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ENTROPY COHERENT AND ENTROPY CONVEX MEASURES OF RISK

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Entropy Coherent and Entropy Convex Measures of Risk[*]

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Abstract

We introduce two subclasses of convex measures of risk, referred to as entropy coherent and entropy convex measures of risk. We prove that convex, entropy convex and entropy coherent measures of risk emerge as certainty equivalents under variational, homothetic and multiple priors preferences, respectively, upon requiring the certainty equivalents to be translation invariant. In addition, we study the properties of entropy coherent and entropy convex measures of risk, derive their dual conjugate function, and prove their distribution invariant representation. Some financial applications and examples of entropy coherent and entropy convex measures of risk are also investigated.

Keywords: Multiple priors; Variational and homothetic preferences; Robustness; Convex risk measures; Exponential utility; Relative entropy; Translation invariance; Convexity; Indifference valuation.

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1 Introduction

Among the most popular theories for decision-making under uncertainty is the multiple priors model, postulating that an economic agent evaluates the consequences (payoffs) of a decision alternative (financial position) X, defined on a measurable space $(\Omega, \mathcal{F})$, according to

$$U(X) = \inf_{Q \in \mathcal{Q}} \mathrm{E}_Q \left[u(X) \right], \tag{1.1}$$

where $u : \mathbb{R} \to \mathbb{R}$ is an increasing function, and $\mathcal{Q}$ is a set of probability measures (priors) on $(\Omega, \mathcal{F})$. The function u, referred to as a utility function, represents the agent's attitude towards wealth, and the set $\mathcal{Q}$ represents the agent's uncertainty about the correct probabilistic model. Gilboa and Schmeidler [20] establish a preference axiomatization of this *robust* Savage representation, generalizing Savage [40] in the framework of Anscombe and Aumann [1]. The representation of Gilboa and Schmeidler [20], also referred to as maxmin expected utility, is a decision-theoretic foundation of the classical decision rule of Wald [45] — see also Huber [29] — that had long seen little popularity outside (robust) statistics.

The multiple priors model is a special case of interest in the class of variational preferences axiomatized by Maccheroni, Marinacci and Rustichini [33]. Under variational preferences, the numerical representation takes the form

$$U(X) = \inf_{Q \in \mathcal{Q}} \{ \mathrm{E}_Q \left[u(X) \right] + \alpha(Q) \}, \tag{1.2}$$

where α is an ambiguity index (penalty function) on probability measures on $(\Omega, \mathcal{F})$. Multiple priors occurs when α is an indicator function that takes the value zero if $Q \in \mathcal{Q}$ and ∞ otherwise. Under multiple priors, the degree of ambiguity is reflected by the multiplicity of the priors. Under variational preferences, the degree of ambiguity is reflected by the multiplicity of the priors and the esteemed plausibility of the prior according to the ambiguity index. Recently, Chateauneuf and Faro [9] and, slightly more generally, Cerreia-Vioglio *et al.* [8] axiomatized a multiplicative analog of variational preferences, henceforth referred to as homothetic preferences, represented as

$$U(X) = \inf_{Q \in \mathcal{Q}} \{ \beta(Q) \mathrm{E}_Q \left[u(X) \right] \}, \tag{1.3}$$

with β a penalty function on probability measures on $(\Omega, \mathcal{F})$; it also includes multiple priors as a special case ($\beta(Q) \equiv 1$).

To measure the 'risk' related to a financial position X, the theories of variational and homothetic preferences sketched above would lead to the definition of a loss functional $L(X) = -U(X)$, satisfying

$$L(X) = \sup_{Q \in \mathcal{Q}} \{ \mathrm{E}_Q \left[\phi(-X) \right] - \alpha(Q) \} \quad \text{and} \quad L(X) = \sup_{Q \in \mathcal{Q}} \{ \beta(Q) \mathrm{E}_Q \left[\phi(-X) \right] \},$$

respectively, where $\phi(x) = -u(-x)$. The disutility (or loss) function ϕ describes how much a loss in wealth hurts. One could, then, look at the capital amount $\bar{m}_X$ that is 'equivalent' to the potential loss of X, solving for $\bar{m}_X$ in $L(\bar{m}_X) = L(X)$; this number is commonly referred to as the certainty equivalent of X; see, e.g., Gollier [21]. However, because we are interested in the amount of capital one needs to hold in response to the financial position X, we will rather look at the negative certainty equivalent of X, m_X, given by $-\bar{m}_X$, satisfying $L(-m_X) = \phi(m_X) = L(X)$, or equivalently,

$$m_X = \phi^{-1} \left(\sup_{Q \in \mathcal{Q}} \{ \mathrm{E}_Q \left[\phi(-X) \right] - \alpha(Q) \} \right) \text{ and } m_X = \phi^{-1} \left(\sup_{Q \in \mathcal{Q}} \{ \beta(Q) \mathrm{E}_Q \left[\phi(-X) \right] \} \right). \tag{1.4}$$

In a related strand of the literature, convex risk measures have played an increasingly important role since their introduction by Föllmer and Schied [15], Fritelli and Rosazza Gianin

[18] and Heath and Ku [28], generalizing Artzner *et al.* [2]; see also the early work of Deprez and Gerber [13] and Ben-Tal and Teboulle [4, 5], and the more recent Ben-Tal and Teboulle [6] and Ruszczyński and Shapiro [38, 39]. For a given financial position X that an economic agent holds, a convex risk measure ρ returns the minimal amount of capital the agent is required to commit and add to the financial position in order to make it 'safe': the theory of convex risk measures postulates that from the viewpoint of the supervisory authority, the financial position $X + \rho(X)$ is acceptably insured against adverse shocks. Convex risk measures are characterized by the axioms of monotonicity, translation invariance and convexity. They can — under additional assumptions on the space of financial positions and on continuity properties of the risk measures; see Section 2 — be represented in the form

$$\rho(X) = \sup_{Q \in \mathcal{Q}} \{ \mathrm{E}_Q [-X] - \alpha(Q) \}, \tag{1.5}$$

where α is a penalty function defined on probability measures on $(\Omega, \mathcal{F})$. With

$$\alpha(Q) = \begin{cases} 0, & \text{if } Q \in \mathcal{Q}; \\ \infty, & \text{otherwise}; \end{cases}$$

we obtain the particular subclass of coherent measures of risk, represented in the form

$$\rho(X) = \sup_{Q \in \mathcal{Q}} \mathrm{E}_Q [-X].$$

The contribution of this paper is twofold. First we derive precise connections between risk measurement under the theories of variational, homothetic and multiple priors preferences — (1.4) — and risk measurement using convex measures of risk — (1.5). In particular, we identify two subclasses of convex risk measures that we call entropy coherent and entropy convex measures of risk, and that include all coherent risk measures. We show that, under technical conditions, negative certainty equivalents under variational, homothetic, and multiple priors preferences are translation invariant if and only if they are convex, entropy convex, and entropy coherent measures of risk, respectively. It entails that convex, entropy convex and entropy coherent measures of risk induce linear or exponential utility functions in the theories of variational, homothetic and multiple priors preferences. We show further that, under a normalization condition, this characterization remains valid when the condition of translation invariance is replaced by requiring convexity. The mathematical details in the proofs of these characterization results are delicate.

Second we study the classes of entropy coherent and entropy convex measures of risk. We show that they satisfy many appealing properties. We prove various results on the dual conjugate function for entropy coherent and entropy convex measures of risk. We show in particular that, quite exceptionally, the dual conjugate function can explicitly be identified under some technical conditions. We also study entropy coherent and entropy convex measures of risk under the assumption of distribution invariance. Due to their convex nature, a feature that singles out entropy convex measures of risk in the class of negative certainty equivalents under homothetic preferences, we can obtain explicit representation results in this setting. Some financial applications and examples of entropy coherent and entropy convex measures of risk are also provided, explicitly utilizing some of the representation results derived.

In the traditional setting of Von Neumann and Morgenstern [44], where the probability measure is known and given so that simply $U(X) = \mathrm{E}[u(X)]$, analogs of the characterization

3

results established in this paper are relatively easy to obtain; see Hardy, Littlewood and Pólya [27] (p. 88, Theorem 106), Gerber [19] (Chapter 5) and Goovaerts, De Vylder and Haezendonck [22] (Chapter 3). It is intriguingly more complicated for the variational, homothetic and multiple priors preferences considered here, and we will show that without richness assumptions on the probability space and subdifferentiability conditions on ρ, our representation theorems in fact break down. In recent work, Cheridito and Kupper [10] (Example 3.6.3) suggest (without formal proof) a connection between certainty equivalents in the pure multiple priors model and convex measures of risk. They restrict, however, to a specific and simple probabilistic setting which, as we will see below, can be viewed as supplementary (and non-overlapping) to a special case of the general setting considered here. While there is a rich literature on both theories (1.4) and (1.5), to the best of our knowledge, we are not aware of other work establishing precise connections between these prominent paradigms.

The rest of this paper is organized as follows: in Section 2, we review some preliminaries for coherent and convex measures of risk. In Section 3, we introduce entropy coherent and entropy convex measures of risk and discuss some of their basic properties. In Section 4, we prove axiomatic characterization results for convex, entropy convex and entropy coherent measures of risk. Section 5 studies the dual conjugate function for entropy coherent and entropy convex measures, and Section 6 proves their distribution invariant representation. Section 7 presents some financial applications and examples of entropy coherent and entropy convex measures of risk. Conclusions are in Section 8.

2 Preliminaries

We fix a probability space $(\Omega, \mathcal{F}, P)$. Throughout this paper, equalities and inequalities between random variables are understood in the P-almost sure sense. We let $L^\infty(\Omega, \mathcal{F}, P) \equiv L^\infty$ denote the space of all real-valued random variables X on $(\Omega, \mathcal{F}, P)$ for which $||X||_\infty := \inf\{c > 0 | P[|X| \leq c] = 1\} < \infty$, where two random variables are identified if they are P-almost surely equal. We denote $]0, \infty[$ by $\mathbb{R}^+$ and $]-\infty, 0]$ by $\mathbb{R}_0^-$.

Definition 2.1 *We call a mapping $\rho : L^\infty \to \mathbb{R}$ a convex risk measure if it has the following properties:*

- *Normalization: $\rho(0) = 0$*

- *Translation Invariance: $\rho(X + m) = \rho(X) - m$ for all $m \in \mathbb{R}$*

- *Monotonicity: If $X \leq Y$, then $\rho(X) \geq \rho(Y)$*

- *Convexity: $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$ for $\lambda \in [0, 1]$*

- *Continuity from above: If $X_n \in L^\infty$ is a decreasing sequence converging to $X \in L^\infty$, then*

$$\rho(X_n) \uparrow \rho(X).$$

Furthermore, ρ is called a coherent risk measure if additionally it is positively homogeneous, i.e.,

- *Positive Homogeneity: For $\lambda > 0 : \rho(\lambda X) = \lambda\rho(X)$.*

We denote by $\mathcal{Q}(P) \equiv \mathcal{Q}$ all probability measures that are absolutely continuous with respect to P. If $Q \in \mathcal{Q}$, we also write $Q \ll P$. It is well-known that if ρ is a convex risk measure then there exists a unique lower-semicontinuous and convex function $\alpha : \mathcal{Q} \to \mathbb{R} \cup \{\infty\}$, referred to as the *dual conjugate* of ρ, such that the following dual representation holds:

$$\rho(X) = \sup_{Q \in \mathcal{Q}} \left\{ \mathrm{E}_Q\left[-X\right] - \alpha(Q) \right\}. \tag{2.1}$$

Furthermore,

$$\alpha(Q) = \sup_{X \in L^\infty} \left\{ \mathrm{E}_Q\left[-X\right] - \rho(X) \right\}; \tag{2.2}$$

α is minimal in the sense that for every other (possibly non-convex or non-lower-semicontinuous) function α' satisfying (2.1), $\alpha \le \alpha'$; see, for instance, Föllmer and Schied [16] and Ruszczyński and Shapiro [38, 39]. We define the subdifferential of ρ by

$$\partial \rho(X) = \{ Q \in \mathcal{Q} | \rho(X) = \mathrm{E}_Q\left[-X\right] - \alpha(Q) \}. \tag{2.3}$$

We say that ρ is subdifferentiable if for every $X \in L^\infty$, $\partial\rho(X) \ne \emptyset$. In this paper, we furthermore denote by $C^n(E)$ the space of all functions from $\mathbb{R}$ to $\mathbb{R}$ for which the first n-derivatives exist and which are continuous in an open set E. Finally, for a set $M \subset \mathcal{Q}$, we denote by $\bar{I}_M$ the penalty function that is zero if $Q \in M$ and ∞ otherwise.

3 Entropy Coherence and Entropy Convexity: Definitions and Basic Properties

Throughout this section we suppose that $\gamma \in [0, \infty]$ is fixed. A risk measure that is particularly popular in insurance and financial mathematics (Gerber [19], Föllmer and Schied [16] and Mania and Schweizer [34]), macroeconomics (Hansen and Sargent [25, 26]), and decision theory (Gollier [21] and Strzalecki [43]), is the (standard) entropic risk measure defined by

$$e_\gamma(X) = \gamma \log \left(\mathrm{E}\left[\exp\left\{ -\frac{X}{\gamma} \right\} \right] \right),$$

with $e_0(X) = \lim_{\gamma \downarrow 0} e_\gamma(X) = -\operatorname{ess\,inf} X$ and $e_\infty(X) = \lim_{\gamma \uparrow \infty} e_\gamma(X) = -\mathrm{E}\left[X\right]$. In a setting with distribution invariance, it is commonly referred to as the exponential premium; see Gerber [19] and Goovaerts *et al.* [23]. As is well-known (Csiszár [11]),

$$e_\gamma(X) = \sup_{\bar{P} \ll P} \left\{ \mathrm{E}_{\bar{P}}\left[-X\right] - \gamma H(\bar{P}|P) \right\},$$

where $H(\bar{P}|P)$ is the relative entropy, i.e.,

$$H(\bar{P}|P) = \begin{cases} \mathrm{E}_{\bar{P}}\left[\log\left(\dfrac{d\bar{P}}{dP} \right) \right], & \text{if } \bar{P} \ll P; \\ \infty, & \text{otherwise.} \end{cases}$$

The relative entropy is also known as the Kullback-Leibler divergence; it measures the distance between the measures $\bar{P}$ and P.

Risk measurement with the relative entropy is natural in the following setting: the economic agent has a reference measure P; the measure P is, however, an approximation to the probabilistic model of the payoff X rather than the true model. The agent therefore does not fully trust the measure P and considers many measures $\bar{P}$, with esteemed plausibility decreasing proportionally to their distance from the approximation P. Note that for every given X, the mapping $\gamma \to e_\gamma(X)$ is increasing. Consequently, the parameter γ may be viewed as measuring the degree of trust the agent puts in the reference measure P. If $\gamma = 0$, then $e_0(X) = -\operatorname{ess\,inf} X$, which corresponds to a maximal level of distrust; in this case only the zero sets of the measure P are considered reliable. If, on the other hand, $\gamma = \infty$, then $e_\infty(X) = -\mathrm{E}[X]$, which corresponds to a maximal level of trust in the measure P. In the case that $\gamma \in \mathbb{R}^+$, it is well-known that $\partial e_\gamma(X)$ is given by the *Esscher density* with respect to P: $\exp\left\{-\frac{X}{\gamma}\right\}/\mathrm{E}\left[\exp\left\{\frac{-X}{\gamma}\right\}\right]$.

In certain situations the agent can possibly consider other reference measures $Q \ll P$. Then we define the entropy $e_{\gamma,Q}$ with respect to Q as

$$e_{\gamma,Q}(X) = \gamma \log\left(\mathrm{E}_Q\left[\exp\left\{\frac{-X}{\gamma}\right\}\right]\right).$$

Consider the following example:

Example 3.1 Suppose that the agent is only interested in downside tail risk. The standard risk measure focusing on tail risk is the Tail-Value-at-Risk ($TV@R$), also referred to as Conditional-Value-at-Risk or Average-Value-at-Risk (Rockafellar and Uryasev [36] and Rockafellar, Uryasev and Zabarankin [37]). $TV@R$ is defined by

$$TV@R^\alpha(X) = \frac{1}{\alpha}\int_0^\alpha V@R^\lambda(X)d\lambda, \quad \alpha \in\,]0,1],$$

with $V@R^\lambda(X) = -q_X^+(\lambda)$, where q_X^+ is the upper quantile function of X: $q_X^+(\lambda) = \inf\{x | P[X \le x] > \lambda\}$. If the distribution of X is continuous, $TV@R^\alpha(X) = \mathrm{E}\left[-X|X \le q_X^+(\alpha)\right]$, so that $TV@R$ computes the average over the left tail of the distribution of X up to $q_X^+(\alpha)$. It is well-known that

$$TV@R^\alpha(X) = \sup_{Q \in M_\alpha} \mathrm{E}_Q\left[-X\right],$$

where M_α is the set of all probability measures $Q \ll P$ such that $\frac{dQ}{dP} \le \frac{1}{\alpha}$. Let $\frac{dQ}{dP} = \frac{1}{\alpha}I_{\{X<q_X^+(\alpha)\}} + cI_{\{X=q_X^+(\alpha)\}}$, where c should be chosen such that $\mathrm{E}\left[\frac{dQ}{dP}\right] = 1$. Then one can show that

$$Q \in \arg\max\{\mathrm{E}_{\bar{P}}\left[-X|\bar{P} \in M_\alpha\right]\},$$

i.e., $TV@R^\alpha(X) = \mathrm{E}_Q\left[-X\right]$, and, for continuous distributions, $Q = P[.|X \le q_X^+(\alpha)]$. Thus, the measure Q coincides with the original reference measure P, but concentrated on the left tail of X. The economic agent may, however, not fully trust the probabilistic model of X under P, hence under Q. Therefore, for every fixed Q, the agent considers the supremum over all measures absolutely continuous with respect to Q, where measures that are 'close' to Q are esteemed more plausible than measures that are 'distant' from Q. This leads to a risk measure ρ given by

$$\rho(X) = \sup_{\bar{P} \ll Q}\sup_{Q \in M_\alpha}\{\mathrm{E}_{\bar{P}}\left[-X\right] - \gamma H(\bar{P}|Q)\} = \sup_{Q \in M_\alpha}\sup_{\bar{P} \ll P}\{\mathrm{E}_{\bar{P}}\left[-X\right] - \gamma H(\bar{P}|Q)\} = \sup_{Q \in M_\alpha} e_{\gamma,Q}(X)$$

$$= \gamma \log\left(\sup_{Q \in M_\alpha}\mathrm{E}_Q\left[\exp\left\{\frac{-X}{\gamma}\right\}\right]\right) = \gamma \log\left(TV@R^\alpha\left(-\exp\left\{\frac{-X}{\gamma}\right\}\right)\right),$$

6

where we have used in the second and third equalities that $H(\bar{P}|Q) = \infty$ if $\bar{P}$ is not absolutely continuous with respect to Q. The risk measure given by $\rho(X) = \gamma \log \left(TV@R^\alpha \left(-\exp\left\{ \frac{-X}{\gamma} \right\} \right) \right)$ accounts for tail risk and model uncertainty. Furthermore, it is computationally attractive because all one needs is a reference model P for the payoff X, which in a probabilistic approach seems a mild presumption.

