this theory have not been very successful in explaining investment.[14] There may be several reasons for this, but one possibility is that in such models the cost of a marginal unit of capital typically includes only the purchase and installation cost. As we have seen, this can grossly understate the true cost of the unit.

In empirical applications (e.g. Abel and Blanchard), the change in market value from a marginal unit of capital is measured as the NPV of the expected profit flow from the unit, i.e. the present value of the unit. In our notation, $q = \Delta V(K)/k$. But this ignores the opportunity cost $\Delta F(K)$; clearly $q > 1$ does not imply that K should be increased. The correct measure for q is found by noting that the addition of a marginal unit of capital increases the value of the firm's capital in place by $\Delta V(K)$, but decreases the value of the firm's growth options by $\Delta F(K)$. The net change in market value is thus $\Delta V(K) - \Delta F(K)$, so the correct measure for q is:

$$q^* = [\Delta V(K) - \Delta F(K)]/k \qquad (11)$$

Setting $q^* = 1$ yields the optimal K^*.

It is easy to see why marginal q as it is usually measured may fail to explain investment. Note that an increase in demand (i.e. in θ) will increase $\Delta V(K)$, but it will also increase $\Delta F(K)$, although by a smaller amount. And if σ increases, both $\Delta F(K)$ and $\Delta V(K)$ rise, but $\Delta V(K)$ by less, so that the desired capital stock falls. But the market value of the firm increases, so that even if market value data is used to measure q, we will see an increase in q but a decrease in investment.

As mentioned earlier, $\Delta F(K)$ is unfortunately difficult to measure. But

in some cases it may be possible to construct a proxy for $\Delta F(K)$ based on the sample variance of demand fluctuations. The use of such a proxy might improve the explanatory power of q theory models.

7. Conclusions.

The model presented here is a simple one that ignores complications such as adjustment costs, delivery lags, and lumpiness of capital expenditures. It can easily be extended to account for these factors, but numerical methods may then be required to obtain solutions. Of course once numerical methods are used, other aspects of the model can also be generalized. For example, demand can be a nonlinear function of θ, or θ could follow some alternative stochastic process, perhaps including jumps.

By treating capital in place as homogeneous, we have been able to focus on the marginal investment decision, and clarify the ways in which irreversibility of investment and uncertainty over future demand affect both the value and cost of a marginal unit of capacity. Besides yielding a relatively simple solution to the problem of capacity choice, the model provides a straightforward method for calculating the firm's market value and its components. It also provides insight into the measurement of long-run marginal cost, and Tobin's q.

In many markets output prices fluctuate with annual standard deviations in excess of 20 percent. Our results show that in such markets, firms should hold much less capacity than would be the case if investment were reversible or future demand were known with certainty. Also, much of the market value of these firms is due to the possibility (as opposed to the expectation) of stronger demands in the future. This value may result from

patents and technical knowledge, but it also arises from the managerial expertise and infrastructure, and market position that gives these firms (as opposed to potential entrants) the option to economically expand capacity.

One might ask whether firms correctly compute and take into account the opportunity cost of investing when making capacity expansion decisions. Ignoring such costs would lead to overinvestment. McConnell and Muscarella (1986) have found that for manufacturing firms, market value tends to increase (decrease) when managers announce an increase (decrease) in planned investment expenditures, which is inconsistent with a systematic tendency to overinvest.[15] But there is anecdotal evidence that managers often base investment decisions on present values computed with discount rates that far exceed those that would be implied by the CAPM -- diversifiable and non-diversifiable risk are sometimes confused, and an arbitrary "risk factor" is often added to the discount rate. It may be, then, that managers use the wrong method to get close to the right answer.

APPENDIX

Here we derive eqn. (5) for $\Delta V(K;\theta)$ and eqns. (6) and (7) for the optimal investment rule and value of the investment option $\Delta F(K;\theta)$.

