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**UNIVERSAL ADJACENCY MATRICES WITH TWO
EIGENVALUES**

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Universal adjacency matrices with two eigenvalues

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Abstract

Consider a graph Γ on n vertices with adjacency matrix A and degree sequence (d_1, \dots, d_n) . A universal adjacency matrix of Γ is any matrix in $\text{Span}\{A, D, I, J\}$ with a nonzero coefficient for A , where $D = \text{diag}(d_1, \dots, d_n)$ and I and J are the $n \times n$ identity and all-ones matrix, respectively. Thus a universal adjacency matrix is a common generalization of the adjacency, the Laplacian, the signless Laplacian and the Seidel matrix. We investigate graphs for which some universal adjacency matrix has just two eigenvalues. The regular ones are strongly regular, complete or empty, but several other interesting classes occur.

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1 Introduction

Throughout Γ will denote a simple undirected graph with vertex set $\{v_1, \dots, v_n\}$. The *adjacency matrix* of Γ is the $n \times n$ matrix A_Γ , whose (i, j) -entry is 1 if v_i is adjacent to v_j and is 0, otherwise. Let I be the identity matrix, let J denote the all-ones matrix, and define D_Γ to be the $n \times n$ diagonal matrix whose (i, i) -entry equals the degree d_i of vertex v_i . Any matrix of the form

$$U = U_\Gamma(\alpha, \beta, \gamma, \delta) = \alpha A_\Gamma + \beta I + \gamma J + \delta D_\Gamma,$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, $\alpha \neq 0$ is called a *universal adjacency matrix* of Γ . The matrices $L_\Gamma = U_\Gamma(-1, 0, 0, 1)$ and $Q_\Gamma = U_\Gamma(1, 0, 0, 1)$ are better known as the *Laplacian matrix*, and *signless*

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Laplacian of Γ , respectively. The matrix $S_\Gamma = U_\Gamma(-2, -1, 1, 0)$ is the *Seidel matrix* of Γ , and $A_{\bar{\Gamma}} = U_\Gamma(-1, -1, 1, 0)$ is the adjacency matrix of $\bar{\Gamma}$, the complement of Γ . If $\delta = 0$, a universal matrix is often called a *generalized adjacency matrix*; see for example [3],[11]. In this paper we investigate graphs for which some universal adjacency matrix has at most two distinct eigenvalues. The only symmetric matrices with just one eigenvalue are the multiples of I , therefore only the complete graph K_n , and its complement the empty graph nK_1 admit a universal adjacency matrix with only one eigenvalue. Note that for $x, y \in \mathbb{R}$, $x \neq 0$, ρ is an eigenvalue of $U_\Gamma(\alpha, \beta, \gamma, \delta)$, if and only if $x\rho - y$ is an eigenvalue of $U_\Gamma(x\alpha, x\beta - y, x\gamma, x\delta)$, so without loss of generality we may assume that $\alpha = 1$ and $\beta = 0$, but it is not always convenient to do so.

Graphs with at most three eigenvalues for the adjacency, the Laplacian, the signless Laplacian and the Seidel matrix have been investigated in a number of papers. Many of these graphs admit a universal adjacency matrix with two eigenvalues. All these cases can be seen as a generalization of strongly regular graphs. For results on graphs with few distinct adjacency eigenvalues, we refer the reader to [2, 4, 5, 7, 8, 9, 12, 14]; for few Laplacian eigenvalues see [8, 10, 17] and for few signless Laplacian eigenvalues see [1]. For the Seidel matrix we refer to [16]. A short survey is given in [3].

2 Strong graphs

Recall that Γ is *strongly regular* with parameters (n, k, λ, μ) , whenever Γ is k -regular with $0 < k < n - 1$, and the number of common neighbors of any two distinct vertices equals λ if the vertices are adjacent and μ otherwise (see for example [3]). In terms of the adjacency matrix $A = A_\Gamma$, this means that $A\mathbf{1} = k\mathbf{1}$ ($\mathbf{1}$ is the all-ones vector), and $A^2 = kI + \lambda A + \mu(J - A - I)$. This implies that $\mathbf{1}$ is an eigenvector with eigenvalue k , and each other eigenvalue has its eigenvectors orthogonal to $\mathbf{1}$ and equals one of the two roots r and s (say) of the equation $\rho^2 + (\mu - \lambda)\rho + \mu - k = 0$. Since A and J have a common basis of eigenvectors, it follows that both $A + \frac{r-k}{n}J$ and $A + \frac{s-k}{n}J$ are universal matrices with two distinct eigenvalues. Conversely, if Γ is k -regular with $0 < k < n - 1$, and admits a universal matrix $U_\Gamma(\alpha, \beta, \gamma, \delta)$ with two eigenvalues, then also $U_\Gamma(1, 0, \frac{\gamma}{\alpha}, 0) = A + \frac{\gamma}{\alpha}J$ has two eigenvalues and it follows that $A^2 \in \text{Span}\{A, I, J\}$ which proves that Γ is strongly regular. Thus we have:

Proposition 1 *A regular graph Γ , which is not complete or empty, has a universal adjacency matrix with two eigenvalues if and only if Γ is strongly regular.*