This example motivates the following definition:

Definition 3.2 *We call a mapping $\rho : L^\infty \to \mathbb{R}$ γ-entropy coherent, $\gamma \in [0, \infty]$, if there exists a set $M \subset \mathcal{Q}$ such that*

$$\rho(X) = \sup_{Q \in M} e_{\gamma, Q}(X).$$

It will be interesting to consider as well a more general class of risk measures:

Definition 3.3 *The mapping $\rho : L^\infty \to \mathbb{R}$ is γ-entropy convex, $\gamma \in [0, \infty]$, if there exists a penalty function $c : \mathcal{Q} \to [0, \infty]$ with $\inf_{Q \in \mathcal{Q}} c(Q) = 0$, such that*

$$\rho(X) = \sup_{Q \in \mathcal{Q}} \{e_{\gamma, Q}(X) - c(Q)\}. \tag{3.1}$$

Henceforth, we call a mapping entropy coherent (convex) if there exists a $\gamma \in [0, \infty]$ such that ρ is γ-entropy coherent (convex).

Considering

$$-\rho(X) = \inf_{Q \in \mathcal{Q}} \left\{ -\gamma \log \left(E_Q \left[\exp\left\{ \frac{-X}{\gamma} \right\} \right] \right) + c(Q) \right\},$$

the definition of entropy convexity (whence the special case of entropy coherence as well) can also be motivated as follows: an economic agent with a CARA (exponential) utility function $u(x) = 1 - e^{-x/\gamma}$ computes the certainty equivalent to the payoff X with respect to the reference measure P. The agent is, however, uncertain about the probabilistic model under the reference measure, and therefore takes the infimum over all probability measures Q absolutely continuous with respect to P, where the penalty function $c(Q)$ represents the esteemed plausibility of the probabilistic model under Q. The robust certainty equivalent thus computed is precisely $-\rho(X)$.

Proposition 3.4 *Every γ-entropy convex functional is a convex risk measure.*

Proof. For every fixed Q with $Q \ll P$ we have that if $X = Y$ P-a.s. then also $X = Y$ Q-a.s., hence, $e_{\gamma, Q}(X) = e_{\gamma, Q}(Y)$ and therefore $\sup_{Q \in \mathcal{Q}} \{e_{\gamma, Q}(X) - c(Q)\} = \sup_{Q \in \mathcal{Q}} \{e_{\gamma, Q}(Y) - c(Q)\}$ as well. Furthermore, $e_{\gamma, Q}(X) - c(Q)$ is translation invariant, monotone, convex and lower-semicontinuous (hence, continuous from above). Thus, also $\sup_{Q \in \mathcal{Q}} \{e_{\gamma, Q}(X) - c(Q)\}$ is translation invariant, monotone, convex and continuous from above. Normalization follows because $\inf_{Q \in \mathcal{Q}} c(Q) = 0$ by assumption. $\square$

In principle, one might consider as well more general classes of risk measures, replacing $e_{\gamma, Q}(X)$ in (3.1) by an arbitrary functional on $L^\infty \times \mathcal{Q}$, for example, the certainty equivalent of an agent with CRRA (power) utility. In that case, however, one generally loses the translation invariance property and the resulting ρ will no longer be a convex risk measure. We therefore do not pursue this route.

As $e_{\infty, Q}(X) = E_Q[-X]$, (2.1) implies that ρ is a convex risk measure if and only if it is ∞-entropy convex. As we will see later (for example, Theorem 5.2 below), however, with $\gamma < \infty$,

not every convex risk measure is γ-entropy convex. This is important: in Theorem 4.1 below we will see that, under some technical conditions, negative certainty equivalents under homothetic preferences are translation invariant if and only if they are γ-entropy convex with $\gamma \in \mathbb{R}^+$ or ∞-entropy coherent, ruling out the general ∞-entropy convex case. But the following result is available:

Proposition 3.5 *Let ρ be a convex risk measure. Then for every $\gamma \in [0, \infty]$ there exists a γ-entropy convex risk measure $\rho_{\gamma,dom}$ dominating ρ.*

Proof. We have $e_{\gamma,Q}(X) = \sup_{\bar{P} \ll Q}\{\mathrm{E}_{\bar{P}}[-X] - \gamma H(\bar{P}|Q)\} \geq \mathrm{E}_Q[-X]$. Thus, setting $\alpha = c$, $\rho(X) = \sup_{Q \ll P}\{\mathrm{E}_Q[-X] - \alpha(Q)\} \leq \sup_{Q \ll P}\{e_{\gamma,Q}(X) - c(Q)\} = \rho_{\gamma,dom}(X)$. $\quad\square$

For a risk measure ρ we define

$$\rho^*(Q) = \sup_{X \in L^\infty} \{e_{\gamma,Q}(X) - \rho(X)\}$$

and

$$\rho^{**}(X) = \sup_{Q \ll P} \{e_{\gamma,Q}(X) - \rho^*(Q)\}.$$

Lemma 3.6 *If ρ is γ-entropy convex, then for every $X \in L^\infty$,*

$$\rho^{**}(X) \leq \rho(X). \tag{3.2}$$

Proof. As $\rho^*(Q) = \sup_{X \in L^\infty}\{e_{\gamma,Q}(X) - \rho(X)\}$ it follows that $e_{\gamma,Q}(X) - \rho^*(Q) \leq \rho(X)$ for all $X \in L^\infty$. Taking the supremum over all measures Q which are absolutely continuous with respect to P yields (3.2). $\quad\square$

The next proposition establishes a basic duality result for γ-entropy convex risk measures:

Proposition 3.7 *A normalized mapping ρ is γ-entropy convex if and only if $\rho^{**} = \rho$. Furthermore, ρ^* is the minimal penalty function.*

Proof. This duality result follows in principle from the general duality results in Moreau [35]. We provide a short proof to be self-contained. The 'if' part holds because if $\rho(X) = \rho^{**}(X) = \sup_{Q \ll P}\{e_{\gamma,Q}(X) - \rho^*(Q)\}$ then by virtue of the equalities

$$0 = -\rho(0) = -\sup_{Q \in \mathcal{Q}} -\rho^*(Q) = \inf_{Q \in \mathcal{Q}} \rho^*(Q),$$

ρ is γ-entropy convex. Let us prove the 'only if' direction. We already know from Lemma 3.6 that $\rho^{**} \leq \rho$. We will prove that $\rho^{**} \geq \rho$. If ρ is γ-entropy convex there exists a penalty function c such that

$$\rho(X) = \sup_{Q \in \mathcal{Q}} \{e_{\gamma,Q}(X) - c(Q)\}.$$

Thus, for every $Q \ll P$ we have $c(Q) \geq e_{\gamma,Q}(X) - \rho(X)$. By the definition of ρ^* this yields $c(Q) \geq \rho^*(Q)$. This proves that every penalty function ρ is dominating ρ^*. Moreover,

$$\rho^{**}(X) = \sup_{Q \ll P} \{e_{\gamma,Q}(X) - \rho^*(Q)\} \geq \sup_{Q \ll P} \{e_{\gamma,Q}(X) - c(Q)\} = \rho(X).$$

$\quad\square$

Proposition 3.7 suggests a way to find out whether a risk measure ρ is γ-entropy convex: compute ρ^* and ρ^{**}, and verify whether $\rho^{**} = \rho$.

Remark 3.8 ρ^* measures how much ρ deviates from below from the Q-entropy. If there exists a $Q \ll P$ such that $\rho(X) \leq e_{\gamma,Q}(X)$ then $\rho^*(Q) \geq e_{\gamma,Q}(X) - \rho(X) \geq 0$. This and the convexity of ρ^* jointly imply that ρ is entropy coherent if and only if $\rho^* = \bar{I}_M$ for a set $M \subset \mathcal{Q}$.

Remark 3.9 Let $\mathcal{A}$ be the acceptance set of ρ, i.e., $\mathcal{A} = \{X \in L^\infty | \rho(X) \leq 0\}$. ρ^* can be represented as

$$\rho^*(Q) = \sup_{X \in \mathcal{A}} e_{\gamma,Q}(X).$$

To see this, note that clearly,

$$\rho^*(Q) = \sup_{X \in L^\infty} \{e_{\gamma,Q}(X) - \rho(X)\} \geq \sup_{X \in \mathcal{A}} \{e_{\gamma,Q}(X) - \rho(X)\} \geq \sup_{X \in \mathcal{A}} e_{\gamma,Q}(X).$$

On the other hand, if $X \in \mathcal{A}$ then $X + \rho(X) \in \mathcal{A}$, which implies that

$$\rho^*(Q) = \sup_{X \in L^\infty} \{e_{\gamma,Q}(X) - \rho(X)\} = \sup_{X \in L^\infty} \{e_{\gamma,Q}(X + \rho(X))\}$$

$$= \sup_{X + \rho(X) = Y \in L^\infty} e_{\gamma,Q}(Y) \leq \sup_{Y \in \mathcal{A}} e_{\gamma,Q}(Y).$$

Definition 3.10 *For a γ-entropy convex function ρ we denote by*

$$\partial_{entropy}\rho(X) = \{Q^* \in \mathcal{Q} | \rho(X) = e_{\gamma,Q^*}(X) - c(Q^*)\}$$

the entropy subdifferential. Furthermore, if for every $X \in L^\infty$, $\partial_{entropy}\rho(X) \neq \emptyset$, then we say that ρ is entropy subdifferentiable.

Remark 3.11 If $\gamma \in \mathbb{R}^+$ and $Q^* \in \partial_{entropy}\rho(X)$, then $\dfrac{\exp\left\{-\frac{X}{\gamma}\right\}}{\mathrm{E}_{Q^*}\left[\exp\left\{-\frac{X}{\gamma}\right\}\right]} \in \partial\rho(X)$, where $\partial\rho(X)$ is the usual subdifferential defined by (2.3). In the case that there exists a c such that (3.1) holds and such that the domain of c is a separated compact space it follows directly from Theorem 2.4.18, Zalinscu [47] that every $\bar{P}$ in $\partial\rho(X)$ can be written as the L^1 limit of convex combinations of measures $\bar{P}^n$ given by $\dfrac{d\bar{P}^n}{dP} = \dfrac{\exp\left\{-\frac{X}{\gamma}\right\}}{\mathrm{E}_{Q_n^*}\left[\exp\left\{-\frac{X}{\gamma}\right\}\right]}$ with $Q_n^* \in \partial_{entropy}\rho(X)$. In particular, in this case $\partial_{entropy}\rho(X) \neq \emptyset$ if and only if $\partial\rho(X) \neq \emptyset$.

Proposition 3.12 *Suppose that ρ is a γ-entropy coherent risk measure with $\gamma \in]0,\infty]$. Then the following statements are equivalent:*

(a) *For every $X \in L^\infty$,*

$$\rho(X) = \max_{Q \in M} e_{\gamma,Q}(X).$$

(b) *$M \subset \mathcal{Q}$ is weakly compact.*

(c) *ρ is continuous from below, i.e., $X_n \uparrow X \Rightarrow \rho(X_n) \downarrow \rho(X)$.*

Proof. Let

$$\bar{\rho}(X) = \sup_{Q \in M} \mathrm{E}_Q[-X]. \tag{3.3}$$

First of all, notice that by Corollary 4.35 in Föllmer and Schied [16] and the translation invariance of $\bar{\rho}$, M being weakly compact is equivalent to the maximum in (3.3) being attained for every $X < 0$.

(a)$\Rightarrow$(b): Suppose that $X < 0$. Then

$$\bar{\rho}(X) = \exp\left\{\frac{1}{\gamma}\rho(-\gamma\log(-X))\right\} = \exp\left\{\frac{1}{\gamma}\max_{Q\in M}\gamma\log(\mathrm{E}_Q\left[-X\right])\right\} = \max_{Q\in M}\mathrm{E}_Q\left[-X\right].$$

(b)$\Rightarrow$(a): We write

$$\rho(X) = \gamma\log\left(\sup_{Q\in M}\mathrm{E}_Q\left[\exp\left\{\frac{-X}{\gamma}\right\}\right]\right) = \gamma\log\left(\max_{Q\in M}\mathrm{E}_Q\left[\exp\left\{\frac{-X}{\gamma}\right\}\right]\right) = \max_{Q\in M}e_{\gamma,Q}(X).$$

(b)$\Leftrightarrow$(c): Corollary 4.35 in Föllmer and Schied [16] implies also that M being weakly compact is equivalent to $\bar{\rho}$ being continuous from below. Now clearly $\bar{\rho}$ being continuous from below implies that ρ is continuous from below. On the other hand, suppose that $X_n \uparrow X$. Since $\bar{\rho}$ is translation invariant we may assume without loss of generality that $X_n < 0$. Define $Y_n := -\gamma\log(-X_n) \uparrow Y =: -\gamma\log(-X)$. Then the continuity from below of ρ implies that

$$\bar{\rho}(X_n) = \exp\left\{\frac{\rho(Y_n)}{\gamma}\right\} \downarrow \exp\left\{\frac{\rho(Y)}{\gamma}\right\} = \bar{\rho}(X).$$

$\square$

4 Axiomatic Characterizations

In this section, we axiomatize convex, entropy convex and entropy coherent measures of risk, showing that they emerge as certainty equivalents under variational, homothetic and multiple priors preferences, respectively, upon requiring the certainty equivalents to be translation invariant.

In the main characterization theorems (Theorem 4.1, Corollary 4.10 and Theorem 4.12), we consider, more specifically, negative certainty equivalents of the form $\rho(X) = \phi^{-1}(\bar{\rho}(-\phi(-X)))$, with

$$\bar{\rho}(X) = \begin{cases} \sup_{Q\in\mathcal{Q}}\beta(Q)\mathrm{E}_Q\left[-X\right], \\ \sup_{Q\in M}\mathrm{E}_Q\left[-X\right], \\ \sup_{Q\in\mathcal{Q}}\{\mathrm{E}_Q\left[-X\right] - \alpha(Q)\}, \end{cases}$$

respectively. These constitute the negative certainty equivalents in the theories of homothetic, multiple priors and variational preferences, respectively; cf. (1.4), and also Section I.3 in Föllmer, Schied and Weber [17]. Recall that a rich probability space supports a random variable with a uniform distribution.

4.1 Homothetic Preferences and Entropy Convex Measures of Risk

We state the following theorem:

Theorem 4.1 *Suppose that the probability space is rich and that $\bar{\rho} : L^\infty \to \mathbb{R}$ is monotone, convex, positively homogeneous and continuous from above and for all $m \in \mathbb{R}_0^-$, $\bar{\rho}(m) = -m$. Let ϕ be a strictly increasing and continuous function satisfying $0 \in closure(Image(\phi))$, $\phi(\infty) = \infty$ and $\phi \in C^3(]\phi^{-1}(0),\infty[)$. Then the following statements are equivalent:*

10

(i) $\rho(X) = \phi^{-1}(\bar{\rho}(-\phi(-X)))$ *is translation invariant and the subdifferential of $\bar{\rho}$ is always nonempty.*

(ii) ρ *is γ-entropy convex with $\gamma \in \mathbb{R}^+$ or ρ is ∞-entropy coherent, and the entropy subdifferential is always nonempty.*

Remark 4.2 In the proof of Theorem 4.1 (see also Corollary 4.10 below), it will become apparent that $\rho(X) = \phi^{-1}(\bar{\rho}(-\phi(-X)))$ is entropy coherent if and only if $\bar{\rho}$ is a coherent risk measure. In this case, $\bar{\rho}(X) = \sup_{Q \in M} \mathrm{E}_Q[-X]$ for a set $M \subset \mathcal{Q}$, and $\rho(X) = \phi^{-1}(\bar{\rho}(-\phi(-X)))$ is a negative certainty equivalent in the multiple priors model. In addition, it will turn out that in this case ϕ is linear or exponential.

Furthermore, we will see that the case that $\rho(X) = \phi^{-1}(\bar{\rho}(-\phi(-X)))$ is entropy convex corresponds to ρ being the negative certainty equivalent under homothetic preferences, with $\bar{\rho}(X) = \sup_{Q \in M} \beta(Q)\mathrm{E}_Q[-X]$, where $\beta : M \to [0, 1]$ can be viewed as a discount factor, and with ϕ being linear (implying $\beta(Q) \equiv 1$) or exponential. In this case, every probabilistic model Q is discounted by a factor $\beta(Q)$ corresponding to its esteemed plausibility. If $\beta(Q) = 1$ for all $Q \in M$, we are back in the multiple priors framework. However, if there exists a $Q \in M$ such that $\beta(Q) < 1$, ρ is entropy convex with $\gamma \in \mathbb{R}^+$ but not entropy coherent.

Remark 4.3 The direction (i)$\Rightarrow$(ii) in Theorem 4.1 does not hold (even not in the case that we additionally assume that $\bar{\rho}$ is translation invariant as in Corollary 4.10 below) if the probability space is not rich, or if the assumption on the subdifferential of $\bar{\rho}$ is omitted.

Suppose, for instance, that $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$ and that, without loss of generality, $P[\{\omega_i\}] = p_i > 0$, $i = 1, \ldots, n$. Then for a payoff X we can define $\bar{\rho}(X) = \max_{i=1,\ldots,n} -X(\omega_i)$, where the maximum is attained in the measure Q that sets $Q[\{\omega_{i_0}\}] = 1$, where $\omega_{i_0} = \arg\max_\omega -X(\omega)$. Such a discrete worst case measure of risk is popular in robust optimization. Let ϕ be a strictly increasing and continuous function. Then it *always* holds that

$$\phi^{-1}(\bar{\rho}(-\phi(-X))) = \phi^{-1}(\max_i \phi(-X(\omega_i)))$$
$$= \phi^{-1}(\phi(-X(\omega_{i_0}))) = -X(\omega_{i_0}) = \bar{\rho}(X).$$

In particular, $\rho(X) = \phi^{-1}(\bar{\rho}(-\phi(-X))) = \bar{\rho}(X)$ is translation invariant for every function ϕ that is strictly increasing and continuous. This shows that (i)$\Rightarrow$(ii) in Theorem 4.1 does not hold if the probability space is finite.

