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# Excess Based Allocation of Risk Capital

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**Abstract**

In this paper we propose a new rule to allocate risk capital to portfolios or divisions within a firm. Specifically, we determine the capital allocation that minimizes the excesses of sets of portfolios in lexicographical sense. The excess of a set of portfolios is defined as the expected loss of that set of portfolios in excess of the amount of risk capital allocated to them. The underlying idea is that large excesses are undesirable, and therefore the goal is to determine the allocation for which the largest excess is as small as possible. We show that this allocation rule yields a unique allocation, and that it satisfies some desirable properties. We also show that the allocation can be determined by solving a series of linear programming problems.

**Keywords**: risk capital, capital allocation, excesses, lexicographic minimum.
**JEL-codes**: G10, C61, C71.

## 1 Introduction

Regulators require that in order to be able to hold a risky position, financial institutions withhold a level of capital, referred to as *risk capital*. The risk capital needs to be added to the risky position and invested safely in order to act as a buffer and reduce the adverse effects of unfavorable events on the solvency of the firm. The focus in this paper is on the allocation of the total amount of risk capital to different subportfolios, divisions, or lines of business. As argued by, e.g., Tasche (1999), withholding risk capital is costly, and therefore allocating risk capital to individual investments is important for performance evaluation as well as for pricing decisions. The allocation problem is nontrivial because the amount of risk capital allocated to a portfolio consisting of multiple subportfolios is typically less than the sum of the amounts of risk capital that would need to be withheld for each subportfolio separately. The underlying intuition is that because the risks of different subportfolios are typically not perfectly correlated, some hedge potential may

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arise from combining different subportfolios. The issue then is what is a "fair" division of these diversification gains over the subportfolios.

The allocation problem described above has received considerable attention in the literature. Tasche (1999) considers allocation of risk capital to financial instruments in a portfolio, and argues that the only "appropriate" way to allocate risk capital for performance measurement purposes is to determine the *marginal risk contribution* of each investment. The marginal risk contribution is defined as the derivative of the aggregate risk capital with respect to the weight of the financial instrument in the portfolio. Denault (2001) instead proposes a game-theoretic approach to determine risk capital allocations for companies with multiple business divisions. He focuses on risk capital allocations that are "fair" in the sense that no set of divisions is allocated more risk capital than the amount of risk capital that they would need to withhold if they were on their own. He shows that when business divisions are infinitely divisible, the only allocation that satisfies this fairness condition is the marginal risk contribution defined above. In game-theoretic terms, this allocation is referred to as the Aumann-Shapley value. For the special case where the risk measure is Expected Shortfall, the corresponding allocation rule is also referred to as Conditional Tail Expectation (CTE) rule (see, e.g., Overbeck 2000, Panjer 2002, and Dhaene et al. 2008, 2009).

The literature on risk capital allocation has then evolved in several directions. First, there is some literature that considers the risk capital allocation that results from using the Aumann-Shapley value when a specific risk measure is used (Tsanakas and Barnett 2003, and Tsanakas 2009), or when the portfolio losses have a specific probability distribution (Panjer 2001, and Landsman and Valdez 2003). Second, there is some literature that focuses on generalizations or extensions of the Aumann-Shapley value (e.g., Fisher 2003, Tsanakas 2004, Powers 2007, and Furman and Zitikis 2008), or on capital allocations that result when alternative game-theoretic concepts are used (Csòka 2009). Third, Myers and Read Jr. (2001), Sherris (2006), and Kim and Hardy (2009) consider alternative capital allocation rules based on solvency ratios or expected return. Finally, there is a stream of literature in which the capital allocation is determined as the solution of an optimization problem (Dhaene, Goovaerts, and Kaas 2003, Laeven and Goovaerts 2004, Goovaerts, Van den Borre, and Laeven 2005, and Dhaene, Tsanakas, Valdez, and Vanduffel 2009). Specifically, they consider capital allocations such that the (weighted) sum of a measure for the deviation of the business unit's losses from its allocated risk capital is minimized. Dhaene, Tsanakas, Valdez, and Vanduffel (2009) show that by choosing specific deviation measures and/or specific weights, one can reproduce several of the allocation techniques proposed in the literature, including, e.g., the CTE rule.