Seidel (see for example [15]) considers the matrix $S_\Gamma = U_\Gamma(-2, -1, 1, 0)$ of a graph Γ . This matrix has zero diagonal, off diagonal entries ± 1 , and is now known as the Seidel matrix of Γ . If Γ is strongly regular then it follows that there exist $\rho_1, \rho_2 \in \mathbb{R}$ such that

$$(S_\Gamma - \rho_1 I)(S_\Gamma - \rho_2 I) = (n - 1 + \rho_1 \rho_2)J.$$

However the converse need not be true. Seidel called a graph *strong* if the Seidel matrix satisfies the above equation. It easily follows that a strong graph with $n - 1 + \rho_1\rho_2 \neq 0$ is regular, and that a graph Γ is strong and regular if and only if Γ is strongly regular, complete or empty. Assume Γ is strong with $n - 1 + \rho_1\rho_2 = 0$. Then Γ has two eigenvalues ρ_1 and ρ_2 . Multiplying some rows and the corresponding columns of S_Γ by -1 is a similarity operation on S_Γ which doesn't change the eigenvalues, but the result is again the Seidel matrix $S_{\Gamma'}$ of a graph Γ' . So clearly also Γ' is strong with $n - 1 + \rho_1\rho_2 = 0$. The graph operations that transforms Γ into Γ' is called *Seidel switching*, and Γ and Γ' are said to be *switching equivalent* (indeed, the relation defined by Seidel switching is an equivalence relation). If one switches in Γ with respect to the neighbors of a given vertex v , then the obtained graph Γ' has the vertex v isolated, and if we also switch with respect to v , we will obtain a graph Γ'' in which v is adjacent to all other vertices. Let Γ_v be the graph obtained from Γ' (or Γ'') by deleting v . The graph Γ'' is called the *cone* over Γ_v . It can be shown that Γ_v is the complete graph, the empty graph, or a strongly regular graph with parameters $(n - 1, k, \lambda, \mu)$ with $k = 2\mu$. And conversely, the cone over a strongly regular graph with $k = 2\mu$ is a strong graph whose Seidel matrix has two eigenvalues. For example the pentagon C_5 is strongly regular with $k = 2\mu = 2$. So $C_5 + K_1$ and W_6 (the wheel with 5 spokes) are strong; the Seidel eigenvalues are $\pm\sqrt{5}$. Also the Petersen graph is strong with two Seidel eigenvalues, being ± 3 . If we isolate a vertex by switching, the graph on the remaining vertices is the (unique) strongly regular graph with parameters $(9, 4, 1, 2)$. Note that also the complete bipartite graphs $K_{\ell, m}$ and their complements are strong. Indeed, they can be switched into the empty graph with Seidel matrix $J - I$, and the complete graph with Seidel matrix $I - J$, respectively. An equivalence class of strong graphs with two Seidel eigenvalues is the same as a *regular two-graph*; see [16].

It was observed in [3] that strong graphs are precisely the graphs that admit a generalized adjacency matrix with two eigenvalues. Recall that a generalized adjacency matrix is a universal matrix $U_\Gamma(\alpha, \beta, \gamma, \delta)$ with $\delta = 0$, so we get:

Proposition 2 *A graph Γ with at least two vertices admits a universal adjacency matrix $U_\Gamma(\alpha, \beta, \gamma, 0)$ with two eigenvalues if and only if Γ is strong.*

Correctness of the ‘if’ of this statement has already been established. The ‘only if’ will be proved below.

3 Graphs with constant μ and $\bar{\mu}$

Consider the Laplacian matrix $L_\Gamma = U_\Gamma(-1, 0, 0, 1)$ of Γ . The all-ones vector $\mathbf{1}$ is an eigenvector of L_Γ with eigenvalue 0. We call an eigenvalue *restricted* if it has an eigenvector orthogonal to $\mathbf{1}$. Suppose L_Γ has exactly two distinct restricted eigenvalues ρ_1 and ρ_2 . Then $L_\Gamma + \frac{\rho_1}{n}J$ and

$L_\Gamma + \frac{\rho^2}{n}J$ are universal adjacency matrices with two eigenvalues. Conversely, if a universal matrix $U_\Gamma(\alpha, \beta, \gamma, \delta)$ with $\alpha = -\delta$ has two eigenvalues, then $L_\Gamma = -\frac{1}{\alpha}U + \frac{\beta}{\alpha}I + \frac{\gamma}{\alpha}J$ is the Laplacian matrix of Γ , and has at most two restricted eigenvalues. Graphs with the above property have been investigated in [10] (see also [8, 3]). They have an easy characterization in terms of the following concept. A graph Γ is said to have *constant* $\mu(\Gamma)$ if Γ is not complete, and any two nonadjacent vertices have the same number of common neighbors (equal to $\mu(\Gamma)$).

Theorem 1[10] *For a graph Γ the following are equivalent:*

- (i) *The Laplacian matrix of Γ has exactly two restricted eigenvalues.*
- (ii) *Γ has constant $\mu = \mu(\Gamma)$ and its complement $\bar{\Gamma}$ has constant $\bar{\mu} = \mu(\bar{\Gamma})$.*
- (iii) *Γ is not complete or empty and has a universal adjacency matrix $U_\Gamma(\alpha, \beta, \gamma, -\alpha)$ with two eigenvalues.*

A regular graph with constant μ and $\bar{\mu}$ is strongly regular (this follows for example from Proposition 1). However many non-regular graphs with constant μ and $\bar{\mu}$ are known. For example, if N is a symmetric incidence matrix of a projective plane of order ℓ with a polarity, then $L = (\ell + 1)I - N$ is a Laplacian matrix with two eigenvalues, and because N cannot have a constant diagonal, the corresponding graph has vertex degrees ℓ and $\ell + 1$. It can be seen that graphs with constant μ and $\bar{\mu}$ have at most two degrees. In fact, this property holds more generally as we shall see in the next section.