If, on the other hand, the probability space is rich but we omit the assumption that $\bar{\rho}$ is subdifferentiable, then the coherent risk measure $\bar{\rho}(X) = \mathrm{ess\,sup} -X$ satisfies for *every* strictly increasing and continuous function ϕ that $\rho(X) = \phi^{-1}(\bar{\rho}(-\phi(-X))) = \bar{\rho}(X)$ is a convex risk measure. The equality may be seen to hold as

$$\phi^{-1}(\bar{\rho}(-\phi(-X))) = \mathrm{ess\,sup}\, \phi^{-1}(\phi(-X)) = \mathrm{ess\,sup} -X = \bar{\rho}(X).$$

Remark 4.4 In financial mathematics, translation invariance is typically motivated by the interpretation of a risk measure on L^∞ as a minimal amount of risk capital. It ensures that $\rho(X + \rho(X)) = 0$.

Remark 4.5 Notice that since ϕ is positive somewhere and $0 \in closure(Image(\phi))$ we have that $\phi^{-1}(\delta)$ is well-defined for all $\delta > 0$ small enough and we can define $\phi^{-1}(0) = \lim_{\delta \downarrow 0} \phi^{-1}(\delta)$. The common condition that $\phi(\infty) = \infty$ implies that ρ remains loss sensitive.

Before proving Theorem 4.1, we first present the following two lemmas:

Lemma 4.6 *Suppose that* $\bar{\rho} : L^{\infty} \to \mathbb{R}$ *is monotone, convex, positively homogeneous and continuous from above and for all* $m \in \mathbb{R}_0^-$, $\bar{\rho}(m) = -m$. *Then there exists a function* $\beta : \mathcal{Q} \supset M \to [0, 1]$ *with* $\sup_{Q \in M} \beta(Q) = 1$, *such that for all* $X \in L^{\infty}$ *with* $X \le 0$,

$$\bar{\rho}(X) = \sup_{Q \in M} \beta(Q) \mathrm{E}_Q [-X] . \tag{4.1}$$

Furthermore, if additionally we have $\bar{\rho}(1) = -1$ *then* M *can be chosen such that* $\beta(Q) = 1$ *for all* $Q \in M$.

Proof. By standard arguments (see, for example, Lemma A64 in the appendix of Föllmer and Schied [16]), we may conclude that $\bar{\rho}$ is weak* lower-semicontinuous. Proposition 3.1.2 in Dana [12] implies that

$$\bar{\rho}(X) = \sup_{X' \in L_+^1} \{ \mathrm{E} \left[-X'X \right] - \hat{\bar{\rho}}(X') \},$$

and it follows from standard results in convex analysis that the positive homogeneity of $\bar{\rho}$ entails that $\hat{\bar{\rho}}$ is an indicator function of a convex nonempty set, say $H \subset L_+^1$. Hence,

$$\bar{\rho}(X) = \sup_{X' \in H} \mathrm{E} \left[-X'X \right]$$
$$= \sup_{X' \in H} \mathrm{E} \left[X' \right] \mathrm{E} \left[-\frac{X'}{\mathrm{E}\left[X'\right]} X \right] = \sup_{X' \in H} \mathrm{E} \left[X' \right] \mathrm{E}_{Q^{X'}} [-X] , \tag{4.2}$$

where in the case that $X' \equiv 0$, we set $0/0 = 1$ and $Q^{X'} = P$. Now set $M = \left\{ Q \in \mathcal{Q} \mid \text{there exists a } \lambda \ge 0 \text{ such that } \lambda \frac{dQ}{dP} \in H \right\}$. Then (4.2) entails that for all $X \in L^{\infty}$ with $X \le 0$,

$$\bar{\rho}(X) = \sup_{Q \in M} \beta(Q) \mathrm{E}_Q [-X] ,$$

where for $Q \in M$, $\beta(Q) = \sup\{\lambda \ge 0 \mid \lambda \frac{dQ}{dP} \in H\}$. This shows (4.1). Furthermore,

$$\sup_{Q \in M} \beta(Q) = \bar{\rho}(-1) = 1.$$

To see the last part of the lemma note that if $\bar{\rho}(1) = -1$ then we must have $-1 = \bar{\rho}(1) = \sup_{X' \in H} \mathrm{E} \left[-X' \right]$. This implies that

$$\inf_{X' \in H} \mathrm{E} \left[X' \right] = 1.$$

On the other hand, since $\bar{\rho}(-1) = 1$, we also have that $\sup_{X' \in H} \mathrm{E} \left[X' \right] = 1$. Hence, for every $X' \in H$ we get that $\mathrm{E} \left[X' \right] = 1$ and by the definition of β we obtain that $\beta(Q) = 1$ for every $Q \in M$. $\quad\square$

Subsequently, we will identify the measure $\beta(Q)Q$ (given by $(\beta(Q)Q)(A) = \beta(Q)Q(A)$ for every $A \in \mathcal{F}$) with its density $\beta(Q)\frac{dQ}{dP}$. We recall that an element $X' \in H \subset L_+^1$ is in $\partial\bar{\rho}(X)$ if it attains the supremum in (4.2), i.e., $\bar{\rho}(X) = \mathrm{E} \left[-X'X \right]$.

Lemma 4.7 *Suppose that* $\bar\rho : L^\infty \to \mathbb{R}$ *is monotone, convex, positively homogeneous and continuous from above and for all* $m \in \mathbb{R}_0^-$, $\bar\rho(m) = -m$. *Let* $X \in L^\infty$ *with* $X > 0$. *Then for every* Q *with* $\beta(Q)Q \in \partial\bar\rho(-X)$ *we have that*

$$\beta(Q) \geq \frac{\operatorname{ess\,inf} X}{\operatorname{ess\,sup} X}.$$

Proof. Choose $Q \in M$ such that $\beta(Q)Q \in \partial\bar\rho(-X)$. Then by (4.1) and the monotonicity of $\bar\rho$

$$\operatorname{ess\,inf} X = \bar\rho(-\operatorname{ess\,inf} X) \leq \bar\rho(-X) = \beta(Q)\mathrm{E}_Q\left[X\right] \leq \beta(Q)\operatorname{ess\,sup} X,$$

where the last inequality holds as $\beta(Q) \geq 0$. Dividing both sides by $\operatorname{ess\,sup} X$ completes the proof. $\qquad\square$

Proof of Theorem 4.1. (i)$\Rightarrow$(ii):
Since ϕ is positive somewhere and $0 \in closure(Image(\phi))$, there are two cases:

(H1) There exists an x_0 such that $\phi(x_0) = 0$.

(H2) $\lim_{x\to-\infty} \phi(x) = 0$ and for every $x \in \mathbb{R}$ we have $\phi(x) > 0$.

Let $\phi_z(\cdot) := \phi(\cdot + z)$ for $z \in \mathbb{R}$. By translation invariance,

$$\phi_z^{-1}(\bar\rho(-\phi_z(-X))) = \phi^{-1}(\bar\rho(-\phi_z(-X))) - z = \phi^{-1}(\bar\rho(-\phi(-X))).$$

Thus, by considering ϕ_z instead of ϕ, we may assume without loss of generality that:

- If (H1) holds then $\phi(0) = 0$ and $\phi \in C^3(]\phi^{-1}(0), \infty[) = C^3(\mathbb{R}^+)$.

- If (H2) holds then $\phi(0) > 0$ and $\phi \in C^3(]\phi^{-1}(0), \infty[) = C^3(]-\infty, \infty[)$.

In particular, we may always assume that $\phi^{-1}(0) \in \{-\infty, 0\}$ and

$$\phi(0) \geq 0. \tag{4.3}$$

Next, let us look at $X \in L^\infty$ such that $X < 0$. By assumption, $\partial\bar\rho(-\phi(-X)) \neq \emptyset$. As $-\phi(-X) < 0$ (since $\phi(0) \geq 0$ and ϕ is strictly increasing), by (4.1) and the assumption that the subdifferential of $\bar\rho$ is always nonempty we have that

$$\bar\rho(-\phi(-X)) = \max_{\beta(Q)Q\in\partial\bar\rho(-\phi(-X))} \beta(Q)\mathrm{E}_Q\left[\phi(-X)\right]. \tag{4.4}$$

Now we need the following lemma:

Lemma 4.8 *Let* $X \in L^\infty$ *with* $X < 0$. *Under the assumptions of Theorem 4.1 (i) we have that*

$$\phi' \circ \phi^{-1}\left(\max_{\beta(Q)Q\in\partial\bar\rho(-\phi(-X))} \beta(Q)\mathrm{E}_Q\left[\phi(-X)\right]\right) = \max_{\beta(Q)Q\in\partial\bar\rho(-\phi(-X))} \beta(Q)\mathrm{E}_Q\left[\phi'(-X)\right]. \tag{4.5}$$

Proof. Note that as ϕ is in $C^3(]\phi^{-1}(0), \infty[)$ we have for $|m| < \operatorname{ess\,inf} -X$,

$$\bar\rho(-\phi(-X + m)) = \bar\rho(-\phi(-X) - \phi'(-X)m + O(m^2)).$$

13

As a result, we will find that

$$\lim_{m \to 0} \frac{\bar{\rho}(-\phi(-X+m)) - \bar{\rho}(-\phi(-X))}{m} = \lim_{m \to 0} \frac{\bar{\rho}(-\phi(-X) - \phi'(-X)m + O(m^2)) - \bar{\rho}(-\phi(-X))}{m}$$
$$= \max_{\beta(Q)Q \in \partial\bar{\rho}(-\phi(-X))} \beta(Q)\mathrm{E}_Q\left[\phi'(-X)\right]. \qquad (4.6)$$

That the last equality holds is seen as follows: For arbitrary $\epsilon > 0$ we have for small m that $|\frac{O(m^2)}{m}| \leq \epsilon$. Therefore,

$$\limsup_{m \downarrow 0} \frac{\bar{\rho}(-\phi(-X) - \phi'(-X)m + O(m^2)) - \bar{\rho}(-\phi(-X))}{m}$$
$$\leq \limsup_{m \downarrow 0} \frac{\bar{\rho}(-\phi(-X) - (\phi'(-X) + \epsilon)m) - \bar{\rho}(-\phi(-X))}{m} = \max_{\beta(Q)Q \in \partial\bar{\rho}(-\phi(-X))} \beta(Q)\mathrm{E}_Q\left[\phi'(-X) + \epsilon\right],$$

where the inequality holds by the monotonicity of $\bar{\rho}$ while the equality holds by Theorem 2.4.9 Zalinescu [47]. As ϵ can be chosen to be arbitrary small we find that

$$\limsup_{m \downarrow 0} \frac{\bar{\rho}(-\phi(-X) - \phi'(-X)m + O(m^2)) - \bar{\rho}(-\phi(-X))}{m} \leq \max_{\beta(Q)Q \in \partial\bar{\rho}(-\phi(-X))} \beta(Q)\mathrm{E}_Q\left[\phi'(-X)\right].$$

Similarly, one can prove (with ϵ replaced by $-\epsilon$) that the same inequality holds when $\limsup_{m \downarrow 0}$ on the left-hand side is replaced by $\limsup_{m \uparrow 0}$. It means that

$$\limsup_{m \to 0} \frac{\bar{\rho}(-\phi(-X) - \phi'(-X)m + O(m^2)) - \bar{\rho}(-\phi(-X))}{m} \leq \max_{\beta(Q)Q \in \partial\bar{\rho}(-\phi(-X))} \beta(Q)\mathrm{E}_Q\left[\phi'(-X)\right].$$

The reverse inequality

$$\liminf_{m \to 0} \frac{\bar{\rho}(-\phi(-X) - \phi'(-X)m + O(m^2)) - \bar{\rho}(-\phi(-X))}{m} \geq \max_{\beta(Q)Q \in \partial\bar{\rho}(-\phi(-X))} \beta(Q)\mathrm{E}_Q\left[\phi'(-X)\right]$$

is proven analogously. Hence, indeed (4.6) holds. In particular, the mapping $g(m) = \bar{\rho}(-\phi(-X+m))$ is differentiable in $m = 0$ and

$$g'(0) = \max_{\beta(Q)Q \in \partial\bar{\rho}(-\phi(-X))} \beta(Q)\mathrm{E}_Q\left[\phi'(-X)\right]. \qquad (4.7)$$

Now by assumption, $\phi^{-1}(\bar{\rho}(-\phi(-X)))$ is translation invariant and for all $m \in \mathbb{R}$,

$$\frac{\phi^{-1}(\bar{\rho}(-\phi(-X+m))) - \phi^{-1}(\bar{\rho}(-\phi(-X)))}{m} = 1. \qquad (4.8)$$

Letting m converge to zero in (4.8) we get that

$$(\phi^{-1} \circ g)'(0) = 1. \qquad (4.9)$$

14

On the other hand, applying the chain rule to $\phi^{-1} \circ g$, we obtain

$$(\phi^{-1} \circ g)'(0) = \frac{\partial\left[\phi^{-1}(\bar{\rho}(-\phi(-X+m)))\right]}{\partial m}\bigg|_{m=0} = \frac{g'(m)}{\phi' \circ \phi^{-1}\left(\bar{\rho}(-\phi(-X+m))\right)}\bigg|_{m=0}$$

$$= \frac{\max_{\beta(Q)Q \in \partial\bar{\rho}(-\phi(-X))} \beta(Q)\mathrm{E}_Q\left[\phi'(-X)\right]}{\phi' \circ \phi^{-1}\left(\bar{\rho}(-\phi(-X))\right)}$$

$$= \frac{\max_{\beta(Q)Q \in \partial\bar{\rho}(-\phi(-X))} \beta(Q)\mathrm{E}_Q\left[\phi'(-X)\right]}{\phi' \circ \phi^{-1}\left(\max_{\beta(Q)Q \in \partial\bar{\rho}(-\phi(-X))} \beta(Q)\mathrm{E}_Q\left[\phi(-X)\right]\right)}, \qquad (4.10)$$

where we applied (4.7) in the third and (4.4) in the last equality. Finally, (4.9) together with (4.10) entail that (4.5) holds true. □

Continuation of the Proof of Theorem 4.1. (i)$\Rightarrow$(ii):
Next, we will show that Lemma 4.8 implies that there exists p, γ, q such that, for all $x \in$ $]\phi^{-1}(0), \infty[$, $\phi(x) = p\exp\{\frac{x}{\gamma}\} + q$ or $\phi(x) = px + q$. We state the following lemma:

Lemma 4.9 *In the setting of Theorem 4.1, suppose that there does not exist p, γ, q such that, for all $x \in]\phi^{-1}(0), \infty[$, $\phi(x) = \gamma\exp\{\frac{x}{\gamma}\} + q$ or $\phi(x) = px + q$. Then the function $\phi' \circ \phi^{-1}$ is not linear on $\phi(]\phi^{-1}(0), \infty[) = \mathbb{R}^+$.*

Proof. Suppose that there exists c, d such that $\phi' \circ \phi^{-1}(x) = cx + d$ for all $x \in \mathbb{R}^+$. As $\phi' \circ \phi^{-1} = \frac{1}{(\phi^{-1})'}$ we get that

$$(\phi^{-1})'(x) = \frac{1}{cx + d}.$$

If $c = 0$ then ϕ is linear on $]\phi^{-1}(0), \infty[$ contrary to our assumptions. As ϕ^{-1} is strictly increasing on $\mathbb{R}^+$, we must have that $c > 0$. This entails $\phi^{-1}(x) = \frac{1}{c}\log(cx + d)$, which yields that $\phi(x) = \frac{1}{c}\exp\{cx\} - \frac{d}{c}$ on $]\phi^{-1}(0), \infty[$. This contradicts again our assumptions. Hence, under the stated assumptions, $\phi' \circ \phi^{-1}$ is not linear on $\mathbb{R}^+$. □

Continuation of the Proof of Theorem 4.1. (i)$\Rightarrow$(ii):
Now assume that (i)$\Rightarrow$(ii) does not hold, i.e., there does not exist p, γ, q such that, for all $x \in]\phi^{-1}(0), \infty[$, $\phi(x) = p\exp\{\frac{x}{\gamma}\} + q$ or $\phi(x) = px + q$. We will then prove that we obtain a contradiction to Lemma 4.8. By Lemma 4.9, this assumption implies that $\phi' \circ \phi^{-1}$ is not linear on $\phi(]\phi^{-1}(0), \infty[) = \mathbb{R}^+$. As ϕ is in $C^3(]\phi^{-1}(0), \infty[)$, $\phi' \circ \phi^{-1}$ is in $C^2(\mathbb{R}^+)$. Now the second derivative of $\phi' \circ \phi^{-1}$ cannot be constantly zero on $\mathbb{R}^+$ as $\phi' \circ \phi^{-1}$ is not linear. Let $u = \inf\left\{t > 0 \mid \left(\phi' \circ \phi^{-1}\right)''(t) \neq 0\right\} \geq 0$. There are two cases:

(i) There exists a nonempty interval $J =]u, t[\subset \mathbb{R}^+$ such that $\left(\phi' \circ \phi^{-1}\right)'' < 0$, i.e., $\phi' \circ \phi^{-1}$ is strictly concave on J.

(ii) There exists a nonempty interval $J =]u, t[\subset \mathbb{R}^+$ such that $\left(\phi' \circ \phi^{-1}\right)'' > 0$, i.e., $\phi' \circ \phi^{-1}$ is strictly convex on J.

15

As $\phi' \circ \phi^{-1}$ is continuously differentiable on $]0,t[$ and linear on $]0,u]$ (by the definition of u), $\phi' \circ \phi^{-1}$ in case (i) is concave on $]0,t[$ and in case (ii) is convex on $]0,t[$. Let $\epsilon > 0$ such that $(1-\epsilon)^2 t > u$. Since the probability space is rich we may choose $X \in L^\infty$ satisfying both of the following two properties:

(a) $-X \in \phi^{-1}(](1-\epsilon)t, t[) \subset \phi^{-1}(J)$.

(b) $-X$ is diffuse.