The value of a marginal unit of capacity, $\Delta V(K;\theta)$, is found by valuing an equivalent "incremental project" that produces 1 unit of output per period at cost $(2\gamma+c_2) + c_1$, which is sold at price $\theta(t)$, and where the firm can (temporarily and costlessly) shut down if price falls below cost. To value this, create a portfolio that is long the project and short ΔV_θ units of the output, or equivalently the asset or portfolio of assets perfectly correlated with θ. Because the expected rate of growth of θ is only $\alpha = \mu - \delta$, the short position requires a payment of $\delta\theta\Delta V_\theta$ per unit of time (or no rational investor would hold the corresponding long position). The value of this portfolio is $\Phi = \Delta V - \Delta V_\theta\theta$, and its instantaneous return is:

$$d(\Delta V) - \Delta V_\theta d\theta - \delta\theta\Delta V_\theta + j[\theta - (2\gamma+c_2)K - c_1] \qquad (A.1)$$

The last term in (A.1) is the cash flow from the "incremental project;" j is a switching variable: $j = 1$ if $\theta(t) \geq (2\gamma+c_2)K + c_1$, and 0 otherwise.

By Ito's Lemma, $d(\Delta V) = \Delta V_\theta d\theta + (1/2)\Delta V_{\theta\theta}(d\theta)^2$. Substitute eq. (2) for $d\theta$ and observe that the return (A.1) is riskless. Setting that return equal to $r\Phi = r\Delta V - r\Delta V_\theta\theta$ yields the following equation for ΔV:

$$\frac{1}{2}\sigma^2\theta^2\Delta V_{\theta\theta} + (r-\delta)\theta\Delta V_\theta + j[\theta - (2\gamma+c_2)K - c_1] - r\Delta V = 0 \qquad (A.2)$$

The solution must satisfy the following boundary conditions:

$$\Delta V(\theta=0) = 0.$$

$$\lim_{\theta\to\infty} \Delta V = \theta/\delta - [(2\gamma+c_2) + c_1]/r$$

$$\lim_{\theta\to\infty} \Delta V_\theta = 1/\delta$$

and ΔV and ΔV_θ continuous at the switch point $\theta = (2\gamma+c_2)K + c_1$. The reader can verify that (5) is the solution to (A.2) and its boundary conditions.

Eqn. (A.2) can also be obtained by dynamic programming. Consider the optimal operating policy ($j = 0$ or 1) that maximizes the value Φ of the above portfolio. The Bellman equation is:

$$r\Phi = \max_{j=(0,1)} \{j[\theta - (2\gamma+c_2) - c_1] - \delta\theta\Delta V_\theta + \frac{1}{dt}E_t d\Phi\} \tag{A.3}$$

i.e. the competitive return $r\Phi$ has two components, the cash flow given by the first two terms in the maximand, and the expected rate of capital gain. Expanding $d\Phi = d\Delta V - \Delta V_\theta d\theta$ and substituting into (A.3) gives (A.2).

Finally, note that ΔV must be the solution to (A.2) and the boundary conditions even if the unit of capacity (the "incremental project") did not exist, and could not be included in a hedge portfolio. All that is required is an asset or portfolio of assets (x) that replicates the stochastic dynamics of θ. As Merton (1977) has shown, one can replicate the value function with a portfolio consisting only of the asset x and risk-free bonds, and since the value of this portfolio will have the same dynamics as ΔV, the solution to (A.2), ΔV must be the value function to avoid dominance.

Eqn. (6) for $\Delta F(K;\theta)$ can be derived in the same way. Using the same arguments as above, it is easily shown that ΔF must satisfy the equation:

$$\frac{1}{2}\sigma^2\theta^2\Delta F_{\theta\theta} + (r-\delta)\theta\Delta F_\theta - r\Delta F = 0 \tag{A.4}$$

with boundary conditions:

$$\Delta F(\theta=0) = 0$$

$$\Delta F(\theta=\theta^*) = \Delta V(\theta=\theta^*) - k$$

$$\Delta F_\theta(\theta=\theta^*) = \Delta V_\theta(\theta=\theta^*)$$

where $\theta^* = \theta^*(K)$ is the exercise point, and $\Delta V(\theta^*) - k$ is the net gain from exercising. The reader can verify that eqns. (6) and (7) are the solution to (A.4) and the associated boundary conditions.

