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MONOTONIC STABLE SOLUTIONS FOR MINIMUM COLORING GAMES

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Abstract

For the class of minimum coloring games (introduced by Deng et al. (1999)) we investigate the existence of population monotonic allocation schemes (introduced by Sprumont (1990)). We show that a minimum coloring game on a graph G has a population monotonic allocation scheme if and only if G is $(P_4, 2K_2)$-free (or, equivalently, if its complement graph $\bar{G}$ is quasi-threshold). Moreover, we provide a procedure that for these graphs always selects an integer population monotonic allocation scheme.

Keywords: Minimum coloring game, population monotonic allocation scheme, $(P_4, 2K_2)$-free graph, quasi-threshold graph.

JEL-code: C71

1 Introduction

Minimum coloring problems deal with situations where the agents present are interested in having access to some facility but can be involved in conflict. Conflicts between agents are modeled by an undirected conflict graph in which two agents are connected if and only if they are in conflict. Agents who are in conflict can not have access to the same facility and the problem is to find the minimum number of facilities that can serve all agents, or, equivalently, to find the chromatic number of the conflict graph.

The cost allocation problem arising from such a situation can be tackled using cooperative game theory. A corresponding cooperative game is constructed where the value of any coalition of agents is equal to the chromatic

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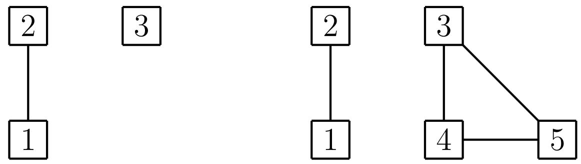
number of the conflict subgraph induced by this coalition. For this cooperative coloring game the existence of core elements, i.e. cost allocation vectors which are such that no group of agents has an incentive to deviate, is investigated. Deng et al. (1999) did this job for the more general class of combinatorial optimization games. They showed that such games have core elements if and only if an associated linear program has an integer optimal solution.

Combinatorial optimization games, and hence in particular minimum coloring games, can be viewed as examples of so-called operations research games (OR games for short). OR games can be classified via the nature of the underlying operations research problem. It can be, for example, a scheduling problem (cf. Curiel et al. (1989)), linear production problem (cf. Owen (1975)), inventory problem (cf. Guardiola et al. (2009)) or a network problem (cf. Curiel et al. (1992)). A survey on OR games can be found in Borm et al. (2001).

For OR games arising from network problems, which include the minimum coloring games, a line of research exists in which properties of the OR game are characterized by properties of the underlying network (graph). For example, Granot et al. (1999) showed that a graph is weakly Eulerian (respectively weakly cyclic) if and only if the induced Chinese postman game is balanced (respectively submodular). Herer and Penn (1995) showed that graphs which are obtained as a 1-sum of K_4 and outerplanar graphs characterize submodular Steiner-traveling salesman games. Okamoto (2003) showed that minimum vertex cover games are submodular if and only if the underlying graph is (K_3, P_3)-free, i.e., no subgraph is isomorphic to K_3 or P_3.

For the subclass of minimum coloring games it seems impossible to characterize balancedness in terms of properties of the underlying graph. Deng et al. (2000) succeeded however in providing a nice characterization of the somewhat smaller collection of minimum coloring games that are totally balanced: a minimum coloring game on a graph is totally balanced if and only if the graph is perfect. Totally balancedness is an attractive property that states that the game, together with all its subgames, has a core element. Okamoto (2003a) characterized the core of a minimum coloring game on a perfect graph as the convex hull of the characteristic vectors of maximum cliques of the graph.

Despite its attractiveness, totally balancedness does not guarantee in all situations the existence of a reasonable allocation. Consider for example the very simple 3-person conflict graph G_1 in Figure 1 and its associated minimum coloring game. It is straightforward to see that $(1, 1, 0)$ is the only core element in this minimum coloring game, as $\{1, 2\}$ is the only maximum clique in perfect graph G_1. So agent 3 does not have to pay at all and is rewarded for not being in conflict with any of the other two agents. Now suppose that agents 1, 2 and 3 consider the possibility of cooperation with agents 4 and 5, who are in conflict with agent 3 and with each other but not with agents 1 and 2 (see graph G_2 in Figure 1). In order to establish cooperation it seems reasonable to



G_1 G_2

Figure 1: Graph G_1 and the extended graph G_2.

require that all agents should benefit in this new cooperative situation. Hence agents 1, 2, 4 and 5 are willing to pay 1 unit at most and agent 3 still does not want to pay anything. However, the minimum coloring game associated with G_2 only has one core element, namely $(0, 0, 1, 1, 1)$ ($\{3, 4, 5\}$ is the only maximum clique in G_2). In this core element agent 3 has to pay 1 unit! In spite of the fact that this 5-person minimum coloring game is totally balanced it lacks a form of dynamic stability. As soon as new agents enter the scene, some agents already present may have disadvantage of it.

The problem with the minimum coloring game on G_2 is that it is impossible to find core elements in the game and all its subgames in a monotonic way. It is the existence of such monotonic schemes that leads to a stable setting in situations where the set of agents may increase over time. Such monotonic schemes were introduced for any cooperative game by Sprumont (1990) as population monotonic allocation schemes (pmas-es for short).

In this paper we characterize the class of minimum coloring games with a pmas. Our main result is that a minimum coloring game has a pmas if and only if the underlying conflict graph is $(P_4, 2K_2)$-free. The collection of $(P_4, 2K_2)$-free graphs is well-known in graph theory: these are the graphs that are complements of (P_4, C_4)-free graphs, which are also known as quasi-threshold graphs or comparability graphs of arborescence orders (see Wolk (1962, 1965)). Yan et al. (1996) studied the class of quasi-threshold graphs in more detail and showed that they can be recognized in linear time (hence the same holds for $(P_4, 2K_2)$-free graphs as well). Our characterization result is in the same spirit as the result of Deng (2000). Also the results of Bietenhader and Okamoto (2006) perfectly fit in this context. For minimum coloring games on perfect graphs they show that exactness, extendability and largeness of the core are all equivalent to the statement that every clique in the conflict graph is contained in some maximum clique. The properties exactness, extendability and largeness of the core are not related with the property of having a pmas, even not in the special case of minimum coloring games. For example consider perfect graph G_3 (see Figure 2) which is $(P_4, 2K_2)$-free but which does not have the property that every clique is contained in a maximum clique, and perfect graph G_4 where the opposite statements are true. Okamoto (2003b) characterized the much smaller class of concave or submodular minimum coloring games by complete multipartiteness of the underlying graph. Concave

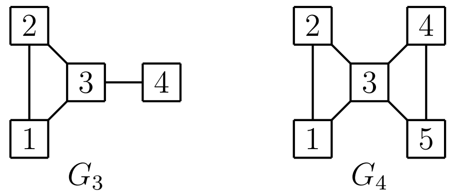


Figure 2: Graphs G_3 and G_4.

games always admit pmas-es (see Sprumont (1990)) and complete multipartite graphs are indeed instances of $(P_4, 2K_2)$-free graphs.