4 Regularity

For two adjacent vertices v_i and v_j of Γ , $\lambda(v_i, v_j)$ is the number of common neighbors of v_i and v_j . Similarly, $\mu(v_i, v_j)$ denotes the number of common neighbors of two distinct nonadjacent vertices v_i and v_j . Recall that d_i denotes the degree of v_i .

Lemma 1 *A graph Γ with at least two vertices admits a universal adjacency matrix with two eigenvalues if and only if there exist $\beta, \gamma, \delta, \rho \in \mathbb{R}$ such that*

- (i) $\delta^2 d_i^2 + d_i(2\gamma\delta + 2\beta\delta + 2\gamma + 1) + \gamma^2 n + 2\beta\gamma + \beta^2 = \rho^2$, for $i = 1, \dots, n$,
- (ii) $\lambda(v_i, v_j) = -(d_i + d_j)(\gamma\delta + \gamma + \delta) - 2\beta(\gamma + 1) - n\gamma^2$, if v_i and v_j are adjacent,
- (iii) $\mu(v_i, v_j) = -(d_i + d_j)(\gamma\delta + \gamma) - 2\beta\gamma - n\gamma^2$ if v_i and v_j are distinct and nonadjacent.

Proof Let $U = U_\Gamma(\alpha, \beta, \gamma, \delta)$ be a universal adjacency matrix of Γ with two eigenvalues. Without loss of generality we take $\alpha = 1$, and choose β such that the two eigenvalues get opposite sign: $\pm\rho$. Then $U^2 = \rho^2 I$. From this the result follows straightforwardly. \square

Note that in case $\delta = 0$, Equation (i) implies that Γ is regular, or $2\gamma + 1 = 0$. In the latter case $S = \beta I - U_\Gamma(1, \beta, -\frac{1}{2}, 0)$ is the Seidel matrix of Γ with two eigenvalues. Thus we have the promised proof of the ‘only if’ part of Proposition 2.

Corollary 1 *Assume Γ admits a universal adjacency matrix $U_\Gamma(\alpha, \beta, \gamma, \delta)$ with two eigenvalues. If Γ is not strong, then Γ has two degrees. Moreover, the number of common neighbors of any pair of vertices $\{v_i, v_j\}$ only depends on d_i, d_j and whether v_i and v_j are adjacent.*

Proof First observe that Γ is not strongly regular, so by Proposition 1, Γ has at least two degrees. Proposition 2 implies that $\delta \neq 0$, so (i) of the lemma gives a quadratic equation in d_i . The second part of the statement follow from (ii) and (iii) of the above lemma. \square

In case Γ has two degrees, k_1 , and k_2 say, then we write $\lambda_{\ell,m} = \lambda(v_i, v_j)$, and $\mu_{\ell,m} = \mu(v_i, v_j)$, if v_i has degree k_ℓ and v_j has degree k_m ($\ell, m = 1, 2$).

We remark that strong graphs may have more than two degrees. Indeed, the switching class of the Petersen graph (a strong graph with two Seidel eigenvalues) contains graphs with more than two different degrees.

Lemma 1 provides a fast tool for testing whether a given graph Γ admits a universal adjacency matrix with two eigenvalues. It works as follows. First check if Γ is strong. This is the case if and only if the Seidel matrix S satisfies $S^2 \in \text{Span}\{S, I, J\}$. If Γ is not strong, then we check if Γ has two degrees, and if $\lambda(v_i, v_j)$ and $\mu(v_i, v_j)$ only depend on the degrees of v_i and v_j . If so then (i)–(iii) of the lemma give (at most) seven equations for β, γ and δ , and it is easily checked if there is a solution. For example if Γ is the path P_4 , then Γ is not strong, $k_1 = 1, k_2 = 2, \lambda_{1,2} = \lambda_{2,2} = \mu_{1,1} = 0, \mu_{1,2} = 1$, whilst $\lambda_{1,1}$ and $\mu_{2,2}$ are undefined. Now we easily find that $\beta = -1, \gamma = -\frac{1}{2}, \delta = 1$ satisfies all conditions (with $\rho = \pm 2$). Thus $U_{P_4}(1, -1, -\frac{1}{2}, 1)$ has two eigenvalues. In the next section we shall generalize this example to infinite families.

In the previous sections we have classified the cases $\delta = 0, \delta = -\alpha$, and Γ is regular. Let us call a universal adjacency matrix *proper* if it does not belong to one of the three classified cases. For example, the universal adjacency matrix $U_{P_4}(1, -1, -\frac{1}{2}, 1)$ of the path P_4 is proper. By Corollary 1, a graph admitting a proper universal adjacency matrix with two eigenvalues has exactly two degrees. But we can prove more.

Theorem 2 *If Γ admits a proper universal adjacency matrix with two eigenvalues, then Γ has two degrees and for both degrees, the subgraph induced by the set of vertices with that degree is regular.*

Proof From the above we know that Γ has exactly two different degrees. Assume again that $U = U_\Gamma(\alpha, \beta, \gamma, \delta)$ satisfies $U^2 = \rho^2 I$, and let \mathbf{w}_1 and \mathbf{w}_2 be characteristic vectors of the sets of

vertices with the same degree. Define $W = \text{Span}\{\mathbf{w}_1, \mathbf{w}_2\}$ and $\mathbf{w} = U\mathbf{1}$, then $U\mathbf{w} = \rho^2\mathbf{1}$ and $\mathbf{w} \in W$. Since $\alpha + \delta \neq 0$, the row sums of U are not constant, so \mathbf{w} and $\mathbf{1}$ are independent, so $\text{Span}\{\mathbf{w}, \mathbf{1}\} = W$. Therefore $U\mathbf{w}_1 \in W$ and $U\mathbf{w}_2 \in W$, which reflects that the subgraphs are regular. \square

In case Γ has two degrees, and $U_\Gamma(\alpha, \beta, \gamma, -\alpha)$ (which is not proper) has two eigenvalues, the two subgraphs are not necessarily regular. See [10] for counter examples.