In this paper we propose an alternative approach to allocate risk capital that falls into the latter stream of literature. Our approach is inspired by the fact that allocating risk capital on the basis of the Aumann-Shapley value can lead to undesirable allocations in the sense that the expected *excess loss*, i.e., the loss of the subportfolio in excess of the amount of risk capital allocated to it, can differ substantially across subportfolios. Large differences in expected excess losses could be perceived as unfair by managers who are evaluated based on the risk capital allocated to their portfolios. Therefore, we propose an alternative allocation rule in which the goal is to determine the capital allocation that minimizes the excesses of all subportfolios in lexicographical sense. This implies that, from a set of feasible allocations, one first selects those allocations for which the highest excess is minimized. Within the set of allocations for which the highest excess is minimized, one then determines those allocations for which the second highest excess is minimized, and so on. This approach differs from the existing optimization approaches in two ways. First, whereas the existing literature focuses on minimizing the aggregate (weighted) excess over all portfolios, we consider each excess separately. Second, we do not only take into account the excesses of the individual portfolios, but also of all possible subsets of portfolios.

The paper is organized as follows. In section 2 we formally define risk capital allocation problems, and show that the Aumann-Shapley value can lead to allocations in which the excess losses are significantly different across portfolios. We then define the alternative risk capital allocation rule that we propose, which we will refer to as the Excess Based Allocation (EBA). In section 3 we define desirable properties of a risk capital allocation rule, and show that EBA satisfies these properties. In section 4 we show how EBA can be determined by solving a sequence of linear programming problems. We conclude in section 5.

# 2 Model

In this section, we first define risk capital allocation problems and risk capital allocation rules. Next, we discuss an allocation rule that has received considerable attention in the literature. We then propose an alternative risk capital allocation rule.

## 2.1 Risk Capital Allocation Problems

Our focus in this paper is on the allocation of risk capital to subportfolios. We consider a portfolio consisting of $n$ subportfolios, indexed by $i = 1, \cdots, n$. Subportfolio $i$ generates a random loss $X_i$ at a given future date, and so the aggregate loss is given by $\sum_{i=1}^{n} X_i$. Regulators require that, in order to be able to hold this risky position, an amount of risk capital should be withheld and invested safely. The total required amount of risk capital is determined by means of a risk measure, and the issue is how to allocate this total amount to the $n$ subportfolios.

Throughout this paper we will use this following notation:

- $N = \{1, \cdots, n\}$ denotes the set of subportfolios;
- $\Omega = \{\omega_1, \cdots, \omega_m\}$ denotes the finite set of states of the world;
- $\pi(\omega) > 0$ denotes the probability that state $\omega \in \Omega$ occurs;[1]

[1] We assume that for each state $\omega \in \Omega$, it holds that $\pi(\omega) > 0$, i.e., the probability that the state occurs is strictly positive. Because we consider a finite state space, this is without loss of generality; states that occur with zero probability can be omitted from the state space.

- $X_i(\omega)$ denotes the loss from subportfolio $i$ in state $\omega \in \Omega$;
- $X = \{X_1, \cdots, X_n\}$ denotes the losses of the $n$ subportfolios;
- $\mathcal{V}$ denotes the set of random variables on $\Omega$;
- $\mathbf{1} \in \mathcal{V}$ denotes the random variable given by: $\mathbf{1}(\omega) = 1$ for all $\omega \in \Omega$.