This paper is organized as follows. Section 2 is a preliminary section introducing the relevant concepts of game and graph theory. In section 3 the class of quasi-threshold graphs and the complementary $(P_4, 2K_2)$-free graphs are discussed in detail. An alternative description of the class of $(P_4, 2K_2)$-free graphs is provided using the concept of 'degree consistency'. Section 4 contains the main results of this paper. Here we prove that every minimum coloring game has a pmas if and only if the underlying graph is $(P_4, 2K_2)$-free. Moreover, in case the graph is $(P_4, 2K_2)$-free we provide a description of the complete set of pmas-es and we present a procedure that always selects an integer pmas.

2 Preliminaries

In this section we present some concepts and results from game theory and recall some notions from graph theory.

2.1 Game Theory

A *cooperative (cost) game* is a tuple (N, c) where $N = \{1, 2, ..., n\}$ is the set of agents, and $c : 2^N \to \mathbb{R}$ its characteristic (cost) function, with the convention that $c(\emptyset) = 0$. Here $c(S)$ is interpreted as the total cost coalition $S \in 2^N$ faces when fulfilling their objectives. A game (N, c) is called *monotonic* if $c(S) \leq c(T)$ for all $S, T \in 2^N$ with $S \subset T$. The *core* of a cost game (N, c) is the set

$$C(c) := \big\{x \in \mathbb{R}^N | \sum_{i \in N} x_i = c(N) \text{ and } \sum_{i \in S} x_i \leq c(S) \text{ for all } S \in 2^N \backslash \{\emptyset\}\big\}.$$

If $x \in C(c)$, then no coalition $S \in 2^N \backslash \{\emptyset\}$ has an incentive to split off from the grand coalition N if x is the proposed vector of cost shares. A game (N, c) is called *balanced* if $C(c) \neq \emptyset$ and *totally balanced* if the core of every subgame is nonempty, where the subgame corresponding to some coalition $T \in 2^N \backslash \{\emptyset\}$ is the game (T, c_T) with $c_T(S) = c(S)$ for all $S \in 2^T$.

By introducing population monotonic allocation schemes Sprumont (1990) shifted the attention from allocation vectors for the grand coalition only to allocation schemes. Such schemes provide an allocation vector for *any* coalition. Given a cost game (N, c), the table $x = (x_{S,i})_{S \in 2^N \setminus \{\emptyset\}, i \in S}$ is said to be a *population monotonic allocation scheme* (pmas for short) if the following two conditions hold:

1. *efficiency*: for all $S \in 2^N \setminus \{\emptyset\}$ we have $\sum_{i \in S} x_{S,i} = c(S)$;
2. *monotonicity*: for all $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subset T$ and for all $i \in S$ we have $x_{S,i} \geq x_{T,i}$.

A pmas provides for every coalition a core element in the corresponding subgame in a monotonic way. The collection of all pmas-es of (N, c) is denoted by $P(N, c)$. The following proposition shows that pmas-es of monotonic, nonnegative games are always nonnegative.

Proposition 1. *Let (N, c) be a monotonic game and let $x \in P(N, c)$. Then $x_{S,i} \geq 0$ for every $S \in 2^N \setminus \{\emptyset\}$ and $i \in S$.*

Proof. Let $S \in 2^N \setminus \{\emptyset\}$ and $i \in S$. If $|S| = 1$ then $x_{S,i} = c(S) \geq c(\emptyset) = 0$ by efficiency. If $|S| > 1$ then

$$\begin{aligned} x_{S,i} &= c(S) - \sum_{j \in S \setminus \{i\}} x_{S,j} \\ &\geq c(S) - \sum_{j \in S \setminus \{i\}} x_{S \setminus \{i\},j} \\ &= c(S) - c(S \setminus \{i\}) \\ &\geq 0, \end{aligned}$$

where the first inequality follows from the fact that x satisfies monotonicity, the second inequality from the fact that (N, c) is monotonic and the equalities from the fact that x satisfies efficiency. $\square$

A very simple game is the *unit game* $(N, \mathbf{1})$ defined by $\mathbf{1}(S) = 1$ for every $S \in 2^N \setminus \{\emptyset\}$. It is straightforward to see that this game admits a pmas. The collection of pmas-es of (subgames $\mathbf{1}_T$ of) this unit game play an important role later on, when describing pmas-es of coloring games. Next proposition describes the collection of integer pmas-es of $\mathbf{1}_T$. For this we need the concept of 'orders'. An *order on* $T \in 2^N \setminus \{\emptyset\}$ is a bijection from T to $\{1, \ldots, |T|\}$. The collection of all orders on T is denoted by Σ_T.

Proposition 2. *Let $T \in 2^N \setminus \{\emptyset\}$.*

(i) Let $\sigma \in \Sigma_T$. Define the scheme $y^\sigma = (y^\sigma_{S,i})_{S \in 2^T \setminus \{\emptyset\}, i \in S}$ by

$$y^\sigma_{S,i} = \begin{cases} 1 & \text{if } \sigma(i) > \sigma(j) \text{ for all } j \in S \setminus \{i\} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

for every $S \in 2^T \setminus \{\emptyset\}$ and $i \in S$. Then $y^\sigma \in P(T, \mathbf{1}_T)$.

(*ii*) *If* $y \in P(T, \mathbf{1}_T)$ *and* y *is integer-valued, then there exists an order* $\sigma \in \Sigma_T$ *such that* $y = y^\sigma$.

Proof. (*i*) First, let's check efficiency. For every $S \in 2^T \backslash \{\emptyset\}$ there is precisely one $k \in S$ such that $\sigma(k) > \sigma(j)$ for all $j \in S \backslash \{k\}$. Hence $\sum_{i \in S} y^\sigma_{S,i} = 1 = \mathbf{1}_T(S)$.
Now we check monotonicity. Let $S, S' \in 2^T \backslash \{\emptyset\}$ with $S \subset S'$ and $i \in S$. In order to show that $y^\sigma_{S,i} \geq y^\sigma_{S',i}$ it is sufficient to show that $y^\sigma_{S',i} = 1$ implies $y^\sigma_{S,i} = 1$, as y^σ is $\{0,1\}$-valued. So, assume $y^\sigma_{S',i} = 1$. Then $\sigma(i) > \sigma(j)$ for all $j \in S' \backslash \{i\}$. Hence $\sigma(i) > \sigma(j)$ for all $j \in S \backslash \{i\}$ and therefore $y^\sigma_{S,i} = 1$.
(*ii*) Let $y \in P(T, \mathbf{1}_T)$ be integer-valued. According to Proposition 1 scheme y is nonnegative. Nonnegativity, efficiency, and the fact that y is integer-valued together imply that y is $\{0,1\}$-valued.
There is a unique $i \in T$ with $y_{T,i} = 1$. Denote this agent by i_1. If $|T| > 1$ there is a unique $i \in T \backslash \{i_1\}$ with $y_{T \backslash \{i_1\},i} = 1$. Denote this agent by i_2. If $|T| > 2$ there is a unique $i \in T \backslash \{i_1, i_2\}$ with $y_{T \backslash \{i_1,i_2\},i} = 1$. Denote this agent by i_3, etcetera. In this way we get an exhaustive sequence $i_1, \ldots, i_{|T|}$ of agents in T. Now define $\sigma \in \Sigma_T$ by $\sigma(i_k) = |T| + 1 - k$ for every $k \in \{1, \ldots, |T|\}$. We will show that $y = y^\sigma$.