5 Constructions

The necessary conditions of Corollary 1 and Theorem 2 for proper universal adjacency matrices are by no means sufficient. For example, the disjoint union of two connected strongly regular graphs with different degrees satisfies these conditions, but $\mu_{1,2} = 0$ and (iii) of Lemma 1 yields $\mu_{1,1} + \mu_{2,2} = 2\mu_{1,2}$, which is impossible. However, if one of the two graphs is the complete graph, then it may work. We deal with this in the following two theorems.

Theorem 3 *A strongly regular graph extended by an isolated vertex, admits a universal adjacency matrix with two eigenvalues.*

Proof Let A be the adjacency matrix of the strongly regular graph with eigenvalues k, r and s . It suffices to show that there are real numbers β and γ , such that the following matrix has two eigenvalues.

$$U = \begin{bmatrix} \beta & \mathbf{0}^\top \\ \mathbf{0} & A \end{bmatrix} + \gamma J.$$

Define $\mathbf{w}_1 = [1 \ 0 \ \cdots \ 0]^\top$, $\mathbf{w}_2 = [0 \ 1 \ \cdots \ 1]^\top$, and $W = \text{Span}\{\mathbf{w}_1, \mathbf{w}_2\}$. Then the eigenvectors of U are in W , or orthogonal to W . The latter eigenvectors all correspond to an eigenvalue r or s . The two remaining eigenvalues are eigenvalues of the quotient matrix

$$R = \begin{bmatrix} \beta + \gamma & (n-1)\gamma \\ \gamma & k + (n-1)\gamma \end{bmatrix}.$$

So it suffices to show that we can choose β and γ such that R has eigenvalues r and s . From $\text{tr} R = r + s$ we obtain $\beta = r + s - k - n\gamma$, and $\det R = rs$ leads to the following quadratic equation in γ :

$$(n-1)(n-2)\gamma^2 + (n-1)(r+s)\gamma - (k-r)(k-s) = 0,$$

which has a real solution because $(n-1)(n-2)(k-r)(k-s) \geq 0$. \square

In fact, the above quadratic equation has two solutions so there are two choices of (β, γ, δ) for which $U_\Gamma(1, \beta, \gamma, \delta)$ has two eigenvalues. If $k = r$, the strongly regular graph is the disjoint union of complete graphs K_{k+1} . Then $\gamma = 0$ and $\beta = k$ gives a third possibility.

Note that Γ has a universal adjacency matrix with two eigenvalues if and only if its complement has one. Therefore a cone over a strongly regular graph also admits a universal adjacency matrix with two eigenvalues. A similar remark holds for the next result.

Theorem 4 *Suppose Γ is the disjoint union of the complete graph K_m ($m \geq 2$) with a strongly regular graph with adjacency eigenvalues k ($=$ degree), r and s . Then Γ has a universal adjacency matrix with two eigenvalues if and only if $k \leq -rs - s + r - 2$. If this is the case then r and s are integers, $m = r - s - rs - k$ and $U_\Gamma(1, 0, \frac{s+rs}{n}, \frac{s+1}{r-s-rs-2k-1})$ has two eigenvalues.*

Proof Let A be the adjacency matrix of the strongly regular graph. Then there exist a universal adjacency matrix for Γ if and only if there exist real numbers x and γ , such that the following matrix has two eigenvalues:

$$U = \begin{bmatrix} J + xI & O \\ O & A \end{bmatrix} + \gamma J.$$

Again let \mathbf{w}_1 and \mathbf{w}_2 be the characteristic vectors of the partition, and define $W = \text{Span}\{\mathbf{w}_1, \mathbf{w}_2\}$. Then the eigenvectors of U are in W , or orthogonal to W . The latter eigenvectors all correspond to an eigenvalue r , s , or x . Therefore $x = r$, or $x = s$. We forget the convention that $r > s$, and take $x = s$ without loss of generality. The two remaining eigenvalues of U are the eigenvalues of the quotient matrix

$$R = \begin{bmatrix} s + m + m\gamma & \ell\gamma \\ m\gamma & k + \ell\gamma \end{bmatrix},$$

where $\ell = n - m$ is the order of the strongly regular graph. So we have to choose m and γ such that R has eigenvalues r and s . Using $\text{tr} R = r + s$, and $\det R = rs$, we get $m = r - s - \frac{1}{\ell}(k - r)(k - s)$. The eigenvalues of a strongly regular graph satisfy $(k - r)(k - s) = \ell(rs + k)$, therefore $m = r - s - rs - k$. Note that m is negative if $s \geq 0 > r$, so s is the negative eigenvalue as is conventional. Also if r and s are not integral, then $k = \frac{1}{2}(\ell - 1)$, $r, s = \frac{1}{2}(-1 \pm \sqrt{\ell})$ which implies that $m < 2$. \square

For example let Γ' be a strongly regular graph with parameters $(16, 6, 2, 2)$. Then Γ' has adjacency eigenvalues $k = 6$, $r = -s = 2$. Theorem 4 gives $m = 2$. Therefore $\Gamma = \Gamma' + K_2$ admits a universal adjacency matrix with two eigenvalues. Indeed, $U_\Gamma(1, -\frac{13}{15}, -\frac{1}{3}, \frac{1}{5})$ has eigenvalues ± 2 . There are infinitely many strongly regular graphs satisfying the condition of Theorem 4. For example the collinearity graphs of generalized quadrangles with $p > 2$ points on a line gives $m = p - 2$ (see for example [3]). In most cases the universal adjacency matrix of Theorem 4 is proper. An improper example is obtained if the involved strongly regular graph is ℓK_{k+1} . Then $m = k + 1$, and $\Gamma = (\ell + 1)K_m$.