TABLE 1: VALUE OF FIRM
$(c_1 = c_2 = 0, \ \gamma = .5, \ r = \delta = .05, \ k = 10)$

σ	θ	K*	V(K*)	F(K*)	VALUE
0	.5	0	0	0	0
	1	0.5	7.5	0	7.5
	2	1.5	37.5	0	37.5
	3	3.5	87.5	0	87.5
	4	3.5	157.5	0	157.5
.1	.5	0	0	0.4	0.4
	1	0.23	4.3	3.4	7.7
	2	1.00	33.1	9.1	42.2
	3	1.82	80.4	20.3	100.7
	4	2.65	147.3	35.5	182.8
.2	.5	0	0	3.1	3.1
	1	0.04	0.8	13.5	14.3
	2	0.67	24.0	49.2	73.2
	3	1.37	67.1	94.6	161.7
	4	2.09	134.1	143.7	277.8
.4	.5	0	0	25.8	25.8
	1	0	0	69.7	69.7
	2	0.15	5.9	182.6	188.7
	3	0.64	36.2	307.2	343.4
	4	1.22	91.3	427.5	518.8

FOOTNOTES

1. Other examples of this literature include the papers by Bernanke (1983), Cukierman (1980), Baldwin (1982), and Paddock, Siegel, and Smith (1984). In the papers by Bernanke and Cukierman, uncertainty over future market conditions is reduced as time passes, so firms have an incentive to delay investing when markets are volatile (e.g. during recessions). In the other papers cited above and in the model I present here, future market conditions are <u>always</u> uncertain. But because access to the investment opportunity is analogous to holding a call option on a dividend-paying stock, for any positive amount of risk, an expenditure should be made only when the value of the resulting project exceeds its cost by a positive amount, and increased uncertainty will increase the incentive to delay the investment expenditure. Thus the results are similar to those in Bernanke and Cukierman, but for different reasons.

2. In Abel (1983) and Hartman (1972) uncertainty over future prices leads to an increase in the firm's optimal capital stock when the production function is linear homogeneous and there are convex adjustment costs. The reason is that the marginal revenue product of capital is a convex function of price, so that as in our model, a marginal unit of capital is worth more when price is stochastic. However in Abel and Hartman, investment is reversible, so that no options are closed when the firm invests, and the opportunity cost of investing is zero.

3. A complete model of industry evolution would also describe the competitive processes through which firms obtain these options. Such a model is beyond the scope of this paper.

4. Analytic solutions for this model can be obtained for any demand function linear in θ, i.e. $P = \theta(t) + f(Q)$. I use (1) for simplicity.

5. It is straightforward to also allow for uncertainty over future operating cost. The qualitative results would be the same.

6. Relaxing these assumptions makes no qualitative difference in the results. In fact, allowing for lead times in the construction and installation of new capacity will magnify the effects of uncertainty. For a related model that looks specifically at the effects of lead times, see Majd and Pindyck (1985).

7. Still another way to see this is to note that the NPV of investing in a marginal unit of·capital might be positive because of possible future states of nature that are profitable. But that NPV might be made even larger by investing at a later date (and discounting the cost).

8. The valuation of a factory that can be temporarily shut down has been studied by Brennan and Schwartz (1985) and McDonald and Siegel (1985). Observe from eqn. (3) that the present value of an incremental profit at future time t is the value of a European call option, with expiration date t and exercise price $(2\theta + c_2)K + c_1$, on a stock whose price is

θ, paying a proportional dividend δ. This point was made by McDonald and Siegel (1985). Thus ΔV(K), the value of our "equivalent factory," can be found by summing the values of these call options for every future t. However this does not readily yield a closed form solution, and I use an approach similar to that of Brennan and Schwartz (1985).

9. With r and δ equal, α = 0 if stochastic changes in θ are completely diversifiable (i.e. $\rho_{\bullet m}$ = 0 so μ = r), but α > 0 otherwise. Also, as can be seen from eqn. (5), (Y=0, c_2=1) and (Y=.5, c_2=0) give the same marginal value of capital, and the same optimal behavior of the firm. In the first case the firm is a price-taker but earns infra-marginal rent, and in the second case it has monopoly power.