Let $S \in 2^T \backslash \{\emptyset\}$ and $i \in S$. Let k^* be the minimal number in $\{1, \ldots, |T|\}$ such that $i_{k^*} \in S$ and let $S^* = \{i_{k^*}, i_{k^*+1}, \ldots, i_{|T|}\} (= T \backslash \{i_1, i_2, \ldots, i_{k^*-1}\})$. Clearly $S \subset S^*$. If $i = i_{k^*}$ we have $\sigma(i) > \sigma(j)$ for every $j \in S \backslash \{i\}$ and hence $y^\sigma_{S,i} = 1$. Moreover, we have $1 = y_{S^*, i_{k^*}} = y_{S^*,i} \leq y_{S,i} \leq y_{\{i\},i} = 1$ and therefore also $y_{S,i} = 1$. So, $y_{S,i} = y^\sigma_{S,i}$. If $i \neq i_{k^*}$ we have $\sigma(i_{k^*}) > \sigma(i)$ and hence $y^\sigma_{S,i} = 0$. Since $1 = y_{S^*, i_{k^*}} \leq y_{S, i_{k^*}} \leq y_{\{i_{k^*}\}, i_{k^*}} = 1$ we get $y_{S, i_{k^*}} = 1$. Since y is $\{0,1\}$-valued we must have by efficiency that $y_{S,i} = 0$. Again we get $y_{S,i} = y^\sigma_{S,i}$. So $y = y^\sigma$. □

For every cooperative game the collection of pmas-es constitute a bounded polyhedral set and can hence be 'computed' as the convex hull of its extreme points. In the special case of unit (sub)games $(T, \mathbf{1}_T)$ it is easy to see that the schemes x^σ are extreme points of $P(T, \mathbf{1}_T)$ for every $\sigma \in \Sigma_T$. If $|T| \leq 3$ one can show that all extreme points are obtained in this way. If $|T| \geq 4$ however, this is unfortunately not the case, as the following example illustrates.

Example 3. Consider the game $(N, \mathbf{1})$ where $N = \{1,2,3,4\}$. Let $x = (x_{S,i})_{S \in 2^N \backslash \{\emptyset\}, i \in S}$ be the scheme, given by the following table:

S	1	2	3	4
$\{1\}$	1	–	–	–
$\{2\}$	–	1	–	–
$\{3\}$	–	–	1	–
$\{4\}$	–	–	–	1
$\{1,2\}$	0.5	0.5	–	–
$\{1,3\}$	0.5	–	0.5	–
$\{1,4\}$	0.5	–	–	0.5
$\{2,3\}$	–	0.5	0.5	–
$\{2,4\}$	–	0.5	–	0.5
$\{3,4\}$	–	–	0.5	0.5
$\{1,2,3\}$	0	0.5	0.5	–
$\{1,2,4\}$	0	0.5	–	0.5
$\{1,3,4\}$	0	–	0.5	0.5
$\{2,3,4\}$	–	0.5	0.5	0
$\{1,2,3,4\}$	0	0.5	0.5	0

Clearly, $x \in P(N, \mathbf{1})$ and x is not integer-valued, so there is no $\sigma \in \Sigma_N$ with $x = x^\sigma$. In order to show that nevertheless x is an extreme point of $P(N, \mathbf{1})$ write $x = \frac{1}{2}y + \frac{1}{2}z$ with $y, z \in P(N, \mathbf{1})$. We will show that $x = y = z$. According to Proposition 1 schemes y and z are taking nonnegative values only. Hence we have $0 \leq y_{S,i} \leq 1$ and $0 \leq z_{S,i} \leq 1$ for all S and i. Therefore $x_{S,i} = y_{S,i} = z_{S,i}$ for all S and i with $x_{S,i} \in \{0, 1\}$. Let $a = y_{N,2}$, then $y_{N,3} = 1 - a$ (recall that $y_{N,1} = y_{N,4} = 0$). Since $x_{\{2,3,4\},2} = x_{N,2}$ we get $\frac{1}{2}(y_{\{2,3,4\},2} - y_{N,2}) + \frac{1}{2}(z_{\{2,3,4\},2} - z_{N,2}) = 0$ and hence $y_{\{2,3,4\},2} = y_{N,2} = a$ (and $z_{\{2,3,4\},2} = z_{N,2}$). The same argument yields $y_{S,2} = y_{N,2} = a$ for all $S \in 2^N$ with $2 \in S$ and $S \neq \{2\}$ and $y_{S,3} = 1 - a$ for all $S \in 2^N$ with $3 \in S$ and $S \neq \{3\}$. Using efficiency we get $y_{\{1,3\},1} = y_{\{3,4\},4} = y_{\{1,3,4\},4} = a$ and $y_{\{1,2\},1} = y_{\{2,4\},4} = y_{\{1,2,4\},4} = 1 - a$. Since $x_{\{1,4\},4} = x_{\{1,2,4\},4} = x_{\{1,3,4\},4}$ we get $y_{\{1,4\},4} = y_{\{1,2,4\},4} = y_{\{1,3,4\},4}$. So $a = 1 - a$ and hence $a = \frac{1}{2}$. We conclude that $y = x$ (and hence $z = x$). $\triangleleft$

2.2 Graph Theory

An (undirected) *graph* is a pair (N, E) where N is a finite vertex set and E is the edge set which is a subset of the collection of 2-element subsets of N. Edges will be denoted by ij instead of $\{i, j\}$. If $G = (N, E)$ is a graph and $S \in 2^N \backslash \{\emptyset\}$ the *subgraph* of G induced by S is the graph $G[S] = (S, E_S)$ where $E_S = \{ij \in E : i \in S, j \in S\}$. The *complement* of $G = (N, E)$ is the graph $\bar{G} = (N, \bar{E})$ where $\bar{E} = \{ij : i, j \in N, i \neq j, ij \notin E\}$.

For a graph $G = (N, E)$ and a vertex $i \in N$ the number $d_N(i) = |\{j \in N \backslash \{i\} : ij \in E\}|$ denotes the number of adjacent vertices of i in G and is called the *degree* of i in G. Similarly, for every $S \in 2^N \backslash \{\emptyset\}$ and $i \in S$ the degree of i in $G[S]$ is denoted by $d_S(i)$.