The amount of risk capital that needs to be withheld in order to be able to hold a risky position $Y \in \mathcal{V}$ is determined by means of a risk measure $\rho : \mathcal{V} \to \mathbb{R}$. Following Artzner, Delbaen, Eber, and Heath (1999), we consider the case where risk capital is determined by means of a coherent risk measure. A risk measure $\rho : \mathcal{V} \to \mathbb{R}$ is coherent if and only if it satisfies the following four properties:[2]

**(i)** *Subadditivity*: for all $Y_1, Y_2 \in \mathcal{V}$

$$\rho(Y_1 + Y_2) \leq \rho(Y_1) + \rho(Y_2).$$

**(ii)** *Monotonicity*: for all $Y_1, Y_2 \in \mathcal{V}$ such that $Y_1(\omega) \geq Y_2(\omega)$ for all $\omega \in \Omega$

$$\rho(Y_1) \geq \rho(Y_2).$$

**(iii)** *Positive Homogeneity*: for all $Y \in \mathcal{V}$ and $c \geq 0$

$$\rho(c \cdot Y) = c \cdot \rho(Y).$$

**(iv)** *Translation Invariance*: for all $Y \in \mathcal{V}$ and $c \in \mathbb{R}$

$$\rho(Y + c \cdot \mathbf{1}) = \rho(Y) + c.$$

While all the results presented in this paper hold true for any coherent risk measure, in our numerical examples we will use Expected Shortfall as risk measure. In the literature, Expected Shortfall has been defined in different ways and not all definitions result in a coherent risk measure. This is the case in particular if the distribution of the loss has point masses. To ensure coherence, we use the definition which is due to Acerbi and Tasche (2002). First, let $q_{1-\alpha}(Y)$ denote the upper $100 \cdot (1-\alpha)\%$-quantile of $Y \in \mathcal{V}$, i.e.,

$$q_{1-\alpha}(Y) = \inf\{y \in \mathbb{R} | P(Y \leq y) > 1 - \alpha\}.$$

[2] Note that whereas Artzner et al. (1999) considers random gains, $X \in \mathcal{V}$ in our case is defined as a loss. This affects the *Monotonicity* and the *Translation Invariance* property.

Then, Expected Shortfall is defined as[3]

$$\rho_\alpha^{ES}(Y) = \frac{E\left(Y \cdot 1_{Y \geq q_{1-\alpha}(Y)}\right) - q_{1-\alpha}(Y) \cdot [P\left(Y \geq q_{1-\alpha}(Y)\right) - \alpha]}{\alpha}. \quad (1)$$

For any given coherent risk measure $\rho$, the amount of risk capital that needs to be withheld on aggregate is given by $\rho\left(\sum_{i \in N} X_i\right)$, and *Subadditivity* implies that

$$\rho\left(\sum_{i \in N} X_i\right) \leq \sum_{i \in N} \rho(X_i),$$

i.e., due to diversification effects, the amount of risk capital for the aggregate portfolio is weakly less than the sum of the amounts of risk capital that would need to be withheld for every individual subportfolio. This implies that allocating the aggregate amount to the subportfolios is non-trivial; one cannot simply allocate to each subportfolio the risk capital that it would need to hold on its own. There are gains from diversification, and the issue is how to divide these gains over the subportfolios. We refer to this problem as a *risk capital allocation problem*. In the sequel we will refer to a subportfolio as a portfolio. Moreover, throughout the paper we assume that $\Omega$ and $N$ are fixed, and denote a risk capital allocation problem as a tuple $(X, \pi, \rho)$. We denote the set of all risk capital allocation problems by $\mathcal{R}$.