Suppose N is the incidence matrix of a symmetric 2 - (v, k, λ) design \mathcal{D} . That is, N is a $v \times v$ matrix satisfying $NN^\top = N^\top N = \lambda J + (k - \lambda)I$ ($v > k > \lambda \geq 0$). Let Γ be the graph with

adjacency matrix

$$A = \begin{bmatrix} O & N \\ N^\top & J - I \end{bmatrix}.$$

Then Γ is called the *split graph* of \mathcal{D} . Note that if Γ is the split graph of \mathcal{D} , then $\bar{\Gamma}$ (the complement of Γ) is the split graph of the complement of \mathcal{D} (that is, the design with incidence matrix $J - N$).

Theorem 5 *The split graph of a symmetric design admits a universal adjacency matrix with two eigenvalues.*

Proof Let N be the incidence matrix of the design. It suffices to prove that there exist numbers x and γ such that

$$U = \begin{bmatrix} O & N \\ N^\top & J + xI \end{bmatrix} + \gamma J$$

has two eigenvalues. Let $W = \text{Span}\{\mathbf{w}_1, \mathbf{w}_2\}$, where (as before) \mathbf{w}_1 and \mathbf{w}_2 are the characteristic vectors of the partition. We easily have $U^2 = xU + (k - \lambda)I + K$, where K is a symmetric matrix with columns in W . Therefore the eigenvalues of U with eigenvectors orthogonal to W are $(x \pm \sqrt{x^2 + 4(k - \lambda)})/2$. The eigenvalues with eigenvectors in W are the eigenvalues of the quotient matrix

$$R = \begin{bmatrix} \gamma v & \gamma v + k \\ \gamma v + k & \gamma v + v + x \end{bmatrix}.$$

So the eigenvalues of R must be equal to the two eigenvalues obtained above. This is the case if $\text{tr } R = x$, and $\det R = -k + \lambda$, which leads to $\gamma = -\frac{1}{2}$, and $x = -1 - 2k + 2(k^2 + k - \lambda)/v$. \square

The symmetric design mentioned in Theorem 5 may be degenerate, in which case $(v, k, \lambda) = (v, 1, 0)$, or $(v, v - 1, v - 2)$. Then Γ , or $\bar{\Gamma}$ is a complete graph K_v with a pendant edge attached to each vertex. The smallest example has $v = 2$. Then Γ (and $\bar{\Gamma}$) is the path P_4 .

Other examples can be made from special Hadamard matrices. A Hadamard matrix H is an $n \times n$ matrix with entries ± 1 , such that $HH^\top = nI$.

Theorem 6 *Suppose H is a symmetric Hadamard matrix with the property that the sum of the entries of row i only depends on the diagonal value $(H)_{ii}$. Then H is a universal adjacency matrix with two eigenvalues, which is proper if the diagonal of H is not constant.*

Proof Let A be the off-diagonal part of $\frac{1}{2}(H + J)$, and let D' be the diagonal part of $\frac{1}{2}(H + J)$, thus $A + D' = \frac{1}{2}(H + J)$. If Γ is the graph with adjacency matrix $A = A_\Gamma$, then $D' \in \text{Span}\{I, D_\Gamma\}$, and hence $H \in \text{Span}\{A_\Gamma, I, J, D_\Gamma\}$. If the diagonal of $H = U_\Gamma(\alpha, \beta, \gamma, \delta)$ is not constant also the row sums are not constant. Therefore $\delta \neq 0$, and $\alpha \neq -\delta$ so H is proper. \square

A Hadamard matrix is *regular* if all its row sums are equal, and *graphical* if it is symmetric with constant diagonal (which we will assume to be positive). If H is a regular graphical Hadamard matrix of order n with row sum ℓ , then $\ell = \pm\sqrt{n}$, and $S = H - I$ is the Seidel matrix of a strongly regular graph with parameters $(\ell^2, \frac{1}{2}(\ell^2 - \ell), \frac{1}{4}(\ell^2 - 2\ell), \frac{1}{4}(\ell^2 - 2\ell))$. And conversely, if S is the Seidel matrix of such a graph, then $S + I$ is a regular graphical Hadamard matrix (see for example [3]). Many such Hadamard matrices exist, for example if $\ell = \pm 2m^2$, $m > 0$ (see [13]). If H is a regular graphical Hadamard matrix, then it easily follows that

$$U = \begin{bmatrix} H & H \\ H & -H \end{bmatrix}$$

is a symmetric Hadamard matrix that satisfies the conditions of the above proposition. Other Hadamard matrices with the required property can be constructed from strongly regular graphs with parameters $(\ell^2 + 1, \frac{1}{2}(\ell^2 - \ell), \frac{1}{2}(\ell - 3), \frac{1}{2}(\ell - 1))$. The Seidel matrix S of such a graph satisfies $S^2 = \ell^2 I$ and $S\mathbf{1} = \ell\mathbf{1}$, and is called a *regular conference* matrix. It is easily checked that

$$U = \begin{bmatrix} S + I & S - I \\ S - I & -S - I \end{bmatrix}$$

is a Hadamard matrix with the required structure, and therefore a universal adjacency matrix with two eigenvalues. Regular conference matrices exist if ℓ is an odd prime power. Also the degenerate case $\ell = 1$ works and leads to $U_{P_4}(-2, 2, 1, -2)$ which is a multiple of $U_{P_4}(1, -1, -\frac{1}{2}, 1)$, mentioned in the previous section.