A graph $G = (N, E)$ is *complete* if for all $i, j \in N, i \neq j$ we have $ij \in E$. A *clique* in G is an $S \in 2^N \backslash \{\emptyset\}$ such that $G[S]$ is complete. A clique is *maximum* if there are no cliques containing more elements. The *clique number* of a graph G is the number of elements in a maximum clique in G. An *independent set* is an $S \in 2^N \backslash \{\emptyset\}$ such that $G[S]$ has no edges. An independent set is *maximal* if it is not a strict subset of another independent set.

A *directed graph* (digraph) is a tuple (V, D), where V is the finite vertex set and $D \subset \{(i, j) : i, j \in V, i \neq j\}$ is a collection of directed arcs. A *rooted tree* is a digraph (V, D) for which a special vertex $r \in V$ (the root) exists such, that for each vertex $v \in V$ there is a unique directed path from r to v. A *rooted forest* is a digraph which is a disjoint union of rooted trees. If (V, D) is a rooted forest then for every $v \in V$ there is a unique directed path from some root to v. The collection of vertices on this path will be denoted by $P(v)$. The precedence relation $(V, \prec)$ on V is defined by $v \prec v'$ if $v \in P(v')$ and $v \neq v'$. The set of *followers* of a vertex $v \in V$ is the set $F(v) = \{v' \in V : v \prec v'\}$. A vertex v is called a *leaf* if $F(v) = \emptyset$. If $V' \subseteq V$, then a vertex $v \in V'$ is said to be *maximal in* V' if $F(v) \cap V' = \emptyset$. For each $V' \subseteq V$, $\max(V') = \{v \in V' : v \text{ is maximal in } V'\}$.

3 Quasi-threshold and $(P_4, 2K_2)$-free graphs

In this section we first discuss the class of *quasi-threshold graphs* (see Yan et al. (1996)), which were formerly known as *comparability graphs of arborescence orders* (see Wolk (1962, 1965) who was the first to study this class of graphs). Quasi-threshold graphs are defined in a constructive way: they are the graphs that can be formed, starting from one-vertex graphs, by the following operations:

1) adding a new vertex that is adjacent to all vertices of a quasi-threshold graph;

2) taking the disjoint union of two quasi-threshold graphs.

In Yan et al. (1996) (in fact, already in Wolk (1965)) quasi-threshold graphs are characterized by the fact that they are the graphs induced by a rooted forest. To be more precise, if $F = (N, D)$ is a rooted forest then the induced graph $G = (N, E)$ is the graph where $ij \in E$ $(i, j \in N, i \neq j)$ if and only if there is a directed path in F from i to j or from j to i. This induced graph G is quasi-threshold and all quasi-threshold graphs can be constructed in this way. We refer to F as a rooted forest representation of G.

Example 4. Consider the rooted forest $F = (N, D)$ in Figure 3. The corresponding quasi-threshold graph (N, E) is given by $E = \{12, 13, 14, 15, 16, 23, 24, 25, 34, 35, 78, 79, 89\}$. Note that interchanging the positions of 2 and 3

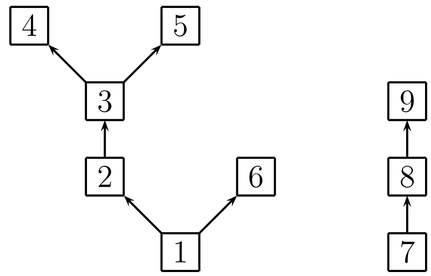


Figure 3: The rooted forest of Example 4.

in F yields the same quasi-threshold graph. Also interchanging 7 and 8 (or 7 and 9 or 8 and 9) results in the same graph. $\triangleleft$

In this paper we need a slightly different rooted forest representation of quasi-threshold graphs. Note that the rooted forest representation of the quasi-threshold graph in Example 4 is not unique. This is due to the fact that vertices exist that have exactly one direct follower. Interchanging such a vertex with its direct follower yields another rooted forest representation of the same graph. A solution to this non-unicity representation problem is to merge such a vertex with its direct follower and repeat this action if necessary. In this way a rooted forest results where 'new' vertices contain one or more 'old' vertices and where such 'new' vertices do not have exactly one direct follower. The corresponding quasi-threshold graph has edges between 'old' vertices i and j ($i \neq j$) if and only if i and j are contained in the same 'new' vertex or if they belong to different 'new' vertices and there is a directed path from one 'new' vertex to the other.

Example 5. Reconsider the rooted forest in Example 4. 'Merging' vertices 2 and 3 and vertices 7, 8 and 9 (in any of the two possible orders) leads to the rooted forest in Figure 4. $\triangleleft$

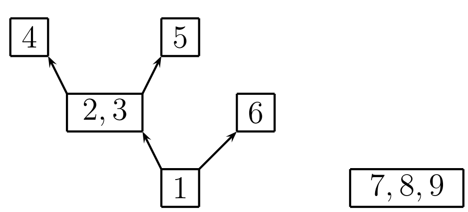


Figure 4: The adjusted rooted forest of Example 4.

On the other hand, consider a rooted forest $F = (V, D)$ without vertices having exactly one follower and where every vertex contains one or more agents. Consider moreover the graph (N, E) where the vertex set N consists of all agents present in F and where for every $i, j \in N, i \neq j$ we have that $ij \in E$ if and only if i and j belong to the same vertex in V or if they belong to different vertices and there is a directed path in F between these vertices. Then (N, E) is a quasi-threshold graph and a 'standard' rooted forest representation of (N, E) is obtained by replacing in F any vertex with more than one agent by a directed chain containing all agents in the vertex in some arbitrary order.

All these observations lead to the following proposition.

Proposition 6. *Let (N, E) be a graph. Then (N, E) is a quasi-threshold graph if and only if a rooted forest $F = (V, D)$ exists, and for every $v \in V$ a nonempty subset M_v of N, such that*

i) *F has no vertices with exactly one follower;*

ii) *$(M_v)_{v \in V}$ is a partition of N;*

iii) *$E = \{ij : i, j \in N, i \neq j, v(i) = v(j)$ or there is a directed path in F from $v(i)$ to $v(j)$ or vice versa$\}$.*
Here $v(k)$ denotes, for every $k \in N$, the unique vertex $v \in V$ with $k \in M_v$.

In the sequel we will refer to the triple $(V, D, (M_v)_{v \in V})$ as *the* rooted forest representation of quasi-threshold graph (N, E).

Example 7. Reconsider the quasi-threshold graph (N, E) of Example 4. Let $F = (V, D)$ be the rooted forest with $V = \{a, b, c, d, e, f\}$ as depicted in Figure 5 and let $M_a = \{1\}$, $M_b = \{2, 3\}$, $M_c = \{4\}$, $M_d = \{5\}$, $M_e = \{6\}$, and $M_f = \{7, 8, 9\}$. Then $(V, D, (M_v)_{v \in V})$ is the rooted forest representation of (N, E). $\triangleleft$

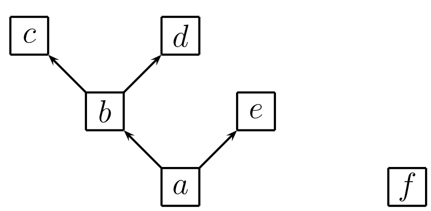


Figure 5: The rooted forest of Example 7.