From (a) it follows in particular that $\phi(-X) \in](1-\epsilon)t, t[\subset J$. Denote

$$
\begin{aligned}
Q_1 &= \arg\max_{\beta(Q)Q \in \partial\bar{\rho}(-\phi(-X))} \beta(Q)\mathrm{E}_Q\left[\phi(-X)\right]. \\
Q_2 &= \arg\max_{\beta(Q)Q \in \partial\bar{\rho}(-\phi(-X))} \beta(Q)\mathrm{E}_Q\left[\phi'(-X)\right].
\end{aligned}
$$

Since $Q_i \ll P$ and $-X$ is diffuse under P we have that $Q_i[-X = x] = 0$ for $i = 1,2$ and every $x \in \phi^{-1}(J)$. Thus, $-X$ is also diffuse under Q_i. As by (a) and (4.3) $\phi(-X) \in J \subset \mathbb{R}^+$ and $\phi(0) \geq 0$, we have that $\phi(-X) > 0$. Since $\beta(Q_i)Q_i \in \partial\rho(-(\phi(-X)))$, Lemma 4.7 gives

$$
\beta(Q_i) \geq \frac{\mathrm{ess\,inf}\,\phi(-X)}{\mathrm{ess\,sup}\,\phi(-X)} \geq \frac{(1-\epsilon)t}{t} = 1 - \epsilon > 0.
$$

Therefore, $\beta(Q_i)\phi(-X)$ is a diffuse random variable under Q_i and

$$
t > \phi(-X) \geq \beta(Q_i)\phi(-X) \geq (1-\epsilon)\phi(-X) \geq (1-\epsilon)^2 t > u,
$$

where the second inequality holds as $\beta(Q_i) \in]0,1]$. In particular, $\beta(Q_i)\phi(-X) \in J$. Finally let us derive the contradiction. Assume case (i) above: Then

$$
\begin{aligned}
\phi' \circ \phi^{-1}\left(\max_{i=1,2}\beta(Q_i)\mathrm{E}_{Q^i}\left[\phi(-X)\right]\right) &= \max_{i=1,2}\phi' \circ \phi^{-1}\left(\mathrm{E}_{Q^i}\left[\beta(Q_i)\phi(-X)\right]\right) \\
&> \max_{i=1,2}\mathrm{E}_{Q^i}\left[\phi' \circ \phi^{-1}\left(\beta(Q_i)\phi(-X)\right)\right] \\
&= \max_{i=1,2}\lim_{\delta \downarrow 0}\mathrm{E}_{Q^i}\left[\phi' \circ \phi^{-1}\left(\beta(Q_i)\phi(-X) + (1 - \beta(Q_i))\delta\right)\right] \\
&\geq \max_{i=1,2}\liminf_{\delta \downarrow 0}\left\{\mathrm{E}_{Q^i}\left[\beta(Q_i)\phi' \circ \phi^{-1}\left(\phi(-X)\right)\right] + (1 - \beta(Q_i))\phi' \circ \phi^{-1}(\delta)\right\} \\
&= \max_{i=1,2}\left\{\beta(Q_i)\mathrm{E}_{Q^i}\left[\phi'(-X)\right] + (1 - \beta(Q_i))\liminf_{\delta \downarrow 0}\phi' \circ \phi^{-1}(\delta)\right\} \\
&\geq \max_{i=1,2}\beta(Q_i)\mathrm{E}_{Q^i}\left[\phi'(-X)\right],
\end{aligned}
$$

where the first inequality holds because of Jensen's inequality for strictly concave functions for the diffuse random variable $\beta(Q_i)\phi(-X) \in J$, with $i = 1,2$, respectively (where we used that $\beta(Q_i)\phi(-X) \in J$ and the *strict* concavity of $\phi' \circ \phi^{-1}$ on J). The second inequality holds by the concavity of the function $\phi' \circ \phi^{-1}$ on $]0,t]$. The third inequality holds because $\phi' \circ \phi^{-1}(\delta) > 0$ for every $\delta > 0$ such that $\phi^{-1}(\delta)$ is well-defined, as ϕ' is positive. The (strict) inequality above is a contradiction to Lemma 4.8, applying to case (i).

Now consider the more challenging case (ii): Then the function $\phi' \circ \phi^{-1}$ is convex on $]0,t]$ and strictly convex on J. Choosing a sequence $\delta_n \downarrow 0$ such that

$$
\liminf_{\delta \downarrow 0}\phi' \circ \phi^{-1}(\delta) = \lim_n \phi' \circ \phi^{-1}(\delta_n),
$$

16

the same argumentation as before yields

$$\phi' \circ \phi^{-1}\left(\max_{i=1,2} \beta(Q_i)\mathrm{E}_{Q^i}\left[\phi(-X)\right]\right) < \max_{i=1,2} \mathrm{E}_{Q^i}\left[\phi' \circ \phi^{-1}\left(\beta(Q_i)\phi(-X)\right)\right]$$

$$\leq \max_{i=1,2}\lim_n\left\{\mathrm{E}_{Q^i}\left[\beta(Q_i)\phi' \circ \phi^{-1}\left(\phi(-X)\right)\right] + (1 - \beta(Q_i))\phi' \circ \phi^{-1}(\delta_n)\right\}$$

$$= \max_{i=1,2}\left\{\beta(Q_i)\mathrm{E}_{Q^i}\left[\phi'(-X)\right] + (1 - \beta(Q_i))\liminf_{\delta\downarrow 0}\phi' \circ \phi^{-1}(\delta)\right\}. \qquad (4.11)$$

Notice that if

$$(1 - \beta(Q_i))\liminf_{\delta\downarrow 0}\phi' \circ \phi^{-1}(\delta) = 0, \qquad (4.12)$$

then (4.11) would imply that

$$\phi' \circ \phi^{-1}\left(\max_{i=1,2} \beta(Q_i)\mathrm{E}_{Q^i}\left[\phi(-X)\right]\right) < \max_{i=1,2} \beta(Q_i)\mathrm{E}_{Q^i}\left[\phi'(-X)\right],$$

which is a contradiction to Lemma 4.8. To see that $(1 - \beta(Q_i))\liminf_{\delta\downarrow 0}\phi' \circ \phi^{-1}(\delta) = 0$ note that there are two cases:

(1.) $\bar{\rho}(1) = -1$,

(2.) $\bar{\rho}(1) \neq -1$.

In the first case the second part of Lemma 4.6 implies that $\beta(Q_i) = 1$ for $i = 1, 2$ and, in particular, (4.12) is satisfied. Let us look at the second case: by positive homogeneity (2.) entails that $\bar{\rho}(m) \neq -m$ for all $m > 0$. Now suppose that there exists $x_0 \in \mathbb{R}$ such that $\phi(-x_0) < 0$. Since by assumption there also exists x_1 such that $\phi(-x_1) > 0$ the continuity of ϕ yields that the assumption (H1) above holds. In particular, $\phi(0) = 0$. By (2.) and the positive homogeneity of $\bar{\rho}$, $\bar{\rho}(-\phi(-x_0)) \neq \phi(-x_0)$. This gives

$$\phi^{-1}(\bar{\rho}(-\phi(-x_0))) \neq -x_0. \qquad (4.13)$$

However, by translation invariance and since $\bar{\rho}(0) = 0$,

$$\phi^{-1}(\bar{\rho}(-\phi(-x_0))) = -x_0 + \phi^{-1}(\bar{\rho}(-\phi(0))) = -x_0 + \phi^{-1}(\bar{\rho}(0)) = -x_0 + \phi^{-1}(0) = -x_0,$$

which is a contradiction to (4.13). Hence, $\phi(x) \geq 0$ for all $x \in \mathbb{R}$ and the assumption (H2) holds, i.e.,

$$\lim_{x\to-\infty}\phi(x) = 0. \qquad (4.14)$$

By construction in (H2) we have $\phi \in C^3(]-\infty,\infty[)$. Now (4.14) implies that the positive function $\phi'(x)$ cannot be bounded constantly away from zero on $(-\infty, z)$ for any $z \in \mathbb{R}$. This means that there is a sequence x_n converging to $-\infty$ such that

$$\liminf_n \phi'(x_n) = 0.$$

Choose $\bar{\delta}_n = \phi(x_n)$. By (4.14) we have that $\lim_n \bar{\delta}_n = 0$ and

$$0 \leq \liminf_{\delta\downarrow 0}\phi' \circ \phi^{-1}(\delta) \leq \lim_n \phi'(\phi^{-1}(\bar{\delta}_n)) = \lim_n \phi'(x_n) = 0.$$

Consequently,
$$\liminf_{\delta \downarrow 0} \phi' \circ \phi^{-1}(\delta) = 0.$$

This proves (4.12). Hence, we have derived a contradiction to Lemma 4.8, applying to case (ii). Furthermore, we have seen that the cases (H1) and (1.), and (H2) and (2.) coincide, respectively.

Hence, (4.5) of Lemma 4.8 implies that the function $\phi' \circ \phi^{-1}$ has to be linear, and by Lemma 4.9 this implies that there exist constants $p, \gamma, q \in \mathbb{R}$ such that $\phi(x) = pe^{x/\gamma} + q$ or $\phi(x) = px + q$ for all $x \in]\phi^{-1}(0), \infty[$ (where in case (H1) $\phi^{-1}(0) = 0$ and in case (H2) $\phi^{-1}(0) = -\infty$).

As ϕ is strictly increasing we have $p > 0$. Now in the case (H2) we must have that $\phi(x) = \exp\{x/\gamma\}$ (with $q = 0$) as only then $\lim_{x \to -\infty} \phi(x) = 0$. On the other hand, in the case (H1), condition (1.) holds and the second part of Lemma 4.6 implies that $\beta(Q) = 1$ for all $Q \in M$. Therefore, $\phi^{-1}(\bar{\rho}(-\phi(-X)))$ is invariant under positive affine transformations of ϕ. Thus, we may *always* assume that $q = 0$. Let us first consider the case that ϕ is not linear, i.e., $\phi(x) = e^{x/\gamma}$. Then

$$\begin{aligned}
\phi^{-1}(\bar{\rho}(-\phi(-X))) &= \phi^{-1}\left(\sup_{Q \in M} \beta(Q) \mathrm{E}_Q \left[\phi(-X)\right]\right) \\
&= \gamma \log\left(\sup_{Q \in M} \beta(Q) \mathrm{E}_Q \left[\exp\left\{\frac{-X}{\gamma}\right\}\right]\right) \\
&= \sup_{Q \in M}\left\{\gamma \log\left(\mathrm{E}_Q \left[\exp\left\{\frac{-X}{\gamma}\right\}\right]\right) + \gamma \log(\beta(Q))\right\} \\
&= \sup_{Q \in M}\{e_{\gamma,Q}(X) - c(Q)\},
\end{aligned}$$

with $c(Q) = -\gamma \log(\beta(Q)) \geq 0$ if $Q \in M$ and $c(Q) = \infty$ else. Thus, indeed $\phi^{-1}(\bar{\rho}(-\phi(-X)))$ is γ-entropy convex. As the supremum on the right-hand side of the first equality is always attained because $\partial\bar{\rho}(-\phi(-X)) \neq \emptyset$, (ii) follows.

Now in the case that ϕ is linear, we may assume that $\phi(x) = px$. But then by our assumptions, $\rho(X) = \phi^{-1}(\bar{\rho}(-\phi(-X))) = \bar{\rho}(X)$ is translation invariant. In particular, ρ is a coherent risk measure attaining its maximum. Thus, ρ is γ-entropy convex (even γ-entropy coherent) with $\gamma = \infty$ and its entropy subdifferential is always nonempty. This completes the proof of the implication (i)$\Rightarrow$(ii) of Theorem 4.1.

Proof of Theorem 4.1. (ii)$\Rightarrow$(i):
To see the direction (ii)$\Rightarrow$(i), we distinguish between two cases: in the case that $\gamma < \infty$, we let $\phi(x) = e^{x/\gamma}$, and $\bar{\rho}(X) = \sup_{Q \in \mathcal{Q}} \beta(Q) \mathrm{E}_Q \left[-X\right]$, with $\beta(Q) = e^{-\rho^*(Q)/\gamma} \geq 0$. Then $\rho(X) = \gamma \log\left(\bar{\rho}\left(-e^{-X/\gamma}\right)\right) = \phi^{-1}(\bar{\rho}(-\phi(-X)))$. Clearly, $\bar{\rho}$ is monotone, convex, positively homogeneous and continuous from above. As $\inf_{Q \in \mathcal{Q}} \rho^*(Q) = 0$, we get that $\sup_{Q \in \mathcal{Q}} \beta(Q) = 1$. This implies that for $m \in \mathbb{R}_0^-$, $\bar{\rho}(m) = -m$. Furthermore, because ρ is entropy convex it is translation invariant.

In the case that $\gamma = \infty$, we let $\phi(x) = x$ and $\bar{\rho}(X) = \rho(X)$. Notice that in both cases we have $\partial\rho(X) \neq \emptyset$ and hence $\partial\bar{\rho}(X) \neq \emptyset$. $\qquad\square$

Corollary 4.10 *In the setting of Theorem 4.1, if $\bar{\rho}$ is additionally assumed to be translation invariant, then statement (i) implies that ρ is γ-entropy coherent with $\gamma \in]0, \infty]$.*

Proof. As $\bar{\rho}$ is assumed to be translation invariant, we have that $\bar{\rho}(m) = -m$ *for all* $m \in \mathbb{R}$. By Lemma 4.6 this implies that in the proof of Theorem 4.1 we can choose $M \subset \mathcal{Q}$ such that $\beta(Q) = 1$ for all $Q \in M$. Hence, we get $c(Q) = \gamma \log(\beta(Q)) = 0$ if $\beta(Q) = 1$ and ∞ else. Thus, indeed $\phi^{-1}(\bar{\rho}(-\phi(-X)))$ is entropy coherent. $\qquad\square$

Remark 4.11 In recent work, Cheridito and Kupper [10] (Example 3.6.3) suggest (without formal proof) a result quite similar to, but essentially different from, Corollary 4.10. Their suggested result can in a way be viewed as supplementary to the statement in Corollary 4.10: they restrict attention to a specific and simple probabilistic setting with a finite outcome space Ω and consider only strictly positive probability measures on Ω. By contrast, in Corollary 4.10, we consider a rich outcome space and allow for weakly positive probability measures.

4.2 Variational Preferences and Convex Measures of Risk

We state the following theorem:

Theorem 4.12 *Suppose that the probability space is rich and that $\bar{\rho} : L^{\infty} \to \mathbb{R}$ is a convex risk measure with dual conjugate α that has uniformly integrable sublevel sets. Let ϕ be a strictly increasing and convex function with $\phi \in C^3(\mathbb{R})$ satisfying either $\phi(-\infty) = -\infty$ or $\phi(x)/x \overset{x \to \infty}{\to} \infty$. Then the following statements are equivalent:*

(i) *$\rho(X) = \phi^{-1}(\bar{\rho}(-\phi(-X)))$ is translation invariant and the subdifferential of $\bar{\rho}$ is always nonempty.*

(ii) *ρ is a convex risk measure and the subdifferential is always nonempty. Furthermore, in the case that $\phi(x)/x \overset{x \to \infty}{\to} \infty$, ρ is γ-entropy coherent with $\gamma \in \mathbb{R}^+$.*

Remark 4.13 Note that under the conditions of Theorem 4.12, $\bar{\rho}(X) = \sup_{Q \in \mathcal{Q}}\{\mathrm{E}_Q[-X] - \alpha(Q)\}$, so that $\rho(X) = \phi^{-1}(\bar{\rho}(-\phi(-X)))$ is a negative certainty equivalent under variational preferences. In the proof of Theorem 4.12, we will see that ϕ is either linear or exponential. In the latter case α, the dual conjugate of $\bar{\rho}$, only takes the values zero and ∞ and ρ is γ-entropy coherent with $\gamma \in \mathbb{R}^+$.

Remark 4.14 Jouini, Schachermayer and Touzi [31] prove that if the probability space is separable, then $\alpha(Q) = \sup_{X \in L^{\infty}}\{\mathrm{E}_Q[-X] - \bar{\rho}(X)\}$ having uniformly integrable sublevel sets is equivalent to $\bar{\rho}$ being continuous from below. But because we do not want to impose any additional assumptions on the probability space, we simply require the sublevel sets of α to be uniformly integrable.

Proof of Theorem 4.12.
The direction (ii)$\Rightarrow$(i) is straightforward. Let us prove (i)$\Rightarrow$(ii). Clearly, for $m' \in \mathbb{R}$ we can consider $\phi(x) + m'$ instead of $\phi(x)$. Hence, we may assume without loss of generality that $\phi(0) = \phi^{-1}(0) = 0$. We need the following lemma, of which the proof is similar to the proof of Lemma 4.8 and therefore omitted.

Lemma 4.15 *Let $X \in L^{\infty}$. Under the assumptions of Theorem 4.12 (i) we have that*

$$\phi' \circ \phi^{-1}\left(\max_{Q \in \partial\bar{\rho}(-\phi(-X))}\{\mathrm{E}_Q[\phi(-X)] - \alpha(Q)\}\right) = \max_{Q \in \partial\bar{\rho}(-\phi(-X))}\mathrm{E}_Q[\phi'(-X)]. \qquad (4.15)$$

19

We also need:

Lemma 4.16 *For any $X \in L^\infty$ such that $Q \in \partial \bar{\rho}(X)$,*

$$0 \leq \alpha(Q) \leq \operatorname{ess\,sup} -X - \operatorname{ess\,inf} -X.$$

Proof. Since $\bar{\rho}(0) = 0$ we must have that $\alpha(Q) \geq 0$. Furthermore, by monotonicity and translation invariance of $\bar{\rho}$

$$\alpha(Q) = \mathrm{E}_Q\left[-X\right] - \bar{\rho}(X) \leq \operatorname{ess\,sup} -X - \operatorname{ess\,inf} -X.$$

$\square$

Continuation of the Proof of Theorem 4.12. (i)$\Rightarrow$(ii):
First of all note that as ϕ is strictly increasing and convex we must have that $\phi(\infty) = \infty$. Assume now that there does not exist p, γ, q such that, for all $x \in]\phi^{-1}(-\infty), \infty[$, $\phi(x) = p \exp\{\frac{x}{\gamma}\} + q$ or $\phi(x) = px + q$, and let us derive a contradiction to Lemma 4.15. By Lemma 4.9, this assumption implies that $\phi' \circ \phi^{-1}$ is not linear on $]\phi(-\infty), \infty[$. As ϕ is in $C^3(\mathbb{R})$ we have that $\phi' \circ \phi^{-1}$ is in $C^2(]\phi(-\infty), \infty[)$. Now the second derivative of $\phi' \circ \phi^{-1}$ cannot be constantly zero on $]\phi(-\infty), \infty[$ as $\phi' \circ \phi^{-1}$ is not linear. Hence, one may see as in the proof of Theorem 4.1 that there are the following two cases:

(a) There exists a nonempty interval $J =]u, t[$ such that $\phi' \circ \phi^{-1}$ is strictly convex on J.

(b) $\phi' \circ \phi^{-1}$ is concave on $]\phi(-\infty), \infty[$. Furthermore, there exists a nonempty interval $J =]u, t[$ such that $\phi' \circ \phi^{-1}$ is strictly concave on J.

Assume case (a). Choose an $\epsilon > 0$ such that $](1-\epsilon)t, t[\subset J$. Since the probability space is rich we may choose $X \in L^\infty$ satisfying both of the following two properties:

(a') $-X \in \phi^{-1}(](1-\frac{2}{3}\epsilon)t, (1-\frac{1}{3}\epsilon)t[) \subset \phi^{-1}(J)$.