6 Characterizations

In the previous sections we classified the graphs for which some universal adjacency matrix has two eigenvalues into the strong graphs, graphs with constant μ and $\bar{\mu}$ and the graphs with a proper universal adjacency matrix. It seems that the characterization of each of these classes is a difficult problem. In this section we give characterizations in some special cases.

Lemma 2 *Suppose Γ is the disjoint union of complete graphs. Then Γ admits a universal adjacency matrix with two eigenvalues if and only if Γ is $K_\ell + K_m$, or $gK_1 + hK_m$ ($\ell, m \geq 1$, $g, h \geq 0$, $g + hm \geq 2$).*

Proof The graph $K_\ell + K_m$ is strong, and therefore the Seidel matrix has two eigenvalues. If $\Gamma = gK_1 + hK_m$, $m > 1$, $h > 0$, then the Laplacian matrix has two eigenvalues 0 and m . If $m = 1$ or $h = 0$, then Γ is empty and $U_\Gamma(1, 0, 1, 0)$ has two eigenvalues.

Suppose $U = U_\Gamma(1, \beta, \gamma, \delta)$ has two eigenvalues. We choose β such that the equations in Lemma 1 hold. If $\delta = -1$, then by Theorem 3 Γ has constant $\bar{\mu}$. This means that, except for

isolated vertices, the cliques in Γ have the same size. Suppose $\gamma = 0$. Then U is a block diagonal matrix $\text{diag}(U_1, \dots, U_t)$ with $U_i = J + (\beta + \delta(m_i - 1))I$ of order m_i . If $m_i > m_j > 1$ for some i, j , then the eigenvalues of B_i and B_j together (which are eigenvalues of U) are $\beta + \delta(m_i - 1)$, $m_i + \beta + \delta(m_i - 1)$, $\beta + \delta(m_j - 1)$, and $m_j + \beta + \delta(m_j - 1)$. At least three of these eigenvalues are mutually different, so we have a contradiction proving that $\Gamma = gK_1 + hK_m$ if $\gamma = 0$. Thus we can assume that $\gamma\delta + \gamma \neq 0$. Now (iii) of Lemma 1 gives that $d_i + d_j$ is constant for any two nonadjacent vertices. So either there exist no three mutually nonadjacent vertices in which case $\Gamma = K_\ell + K_m$, or all vertices have the same degree and $\Gamma = hK_m$. \square

Theorem 7 *A disconnected graph Γ admits a universal adjacency matrix with two eigenvalues if and only if Γ is one of the graphs mentioned in Lemma 2, Theorem 3, or Theorem 4.*

Proof Let Γ be a disconnected graph for which $U_\Gamma(1, \beta, \gamma, \delta)$ has two eigenvalues. We choose β such that the equations in Lemma 1 hold. If Γ is regular, then Γ is strongly regular, complete or empty, hence $\Gamma = \ell K_m$. Assume Γ has three or more different degrees. Then, by Corollary 1, Γ is strong and therefore $\gamma = -1/2$, $\delta = 0$. Choose vertices u, v , and w with different degrees, such that u and v do not lie in the component containing w (note that we can always do so). Since $\mu(u, w) = \mu(v, w) = 0$ and $\gamma\delta + \gamma \neq 0$, (iii) of Lemma 1 implies that u and v have the same degree; a contradiction. So we can assume that Γ has exactly two degrees k_1 and k_2 . Then $n \geq 3$, and we can choose two vertices v and w in different components with different degrees. Therefore $\mu(v, w) = \mu_{1,2} = 0$. Now (iii) of Lemma 1 gives $\mu_{1,1} + \mu_{2,2} = 2\mu_{1,2}$. So if $\mu_{1,1}$ and $\mu_{2,2}$ are defined, then Γ has constant $\mu(\Gamma) = 0$. This implies that Γ is the disjoint union of complete graphs, so we are in the case of Lemma 2. If $\mu_{i,i}$ is not defined, then the vertices of degree k_i induce a clique in Γ . If both $\mu_{1,1}$ and $\mu_{2,2}$ are undefined, then $\Gamma = K_{k_1+1} + K_{k_2+1}$. Assume $\mu_{1,1}$ is not defined. If two vertices of degree k_2 are in different components, then $\mu_{2,2} = 0$, and hence Γ has constant $\mu(\Gamma) = 0$, and Γ is a graph mentioned in Lemma 2. So we can assume that Γ has two components, one of which is a clique. By Lemma 1, the other component is k_2 -regular with constant $\lambda_{2,2}$ and $\mu_{2,2}$, so it is a strongly regular graph and we are in the situation of Theorem 3, or Theorem 4. \square

In Theorem 5 we constructed split graphs with a universal adjacency matrix with two eigenvalues. The next result shows that, except for some trivial cases, there are no other split graph with that property.