The following definition formally defines *risk capital allocations* and *risk capital allocation rules.*

**Definition 2.1**

**(i)** *For any risk capital allocation problem $R = (X, \pi, \rho) \in \mathcal{R}$, a* risk capital allocation *for $R$ is a vector $a \in \mathbb{R}^N$ such that $\sum_{i \in N} a_i = \rho(\sum_{i \in N} X_i)$.*

**(ii)** *For a subclass $\mathcal{R}' \subseteq \mathcal{R}$ of risk capital allocation problems, a* risk capital allocation rule *on $\mathcal{R}'$ is a function $A : \mathcal{R}' \to \mathbb{R}^N$ such that $A(R)$ is a risk capital allocation for $R$ for all $R \in \mathcal{R}'$.*

Throughout the paper, we will use the following shorthand notation: for any given subset of portfolios $S \subseteq N$, and for any given allocation $a \in \mathbb{R}^N$, we denote:

- $X_S := \sum_{i \in S} X_i$ for the aggregate loss of all portfolios in $S$;
- $a_S := \sum_{i \in S} a_i$ for the amount allocated to the portfolios in $S$.

In the following section we first discuss a risk capital allocation rule that has received considerable attention in the literature, and discuss some of its drawbacks.

[3]In case the losses have a continuous distribution, (1) simplifies to $\rho_\alpha^{ES}(Y) = E\left[Y \mid Y \geq q_{1-\alpha}(Y)\right]$. However, if $Y$ is not continuous, this does not yield a coherent risk measure; more specifically, *Subadditivity* is violated.

### 2.2 Marginal risk contribution and the Aumann-Shapley value

An approach that has received considerable attention in the literature is referred to as the *marginal risk contribution rule*. Denault (2001) argues that in order for an allocation to be "fair", it should be such that the amount of risk capital allocated to a set of portfolios is weakly lower than the amount of risk capital that they would have to withhold if they were on their own. This property is referred to as the *no undercut* property.

In game-theoretic terms, the no-undercut property means that the allocation should be in the *core* of the corresponding game, which is given by

$$Core(R) = \{a \in \mathbb{R}^N | a_S \leq \rho(X_S) \text{ for all } S \subseteq N, a_N = \rho(X_N)\}.$$

The core of the game is the set of all risk capital allocations such that each set of portfolios $S \subseteq N$, the amount allocated to those portfolios, $a_S$, is weakly lower than the risk capital they would have incurred on their own, $\rho(X_S)$. Denault (2001) shows that the core of a capital allocation game is non-empty. In order to identify a specific core element, he suggests to require, in addition, that the no undercut property holds also for fractional portfolios. Thus, for any combination of fractions of portfolios $s \in [0,1]^N$, the amount allocated to $\sum_{i \in N} s_i X_i$ should be weakly lower than the risk capital that that portfolio would need to withhold if it was separated from the firm. In game-theoretic terms, this means that the allocation should be an element of the *fuzzy core*, which is given by

$$FCore(R) = \left\{ a \in \mathbb{R}^N \left| \sum_{i \in N} s_i a_i \leq \rho(\sum_{i \in N} s_i X_i) \text{ for all } s \in [0,1]^N, a_N = \rho(X_N) \right. \right\}.$$

Denault (2001) shows that if the function $r : [0,1]^N \to \mathbb{R}$ defined as

$$r(s_1, s_2, \cdots, s_n) = \rho(\sum_{i \in N} s_i X_i),$$

is partially differentiable at $s = (1, 1, \cdots, 1)$, then the fuzzy core consists of one element, which is given by

$$a_i = A_i^{AS}(R) := \frac{\partial r}{\partial s_i}(1, 1, \cdots, 1), \text{ for all } i \in N. \tag{2}$$

This allocation is referred to as the *marginal risk contribution*; the amount of risk capital allocated to a given portfolio is equal to that portfolio's marginal contribution to the aggregate risk capital $\rho(\sum_{i \in N} X_i)$. In game-theoretic terms, the allocation is referred to as the *Aumann-Shapley value*.[4] A special case that has received considerable attention in the literature (see, e.g., Tasche 1999, Overbeck 2000, Panjer 2002, Venter

[4]The Aumann-Shapley value in general is given by $A_i^{AS} = \int_0^1 \frac{\partial r}{\partial s_i}(\gamma, \gamma, \cdots, \gamma) d\gamma$ for all $i$. Due to the positive homogeneity of the risk measure $\rho$, this expression in our case simplifies to (2).