(b') $-X$ is diffuse.

Denote

$$\begin{aligned}
Q_1 &= \arg\max_{Q \in \partial \bar{\rho}(-\phi(-X))} \left\{ \mathrm{E}_Q\left[\phi(-X)\right] - \alpha(Q) \right\}. \\
Q_2 &= \arg\max_{Q \in \partial \bar{\rho}(-\phi(-X))} \mathrm{E}_Q\left[\phi'(-X)\right].
\end{aligned}$$

Similar to the proof of Theorem 4.1, it may be seen that $-X$ is diffuse under each Q_i. From (a') and Lemma 4.16 it follows in particular that $\phi(-X) - \alpha(Q_i)$ are in $](1-\epsilon)t, t[\subset J$ for $i = 1, 2$. Now let us derive the contradiction. We write

$$\begin{aligned}
\phi' \circ \phi^{-1}\left(\max_{i=1,2} \mathrm{E}_{Q^i}\left[\phi(-X)\right] - \alpha(Q)\right) &= \max_{i=1,2} \phi' \circ \phi^{-1}\left(\mathrm{E}_{Q^i}\left[\phi(-X)\right] - \alpha(Q_i)]\right) \\
&< \max_{i=1,2} \mathrm{E}_{Q^i}\left[\phi' \circ \phi^{-1}\left(\phi(-X) - \alpha(Q_i)\right)\right] \\
&\leq \max_{i=1,2} \mathrm{E}_{Q^i}\left[\phi' \circ \phi^{-1}(\phi(-X))\right] = \max_{i=1,2}\left\{\mathrm{E}_{Q^i}\left[\phi'(-X)\right]\right\},
\end{aligned}$$

where the strict inequality holds because of Jensen's inequality for strictly concave functions for the diffuse random variable $\phi(-X) - \alpha(Q_i) \in J$, with $i = 1, 2$, respectively. The second

20

inequality holds since $\alpha(Q_i) \geq 0$. The (strict) inequality above is a contradiction to Lemma 4.15.

Now assume that case (a) does not hold. Then we are in case (b) and $\phi' \circ \phi^{-1}$ is concave on $]\phi(-\infty), \infty[$. Note that, by assumption, $\phi' \circ \phi$ is also increasing and positive (as ϕ is convex and strictly increasing). Since no non-constant concave function having domain equal to $\mathbb{R}$ is bounded from below, $\phi(-\infty) \neq -\infty$. Hence, by our assumptions on ϕ, we must have that in this case $\lim_{x \to \infty} \frac{\phi(x)}{x} = \infty$.

Next note that since the derivative of $\phi' \circ \phi^{-1}$ is decreasing and positive it must converge to a constant, say $c \geq 0$. By the monotonicity of $\phi' \circ \phi^{-1}$ (as ϕ is assumed to be convex) there exists a constant $d \in \mathbb{R}$ such that for every $\epsilon > 0$ there exists $M_\epsilon > 0$ such that

$$cx + d \leq \phi' \circ \phi^{-1}(x) \leq cx + d + \epsilon, \quad \text{for all } x > M_\epsilon.$$

As $\phi' \circ \phi^{-1} = \dfrac{1}{(\phi^{-1})'}$ we get that for any $\epsilon > 0$ there exists a constant $M_\epsilon > 0$ such that for all $x > M_\epsilon$

$$\frac{1}{cx + d} \leq (\phi^{-1})'(x) \leq \frac{1}{cx + d - \epsilon}. \tag{4.16}$$

If $c = 0$ then (4.16) would imply that ϕ grows at most linearly contradicting $\lim_{x \to \infty} \frac{\phi(x)}{x} = \infty$. Hence, $c > 0$ and (4.16) implies

$$\phi^{-1}(M_\epsilon) + \frac{1}{c} \log\left(\frac{cx + d}{cM_\epsilon + d}\right) \leq \phi^{-1}(x) \leq \phi^{-1}(M_\epsilon) + \frac{1}{c} \log\left(\frac{cx + d}{cM_\epsilon + d - \epsilon}\right), \tag{4.17}$$

for all $x \in]M_\epsilon, \infty[$, which yields that

$$\frac{1}{c} \left((cM_\epsilon + d) \exp\{c(x - \phi^{-1}(M_\epsilon))\} - d\right) \geq \phi(x)$$
$$\geq \frac{1}{c} \left((cM_\epsilon + d - \epsilon) \exp\{c(x - \phi^{-1}(M_\epsilon))\} - d\right), \quad \text{for all } x \in]\phi^{-1}(M_\epsilon), \infty[. \tag{4.18}$$

From Lemma 4.17 below we may conclude that (4.17)-(4.18) entail that $\bar{\rho}$ must be coherent. Now it follows from Theorem 4.1 (since $\phi(0) = 0$) that ϕ must be linear or exponential which is a contradiction to our starting assumption that this is not case. Hence, indeed ϕ must be either linear or exponential. Furthermore, if $\phi(-\infty) = -\infty$, we must have that ϕ is linear, while if $\lim_{x \to \infty} \frac{\phi(x)}{x} = \infty$, ϕ is exponential.

Now all what is left to show is that if ϕ has an exponential form, then α, the dual conjugate of $\bar{\rho}$, has to be an indicator function that only takes the values zero and ∞. In this case ρ would be γ-entropy coherent with $\gamma \in \mathbb{R}^+$. However, note that if ϕ has an exponential form, then (4.16) even holds for a certain d and $\epsilon = M_\epsilon = 0$. This implies that (4.17)-(4.18) also hold for every $\epsilon > 0$ and every M_ϵ chosen large enough such that $\epsilon < cM_\epsilon + d$. Hence, the fact that in this case α is an indicator function also follows from Lemma 4.17 below. This completes the proof. $\qquad \square$

Lemma 4.17 *Suppose Theorem 4.12(i) and that there exist $c > 0$ and $d \in \mathbb{R}$ such that for every $\epsilon > 0$ there exists $M_\epsilon > 0$ such that (4.17)-(4.18) hold. Then $\bar{\rho}$ is coherent.*

Proof. The lemma would be proved if we could show that α, the dual conjugate of $\bar{\rho}$, is an indicator function. Let

$$b(\epsilon) := \frac{cM_\epsilon + d - \epsilon}{cM_\epsilon + d},$$

and denote $b^{-1}(\epsilon) = \frac{1}{b(\epsilon)}$. Without loss of generality we may assume that M_ϵ converges to ∞ as ϵ tends to zero so that $b(\epsilon)$ tends to one. We will prove the lemma by contradiction. So assume that there exists Q_0 such that $0 < \alpha(Q_0) < \infty$. Let

$$M = \left\{ Q \in \mathcal{Q} | \alpha(Q) \leq \frac{\alpha(Q_0)}{2} \right\}. \tag{4.19}$$

As M is closed and convex, by the Hahn-Banach Theorem there exists an $X_0' \in L^\infty$ such that

$$\mathrm{E}_{Q_0}\left[-X_0'\right] > \sup_{Q \in M} \mathrm{E}_Q\left[-X_0'\right].$$

By considering $X_0 := X_0' + m$ we may, if we choose m suitably, assume that

$$\mathrm{E}_{Q_0}\left[-X_0\right] > 0 > \sup_{Q \in M} \mathrm{E}_Q\left[-X_0\right].$$

For $\epsilon > 0$ with $\epsilon < cM_\epsilon + d$ let $\lambda^\epsilon := \frac{cb(\epsilon)\alpha(Q_0) + \epsilon}{cM_\epsilon + d - \epsilon} \mathrm{E}_{Q_0}\left[X_0\right]^{-1}$. Then

$$\mathrm{E}_{Q_0}\left[-\lambda^\epsilon X_0\right] > \frac{cb(\epsilon)\alpha(Q_0)}{cM_\epsilon + d - \epsilon} > 0 > \sup_{Q \in M} \mathrm{E}_Q\left[-\lambda^\epsilon X_0\right] \geq \sup_{Q \in M} \left\{ \mathrm{E}_Q\left[-\lambda^\epsilon X_0\right] - \frac{cb(\epsilon)\alpha(Q)}{cM_\epsilon + d - \epsilon} \right\},$$

where we used that $\alpha \geq 0$ in the last inequality. Hence,

$$\mathrm{E}_{Q_0}\left[-\lambda^\epsilon X_0\right] - \frac{cb(\epsilon)\alpha(Q_0)}{cM_\epsilon + d - \epsilon} > 0 > \sup_{Q \in M} \left\{ \mathrm{E}_Q\left[-\lambda^\epsilon X_0\right] - \frac{cb(\epsilon)\alpha(Q)}{cM_\epsilon + d - \epsilon} \right\}. \tag{4.20}$$

Clearly this inequality also holds for $-\lambda^\epsilon X_0 + \bar{m}$ for any constant $\bar{m} \in \mathbb{R}$. Let us choose a suitable constant so that $-Z_\epsilon = -\lambda^\epsilon X_0 + \bar{m} > 1$ and consequently $\frac{\log(-Z_\epsilon)}{c}$ is well-defined and positive. Define

$$Z_\epsilon := \lambda^\epsilon X_0 - ||\lambda^\epsilon X_0||_\infty - \epsilon - 1.$$

Then $\frac{\log(-Z_\epsilon)}{c} > 0$ and

$$||Z_\epsilon||_\infty \leq \frac{cb(\epsilon)\alpha(Q_0) + \epsilon}{cM_\epsilon + d - \epsilon} 2||X_0||_\infty \mathrm{E}_{Q_0}\left[X_0\right]^{-1} + \epsilon + 1. \tag{4.21}$$

Let $\tilde{\rho}(X) := \sup_{Q \in \mathcal{Q}} \left\{ \mathrm{E}_Q\left[-X\right] - \frac{cb(\epsilon)}{cM_\epsilon + d - \epsilon} \alpha(Q) \right\}$. By assumption the sublevel sets of α are weakly compact. This entails that for every ϵ with $\epsilon < cM_\epsilon + d$ there exists $Q_\epsilon^* \in \mathcal{Q}$ such that

$$\tilde{\rho}(Z_\epsilon) = \mathrm{E}_{Q_\epsilon^*}\left[-Z_\epsilon\right] - \frac{cb(\epsilon)\alpha(Q_\epsilon^*)}{cM_\epsilon + d - \epsilon}.$$

It follows from (4.20) that

$$\mathrm{E}_{Q_\epsilon^*}\left[-Z_\epsilon\right] - \frac{cb(\epsilon)\alpha(Q_\epsilon^*)}{cM_\epsilon + d - \epsilon} = \sup_{Q \in \mathcal{Q}} \left\{ \mathrm{E}_Q\left[-Z_\epsilon\right] - \frac{cb(\epsilon)\alpha(Q)}{cM_\epsilon + d - \epsilon} \right\}$$

$$\geq \mathrm{E}_{Q_0}\left[-Z_\epsilon\right] - \frac{cb(\epsilon)\alpha(Q_0)}{cM_\epsilon + d - \epsilon} > \sup_{Q \in M} \left\{ \mathrm{E}_Q\left[-Z_\epsilon\right] - \frac{cb(\epsilon)\alpha(Q)}{cM_\epsilon + d - \epsilon} \right\}.$$

22

Therefore, $Q_\epsilon^* \notin M$ which by (4.19) implies that for every $\epsilon > 0$ we have that $\alpha(Q_\epsilon^*) > \alpha(Q_0)/2 > 0$. Next, choose $\epsilon > 0$ small enough such that $\frac{c}{cM_\epsilon + d} < 1$, $\epsilon < cM_\epsilon + d$ and

$$(b^{-1}(\epsilon) - b(\epsilon))\mathrm{E}_{Q_\epsilon^*}[-Z_\epsilon] = \left((b^{-1}(\epsilon) - 1) + (1 - b(\epsilon))\right)\mathrm{E}_{Q_\epsilon^*}[-Z_\epsilon]$$

$$= \left(\frac{\epsilon}{cM_\epsilon + d - \epsilon} + \frac{\epsilon}{cM_\epsilon + d}\right)\mathrm{E}_{Q_\epsilon^*}[-Z_\epsilon]$$

$$\leq \left(\frac{\epsilon}{cM_\epsilon + d - \epsilon} + \frac{\epsilon}{cM_\epsilon + d}\right)\left(\frac{cb(\epsilon)\alpha(Q_0) + \epsilon}{cM_\epsilon + d - \epsilon}2\|X_0\|_\infty \mathrm{E}_{Q_0}[X_0]^{-1} + \epsilon + 1\right)$$

$$< \frac{c\alpha(Q_0)}{2(cM_\epsilon + d - \epsilon)} < \frac{c\alpha(Q_\epsilon^*)}{cM_\epsilon + d - \epsilon},$$

where we used (4.21) in the first inequality. In particular,

$$(b^{-1}(\epsilon) - b(\epsilon))\mathrm{E}_{Q_\epsilon^*}[-Z_\epsilon] - \frac{c\alpha(Q_\epsilon^*)}{cM_\epsilon + d} < 0.$$

Next choose $m' > 0$ large enough such that

$$(b^{-1}(\epsilon) - b(\epsilon))\mathrm{E}_{Q_\epsilon^*}[-Z_\epsilon] - \frac{c\alpha(Q_\epsilon^*)}{cM_\epsilon + d - \epsilon} < -e^{-cm'}\alpha(Q_\epsilon^*).$$

This is equivalent to

$$b^{-1}(\epsilon)\left\{\mathrm{E}_{Q_\epsilon^*}[-Z_\epsilon] - \frac{cb(\epsilon)\alpha(Q_\epsilon^*)}{cM_\epsilon + d - \epsilon}\right\} < b(\epsilon)\left\{\mathrm{E}_{Q_\epsilon^*}[-Z_\epsilon] - e^{-cm'}b^{-1}(\epsilon)\alpha(Q_\epsilon^*)\right\}. \qquad (4.22)$$

Finally, let us derive a contradiction. We write

$$\rho\left(\frac{-\log(-Z_\epsilon)}{c} - \phi^{-1}(M_\epsilon) - m'\right) = \phi^{-1}\left(\bar\rho\left(-\phi\left(\frac{\log(-Z_\epsilon)}{c} + \phi^{-1}(M_\epsilon) + m'\right)\right)\right)$$

$$\geq \frac{1}{c}\log\left(\frac{1}{cM_\epsilon + d}\left[c\bar\rho\left(-\frac{1}{c}[(cM_\epsilon + d - \epsilon)\right.\right.\right.$$

$$\left.\left.\left.\times \exp\left\{\log(-Z_\epsilon) + c\phi^{-1}(M_\epsilon) + cm' - c\phi^{-1}(M_\epsilon)\right\} - d]\right) + d\right]\right) + \phi^{-1}(M_\epsilon)$$

$$= \frac{1}{c}\log\left(\frac{c}{cM_\epsilon + d}\bar\rho\left(\frac{e^{cm'}}{c}(cM_\epsilon + d - \epsilon)Z_\epsilon\right)\right) + \phi^{-1}(M_\epsilon)$$

$$\geq \frac{1}{c}\log\left(\bar\rho\left(\frac{cM_\epsilon + d - \epsilon}{cM_\epsilon + d}e^{cm'}Z_\epsilon\right)\right) + \phi^{-1}(M_\epsilon)$$

$$= \frac{1}{c}\log\left(\sup_{Q \in \mathcal{Q}}\left\{\mathrm{E}_Q\left[-e^{cm'}b(\epsilon)Z_\epsilon\right] - \alpha(Q)\right\}\right) + \phi^{-1}(M_\epsilon)$$

$$= \frac{1}{c}\log\left(e^{cm'}b(\epsilon)\sup_{Q \in \mathcal{Q}}\left\{\mathrm{E}_Q[-Z_\epsilon] - e^{-cm'}b^{-1}(\epsilon)\alpha(Q)\right\}\right) + \phi^{-1}(M_\epsilon)$$

$$= m' + \frac{1}{c}\log\left(b(\epsilon)\sup_{Q \in \mathcal{Q}}\left\{\mathrm{E}_Q[-Z_\epsilon] - e^{-cm'}b^{-1}(\epsilon)\alpha(Q)\right\}\right) + \phi^{-1}(M_\epsilon)$$

$$\geq m' + \frac{1}{c}\log\left(b(\epsilon)\left\{\mathrm{E}_{Q_\epsilon^*}[-Z_\epsilon] - e^{-cm'}b^{-1}(\epsilon)\alpha(Q_\epsilon^*)\right\}\right) + \phi^{-1}(M_\epsilon), \qquad (4.23)$$

where we have used we have used (4.17)-(4.18) in the first inequality as $\frac{\log(-Z_\epsilon)}{c}+m'+\phi^{-1}(M_\epsilon) > \frac{\log(-Z_\epsilon)}{c} + \phi^{-1}(M_\epsilon) > \phi^{-1}(M_\epsilon)$. In the second equality we have used translation invariance and performed obvious simplifications. In the second inequality we have used that, as $\bar{\rho}$ is convex and $\bar{\rho}(0) = 0$, we must have, for $0 \le \lambda = \frac{c}{cM_\epsilon+d} \le 1$, that $\lambda\bar{\rho}(X) = \lambda\bar{\rho}(X)+(1-\lambda)\bar{\rho}(0) \ge \bar{\rho}(\lambda X)$.

On the other hand we obtain

$$\rho\left(\frac{-\log(-Z_\epsilon)}{c} - \phi^{-1}(M_\epsilon)\right)$$

$$\le \frac{1}{c}\log\left(\frac{1}{cM_\epsilon+d-\epsilon}\left[c\bar{\rho}\left(-\frac{1}{c}\left[(cM_\epsilon+d)\exp\left\{c\left(\frac{\log(-Z_\epsilon)}{c}\right)\right\}-d\right]\right)+d\right]\right) + \phi^{-1}(M_\epsilon)$$

$$= \frac{1}{c}\log\left(\frac{c}{cM_\epsilon+d-\epsilon}\bar{\rho}\left(\frac{cM_\epsilon+d}{c}Z_\epsilon\right)\right) + \phi^{-1}(M_\epsilon)$$

$$= \frac{1}{c}\log\left(\frac{c}{cM_\epsilon+d-\epsilon}\sup_{Q\in\mathcal{Q}}\left\{E_Q\left[-\frac{cM_\epsilon+d}{c}Z_\epsilon\right]-\alpha(Q)\right\}\right) + \phi^{-1}(M_\epsilon)$$

$$= \frac{1}{c}\log\left(b^{-1}(\epsilon)\sup_{Q\in\mathcal{Q}}\left\{E_Q\left[-Z_\epsilon\right]-\frac{cb(\epsilon)\alpha(Q)}{cM_\epsilon+d-\epsilon}\right\}\right) + \phi^{-1}(M_\epsilon)$$

$$= \frac{1}{c}\log\left(b^{-1}(\epsilon)\tilde{\rho}(Z_\epsilon)\right) + \phi^{-1}(M_\epsilon)$$

$$= \frac{1}{c}\log\left(b^{-1}(\epsilon)\left\{E_{Q_\epsilon^*}\left[-Z_\epsilon\right]-\frac{cb(\epsilon)\alpha(Q_\epsilon^*)}{cM_\epsilon+d-\epsilon}\right\}\right) + \phi^{-1}(M_\epsilon), \tag{4.24}$$

where we have used (4.17)-(4.18) in the first inequality.