Yan et al. (1996) show that the class of quasi-threshold graphs coincides with the class of (P_4, C_4)-free graphs, i.e. graphs which do not contain induced subgraphs isomorphic to P_4 or C_4 (see Figure 6). Note that the complement of graph P_4 is isomorphic to P_4 and that the complement of graph C_4 is isomorphic to $2K_2$ (again see Figure 6). This implies that the class of graphs, obtained by taking complements of quasi-threshold graphs, coincides

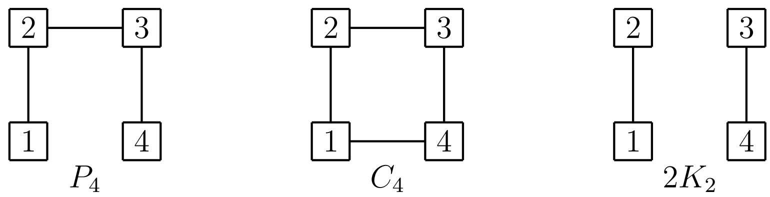


Figure 6: P_4, C_4 and $2K_2$.

with the class of $(P_4, 2K_2)$-free graphs. In fact, it is this class of graphs that plays a central role in this paper. Obviously, the rooted forest representation of a quasi-threshold graph can also be used to describe the complementary $(P_4, 2K_2)$-free graph: edges exist (and only exist) between agents in different vertices between which there is no directed path.

Corollary 8. *Let (N, E) be a graph. Then (N, E) is a $(P_4, 2K_2)$-free graph if and only if a rooted forest $F = (V, D)$ exists, and for every $v \in V$ a nonempty subset M_v of N, such that*

i) *F has no vertices with exactly one follower;*

ii) *$(M_v)_{v \in V}$ is a partition of N;*

iii) *$E = \{ij \,:\, i, j \in N, i \neq j, v(i) \neq v(j)$ and there is no directed path in F from $v(i)$ to $v(j)$ or vice versa$\}$.*

Example 9. Reconsider the quasi-threshold graph (N, E) of Example 4 and let $G = (N, \bar{E})$ be the corresponding $(P_4, 2K_2)$-free graph. In G there are no edges between agents i and j if and only if i and j belong to the same vertex or to different vertices between which a directed path exists. Therefore the only maximal independent subsets in G are $I_1 = \{1, 2, 3, 4\}$, $I_2 = \{1, 2, 3, 5\}$, $I_3 = \{1, 6\}$, and $I_4 = \{7, 8, 9\}$ (the dotted sets in Figure 7).

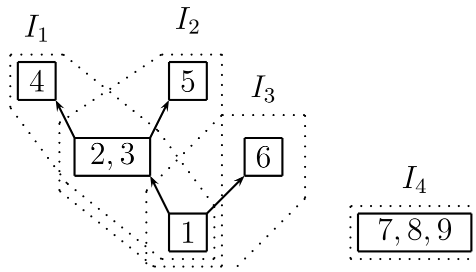


Figure 7: The maximal independent sets of the $(P_4, 2K_2)$-free graph G in Example 9.

It is straightforward to see that the phenomenon observed in Example 9 is true in general: if G is a $(P_4, 2K_2)$-free graph and $(V, D, (M_v)_{v \in V})$ is the rooted forest representation of G then all maximal independent sets in G are generated by taking the union of all agents along directed paths in $F = (V, D)$ from some root to some leaf. Note moreover that, if the rooted forest F consists of trivial singleton trees only, the corresponding graph G is in fact complete multipartite.

The collection of $(P_4, 2K_2)$-free graphs can also be characterized via the concept of 'degree consistency'. Formally a graph $G = (N, E)$ is called *degree consistent* if for all $i, j \in N$, $i \neq j$ with $ij \notin E$ we either have $d_S(i) \leq d_S(j)$ for all $S \subset N$ with $i, j \in S$, or $d_S(i) \geq d_S(j)$ for all $S \subset N$ with $i, j \in S$.

Proposition 10. *Let $G = (N, E)$ be a graph. Then G is $(P_4, 2K_2)$-free if and only if G is degree consistent.*

Proof. First we show the 'if'-part. Suppose that G is degree consistent, but not $(P_4, 2K_2)$-free. Let $S \in 2^N$ be such that $|S| = 4$ and $G[S]$ is either P_4 or $2K_2$. Without loss of generality we can assume that $S = \{1, 2, 3, 4\}$, $12 \in E$, $34 \in E$, $13 \notin E$, $14 \notin E$, and $24 \notin E$ (if $23 \in E$ then $G[S]$ is P_4, otherwise $G[S]$ is $2K_2$). Now consider $i = 1$, $j = 4$, $T = \{1, 2, 4\}$, and $U = \{1, 3, 4\}$. We have $d_T(1) = 1 > 0 = d_T(4)$ and $d_U(1) = 0 < 1 = d_U(4)$, so G is not degree consistent, a contradiction.
Now we show the 'only-if'-part. Suppose that G is $(P_4, 2K_2)$-free and let $(V, D, (M_v)_{v \in V})$ be its rooted forest representation. Let $i, j \in N$, $i \neq j$ be such that $ij \notin E$. First assume that $v(i) = v(j) = v^*$. Now we have for all $k \in N \backslash \{i, j\}$ that $ik \in E$ iff $v(k) \notin P(v^*) \cup F(v^*)$ iff $jk \in E$. So $d_S(i) = d_S(j)$ for every $S \in 2^N \backslash \{\emptyset\}$ with $i, j \in S$. Now assume that $v(i) \neq v(j)$. Then either $v(i) \in P(v(j))$ or $v(j) \in P(v(i))$. Without loss of generality assume that $v(i) \in P(v(j))$. Now for all $k \in N \backslash \{i, j\}$ with $ik \in E$ we have that $v(k) \neq v(i)$ and there is not a directed path between $v(i)$ and $v(k)$. But then $v(k) \neq v(j)$ and there is not a directed path between $v(j)$ and $v(k)$ as well. So $jk \in E$. Consequently we have that $d_S(i) \leq d_S(j)$ for every $S \in 2^N \backslash \{\emptyset\}$ with $i, j \in S$. So G is degree consistent. □

We should check that all this degree consistency-stuff is new

4 Pmas-es for minimum coloring games

A well known problem in graph theory is finding a minimum coloring of a graph. Formally, a *coloring of* $G = (N, E)$ is a mapping $\gamma : N \to \mathbb{N}$ such that $\gamma(i) \neq \gamma(j)$ for every $ij \in E$, i.e. adjacent vertices get different colors. A *minimum coloring of* G is a coloring γ that uses the smallest number of colors, i.e. a coloring for which $|\{\gamma(i) : i \in N\}|$ is minimal. This minimal number of colors needed is called the *chromatic number of* G and is denoted by $\chi(G)$. Of course, the clique number of a graph does not exceed its chromatic number. A graph $G = (N, E)$ is called *perfect* if the clique number of every induced subgraph equals the chromatic number of that subgraph.