2004, Kalkbrener 2005, and Dhaene et al. 2008) is the case where the risk measure is Expected Shortfall. In that case, the marginal risk contribution, if it exists, is given by

$$A_i^{AS}(R) = \frac{1}{\alpha} \cdot \left[ E\left( X_i \cdot 1_{X_N > q_{1-\alpha}(X_N)} \right) + \beta \cdot E\left( X_i \cdot 1_{X_N = q_{1-\alpha}(X_N)} \right) \right], \tag{3}$$

with

$$\begin{aligned} \beta &= \tfrac{\alpha - P(X_N > q_{1-\alpha}(X_N))}{P(X_N = q_{1-\alpha}(X_N))}, && \text{if } P\left(X_N = q_{1-\alpha}(X_N)\right) > 0 \\ &= 0, && \text{if } P\left(X_N = q_{1-\alpha}(X_N)\right) = 0. \end{aligned} \tag{4}$$

This rule is also referred to as CTE (Conditional Tail Expectation) allocation. In case the aggregate loss $X_N$ has a continuous distribution, it holds that $\beta = 0$, and (3) simplifies to

$$A_i^{AS}(R) = E\left( X_i | \, X_N > q_{1-\alpha}(X_N) \right). \tag{5}$$

Using entirely different arguments, both Denault (2001) and Tasche (1999) identify the marginal risk contribution rule as the unique rule that satisfies some desirable properties. Denault shows that it is the only allocation that satisfies the no undercut property if one also allows fractional portfolios. Tasche (1999) argues that it is the only allocation rule that is "suitable" for performance measurement because it is the only allocation that ensures that (marginally) increasing the share of a portfolio improves the overall return of the firm if the risk-adjusted return of that portfolio with respect to the amount of risk capital allocated to it is higher than that of the whole firm. It has been argued in the literature, however, that one of the drawbacks of the Aumann-Shapley value as an allocation rule is that it requires differentiability of $r(\cdot)$ (e.g., Koster 1999, and Fischer 2003). In the remainder of this section we show that even when the differentiability condition is satisfied, allocating risk capital on the basis of the Aumann-Shapley value may lead to undesirable allocations. As can be seen from (3), the Aumann-Shapley value in case Expected Shortfall is used as risk measure (i.e., the CTE rule) results from taking the expectation of the loss of portfolio $i$, conditional on those states of the world in which the aggregate loss weakly exceeds its $(1-\alpha) \cdot 100\%$-quantile. Thus, the amount of risk capital allocated to a portfolio depends only on the distribution of its loss in those particular states of the world. The following example shows that this can lead to allocations for which it is questionable whether they would be perceived as fair.

With slight abuse of notation, we denote $\pi = (\pi(\omega_1), \cdots, \pi(\omega_m))^\top \in \mathbb{R}^m$ for the vector that contains the probabilities of the different states. Likewise, for all portfolios $i = 1, \cdots, n$, we denote $X_i = (X_i(\omega_1), \cdots, X_i(\omega_m))^\top \in \mathbb{R}^m$ for the vector that contains the realizations of $X_i$ in each state of the world.

**Example 2.2** *Consider risk capital problem $R$ defined by $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $N = \{1,2\}$, and*

$$\pi = \begin{bmatrix} \frac{1}{10} \\ \frac{1}{10} \\ \frac{2}{5} \\ \frac{2}{5} \end{bmatrix}, X_1 = \begin{bmatrix} 60 \\ 0 \\ 30 \\ -15 \end{bmatrix}, X_2 = \begin{bmatrix} 6 \\ 60 \\ \gamma \\ 30 \end{bmatrix}, \text{ so } X_N = X_1 + X_2 = \begin{bmatrix} 66 \\ 60 \\ 30+\gamma \\ 15 \end{bmatrix}.$$