Theorem 8 *Suppose Γ is a split graph admitting a universal adjacency matrix with two eigenvalues. Then Γ , or its complement $\bar{\Gamma}$ is $\ell K_1 + K_m$, or Γ is the split graph of a symmetric design.*

Suppose Γ is disconnected. Then by Theorem 7, Γ consists of a strongly regular graph together with some isolated vertices, or Γ is one of the trivial graphs mentioned in Lemma 2. Out of

these only $\ell K_1 + K_m$ is a split graph. Similarly, if $\bar{\Gamma}$ is disconnected then $\bar{\Gamma} = \ell K_1 + K_m$.

Next assume that Γ and $\bar{\Gamma}$ are connected. Let V_1 and V_2 be the vertex sets of the clique and coclique (respectively) corresponding to the split of Γ . Then $|V_i| \geq 2$ ($i = 1, 2$). Without loss of generality we assume $|V_1| \geq |V_2|$ (otherwise we consider the complement $\bar{\Gamma}$).

Suppose Γ has constant $\mu = \mu(\Gamma)$. Then clearly $\mu > 0$. Since $\bar{\Gamma}$ is connected there exists nonadjacent vertices $v_1 \in V_1$ and $v_2 \in V_2$. It follows that every vertex adjacent to v_1 is also adjacent to v_2 , so $d_1 = \mu > 0$. Moreover, for each other vertex $v \in V_1$, we have $\mu(v, v_1) = \mu = d_1$, therefore every neighbor of v_1 is adjacent to every vertex in V_1 . This implies that the neighbors of v_1 are isolated vertices in $\bar{\Gamma}$, which contradicts the connectivity of $\bar{\Gamma}$. So Γ doesn't have constant μ .

Assume Γ is strong. Clearly Γ is not regular, so the Seidel matrix has two eigenvalues. Therefore Γ is switching equivalent with a graph Γ' which is the cone over a strongly regular graph with parameters $(n-1, k, \lambda, \mu)$, where $k = 2\mu$. It follows that Γ' has a clique of size $\omega = |V_1| - 1 \geq \frac{n-1}{2}$. A strongly regular graph with such a large clique can only be a cocktail party graph (complement of mK_2), the disjoint union of two complete graphs, or the pentagon (this is a consequence of Delsarte's clique bound $\omega \leq 1 - k/s$ and the fact that $s \leq -2$ and $k \leq n-4$, except for the mentioned families; see for example [3]). Out of these only the pentagon has $k = 2\mu$. Thus Γ is switching equivalent to the wheel W_6 . The only split graphs in the switching class of W_6 are the split graphs of the (trivial) 2-(3, 1, 0) design, and its complement.

Finally we consider the case that Γ has a proper universal adjacency matrix with two eigenvalues. By Theorem 2, Γ has two degrees k_1 and k_2 and the two subgraphs induced by the vertices of the same degree are regular. It is straightforward that these two vertex sets coincide with V_1 and V_2 . Therefore Γ is the split graph of an incidence structure \mathcal{D} whose incidence matrix N has constant row and column sum. Moreover, (ii) and (iii) of Lemma 1 give that $NN^\top = (\lambda_{1,1} - |V_1| + 2)J + (k_1 - \lambda_{1,1} - 1)I$ and $N^\top N = \mu_{2,2}J + (k_2 - \mu_{2,2})I$. Since $N \neq O$ and $N \neq J$ (otherwise Γ , or $\bar{\Gamma}$ would be disconnected), this implies that \mathcal{D} is a symmetric 2-(v, k, λ) design with $v = \frac{n}{2}$, $k = k_2 = k_1 + 1 - v$, and $\lambda = \mu_{2,2} = \lambda_{1,1} + 2 - v$.

Theorem 9 *A connected bipartite graph Γ with at least two vertices has a universal adjacency matrix with two eigenvalues if and only if Γ is a complete bipartite graph or the path P_4 .*

Proof In the previous sections we saw that if Γ is the complete bipartite graph, the Seidel matrix $U_\Gamma(-1, -1, 1, 0)$ has two eigenvalues, and if $\Gamma = P_4$, then $U_\Gamma(1, -1, -\frac{1}{2}, 1)$ has two eigenvalues.

Let Γ , be a connected bipartite graph with a universal adjacency matrix $U = U_\Gamma(1, \beta, \gamma, \delta)$ with two eigenvalues. We choose β such that the equations in Lemma 1 hold. If Γ is regular, then by Proposition 1, Γ is strongly regular, complete, or empty, and since Γ is bipartite, it must be complete bipartite. If $2 \leq n \leq 4$, then the only connected bipartite graphs are K_2 ,

$K_{1,2}$, $K_{1,3}$, $K_{2,2}$ and the path P_4 .

If $\gamma\delta + \gamma + \delta \neq 0$, then (ii) of Lemma 1 implies that $d_i + d_j$ is constant for every edge $\{i, j\}$. This implies that any two vertices at distance 2 from each other have the same degree. Since Γ is connected, it follows that at most two degrees occur and that vertices in the same part of the bipartition have the same degree. If one of the degrees occurs only once, it follows that $\Gamma = K_{1,n-1}$. So, if Γ is not complete bipartite, then $\mu_{1,1}$, $\mu_{1,2}$ and $\mu_{2,2}$ are defined, $\mu_{1,2} = 0$, and $\mu_{1,1} + \mu_{2,2} = 2\mu_{1,2}$ by (iii) of Lemma 1. Therefore Γ has constant $\mu(\Gamma) = 0$, so Γ contains no $P_3 = K_{1,2}$, and the connectivity of Γ gives $\Gamma = K_2$.