Finally, we may conclude

$$\rho\left(\frac{-\log(-Z_\epsilon)}{c} - \phi^{-1}(M_\epsilon) - m'\right) = m' + \rho\left(\frac{-\log(-Z_\epsilon)}{c} - \phi^{-1}(M_\epsilon)\right)$$

$$\le m' + \frac{1}{c}\log\left(b^{-1}(\epsilon)\left\{E_{Q_\epsilon^*}\left[-Z_\epsilon\right]-\frac{cb(\epsilon)\alpha(Q_\epsilon^*)}{cM_\epsilon+d-\epsilon}\right\}\right) + \phi^{-1}(M_\epsilon)$$

$$< m' + \frac{1}{c}\log\left(b(\epsilon)\left\{E_{Q_\epsilon^*}\left[-Z_\epsilon\right]-e^{-cm'}b^{-1}(\epsilon)\alpha(Q_\epsilon^*)\right\}\right) + \phi^{-1}(M_\epsilon)$$

$$\le \rho\left(\frac{-\log(-Z_\epsilon)}{c} - \phi^{-1}(M_\epsilon) - m'\right), \tag{4.25}$$

where we have used (4.24) in the first inequality, (4.22) in the strict inequality and (4.23) in the last inequality. The equality holds by translation invariance. The strict inequality (4.25) is a contradiction. $\square$

4.3 Convexity Without the Translation Invariance Axiom

In the previous two subsections the axiom of translation invariance played a key role; see Theorems 4.1(i) and 4.12(i). As is well-documented (see, for example, Cheridito and Kupper [10]), the axiom of translation invariance is equivalent to the axiom of convexity for general certainty equivalents under fairly weak conditions (e.g., continuity with respect to the L^∞-norm). In this subsection we adapt and apply this equivalence relation to the present setting, to replace the axiom of translation invariance by the axiom of convexity, which will now play the key role.

Throughout this subsection, we suppose the probability space $(\Omega, \mathcal{F}, P)$ is rich. We state the following theorem:

Theorem 4.18 *Let $\bar{\rho} : L^\infty \to \mathbb{R}$ be monotone, convex, positively homogeneous and continuous from above, and let for all $m \in \mathbb{R}_0^-$, $\bar{\rho}(m) = -m$. Suppose that the subdifferential of $\bar{\rho}$ is always nonempty. Furthermore, suppose that $r : L^\infty \to \mathbb{R}$ is defined by $r(X) = \phi^{-1}\left(\bar{\rho}(-\phi(-X))\right)$, for a strictly increasing and continuous function $\phi \in C^3(]\phi^{-1}(0), \infty[)$. Finally, suppose that $0 \in closure(Image(\phi))$ and that $\phi(\infty) = \infty$. Then the following statements are equivalent:*

(i) r is convex and $r(m) = -m$ for all $m \in \mathbb{R}$.

(ii) r is γ-entropy convex with $\gamma \in \mathbb{R}^+$ or r is ∞-entropy coherent.

Proof. The direction from (ii) to (i) holds by virtue of Proposition 3.4. Let us show the reverse direction. First, notice that r is continuous with respect to the L^∞-norm. This can be seen as follows: from the proof of Lemma 4.6, we have that $\bar{\rho}(X) = \sup_{X' \in H}\{\mathrm{E}\left[-X'X\right]\}$ with $H \subset L_+^1$ and $\sup_{X' \in H}\{\mathrm{E}\left[|X'|\right]\} = \sup_{X' \in H}\{\mathrm{E}\left[X'\right]\} = 1$. Hence, for $X, Y \in L^\infty$,

$$\bar{\rho}(Y) - \bar{\rho}(X) = \sup_{X' \in H}\left\{\mathrm{E}\left[-X'Y\right]\right\} - \sup_{X' \in H}\left\{\mathrm{E}\left[-X'X\right]\right\}$$
$$\leq \sup_{X' \in H}\left\{\mathrm{E}\left[-X'Y\right] - \mathrm{E}\left[-X'X\right]\right\}$$
$$\leq ||Y - X||_\infty \sup_{X' \in H} \mathrm{E}\left[|X'|\right] = ||Y - X||_\infty.$$

Switching the roles of X and Y it follows that $\bar{\rho}$ is indeed continuous with respect to the L^∞-norm. Now as ϕ is continuous we can conclude that r is continuous with respect to the L^∞-norm as well. But then it follows from Proposition 2.5-(8) in Cheridito and Kupper [10] that r is translation invariant. The argument is simple, namely, for $\lambda \in (0, 1)$ we have

$$r(X + m) \leq \lambda r\left(\frac{X}{\lambda}\right) + (1 - \lambda)r\left(\frac{m}{1 - \lambda}\right) = \lambda r\left(\frac{X}{\lambda}\right) - m.$$

Letting λ converge to one and using the continuity of r with respect to the L^∞-norm we find that $r(X + m) \leq r(X) - m$. Replacing X by $X + m$ and m by $-m$ yields the stated result. Therefore, r is indeed translation invariant. Now upon application of Theorem 4.1, the direction from (i) to (ii) follows. $\square$

Using Corollary 4.10, we now obtain directly the following corollary:

Corollary 4.19 *In the setting of Theorem 4.18, suppose that $\bar{\rho}$ is additionally assumed to be translation invariant. Then the following statements are equivalent:*

(i) r is convex.

(ii) r is γ-entropy coherent with $\gamma \in]0, \infty]$.

It is straightforward to adapt the proof of Theorem 4.18 to show that, similarly, the condition of translation invariance in Theorem 4.12 can also be replaced by convexity and the condition that $r(m) = -m$ for all $m \in \mathbb{R}$.

5 The Dual Conjugate

In this section we study the dual conjugate function, defined in (2.2), for entropy coherent and entropy convex measures of risk. Quite unusually, some explicit results on the dual conjugate function can be obtained. Let $\gamma \in [0, \infty]$. We state the following proposition:

Proposition 5.1 *Suppose that ρ is γ-entropy convex. Then*

$$\rho^*(Q) = \sup_{\bar{P} \ll P} \{\alpha(\bar{P}) - \gamma H(\bar{P}|Q)\}. \tag{5.1}$$

Proof. We write

$$\rho^*(Q) = \sup_{X \in L^\infty} \{e_{\gamma,Q}(X) - \rho(X)\} = \sup_{X \in L^\infty} \sup_{\bar{P} \ll P} \{E_{\bar{P}}[-X] - \gamma H(\bar{P}|Q) - \rho(X)\}$$

$$= \sup_{\bar{P} \ll P} \sup_{X \in L^\infty} \{E_{\bar{P}}[-X] - \rho(X) - \gamma H(\bar{P}|Q)\} = \sup_{\bar{P} \ll P} \{\alpha(\bar{P}) - \gamma H(\bar{P}|Q)\}.$$

$\square$

Notice that (5.1) yields that $\alpha(\bar{P}) \leq \rho^*(Q) + \gamma H(\bar{P}|Q)$. Hence,

$$\alpha(\bar{P}) \leq \inf_{Q \in \mathcal{Q}} \{\rho^*(Q) + \gamma H(\bar{P}|Q)\}.$$

The next penalty function duality theorem will show that this inequality is sharp. It also establishes the explicit relationship between the dual conjugate α and the penalty function c for γ-entropy convex measures of risk.

Theorem 5.2 *Suppose that ρ is γ-entropy convex with penalty function c. Then:*

(i) *The dual conjugate of ρ, defined in (2.2), is given by the largest convex and lower-semicontinuous function α being dominated by $\inf_{Q \in \mathcal{Q}}\{\gamma H(\bar{P}|Q) + c(Q)\}$.*

(ii) *If c is convex and lower-semicontinuous, then α is the largest lower-semicontinuous function being dominated by $\inf_{Q \in \mathcal{Q}}\{\gamma H(\bar{P}|Q) + c(Q)\}$.*

(iii) *If c is convex and lower-semicontinuous and for every $r \in \mathbb{R}^+$ the set $B_r = \{Q \in \mathcal{Q}|c(Q) \leq r\}$ is uniformly integrable, then*

$$\alpha(\bar{P}) = \min_{Q \in \mathcal{Q}}\{\gamma H(\bar{P}|Q) + c(Q)\}. \tag{5.2}$$

Proof. If $\gamma = 0$ or $\gamma = \infty$ the theorem follows by standard arguments. Let us therefore assume that $\gamma \in \mathbb{R}^+$.

(i): We write

$$\rho(X) = \sup_{Q \in \mathcal{Q}} \left\{\gamma \log \left(E_Q \left[\exp\left\{\frac{-X}{\gamma}\right\}\right]\right) - c(Q)\right\} = \sup_{Q \in \mathcal{Q}} \sup_{\bar{P} \ll P} \left\{E_{\bar{P}}[-X] - \gamma H(\bar{P}|Q) - c(Q)\right\}$$

$$= \sup_{\bar{P} \ll P} \sup_{Q \in \mathcal{Q}} \left\{E_{\bar{P}}[-X] - \gamma H(\bar{P}|Q) - c(Q)\right\} = \sup_{\bar{P} \ll P} \left\{E_{\bar{P}}[-X] - \inf_{Q \in \mathcal{Q}}\{\gamma H(\bar{P}|Q) + c(Q)\}\right\},$$

26

where we have used in the second equality that $H(\bar{P}|Q) = \infty$ if $\bar{P}$ is not absolutely continuous with respect to Q. Since α is the minimal lower-semicontinuous and convex function satisfying (2.1), statement (i) follows.

(ii): Now assume that c is convex and lower-semicontinuous. We will first show that:

(a) $\gamma H(\bar{P}|Q)$ is jointly convex in $(\bar{P}, Q)$.

(b) If $\bar{P}_n$ and Q_n converge weakly to $\bar{P}$ and $\bar{Q}$, respectively, then $\gamma H(\bar{P}|Q) \leq \liminf_n \gamma H(\bar{P}_n|Q_n)$.

To see (a), note that for every $X \in L^\infty$, $-\gamma \log\left(\mathrm{E}_Q\left[\exp\left\{\frac{-X}{\gamma}\right\}\right]\right)$ is convex in Q, and $\mathrm{E}_{\bar{P}}[-X]$ is convex in $\bar{P}$. Hence, $\mathrm{E}_{\bar{P}}[-X] - \gamma \log\left(\mathrm{E}_Q\left[\exp\left\{\frac{-X}{\gamma}\right\}\right]\right)$ is jointly convex in $(\bar{P}, Q)$ and therefore

$$\gamma H(\bar{P}|Q) = \sup_{X \in L^\infty} \left\{ \mathrm{E}_{\bar{P}}[-X] - \gamma \log\left(\mathrm{E}_Q\left[\exp\left\{\frac{-X}{\gamma}\right\}\right]\right) \right\}$$

is jointly convex in $(\bar{P}, Q)$ as well.

(b) If $Q_n \in \mathcal{Q}$ converges weakly to Q, and $\bar{P}_n \in \mathcal{Q}$ converges weakly to $\bar{P}$, then for every $X \in L^\infty$ we have $\mathrm{E}_{Q_n}[-X] \overset{n \to \infty}{\to} \mathrm{E}_Q[-X]$ and $\mathrm{E}_{\bar{P}_n}[-X] \overset{n \to \infty}{\to} \mathrm{E}_{\bar{P}}[-X]$. Since

$$\mathrm{E}_{\bar{P}}[-X] - \gamma \log\left(\mathrm{E}_Q\left[\exp\left\{\frac{-X}{\gamma}\right\}\right]\right) = \lim_n \left\{ \mathrm{E}_{\bar{P}_n}[-X] - \gamma \log\left(\mathrm{E}_{Q_n}\left[\exp\left\{\frac{-X}{\gamma}\right\}\right]\right) \right\}$$

$$\leq \liminf_n \sup_{X \in L^\infty} \left\{ \mathrm{E}_{\bar{P}_n}[-X] - \gamma \log\left(\mathrm{E}_{Q_n}\left[\exp\left\{\frac{-X}{\gamma}\right\}\right]\right) \right\},$$

it follows that

$$\gamma H(\bar{P}|Q) = \sup_{X \in L^\infty} \left\{ \mathrm{E}_{\bar{P}}[-X] - \gamma \log\left(\mathrm{E}_Q\left[\exp\left\{\frac{-X}{\gamma}\right\}\right]\right) \right\}$$

$$\leq \liminf_n \sup_{X \in L^\infty} \left\{ \mathrm{E}_{\bar{P}_n}[-X] - \gamma \log\left(\mathrm{E}_{Q_n}\left[\exp\left\{\frac{-X}{\gamma}\right\}\right]\right) \right\}$$

$$= \liminf_n \gamma H(\bar{P}_n|Q_n).$$

This proves (b).

(a) and (b) imply that $\gamma H(\bar{P}|Q)$ is jointly convex and lower-semicontinuous in $(\bar{P}, Q)$. Furthermore, $c(Q)$ is convex and lower-semicontinuous. Therefore $\gamma H(\bar{P}|Q) + c(Q)$ is jointly convex and lower-semicontinuous as well. By Theorem 2.1.3 (v) of Zalinescu [47] it follows that $\inf_{Q \in \mathcal{Q}}\{\gamma H(\bar{P}|Q) + c(Q)\}$ is convex in $\bar{P}$. Now (ii) follows since α is the minimal lower-semicontinuous and convex function satisfying (2.1).

(iii): If we could show that

$$\beta(\bar{P}) = \inf_{Q \in \mathcal{Q}}\{\gamma H(\bar{P}|Q) + c(Q)\} \tag{5.3}$$

is also lower-semicontinuous and that the infimum is attained, then (5.2) would follow from the uniqueness of α. First of all let us show that the infimum in (5.3) is attained. Let $Q_k \ll P$ be the minimizing sequence. Since $c \neq \infty$ we have for all $\bar{P}$ that $\beta(\bar{P}) < \infty$. Thus,

$$\limsup_k c(Q_k) \leq \limsup_k \gamma H(\bar{P}|Q_k) + c(Q_k) = \beta(\bar{P}) < \infty.$$

27

In particular, $(c(Q_k))_k$ is a bounded sequence. By our assumptions, Q_k must be a uniformly integrable sequence and by the Theorem of Dunford-Pettis, see for instance Theorem IV.8.9 in Dunford and Schwartz [14], the sequence Q_k is weakly relatively compact. Hence, for fixed $\bar{P}$ we may take the infimum in (5.3) over the weakly compact set $\{Q_1, Q_2, \ldots\}$. As by (b) above $Q \to \gamma H(\bar{P}|Q) + c(Q)$ is lower-semicontinuous we may infer that the infimum is attained.

So suppose that $\bar{P}_n$ converges weakly to $\bar{P}$. For the lower-semicontinuity we have to show that

$$\beta(\bar{P}) \leq \liminf_n \beta(\bar{P}_n). \tag{5.4}$$

If $\liminf_n \beta(\bar{P}_n) = \infty$ then clearly (5.4) holds. So assume that $r := \liminf_n \beta(\bar{P}_n) < \infty$. Denote by $(n_j)_j$ the subsequence such that $\liminf_n \beta(\bar{P}_n) = \lim_j \beta(\bar{P}_{n_j})$. Let

$$\bar{Q}_{n_j} \in \arg\min_{Q \in \mathcal{Q}} \{\gamma H(\bar{P}_{n_j}|Q) + c(Q)\}.$$

As $\limsup_j c(Q_{n_j}) \leq \lim_j \gamma H(\bar{P}_{n_j}|Q_{n_j}) + c(Q_{n_j}) = r$, the sequence Q_{n_j} is uniformly integrable. Again by the Theorem of Dunford-Pettis, Q_{n_j} has a subsequence, denoted by n_{j_k}, converging weakly to a measure $\bar{Q} \in \mathcal{Q}$. Hence, by the lower-semicontinuity of the mapping $(\bar{P}, Q) \to H(\bar{P}|Q)$ proved in (b),

$$\beta(\bar{P}) = \min_{Q \in \mathcal{Q}} \{\gamma H(\bar{P}|Q) + c(Q)\} \leq \gamma H(\bar{P}|\bar{Q}) + c(\bar{Q})$$
$$\leq \liminf_k \gamma H(\bar{P}_{n_k}|\bar{Q}_{n_{j_k}}) + c(Q_{n_{j_k}}) = \liminf_n \beta(\bar{P}_n),$$

where the second equality holds because n_{j_k} was a subsequence of the sequence n_j. Hence, indeed β is lower-semicontinuous and we can conclude that $\beta = \alpha$. $\qquad\square$

Corollary 5.3 *Suppose that*

$$\rho(X) = \sup_{Q \in M} e_{\gamma, Q}(X)$$

for a convex set $M \subset \mathcal{Q}$. Then the dual conjugate of ρ is given by the largest lower-semicontinuous function α being dominated by $\inf_{Q \in M} \gamma H(\bar{P}|Q)$. Furthermore, if M is weakly relatively compact, then

$$\alpha(\bar{P}) = \min_{Q \in M} \gamma H(\bar{P}|Q). \tag{5.5}$$

Proof. The first part of the corollary is precisely (ii) of Theorem 5.2 with $c = \bar{I}_M$. The second part follows as for all $r \in \mathbb{R}^+$ we have $\{Q \in \mathcal{Q}|c(Q) \leq r\} = M$. (5.5) now follows as by the Theorem of Dunford-Pettis, M is weakly relatively compact if and only if M is uniformly integrable. $\qquad\square$

Corollary 5.4 *Suppose that ρ is a convex risk measure with dual conjugate α for which*

$$\alpha(P) = 0 \text{ and } \alpha(Q) > 0 \text{ if } Q \neq P.$$

Then ρ is γ-entropy coherent if and only if $\rho(X) = e_\gamma(X)$ for $\gamma \in]0, \infty]$.