Deng et al. (1999) introduced the class of minimum coloring games. If $G = (N, E)$ is a graph, then the *minimum coloring game* on G is the cost game (N, c^G) defined by $c^G(S) = \chi(G[S])$ for every $S \in 2^N \backslash \{\emptyset\}$. Note that minimum coloring games are nonnegative monotonic games as a minimum coloring for a graph always induces a coloring for some subgraph.

First we show that a graph that induces a minimum coloring game with a population monotonic allocation scheme must be a $(P_4, 2K_2)$-free graph.

Proposition 11. *Let $G = (N, E)$ be a graph and let (N, c^G) be the corresponding minimum coloring game. If $P(N, c^G) \neq \emptyset$ then G is $(P_4, 2K_2)$-free.*

Proof. Let $x \in P(N, c^G)$ and suppose that G is not $(P_4, 2K_2)$-free. Let $S \in 2^N$ be such that $|S| = 4$ and $G[S]$ is either P_4 or $2K_2$. Without loss of generality we can assume again that $S = \{1, 2, 3, 4\}$, $12 \in E$, $34 \in E$, $13 \notin E$, $14 \notin E$, and $24 \notin E$. Note that $c^G(\{1, 2, 4\}) = c^G(\{1, 3, 4\}) = 2$ and $c^G(\{1, 3\}) = c^G(\{1, 4\}) = c^G(\{2, 4\}) = 1$. By efficiency and monotonicity of x we get

$$\begin{aligned}
4 &= c^G(\{1,2,4\}) + c^G(\{1,3,4\}) \\
&= x_{\{1,2,4\},1} + x_{\{1,2,4\},2} + x_{\{1,2,4\},4} + x_{\{1,3,4\},1} + x_{\{1,3,4\},3} + x_{\{1,3,4\},4} \\
&\leq x_{\{1,4\},1} + x_{\{2,4\},2} + x_{\{2,4\},4} + x_{\{1,3\},1} + x_{\{1,3\},3} + x_{\{1,4\},4} \\
&= c^G(\{1,3\}) + c^G(\{1,4\}) + c^G(\{2,4\}) \\
&= 3,
\end{aligned}$$

which yields a contradiction. □

Now we are going to show that the reverse statement in Proposition 11 is true as well. First we show that the rooted forest representation of a $(P_4, 2K_2)$-free graph enables us to provide a simplified expression for the values of the corresponding minimum coloring game.

Proposition 12. *Let $G = (N, E)$ be a $(P_4, 2K_2)$-free graph with rooted forest representation $(V, D, (M_v)_{v \in V})$, and let (N, c^G) be the corresponding minimum coloring game. For every $S \in 2^N \backslash \{\emptyset\}$ we have $c^G(S) = |\max(V_S)|$, where $V_S = \{v \in V \,:\, S \cap M_v \neq \emptyset\}$.*

Proof. Let $S \in 2^N \backslash \{\emptyset\}$ and let $\max(V_S) = \{v_1, v_2, \ldots, v_l\}$. For every $i \in \{1, \ldots, l\}$ choose an agent $s_i \in S \cap M_{v_i}$. For every $i, j \in \{1, \ldots, l\}$ with $i \neq j$ there is no directed path in (V, D) between v_i and v_j, so $s_i s_j \in E$. Therefore $\{s_1, \ldots, s_l\}$ is a clique in $G[S]$ and hence $\chi(G[S]) \geq l$.
Now define $\gamma : S \to \mathbb{N}$ by

$$\gamma(i) = \min\left\{k \in \{1, \ldots, l\} \,:\, v_k \in F(v(i)) \cup \{v(i)\}\right\}$$

for every $i \in S$. We will show that γ is a coloring of $G[S]$. Let $i, j \in S$ be such that $i \neq j$ and $\gamma(i) = \gamma(j) = k^*$. Then $v(i) \in P(v_{k^*})$ and $v(j) \in P(v_{k^*})$ and, consequently, $v(i) \in P(v(j))$ or $v(j) \in P(v(i))$. So, either i and j belong

to the same vertex in V, or they belong to different vertices between which a directed path in (V, D) exists. So $ij \notin E$ and hence γ is a coloring of $G[S]$. It is obvious that γ uses at most l colors, so $\chi(G[S]) \leq l$.
Altogether we conclude that $c^G(S) = \chi(G[S]) = l$. $\square$

Example 13. Consider the $(P_4, 2K_2)$-free graph G of Example 9 with rooted forest representation depicted in Figures 4 and 5. Note that for $S = \{1, 2, 3, 7, 8\}$ we have $V_S = \{a, b, f\}$, $\max(V_S) = \{b, f\}$ and hence $c^G(S) = 2$. In a similar way we get, for example, $c^G(\{1, 6\}) = 1$ and $c^G(N) = 4$.

Now we are able to present one of the main results of this paper: a graph induces a minimum coloring game with a population monotonic allocation scheme if and only if the graph is $(P_4, 2K_2)$-free.

Theorem 14. *Let $G = (N, E)$ be a graph and let (N, c^G) be the corresponding minimum coloring game. Then, $P(N, c^G) \neq \emptyset$ if and only if G is a $(P_4, 2K_2)$-free graph.*

Proof. The 'only-if'-part follows from Proposition 11. Now we prove the 'if'-part. Assume that G is a $(P_4, 2K_2)$-free and let $(V, D, (M_v)_{v \in V})$ be the rooted forest representation of G. For each $v \in V$, select an order σ_v on M_v. Define the $\{0, 1\}$-valued scheme $x = (x_{S,i})_{S \in 2^N \setminus \{\emptyset\}, i \in S}$ by

$$x_{S,i} = \begin{cases} 1 & \text{if } v(i) \in \max(V_S) \text{ and} \\ & \quad \sigma_{v(i)}(i) > \sigma_{v(i)}(j) \text{ for every } j \in (S \cap M_{v(i)}) \setminus \{i\} \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

for every $S \in 2^N \setminus \{\emptyset\}$ and $i \in S$.
We will show that $x \in P(N, c^G)$. First we show efficiency of x. For every $S \in 2^N \setminus \{\emptyset\}$ we have

$$\begin{aligned} \sum_{i \in S} x_{S,i} &= \sum_{v \in V_S} \sum_{i \in S \cap M_v} x_{S,i} = \sum_{v \in \max(V_S)} \sum_{i \in S \cap M_v} x_{S,i} \\ &= \sum_{v \in \max(V_S)} 1 = |\max(V_S)| = c^G(S), \end{aligned}$$