Thus we can assume that Γ is non-regular, $\gamma\delta + \gamma + \delta = 0$ and $n \geq 5$. Clearly $\delta \neq -1$. If $\delta = 0$, then $\gamma = 0$, and $U = A_\Gamma + \beta I$, has two eigenvalues, which implies that $\Gamma = K_2$. Therefore, U is proper, and by Lemma 2, Γ has two degrees k_1 and k_2 , and there exist constants $k_{i,j}$ such that each vertex of degree k_i has exactly $k_{i,j}$ neighbors of degree k_j ($i, j = 1, 2$). Let Γ_i be the subgraph of Γ induced by the vertices of degree k_i ($i = 1, 2$). Then we have that Γ_i is regular and bipartite (Γ_i may be empty). If there is just one vertex of degree k_1 then $k_{2,1} = 1$, so $k_{1,2} = k_1 = n - 1$, and Γ is the star $K_{1,n-1}$. Suppose there are exactly two vertices v and w of degree k_1 . If v and w are adjacent, then $k_{2,1} = 1$, $k_{1,2} = k_1 - 1 = \frac{1}{2}n - 1$. If v and w are nonadjacent, then $k_{2,1} = 2$ would imply that Γ is $K_{2,n-2}$, therefore we can assume that $k_{2,1} = 1$, $k_{1,2} = k_1 = \frac{1}{2}n - 1$. Since $n > 4$, v has at least two neighbors in Γ_2 , which are nonadjacent and hence $\mu_{2,2} > 0$. Two vertices in different classes of the bipartition have no common neighbors. So they have to be adjacent. Hence Γ_2 is complete bipartite. It follows that Γ is complete bipartite, or complete bipartite with one edge deleted. The latter option has $\mu_{1,1} = k_1 > 1$, $\mu_{2,2} = k_2 = k_1 + 1$, $\mu_{1,2} = 0$ which contradicts (iii) of Lemma 1.

So we can assume that Γ is not complete bipartite, and that Γ_1 and Γ_2 both are regular bipartite (possibly empty) with at least three vertices. It follows that all three $\mu_{i,j}$'s are defined. Again we use that $\mu_{1,1} + \mu_{2,2} = 2\mu_{1,2}$. If $\mu_{1,2} = 0$, then all three $\mu_{i,j}$'s are 0, which is impossible in the considered case. Assume $\mu_{1,2} > 0$. Then we can choose three vertices: u and v in Γ_1 , and w in Γ_2 such that u and v lie in one part of the bipartition, whilst w lies in the other part. If w is nonadjacent to u , then $0 = \mu(u, w) = \mu_{1,2}$; a contradiction. So w is adjacent to u and also to v . This implies that $\mu(u, v) = \mu_{1,1} > 0$. Similarly, it follows that $\mu_{2,2} > 0$, so all three $\mu_{i,j}$'s are nonzero which implies that Γ is complete bipartite. \square

The two previous theorems gave characterizations for graphs admitting a universal adjacency matrix with two eigenvalues in terms of the structure of the graph. The next result is a characterization in terms of a matrix property.

Theorem 10 *Suppose Γ admits a universal adjacency matrix with two eigenvalues, one of them being simple. Then Γ , or the complement of Γ is $gK_1 + K_\ell$, or $K_\ell + K_m$ ($g \geq 0$, $\ell, m \geq 1$, $g + \ell \geq 2$).*

Proof Suppose $U = U_\Gamma(\alpha, \beta, \gamma, \delta)$ has two eigenvalues, one of which is simple. Choose β such that $\text{rank } U = 1$. Then $U = \mathbf{u}^\top \mathbf{u}$ for some vector \mathbf{u} . If $(\mathbf{u})_1 = 0$ then the first column of U is $\mathbf{0}$ and the symmetry of U gives that $\mathbf{u} = \mathbf{0}$, so $U = O$; contradiction. Therefore $(\mathbf{u})_1 \neq 0$, and we can choose α such that $(\mathbf{u})_1 = 1$. write $\mathbf{u} = [1 \ a \ \cdots \ a \ b \ \cdots \ b]^\top$. Let k_i be the number of times that the value i occurs in \mathbf{u} ($i = a, b$). Then

$$U = \begin{bmatrix} 1 & a & \cdots & a & b & \cdots & b \\ a & a^2 & \cdots & a^2 & ab & \cdots & ab \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ a & a^2 & \cdots & a^2 & ab & \cdots & ab \\ b & ab & \cdots & ab & b^2 & \cdots & b^2 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ b & ab & \cdots & ab & b^2 & \cdots & b^2 \end{bmatrix}.$$

If Γ is not complete or empty, then $k_a > 0$, $k_b > 0$ and $a \neq b$. Since U admits only two different off-diagonal values, $ab = a$, or $ab = b$. Assume without loss of generality that $ab = b$. Suppose $b = 0$, then $a^2 = 1$ (because at most two different diagonal entries occur). If $k_a \geq 2$, then $a = 1$ and Γ or its complement is $K_{k_a+1} + k_b K_1$. If $k_a = 1$, then $a = \pm 1$ and Γ or its complement equals $K_2 + (n-2)K_1$. If $a = 1$, and $k_b \geq 2$, then $b^2 = b$, or $b^2 = 1$. Since $b \neq a$ we have $b = 0$ or $b = -1$. If $b = 0$ then Γ or its complement is $K_{1+k_a} + k_b K_1$. If $b = -1$ then Γ or its complement is $K_{1+k_a} + K_{k_b}$. If $a = 1$ and $k_b = 1$, then Γ or its complement equals $K_{n-1} + K_1$. \square

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