10. The standard deviations of annual changes in the prices of such commodities as oil, natural gas, copper, and aluminum are in the range of 20 to 50 percent. For manufactured goods these numbers are somewhat lower, but often 20 percent or higher. Thus a value of σ of .2 could be considered "typical" for simulation purposes.

11. If δ = 0, ΔV(K) has an expected rate of growth equal to the risk-adjusted market rate. Since the firm's option to invest is perpetual, there would be no gain from installing capacity now rather than later.

12. If we allow for depreciation, investment will occur more frequently and even when demand is somewhat below historic highs, but it will still occur in spurts.

13. Hall (1986) reports that price significantly exceeds marginal cost for most two-digit industries, and finds no explanation for this disparity consistent with competition. Hall's test of marginal cost pricing is based on the relation between the marginal product of labor and the product wage. If firms set marginal operating cost equal to a (constant) proportion of price, his technique will apply, whatever the capital stock. But as shown in Section 4, there can be a wide range of prices for which the firm is capacity constrained, and the ratio of marginal operating cost to price will vary with price.

14. For example, the model developed by Abel and Blanchard (1986) is one of the most sophisticated attempts to explain investment in a q theory framework; it uses a carefully constructed measure for marginal rather than average q, incorporates delivery lags and costs of adjustment, and explicitly measures expectations of future values of explanatory variables. But Abel and Blanchard conclude that "our data are not sympathetic to the basic restrictions imposed by the q theory, even extended to allow for simple delivery lags." Also, see Summers (1981).

15. But they find the opposite true for firms in the oil industry, where there may be a tendency to overinvest in exploration and development.