Proof. The 'if' direction is trivial. Let us prove the 'only if' direction. If ρ is γ-entropy coherent, then by Corollary 5.3 we must have $\alpha(\bar{P}) \leq \inf_{Q \in M} \gamma H(\bar{P}|Q)$ for a convex set M. Note that if $\bar{P} \in M$ then $0 \leq \alpha(\bar{P}) \leq \inf_{Q \in M} \gamma H(\bar{P}|Q) = 0$. By the assumptions on α this implies that M can at most contain P. Hence, either $\alpha(\bar{P}) = \gamma H(\bar{P}|P)$ for all $\bar{P} \ll P$, or $M = \emptyset$ and $\alpha = \infty$. However, as $\inf_Q \alpha(Q) = \rho(0) = 0$ we must have that $\alpha(\bar{P}) = \gamma H(\bar{P}|P)$. Therefore, by (2.1) indeed

$$\rho(X) = \sup_{\bar{P} \in \mathcal{Q}} \{\mathrm{E}_{\bar{P}}[-X] - \gamma H(\bar{P}|P)\} = e_\gamma(X).$$

$\square$

Corollary 5.5 *Let ρ be a convex risk measure. Then the following statements are equivalent:*

(i) For a convex and lower-semicontinuous function c from $\mathcal{Q}$ to $[0, \infty]$ with $\inf_{Q \in \mathcal{Q}} c(Q) = 0$ and uniformly integrable sublevel sets we have

$$\alpha(\bar{P}) = \min_{Q \in \mathcal{Q}}\{\gamma H(\bar{P}|Q) + c(Q)\}. \tag{5.6}$$

(ii) ρ is γ-entropy convex with a convex and lower-semicontinuous penalty function c which has uniformly integrable sublevel sets.

Proof. The direction from (ii) to (i) is precisely part (iii) of Theorem 5.2. The reverse direction holds since

$$\rho(X) = \sup_{\bar{P} \in \mathcal{Q}} \{\mathrm{E}_{\bar{P}}[-X] - \alpha(\bar{P})\} = \sup_{\bar{P} \in \mathcal{Q}} \{\mathrm{E}_{\bar{P}}[-X] - \min_{Q \in \mathcal{Q}}[\gamma H(\bar{P}|Q) + c(Q)]\}$$

$$= \sup_{Q \in \mathcal{Q}} \sup_{\bar{P} \in \mathcal{Q}} \{\mathrm{E}_{\bar{P}}[-X] - \gamma H(\bar{P}|Q) + c(Q)\} = \sup_{Q \in \mathcal{Q}} \{e_{\gamma,Q}(X) - c(Q)\}.$$

$\square$

In the case that the penalty functions admit uniformly integrable sublevel sets, the next theorem establishes a complete characterization of entropy convexity involving only the dual conjugate α. It shows that entropy convexity is equivalent to a min-max being a max-min.

Theorem 5.6 *Suppose that ρ is a convex risk measure. Furthermore, let c be defined by $c(Q) := \sup_{\hat{P} \ll P}\{\alpha(\hat{P}) - \gamma H(\hat{P}|Q)\}$. Then the following statements are equivalent:*

(i) ρ is γ-entropy convex with ρ^* having uniformly integrable sublevel sets.

(ii) c is convex and lower-semicontinuous with $\inf_{Q \in \mathcal{Q}} c(Q) = 0$ and uniformly integrable sublevel sets, and for every $\bar{P} \in \mathcal{Q}$,

$$\inf_{Q \in \mathcal{Q}} \sup_{\hat{P} \in \mathcal{Q}} \left\{\gamma H(\bar{P}|Q) + \alpha(\hat{P}) - \gamma H(\hat{P}|Q)\right\} \tag{5.7}$$

$$= \sup_{\hat{P} \in \mathcal{Q}} \inf_{Q \in \mathcal{Q}} \left\{\gamma H(\bar{P}|Q) + \alpha(\hat{P}) - \gamma H(\hat{P}|Q)\right\}.$$

Proof. If $\gamma = 0$, both (i) and (ii) cannot hold, so that the theorem holds trivially. Suppose, therefore, that $\gamma \in]0, \infty]$. We can write the right-hand side of (5.7) as

$$\sup_{\hat{P} \in \mathcal{Q}} \inf_{Q \in \mathcal{Q}} \left\{ \gamma \mathrm{E}_Q \left[\log \left(\frac{d\bar{P}}{dQ} \right) - \log \left(\frac{d\hat{P}}{dQ} \right) \right] + \alpha(\hat{P}) \right\} = \sup_{\hat{P} \in \mathcal{Q}} \inf_{Q \in \mathcal{Q}} \left\{ \gamma \mathrm{E}_Q \left[\log \left(\frac{d\bar{P}}{d\hat{P}} \right) \right] + \alpha(\hat{P}) \right\}.$$

If $\frac{d\bar{P}}{d\hat{P}} \neq 1$ on a non-zero set we have that $\log \left(\frac{d\bar{P}}{d\hat{P}} \right) < 0$ on a non-zero set. But then

$$\inf_{Q \in \mathcal{Q}} \gamma \mathrm{E}_Q \left[\log \left(\frac{d\bar{P}}{d\hat{P}} \right) \right] = -\infty.$$

Consequently, we have to choose $\hat{P} = \bar{P}$ in the supremum above, which implies that the right-hand side in (5.7) is equal to $\alpha(\bar{P})$. Moreover, for the left-hand side we have that

$$\inf_{Q \in \mathcal{Q}} \sup_{\hat{P} \in \mathcal{Q}} \left\{ \gamma H(\bar{P}|Q) + \alpha(\hat{P}) - \gamma H(\hat{P}|Q) \right\} = \inf_{Q \in \mathcal{Q}} \left\{ \gamma H(\bar{P}|Q) + \sup_{\hat{P} \in \mathcal{Q}} \{ \alpha(\hat{P}) - \gamma H(\hat{P}|Q) \} \right\}$$

$$= \inf_{Q \in \mathcal{Q}} \left\{ \gamma H(\bar{P}|Q) + c(Q) \right\}.$$

Now the theorem follows from Proposition 5.1 and Corollary 5.5. $\qquad \square$

6   Distribution Invariant Entropy Convex Measures of Risk

In this section, we derive the distribution invariant representation for entropy coherent and entropy convex measures of risk. As a bridge towards the distribution invariant representation, we first present a representation of entropy coherent measures of risk that builds on Schmeidler [41, 42].

Let $\gamma \in [0, \infty]$. For a normalized, monotone and possibly non-additive measure (or, set function) $v : \mathcal{F} \to [0, 1]$ and a (truly) bounded random variable X we define

$$\mathrm{E}_v [X] := \int X \, dv := \int_0^\infty v[X > t] dt + \int_{-\infty}^0 (v[X > t] - 1) dt.$$

We say that v is submodular if

$$v(A \cap B) + v(A \cup B) \leq v(A) + v(B) \text{ for } A, B \in \mathcal{F}.$$

By Schmeidler [41, 42], v is submodular if and only if for every bounded X,

$$\mathrm{E}_v [X] = \max_{Q \in M_v} \mathrm{E}_Q [X], \tag{6.1}$$

with $M_v = \{ Q \text{ is additive on } \mathcal{F} | Q(A) \leq v(A) \text{ for all } A \in \mathcal{F} \}$. M_v is also called the core of v.

We note that every bounded random variable is an element in L^∞, and that every P-almost surely bounded random variable can be identified with a (truly) bounded random variable (by (re)defining $X \in L^\infty$ to be equal to its original value for those $\omega \in \Omega$ for which $|X(\omega)| \leq ||X||_\infty$ and by setting $X(\omega) = 0$ otherwise). Then, for $X \in L^\infty$, we define

$$e_{\gamma, v}(X) := \gamma \log \left(\int \exp \left\{ \frac{-X}{\gamma} \right\} dv \right).$$

In the case that v is continuous from above, that is, if $v(A_n) \downarrow 0$ for any decreasing sequence of events (A_n) such that $\bigcap_n A_n = \emptyset$, we have that (6.1) holds with $M_v = \{Q \in \mathcal{Q} | Q(A) \leq v(A) \text{ for all } A \in \mathcal{F}\}$. We state the following proposition:

Proposition 6.1 *The following statements are equivalent:*

   (i) $\rho(X) = e_{\gamma,v}(X)$ is γ-entropy coherent and continuous from below.

   (ii) v is submodular and continuous from above, and $\rho(X) = \max_{Q \in M_v} e_{\gamma,Q}(X)$ with $M_v = \{Q \in \mathcal{Q} | Q(A) \leq v(A) \text{ for all } A \in \mathcal{F}\}$.

Proof. The direction (ii)⇒(i) follows from (6.1) and the fact that v being continuous from above implies that $M_v \subset \mathcal{Q}$. Furthermore, ρ is continuous from below by virtue of Proposition 3.12.

   To see the reverse direction, let $M \subset \mathcal{Q}$ with $\bar{I}_M = \rho^*$. Since $\gamma \log \left(\int \exp \left\{ \frac{-X}{\gamma} \right\} dv \right) = \rho(X) = \sup_{Q \in M} e_{\gamma,Q}(X)$, we have that

$$\int \exp \left\{ \frac{-X}{\gamma} \right\} dv = \sup_{Q \in M} \mathrm{E}_Q \left[\exp \left\{ \frac{-X}{\gamma} \right\} \right].$$

Now on the one hand, for $A \in \mathcal{F}$, setting $X = -\gamma \log(e^{\frac{-1}{\gamma}} + 1) < 1$ on A and 1 else, we get for $Q \in M$

$$
\begin{aligned}
(e^{-1/\gamma} + 1 - e^{-1/\gamma})v(A) + e^{-1/\gamma} &= \int \exp \left\{ \frac{-X}{\gamma} \right\} dv \\
&\geq \mathrm{E}_Q \left[\exp \left\{ \frac{-X}{\gamma} \right\} \right] = (e^{-1/\gamma} + 1 - e^{-1/\gamma})Q(A) + e^{-1/\gamma}.
\end{aligned}
$$

Thus, $Q(A) \leq v(A)$ and we may infer that $\sup_{Q \in M} Q(A) \leq v(A)$, which implies $M \subset M_v \subset \mathcal{Q}$. On the other hand, if $Q \in M_v$, then

$$\rho(X) = \int \exp \left\{ \frac{-X}{\gamma} \right\} dv \geq \int \exp \left\{ \frac{-X}{\gamma} \right\} dQ = e_{\gamma,Q}(X).$$

Since $Q \in \mathcal{Q}$, this entails that $\rho^*(Q) = 0$. In particular, $Q \in M$. □

   Subsequently, let

$$\Psi = \{\psi : [0,1] \to [0,1] | \psi \text{ is concave, right-continuous at zero with } \psi(0+) = 0 \text{ and } \psi(1) = 1\}.$$

For $\psi \in \Psi$ and $X \in L^\infty$ we define $\mathrm{E}_\psi[X] := \int X d\psi(P)$. Furthermore, we define

$$e_{\gamma,\psi}(X) := \gamma \log \left(\mathrm{E}_\psi \left[\exp \left\{ \frac{-X}{\gamma} \right\} \right] \right) =: e_{\gamma,\psi(P)}(X).$$

We state the following proposition:

Proposition 6.2 *For a given $\psi \in \Psi$, $e_{\gamma,\psi}$ is γ-entropy coherent and its entropy dual $e^*_{\gamma,\psi}$ is given by*

$$e^*_{\gamma,\psi}(Q) = \bar{I}_M(Q),$$

where $M = \{Q \in \mathcal{Q} | Q \leq \psi(P)\}$. Furthermore, the dual conjugate of $e_{\gamma,\psi}$, defined by (2.2), is given by

$$\alpha(\bar{P}) = \min_{Q \in M} \gamma H(\bar{P}|Q).$$

31

Proof. As ψ is concave and right-continuous in zero, the corresponding v, defined by setting $v(A) = \psi(P[A])$ for all $A \in \mathcal{F}$, is submodular and continuous from above. Hence, by (6.1),

$$e_{\gamma,\psi}(X) = \max_{Q \in M} \gamma \log \left(\mathrm{E}_Q \left[\exp \left\{ \frac{-X}{\gamma} \right\} \right] \right),$$

where $M = \{Q \in \mathcal{Q} | Q \leq \psi(P)\}$ and the first part of the proposition follows. To see that the second part holds observe that, as ψ is right-continuous in zero, the set M is weakly compact, see Corollary 4.74, Lemma 4.63, and Corollary 4.35 in Föllmer and Schied [16]. Now Corollary 5.3 yields the proof of the second statement. $\square$

In the remainder of this section, we assume that the probability space is rich. If ρ is distribution invariant we can identify ρ with a functional ρ' on the space of distributions with bounded support by setting $\rho'(q_X^+) = \rho(X)$, with q_X^+ the upper (right-continuous) quantile function of X. Furthermore, we will identify a function $\psi_Q \in \Psi$ with dQ/dP by setting $\psi_Q(t) = \int_0^t q_{\frac{dQ}{dP}}^+(1-s)ds$. For $X \in L^\infty$ with $X \geq 0$, we have, using Fubini's theorem,

$$\mathrm{E}_{\psi_Q}[X] = \int_0^\infty \psi_Q(P[X > t])dt$$

$$= \int_0^\infty \int_0^1 I_{\{F_X(t) \leq 1-s\}} \psi_Q'^+(s)ds\,dt = \int_0^1 q_X^+(1-s)\psi_Q'^+(s)ds \tag{6.2}$$

$$= \int_0^1 q_X^+(1-s)q_{\frac{dQ}{dP}}^+(1-s)ds = \int_0^1 q_X^+(s)q_{\frac{dQ}{dP}}^+(s)ds. \tag{6.3}$$

For a general $X \in L^\infty$, (6.2)-(6.3) hold by translation invariance of $\mathrm{E}_{\psi_Q}[\cdot]$. On the other hand, given a function $\psi \in \Psi$, we can define a measure $Q^\psi \in \mathcal{Q}$ by setting $\frac{dQ^\psi}{dP} = \psi'^+(1-U)$. Note that

$$\{Q \ll P\} = \left\{ Q \ll P | \frac{dQ}{dP} \overset{D}{=} \psi'^+(1-U) \text{ for } \psi \in \Psi \right\}, \tag{6.4}$$

where $\frac{dQ}{dP} \overset{D}{=} \psi_Q'^+(1-U)$ indicates that $\frac{dQ}{dP}$ and $\psi_Q'^+(1-U)$ have the same distribution under P. To see (6.4), note first that for every $\psi \in \Psi$, $\psi'^+(1-U)$ defines a density and thus a measure $Q \in \mathcal{Q}$. On the other hand, for every measure $Q \in \mathcal{Q}$, we have $\frac{dQ}{dP} \overset{D}{=} \psi_Q'^+(1-U)$, with $\psi_Q(t) = \int_0^t q_{\frac{dQ}{dP}}^+(1-s)ds$. Therefore (6.4) holds. Now we can identify ρ^* with a function $(\rho^*)' : \Psi \to \mathbb{R}$ by setting

$$(\rho^*)'(\psi) = \rho^*(Q^\psi).$$

Next, we need Lemma 4.55 of Föllmer and Schied [16]:

Lemma 6.3 *For $X \in L^\infty$ and $Y \in L^1$,*

$$\int_0^1 q_X^+(t)q_Y^+(t)dt = \sup_{\bar{X} \overset{D}{=} X} \mathrm{E}\left[\bar{X}Y\right].$$

Then we state the following theorem, presenting the distribution invariant representation for entropy convex measures of risk (including the special case of entropy coherent measures of risk). It extends the well-known distribution invariant representation results for coherent and convex measures of risk (see, for instance, Dana [12]), which arise whenever $\gamma = \infty$.

Theorem 6.4 *Suppose that ρ is γ-entropy convex. Then the following statements are equivalent:*

(i) ρ *is distribution invariant.*

(ii) ρ^* *is distribution invariant and* $(\rho^*)'(\psi) = \sup_{X \in L^\infty} \{e_{\gamma,\psi}(X) - \rho(X)\}$.

(iii) $\rho(X) = \sup_{\psi \in \Psi} \{e_{\gamma,\psi}(X) - (\rho^*)'(\psi)\}$.

Proof. (i)$\Rightarrow$(ii): We write

$$\rho^*(Q) = \sup_{X \in L^\infty} \left\{ \gamma \log \left(\mathrm{E}\left[\frac{dQ}{dP} \exp\{-X/\gamma\} \right] \right) - \rho(X) \right\}$$

$$= \sup_{X \in L^\infty} \sup_{\bar{X} \overset{D}{=} X} \left\{ \gamma \log \left(\mathrm{E}\left[\frac{dQ}{dP} \exp\{-\bar{X}/\gamma\} \right] \right) - \rho(\bar{X}) \right\}$$

$$= \sup_{X \in L^\infty} \sup_{\bar{X} \overset{D}{=} X} \left\{ \gamma \log \left(\mathrm{E}\left[\frac{dQ}{dP} \exp\{-\bar{X}/\gamma\} \right] \right) - \rho(X) \right\}$$

$$= \sup_{X \in L^\infty} \left\{ \gamma \log \left(\sup_{\bar{X} \overset{D}{=} X} \mathrm{E}\left[\frac{dQ}{dP} \exp\{-\bar{X}/\gamma\} \right] \right) - \rho(X) \right\}$$

$$= \sup_{X \in L^\infty} \left\{ \gamma \log \left(\int_0^1 q^+_{\frac{dQ}{dP}}(s) q^+_{\exp\{-X/\gamma\}}(s) ds \right) - \rho(X) \right\}$$

$$= \sup_{X \in L^\infty} \left\{ \gamma \log \left(\mathrm{E}_{\psi_Q}\left[\exp\{-X/\gamma\} \right] ds \right) - \rho(X) \right\} = \sup_{X \in L^\infty} \left\{ e_{\gamma,\psi_Q}(X) - \rho(X) \right\},$$

where we have used the distribution invariance of ρ in the third, Lemma 6.3 in the fifth, and (6.3) in the sixth equality. In particular, ρ^* is distribution invariant. It follows that

$$(\rho^*)'(\psi) = \rho^*(Q^\psi) = \sup_{X \in L^\infty} \left\{ e_{\gamma,\psi}(X) - \rho(X) \right\}.$$

(ii)$\Rightarrow$(iii): Similar as (i)$\Rightarrow$(ii).