where the second equality follows because $x_{S,i} = 0$ when $i \in S \cap M_v$ such that $v \notin \max(V_S)$ and the third equality follows because for any $v \in \max(V_S)$ the order σ_v ranks precisely one element in $S \cap M_v$ on the first place. So, x satisfies efficiency.
Now we check monotonicity of x. Let $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subset T$ and $i \in S$. In order to show that $x_{S,i} \geq x_{T,i}$ it is sufficient to show that $x_{T,i} = 1$ implies $x_{S,i} = 1$, as x is only taking values 0 and 1. So, assume $x_{T,i} = 1$. Then $v(i) \in \max(V_T)$ and $\sigma_{v(i)}(i) > \sigma_{v(i)}(j)$ for every $j \in (T \cap M_{v(i)}) \setminus \{i\}$. But then obviously $v(i) \in \max(V_S)$ and $\sigma_{v(i)}(i) > \sigma_{v(i)}(j)$ for every $j \in (S \cap M_{v(i)}) \setminus \{i\}$ and hence $x_{S,i} = 1$. So x satisfies monotonicity. $\square$

In the sequel of this section we focus on the structure of the complete set of pmas-es $P(N, c^G)$ of the minimum coloring game (N, c^G) corresponding to $(P_4, 2K_2)$-free graph G. First we show that restricting the minimum coloring game (N, c^G) to some M_v, i.e. to a coalition of agents that are all contained in the same vertex v of the associated rooted forest representation, yields a unit game.

Proposition 15. *Let $G = (N, E)$ be a $(P_4, 2K_2)$-free graph with rooted forest representation $(V, D, (M_v)_{v \in V})$, let (N, c^G) be the corresponding minimum coloring game, and let $v \in V$. Then we have*

i) $c^G(S) = 1$ for every $S \in 2^{M_v} \backslash \{\emptyset\}$, i.e. $c^G_{M_v} = \mathbf{1}_{M_v}$;

ii) if $x \in P(N, c^G)$, then $y = (y_{S,i})_{S \in 2^{M_v} \backslash \{\emptyset\}, i \in S}$, defined by $y_{S,i} = x_{S,i}$ for every $S \in 2^{M_v} \backslash \{\emptyset\}$ and $i \in S$, is such that $y \in P(M_v, \mathbf{1}_{M_v})$.

Proof. i) Let $S \in 2^{M_v} \backslash \{\emptyset\}$. Clearly $V_S = \max(V_S) = \{v\}$. According to Proposition 12 we get $c^G(S) = |\max(V_S)| = 1$.
ii) This is a straightforward consequence of the fact that x satisfies efficiency and monotonicity, and the fact that the restricted game $c^G_{M_v}$ is a unit game. □

In the following proposition we show that if we choose, for every $v \in V$, a pmas of the unit game $(M_v, \mathbf{1}_{M_v})$, this collection of pmas-es can be extended in a unique way to a pmas of the corresponding minimum coloring game.

Proposition 16. *Let $G = (N, E)$ be a $(P_4, 2K_2)$-free graph with rooted forest representation $(V, D, (M_v)_{v \in V})$ and let (N, c^G) be the corresponding minimum coloring game. Let $y^v \in P(M_v, \mathbf{1}_{M_v})$ for every $v \in V$. Then there is a unique $x \in P(N, c^G)$ such that $x_{S,i} = y^v_{S,i}$ for every $v \in V$, $S \in 2^{M_v} \backslash \{\emptyset\}$ and $i \in S$, and this scheme x is given by*

$$x_{S,i} = \begin{cases} y^{v(i)}_{S \cap M_{v(i)}, i} & \text{if } v(i) \in \max(V_S) \\ 0 & \text{if } v(i) \notin \max(V_S) \end{cases} \tag{3}$$

for every $S \in 2^N \backslash \{\emptyset\}$ and $i \in S$.

Proof. First we show that the scheme x, defined by (3), 'extends' the collection of schemes $(y^v)_{v \in V}$. Let $v \in V$, $S \in 2^{M_v} \backslash \{\emptyset\}$ and $i \in S$. Then $V_S = \max(V_S) = \{v\}$ and hence $v(i) = v \in \max(V_S)$. Now $x_{S,i} = y^{v(i)}_{S \cap M_{v(i)}, i} = y^v_{S,i}$.
Now we show that the scheme x is a pmas of (N, c^G). We start with efficiency. Let $S \in 2^N \backslash \{\emptyset\}$. Then

$$\begin{aligned} \sum_{i \in S} x_{S,i} &= \sum_{v \in V_S} \sum_{i \in S \cap M_v} x_{S,i} = \sum_{v \in \max(V_S)} \sum_{i \in S \cap M_v} x_{S,i} \\ &= \sum_{v \in \max(V_S)} \sum_{i \in S \cap M_v} y^v_{S \cap M_v, i} = \sum_{v \in \max(V_S)} 1 \\ &= |\max(V_S)| = c^G(S). \end{aligned}$$

In order to check monotonicity of x let $S,T \in 2^N \backslash \{\emptyset\}$ with $S \subset T$ and let $i \in S$. We distinguish between three cases. If $v(i) \in \max(V_S) \cap \max(V_T)$ then $x_{S,i} = y^{v(i)}_{S \cap M_{v(i)},i} \geq y^{v(i)}_{T \cap M_{v(i)},i} = x_{T,i}$, using monotonicity of $y^{v(i)}$. If $v(i) \in \max(V_S)$ and $v(i) \notin \max(V_T)$ then $x_{S,i} = y^{v(i)}_{S \cap M_{v(i)},i} \geq 0 = x_{T,i}$, using nonnegativity of $y^{v(i)}$ (due to Proposition 1). Finally, if $v(i) \notin \max(V_S)$ and $v(i) \notin \max(V_T)$ then $x_{S,i} = 0 = x_{T,i}$. We conclude that $x \in P(N, c^G)$.

We still have to prove unicity of x. Let $z \in P(N, c^G)$ be such that $z_{S,i} = y^v_{S,i}$ for every $v \in V$, $S \in 2^{M_v} \backslash \{\emptyset\}$, and $i \in S$. We have to show that $z = x$. Let $S \in 2^N \backslash \{\emptyset\}$. We will show that $z_{S,i} = x_{S,i}$ for every $i \in S$. First we show this for every $i \in S$ such that $v(i) \in \max(V_S)$, then for every $i \in S$ with $v(i) \notin \max(V_S)$.
Let $v \in \max(V_S)$. Let $W_1 = V \backslash (F(v) \cup \{v\})$ and $W_2 = V \backslash F(v)$. Obviously v is maximal in W_2. Moreover, it is easy to see that if $v' \neq v$ is maximal in W_2 then v' is maximal in W_1 as well. Finally, if v' is maximal in W_1 it has to be maximal in W_2 as well. For, if v' were not maximal in W_2 we must have $v \in F(v')$, as $W_2 = W_1 \cup \{v\}$. So, v' is not a leaf in (V, D) and has, according to Corollary 8, at least two direct followers in (V, D). Therefore we can find a $w \in V$ such that $w \in F(v')$, and $w \notin F(v) \cup \{v\}$, i.e. $w \in W_1$. This contradicts the fact that v' is maximal in W_1. We conclude that $\max(W_2) = \max(W_1) \cup \{v\}$ and hence $|\max(W_2)| = |\max(W_1)| + 1$.