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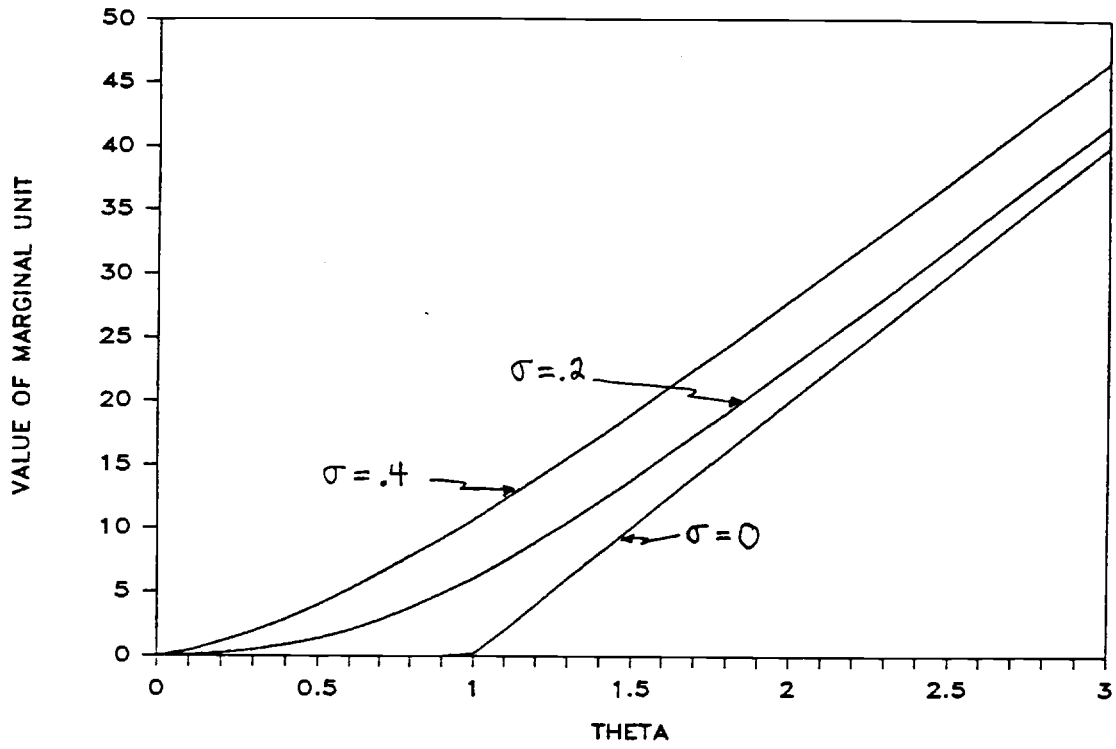
FIGURE 1: VALUE OF A MARGINAL UNIT OF CAPACITY
(K = 1)
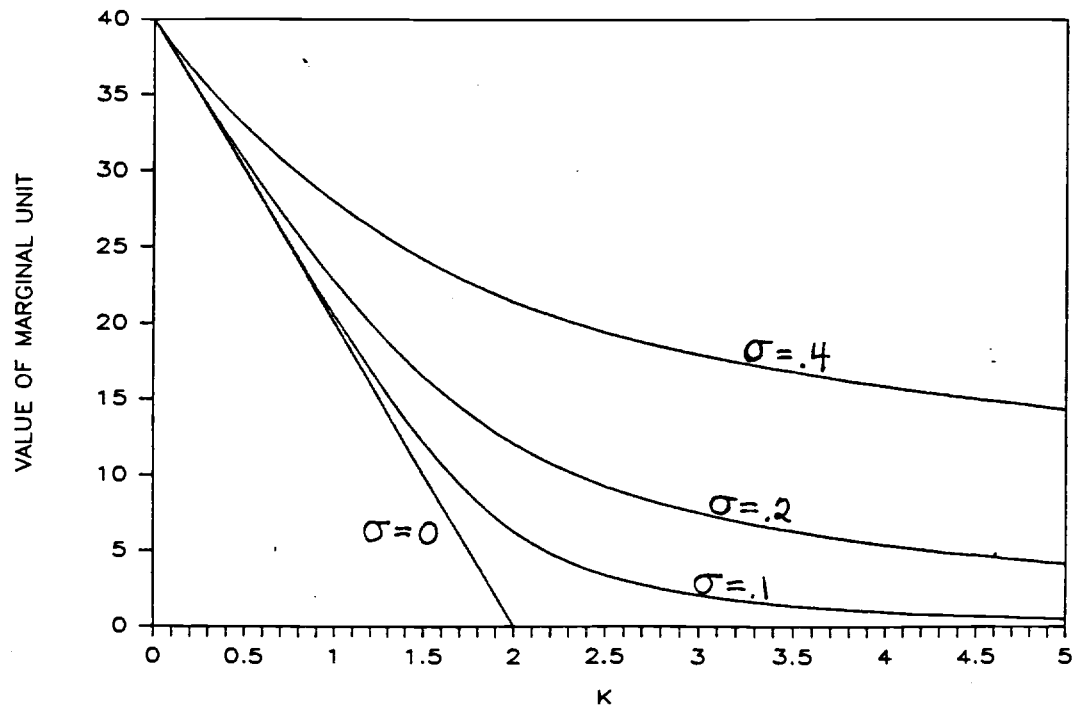

FIGURE 2: MARGINAL VALUE OF INSTALLED CAPACITY
(θ = 2)