(iii)$\Rightarrow$(i): Clearly, ρ is distribution invariant. Set $c(Q) = (\rho^*)'(\psi_Q)$. Notice that if $\frac{dQ}{dP} \overset{D}{=} \psi'^+(1-U)$ then by the definition of $(\rho^*)'$ we have that $c(Q) = (\rho^*)'(\psi)$. We write

$$\rho(X) = \sup_{\psi \in \Psi} \left\{ e_{\gamma,\psi}(X) - (\rho^*)'(\psi) \right\}$$

$$= \sup_{\psi \in \Psi} \left\{ \gamma \log \left(\mathrm{E}_\psi \left[\exp \left\{ -\frac{X}{\gamma} \right\} \right] \right) - (\rho^*)'(\psi) \right\}$$

$$= \sup_{\psi \in \Psi} \left\{ \gamma \log \left(\mathrm{E}\left[\psi'^+(1-U) q^+_{\exp\left\{-\frac{X}{\gamma}\right\}}(U) \right] \right) - (\rho^*)'(\psi) \right\}$$

$$= \sup_{\psi \in \Psi} \sup_{Q, \frac{dQ}{dP} \overset{D}{=} \psi'^+(1-U)} \left\{ \gamma \log \left(\mathrm{E}\left[\frac{dQ}{dP} \exp \left\{ -\frac{X}{\gamma} \right\} \right] \right) - (\rho^*)'(\psi) \right\}$$

$$= \sup_{\psi \in \Psi} \sup_{Q, \frac{dQ}{dP} \overset{D}{=} \psi'^+(1-U)} \left\{ \gamma \log \left(\mathrm{E}\left[\frac{dQ}{dP} \exp \left\{ -\frac{X}{\gamma} \right\} \right] \right) - c(Q) \right\}$$

$$= \sup_{Q \in \mathcal{Q}} \left\{ \gamma \log \left(\mathrm{E}\left[\frac{dQ}{dP} \exp \left\{ -\frac{X}{\gamma} \right\} \right] \right) - c(Q) \right\} = \sup_{Q \in \mathcal{Q}} \left\{ e_{\gamma,Q}(X) - c(Q) \right\},$$

where we applied (6.2) in the third equality. In the fourth equality we used Lemma 6.3. The fifth equality holds by the definition of c and $(\rho^*)'$, and in the sixth equality we applied (6.4). This proves (ii)$\Rightarrow$(i). $\qquad\square$

7 Applications and Examples

In this section, we provide applications of entropy coherent and entropy convex measures of risk to the classical problems of optimal risk sharing, optimal portfolio choice and indifference valuation, and specify two more examples of such measures of risk. Most of the results presented in this section rely on the representation results derived in the previous sections.

7.1 Risk Sharing

Suppose that there are two economic agents A and B measuring risk using a general entropy convex measure of risk ρ^A and ρ^B with $\gamma^A, \gamma^B \in \mathbb{R}^+$. Let $V = -\rho$, $\bar{e}_{\gamma,Q} = -e_{\gamma,Q}$ and $\bar{c} = -c$. Suppose that A owns a financial payoff X^A and B owns a financial payoff X^B. We solve explicitly the problem of optimal risk sharing given by

$$R^{A,B}(X^A, X^B) = \sup_{F \in L^\infty} \{V^A(X^A - F + \Pi^F) + V^B(X^B + F - \Pi^F)\}$$
$$= \sup_{\bar{F} \in L^\infty} \{V^A(X^A + X^B - \bar{F}) + V^B(\bar{F})\} =: V^A \square V^B(X^A + X^B),$$

where Π^F is the agreed price of the financial derivative (risk transfer) F and where we have set $\bar{F} := F + X^B$.

Proposition 7.1 *Assume that*

$$\partial_{entropy} V^A \left(\frac{\gamma^A}{\gamma^A + \gamma^B}(X^A + X^B) \right) \cap \partial_{entropy} V^B \left(\frac{\gamma^B}{\gamma^A + \gamma^B}(X^A + X^B) \right) \neq \emptyset. \qquad (7.1)$$

Then we have that

$$R^{A,B}(X^A, X^B) = \inf_{Q \in \mathcal{Q}} \left\{ \bar{e}_{\gamma^A + \gamma^B, Q}(X^A + X^B) - (\bar{c}^A(Q) + \bar{c}^B(Q)) \right\}.$$

Moreover, the optimal risk sharing is attained in the financial derivative $F^* = \frac{\gamma^B}{\gamma^A + \gamma^B} X^A - \frac{\gamma^A}{\gamma^A + \gamma^B} X^B$.

34

Proof. Let $X = X^A + X^B$ and $\bar{F}^* = F^* + X^B$. We write

$$
\begin{aligned}
R^{A,B}(X^A, X^B) &= \sup_{\bar{F} \in L^\infty} \left\{ \inf_{Q \in \mathcal{Q}} \left(\bar{e}_{\gamma^A, Q}(X - \bar{F}) - \bar{c}^A(Q) \right) + \inf_{Q \in \mathcal{Q}} \left(\bar{e}_{\gamma^B, Q}(\bar{F}) - \bar{c}^B(Q) \right) \right\} \\
&\leq \sup_{\bar{F} \in L^\infty} \inf_{Q \in \mathcal{Q}} \left\{ \bar{e}_{\gamma^A, Q}(X - \bar{F}) + \bar{e}_{\gamma^B, Q}(\bar{F}) - (\bar{c}^A(Q) + \bar{c}^B(Q)) \right\} \\
&\leq \inf_{Q \in \mathcal{Q}} \sup_{\bar{F} \in L^\infty} \left\{ \bar{e}_{\gamma^A, Q}(X - \bar{F}) + \bar{e}_{\gamma^B, Q}(\bar{F}) - (\bar{c}^A(Q) + \bar{c}^B(Q)) \right\} \\
&= \inf_{Q \in \mathcal{Q}} \left\{ \bar{e}_{\gamma^A, Q}(X - F^*) + \bar{e}_{\gamma^B, Q}(F^*) - (\bar{c}^A(Q) + \bar{c}^B(Q)) \right\} \\
&= \inf_{Q \in \mathcal{Q}} \left\{ \bar{e}_{\gamma^A + \gamma^B, Q}(X) - (\bar{c}^A(Q) + \bar{c}^B(Q)) \right\} \\
&= (\gamma_A + \gamma_B) \inf_{Q \in \mathcal{Q}} \left\{ \log \left(\mathrm{E}_Q \left[\exp \left\{ \frac{-X}{\gamma_A + \gamma_B} \right\} \right] \right) - (\bar{c}^A(Q) + \bar{c}^B(Q)) \right\} \\
&= \gamma_A \inf_{Q \in \mathcal{Q}} \left\{ \log \left(\mathrm{E}_Q \left[\exp \left\{ \frac{-X}{\gamma_A + \gamma_B} \right\} \right] \right) - \bar{c}^A(Q) \right\} \\
&\quad + \gamma_B \inf_{Q \in \mathcal{Q}} \left\{ \log \left(\mathrm{E}_Q \left[\exp \left\{ \frac{-X}{\gamma_A + \gamma_B} \right\} \right] \right) - \bar{c}^B(Q) \right\} \\
&= \inf_{Q \in \mathcal{Q}} \left(\bar{e}_{\gamma^A, Q}(X - \bar{F}^*) - \bar{c}^A(Q) \right) + \inf_{Q \in \mathcal{Q}} \left(\bar{e}_{\gamma^B, Q}(\bar{F}^*) - \bar{c}^B(Q) \right) \\
&\leq \sup_{\bar{F} \in L^\infty} \left\{ \inf_{Q \in \mathcal{Q}} \left(\bar{e}_{\gamma^A, Q}(X - \bar{F}) - \bar{c}^A(Q) \right) + \inf_{Q \in \mathcal{Q}} \left(\bar{e}_{\gamma^B, Q}(\bar{F}) - \bar{c}^B(Q) \right) \right\},
\end{aligned}
$$

where the second inequality holds by weak duality. The second equality holds by Borch [7]; see also Barrieu and El Karoui [3]. To verify the fifth equality, note that '$\geq$' clearly holds, while '$\leq$' follows from (7.1). Since the last line is equal to the term we started with, it follows that all inequalities must be equalities. This proves the proposition. $\qquad \square$

The assumption on the entropy subdifferentials in Proposition 7.1 is satisfied in particular if $c^A = c^B$ and for every $d \geq 0$, $\{Q \in \mathcal{Q} | c^A(Q) \leq d\}$ is weakly compact.

7.2 Portfolio Choice and Indifference Valuation

Applying the general results of Laeven and Stadje [32], it is possible to solve explicitly the portfolio choice and indifference valuation problems under entropy coherent measures of risk with $\gamma \in]0, \infty]$ in a general setting. Specifically, consider a probability space $(\Omega, \mathcal{F}, P)$ with two independent stochastic processes:

- A standard d-dimensional Brownian motion W.

- A real-valued Poisson point process p defined on $[0, T] \times \mathbb{R}^{d'} \setminus \{0\}$. Denote by $N_p(ds, dx)$ the associated counting measure such that its compensator is

$$
\hat{N}_p(ds, dx) = n_p(s, dx)ds.
$$

  We assume that for every s the measure $n_p(s, dx)$ is non-negative and satisfies

$$
\sup_s n_p(s, \mathbb{R}^{d'} \setminus \{0\}) < \infty.
$$

Let $\tilde{N}_p(ds, dx) := N_p(ds, dx) - \hat{N}_p(ds, dx)$.

35

We assume that the financial market consists of a bond with interest rate zero and $n \leq d$ stocks. The price process of stock i evolves according to

$$\frac{dS_t^i}{S_{t-}^i} = b_t^i dt + \sigma_t^i dW_t + \int_{\mathbb{R}^{d'}\setminus\{0\}} \tilde{\beta}_t^i(x) \tilde{N}_p(dt, dx), \quad i = 1, \ldots, n,$$

where b^i $(\sigma^i, \tilde{\beta}^i)$ are $\mathbb{R}$ $(\mathbb{R}^d, \mathbb{R})$-valued predictable and uniformly bounded stochastic processes. We also assume that $\tilde{\beta}^i > -1$ for $i = 1, \ldots, n$. We further assume that σ_t satisfies standard assumptions; see Laeven and Stadje [32] for further details. Let $\theta_t := \sigma_t^{\mathsf{T}}(\sigma_t\sigma_t^{\mathsf{T}})^{-1}b_t$ and $\beta_t := \sigma_t^{\mathsf{T}}(\sigma_t\sigma_t^{\mathsf{T}})^{-1}\tilde{\beta}_t$. If $n < d$, the market is incomplete. If $n = d$, the market is still incomplete except if N_p is degenerated. For $i = 1, \ldots, n$, the process π_t^i describes the amount of money invested in stock i at time t. The wealth process X^π of a trading strategy π with initial capital x satisfies

$$X_t^\pi = x + \sum_{i=1}^n \int_0^t \frac{\pi_{i,u}}{S_{i,u-}} dS_{i,u} = x + \int_0^t \pi_u \sigma_u (dW_u + \theta_u du) + \int_0^t \int_{\mathbb{R}^{d'}\setminus\{0\}} \pi_u \sigma_u \beta_u(x) \tilde{N}_p(du, dx).$$

Let $\mathcal{P}$ and $\mathcal{B}(\mathbb{R}^{d'})$ be the predictable σ-algebra on $[0, T] \times \Omega$ and the Borel σ-algebra on $\mathbb{R}^{d'}$, respectively. By Jacod and Shiryaev [30], we can write every density in this setting as $K_t = \mathcal{E}\left((q \cdot W)_t + (\psi \cdot \tilde{N}_p)_t\right)$, where q is measurable with respect to $\mathcal{P}$ and ψ is measurable with respect to $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d'})$. Since K_t is non-negative we must have that $\psi \geq -1$, $dP \times n_p(t, dx) \times dt$-a.s. The process $W_t^Q = W - \int_0^t q_s ds$ is a Brownian motion and the process $\tilde{N}$ has compensator $n^{(\psi)}(s, dx)ds := (1 + \psi_s(x))n_p(s, dx)ds$, under Q. Denote $\tilde{N}_t^Q := \tilde{N}_t - \int_0^t (1 + \psi_s(x))n_p(s, dx)ds$. Then N^Q is a martingale under Q. Notice further that Q is uniquely characterized by q and ψ. We will therefore also write $Q^{(q, \psi)}$.

Let $\tilde{U}$ be a predictable compact set in $\mathbb{R}^{1 \times n}$. The set of admissible trading strategies $\tilde{\mathcal{A}}$ consists of all n-dimensional predictable processes $\pi = (\pi_t)_{0 \leq t \leq T}$ which satisfy $\pi_t \in \tilde{U}$, $dP \times ds$-a.s. Let F be a bounded contingent claim. We are interested in the following optimization problem:

$$\hat{V}^\gamma(x) := \sup_{\pi \in \tilde{\mathcal{A}}} \inf_{Q \in M} -\gamma \log \left(\mathrm{E}_Q \left[\exp \left\{ -\frac{1}{\gamma} \left(x + \int_0^T \pi_t \frac{dS_t}{S_{t-}} + F \right) \right\} \right] \right), \qquad (7.2)$$

where x is the initial wealth and M is a set of measures equivalent to P. We will assume that the set M in (7.2) has the form

$$M = \{Q^{(q, \psi)} | q \text{ and } \psi \text{ are predictable processes with } q_t(\omega) \in C_t(\omega), \text{ and } \psi_t(\omega, x) \in D_t(\omega, x)\},$$

for a predictable, compact and convex set-valued mapping C and a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d'})$-measurable, compact and convex set-valued mapping D with $D_t(\omega, x) \subset]-1, \infty[$ for all (t, ω). Denote by $\bar{D}_s$ all $\mathcal{F}_s \otimes \mathcal{B}(\mathbb{R}^{d'})$-measurable functions ψ such that $\psi \in D_s(x)$. Writing

$$p_t = \pi_t \sigma_t, \qquad t \in [0, T],$$

the set of admissible strategies $\tilde{\mathcal{A}}$ is equivalent to a set $\mathcal{A}$ of $\mathbb{R}^{1 \times d}$-valued predictable processes p, with $p \in \mathcal{A}$ if and only if p is predictable and $p_t \in U_t := \tilde{U}\sigma_t$, $dP \times ds$-a.s.

We will look at a solution $(Y^\gamma, Z^\gamma, \tilde{Z}^\gamma) \in S^\infty \times$ BMO process $\times L^2(dP \times n_p(u, dx)du)$ of the backward SDE

$$Y_t^\gamma = -F + \int_t^T f^\gamma(s, Z_s^\gamma, \tilde{Z}_s^\gamma)ds - \int_t^T Z_s^\gamma dW_s - \int_t^T \int_{\mathbb{R}^{d'}\setminus\{0\}} \tilde{Z}_s^\gamma(x)\tilde{N}_p(ds, dx), \quad t \in [0, T], \ (7.3)$$

with

$$f^\gamma(s, z, \tilde{z}) := \operatorname{ess\,inf}_{p \in U_s} \operatorname{ess\,sup}_{(q,\psi)\in(C_s \times \bar{D}_s)} \left\{ -p\theta_s + \frac{1}{2\gamma}|p - z|^2 + q(z - p) \right.$$
$$+ \gamma \int_{\mathbb{R}^{d'}\setminus\{0\}} \left(\exp\{-\frac{1}{\gamma}(p\beta_s - \tilde{z}(x))\} - 1 + \frac{1}{\gamma}(p\beta_s - \tilde{z}(x)) \right.$$
$$\left. + \psi(x)(-1 + \exp\{-\frac{1}{\gamma}(p\beta_s - \tilde{z}(x))\}) \right) n_p(s, dx) \left. \right\}. \tag{7.4}$$

We prove in Laeven and Stadje [32] that in the case that $\tilde{U}$ is compact (or that D is empty) there exists a BMO $p^* \in \mathcal{A}$, a $q^* \in C$ and (in case D is nonempty) a $\psi^* \in D$ such that

$$f^\gamma(s, z, \tilde{Z}_s^\gamma) = -p_s^*\theta_s + \frac{1}{2\gamma}|p_s^* - z|^2 + q_s^*(z - p_s^*)$$
$$+ \gamma \int_{\mathbb{R}^{d'}\setminus\{0\}} \left(\exp\{-\frac{1}{\gamma}(p_s^*\beta_s - \tilde{Z}_s^\gamma(x))\} - 1 + \frac{1}{\gamma}(p_s^*\beta_s - \tilde{Z}_s^\gamma(x)) \right.$$
$$\left. + \psi_s^*(x)(-1 + \exp\{-\frac{1}{\gamma}(p_s^*\beta_s - \tilde{Z}_s^\gamma(x))\}) \right) n_p(s, dx), \tag{7.5}$$

for $\gamma \in \mathbb{R}^+$. In the case that $\gamma = \infty$ we get a similar representation for f^∞. Specifically:

Theorem 7.2 *For $\gamma \in]0, \infty]$, the (indifference) value of the portfolio optimization problem (7.2) is given by $\hat{V}^\gamma(x) = x - Y_0^\gamma$, and $\hat{V}^\infty(x) = \lim_{\gamma \to \infty} \hat{V}^\gamma(x)$. Furthermore, the optimal strategy is given by p^* if one of the following assumptions holds:*

(i) $\tilde{U}$ (and hence also U) is compact;

(ii) $D = \emptyset$.

The proof of this result, which can be obtained as a special case of the general results in Laeven and Stadje [32], relies on the duality results derived in Section 5 above (Theorem 5.2).

7.3 Further Examples

We complement Example 3.1 with two other examples.

Example 7.3 Let ρ be γ_1-entropy convex, $\gamma_1 \in \mathbb{R}^+$, with penalty function c given by γ_2 times the relative entropy with $\gamma_2 \in \mathbb{R}^+$, that is, $\rho(X) = \sup_{Q \in \mathcal{Q}}\{e_{\gamma_1, Q}(X) - \gamma_2 H(Q|P)\}$. Then, by virtue of Theorem 5.2 (iii), the dual conjugate of ρ is given by

$$\alpha(\bar{P}) = \min_{Q \in \mathcal{Q}}\{\gamma_1 H(\bar{P}|Q) + \gamma_2 H(Q|P)\},$$

and the corresponding discount factor under homothetic preferences takes the form $\beta(Q) = e^{-\frac{\gamma_2}{\gamma_1}H(Q|P)}$.

Example 7.4 Let $\psi(x) = \Phi(\Phi^{-1}(x) + a)$, $a \in \mathbb{R}^+$, with Φ the standard normal cumulative distribution function (see Wang [46] and Goovaerts and Laeven [24]), and consider the corresponding γ-entropy coherent risk measure $e_{\gamma,\psi}$, with $\gamma \in [0, \infty]$. Then, by virtue of Proposition 6.2, the dual conjugate of $e_{\gamma,\psi}$ is given by

$$\alpha(\bar{P}) = \min_{Q \in M} \gamma H(\bar{P}|Q),$$

with $M = \{Q \in \mathcal{Q} | Q \leq \Phi(\Phi^{-1}(P) + a)\}$.

8   Conclusions

In this paper, we have introduced two subclasses of convex risk measures: entropy coherent and entropy convex measures of risk. We have demonstrated that convex, entropy convex and entropy coherent measures of risk emerge as translation invariant certainty equivalents under variational, homothetic and multiple priors preferences, respectively, and induce linear or exponential utility functions in these paradigms. A variety of representation and duality results as well as some applications and examples have made explicit that entropy coherent and entropy convex measures of risk satisfy many appealing properties. The theory developed in this paper is of a static nature. In future research we intend to develop its dynamic counterpart.

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