Now, let $T = \cup_{v' \in W_1} M_{v'}$ and $U = T \cup (S \cap M_v)$. We first show that $S \subset U$. Let $i \in S$. If $i \in M_v$ then obviously $i \in U$. On the other hand, if $i \in M_{v'}$ with $v' \in V_S$ and $v' \neq v$ then $v' \notin F(v) \cup \{v\}$ as v is maximal in V_S. So $v' \in W_1$, and hence $i \in T \subset U$. It is straightforward to see that $V_T = W_1$ and $V_U = W_2$. Therefore

$$\begin{aligned}
1 &= c^G(S \cap M_v) = \sum_{i \in S \cap M_v} z_{S \cap M_v, i} \geq \sum_{i \in S \cap M_v} z_{S,i} \\
&\geq \sum_{i \in S \cap M_v} z_{U,i} = \sum_{i \in U} z_{U,i} - \sum_{i \in T} z_{U,i} = c^G(U) - \sum_{i \in T} z_{U,i} \\
&\geq c^G(U) - \sum_{i \in T} z_{T,i} = c^G(U) - c^G(T) \\
&= |\max(V_U)| - |\max(V_T)| = |\max(W_2)| - |\max(W_1)| = 1.
\end{aligned}$$

Since all inequalities are in fact equalities we get $\sum_{i \in S \cap M_v} z_{S,i} = 1$ and, moreover, for all $i \in S \cap M_v$ we have $z_{S,i} = z_{S \cap M_v, i} = y^v_{S \cap M_v, i} = x_{S,i}$. As $v \in \max(V_S)$ was selected in an arbitrary way we conclude that $\sum_{i \in S \cap M_v} z_{S,i} = 1$ for all $v \in \max(V_S)$ and that $z_{S,i} = x_{S,i}$ for every $i \in S$ with $v(i) \in \max(V_S)$.

In order to complete the proof note that

$$\begin{aligned}
c^G(S) &= \sum_{i\in S} z_{S,i} = \sum_{v\in V_S}\sum_{i\in S\cap M_v} z_{S,i} \\
&= \sum_{v\in\max(V_S)}\sum_{i\in S\cap M_v} z_{S,i} + \sum_{v\notin\max(V_S)}\sum_{i\in S\cap M_v} z_{S,i} \\
&= \sum_{v\in\max(V_S)} 1 + \sum_{v\notin\max(V_S)}\sum_{i\in S\cap M_v} z_{S,i} \\
&= |\max(V_S)| + \sum_{v\notin\max(V_S)}\sum_{i\in S\cap M_v} z_{S,i} \\
&= c^G(S) + \sum_{v\notin\max(V_S)}\sum_{i\in S\cap M_v} z_{S,i}.
\end{aligned}$$

So, $\sum_{v\notin\max(V_S)}\sum_{i\in S\cap M_v} z_{S,i} = 0$. As z is nonnegative (Proposition 1) we derive that $z_{S,i} = 0 = x_{S,i}$ for all $i \in S$ with $v(i) \notin \max(V_S)$. □

Propositions 2 and 16 enable us to construct an integer pmas for a minimum coloring game with a $(P_4, 2K_2)$-free graph as underlying graph. First we select an order σ_v on M_v for every $v \in V$ and we let $\sigma = (\sigma_v)_{v\in V}$ denote the collection of chosen orders. Then we define for every $v \in V$ the $\{0,1\}$-valued scheme $y^v \in P(M_v, \mathbf{1}_{M_v})$ by $y^v = y^{\sigma_v}$, where y^{σ_v} is defined by (1). Now Proposition 16 states that a unique $x \in P(N, c^G)$ exists that 'extends' the collection of schemes $(y_v)_{v\in V}$. We will denote this unique pmas by x^σ. It is not difficult to show that the collection of integer pmas-es of a minimum coloring game is constituted by the collection of x^σ's.

Proposition 17. *Let $G = (N, E)$ be a $(P_4, 2K_2)$-free graph with rooted forest representation $(V, D, (M_v)_{v\in V})$, let (N, c^G) be the corresponding minimum coloring game, and let $x \in P(N, c^G)$.*
Then x is integer-valued if and only if there is a collection of orders $\sigma = (\sigma_v)_{v\in V}$ such that $x = x^\sigma$.

Proof. First we prove the if-part. As x^σ is defined by (3), it is obvious that it is $\{0,1\}$-valued as well. In fact, x^σ is the same scheme as the one defined by (2) in the proof of Theorem 14.
In order to show the only-if part assume that x is integer valued. Let $v \in V$. According to Proposition 15 we have that the scheme $y^v = (y^v_{S,i})_{S\in 2^{M_v}\setminus\{\emptyset\}, i\in S}$, defined by $y^v_{S,i} = x_{S,i}$ for every $S \in 2^{M_v}\setminus\{\emptyset\}$ and $i \in S$, is such that $y^v \in P(M_v, \mathbf{1}_{M_v})$. As y^v is integer-valued we can find, according to Proposition 2, an order σ_v on M_v, such that $y^v = y^{\sigma_v}$. Now let $\sigma = (\sigma_v)_{v\in V}$. As both schemes x and x^σ are pmas-es of (N, c^G) that 'extend' the collection of schemes $(y_v)_{v\in V}$ we derive from Proposition 16 that $x = x^\sigma$. □

Now consider a special $(P_4, 2K_2)$-free graph G, namely one whose rooted forest representation $(V, D, (M_v)_{v\in V})$ is such that $|M_v| = 1$ for every $v \in V$.

Obviously, $P(M_v, \mathbf{1}_{M_v})$ consists of one trivial element for every $v \in V$, and this element is y^{σ_v}, where σ_v is the trivial order on the singleton set M_v. Now Propositions 16 and 17 together yield the following corollary.

Corollary 18. *Let $G = (N, E)$ be a $(P_4, 2K_2)$-free graph with rooted forest representation $(V, D, (M_v)_{v \in V})$, where $|M_v| = 1$ for every $v \in V$ and let (N, c^G) be the corresponding minimum coloring game. Then (N, c^G) has only one pmas and this pmas is integer-valued.*

According to Yan et al. (1996) quasi-threshold graphs, and hence $(P_4, 2K_2)$-free graphs, can be found using a linear time recognition algorithm, that also produces the rooted forest representation of these graphs. Population monotonic allocation schemes can not be computed in linear time, for the simple reason that there is an exponential number of coalitions present. Nevertheless, the allocation to one coalition in a pmas, the grand coalition for example, can of course be computed in linear time.

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