FIGURE 3: VALUE OF OPTION TO INVEST IN MARGINAL UNIT OF CAPACITY
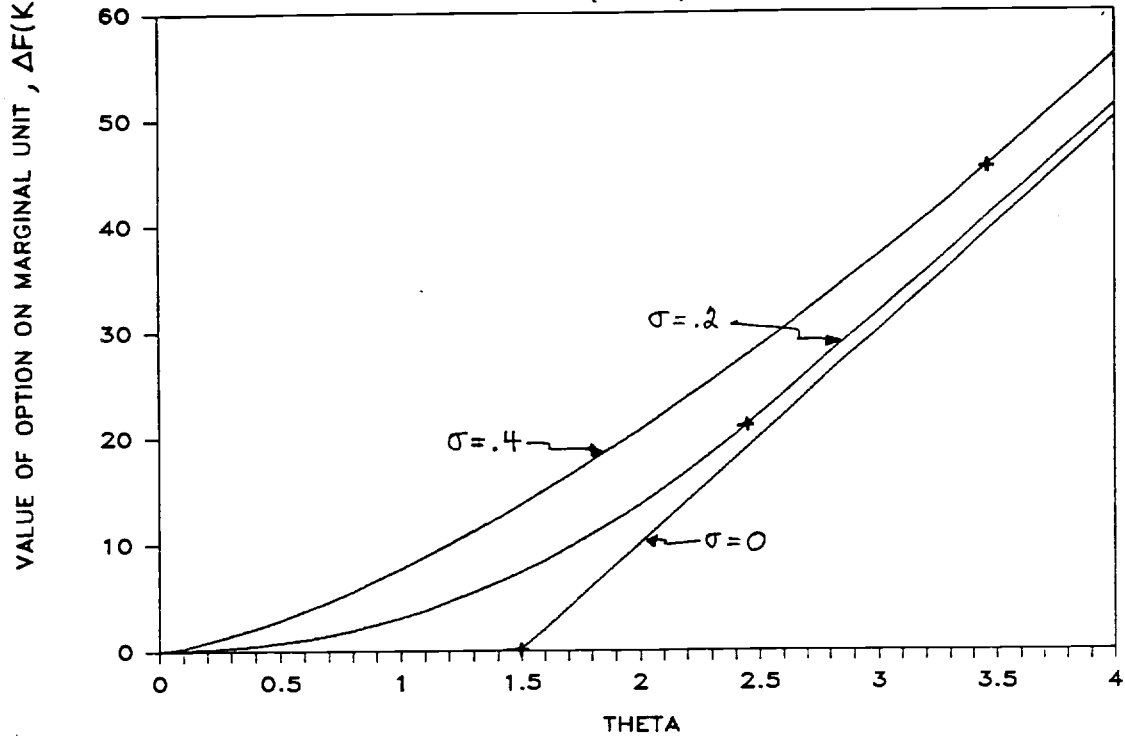(K = 1; + INDICATES θ*(K=1); FOR θ ≥ θ*, ΔF(K) = ΔV(K) − k)
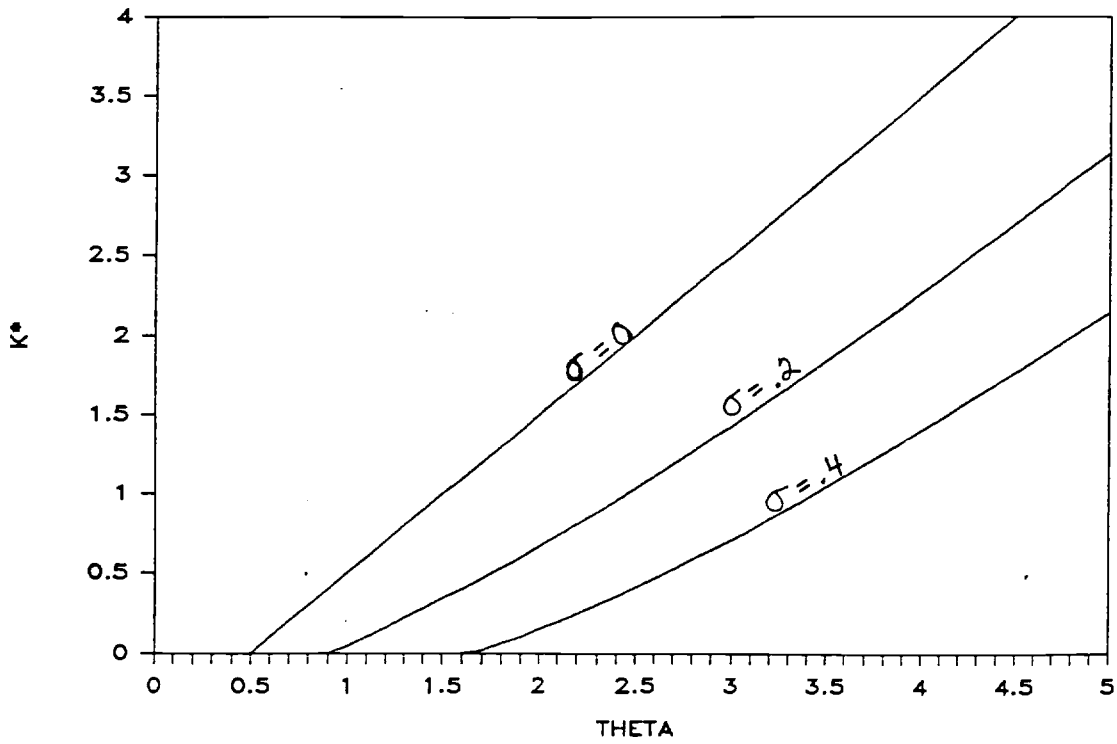

FIGURE 4: OPTIMAL CAPACITY K*(θ)

FIGURE 5: OPTIMAL CAPACITY, NET VALUE OF MARGINAL UNIT OF INSTALLED
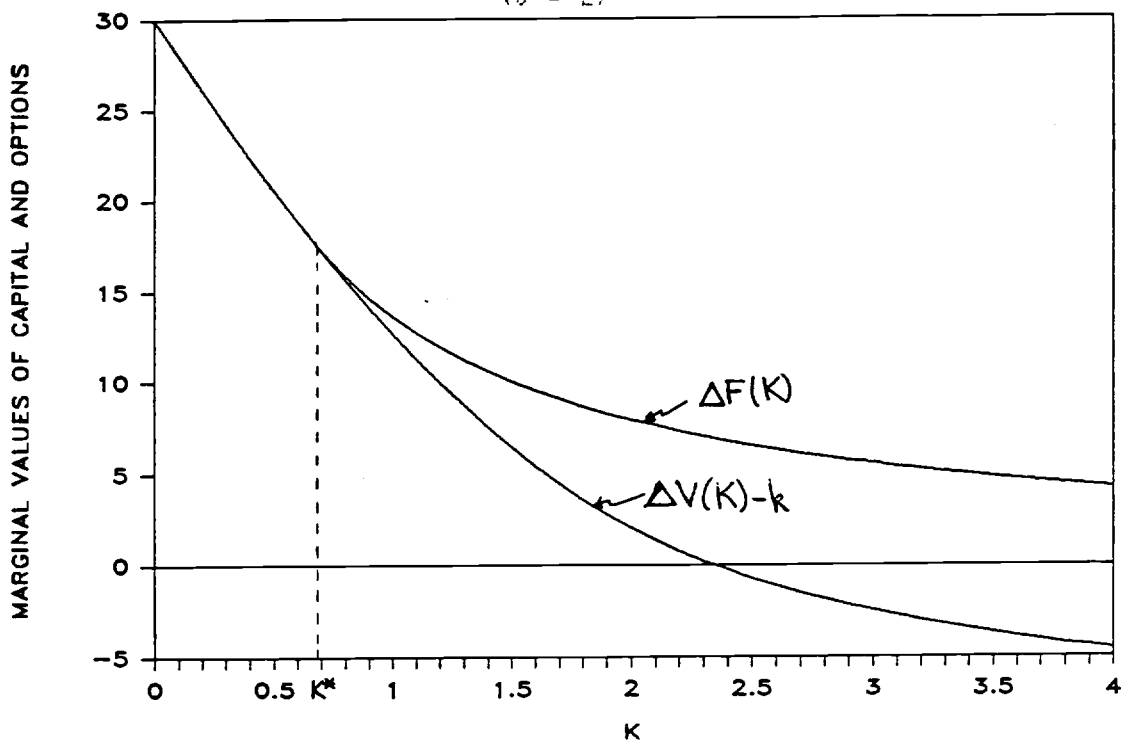CAPACITY, AND VALUE OF OPTION TO INVEST IN MARGINAL UNIT
$(\theta = 2)$

FIGURE 6: BEHAVIOR OF CAPACITY AND CAPACITY UTILIZATION