*We use Expected Shortfall at the 15%-level as a risk measure, i.e., $\rho(Y) = \rho_{0.15}^{ES}(Y)$. Now first consider the case where $\gamma \leq 30$. Then, we find for $X_N$ that*

$$q_{1-\alpha}(X_N) = \inf\{x \in \mathbb{R} | P(X_N \leq x) > 0.85\} = 60.$$

*This implies that the Expected Shortfall of the aggregate risk is given by*

$$\begin{aligned}\rho(X_N) &= \alpha^{-1}(E[X_N \cdot 1_{X_N \geq q_{1-\alpha}(X_N)}] - q_{1-\alpha}(X_N) \cdot [P(X_N \geq q_{1-\alpha}(X_N)) - \alpha]) \\ &= 0.15^{-1}\left(E\left(X_N \cdot 1_{X_N \geq 60}\right) - 60 \cdot [P(X_N \geq 60) - 0.15]\right) \\ &= 0.15^{-1}(60 \cdot \frac{1}{10} + 66 \cdot \frac{1}{10} - 60(\frac{1}{10} + \frac{1}{10} - 0.15)) \\ &= 64.\end{aligned}$$

*The aggregate risk capital in case $\gamma \geq 30$ can be determined in a similar way. This yields*

$$\begin{aligned}\rho(X_N) \quad &= 64, && \text{if } \gamma \leq 30, \\ &= 54 + \tfrac{\gamma}{3}, && \text{if } 30 < \gamma \leq 36, \\ &= 30 + \gamma, && \text{if } \gamma \geq 36.\end{aligned}$$

*Moreover, it can be verified that $\rho(s_1X_1 + s_2X_2)$ is differentiable at $s = (1,1)$ if and only if $\gamma \notin \{30, 36\}$, i.e., the Aumann-Shapley value does not exist for $\gamma \in \{30, 36\}$. For $\gamma \notin \{30, 36\}$, it follows from (4) that*

$$\begin{aligned}\beta \quad &= 0.5, && \text{if } \gamma < 30, \\ &= 0.5, && \text{if } 30 < \gamma < 36, \\ &= \tfrac{0.15}{0.4}, && \text{if } \gamma > 36.\end{aligned}$$

*Combined with (3), this yields*

$$\begin{aligned}A^{AS}(R) \quad &= (40, 24), && \text{if } \gamma < 30, \\ &= (50, 4 + \tfrac{\gamma}{3}), && \text{if } 30 < \gamma < 36, \\ &= (30, \gamma), && \text{if } \gamma > 36.\end{aligned}$$

*We see that:*

- *At $\gamma = 30$ or $\gamma = 36$, a marginal change in the loss distribution of portfolio 2 due to a marginal change in $\gamma$ induces a significant change in the allocation. For example, a marginal increase in the loss of portfolio 2 due to a marginal increase in $\gamma$ from just below 30 to just above 30, implies that the amount of risk capital allocated to portfolio 2 jumps down by 41.7%;*

- *for any $\gamma < 36$, the amount of risk capital allocated to portfolio 2 is strictly lower than the amount allocated to portfolio 1, even though for any $\gamma \geq -15$, portfolio 2 is strictly more risky than portfolio 1 in the sense that $P(X_2 > x) \geq P(X_1 > x)$ for all $x \in \mathbb{R}$, with a strict inequality for all $x \in (0, 6)$.*

One could question whether an allocation rule that yields the above results could be perceived as fair by managers whose performance evaluation is based on the amount of risk capital allocated to them. As can be seen from (3), the risk capital allocated to the portfolios is determined exclusively by those states in which the aggregate loss is weakly higher than its 85% quantile. The allocation does not depend on the possible losses in the other states of the world. In the above example, this implies that the more risky portfolio gets allocated less risk capital.

Next, as argued also by Denault (2001), another drawback of allocating risk capital based on the Aumann-Shapley value is that it can imply that certain portfolios are allocated a negative risk capital. This is illustrated in the next example.

**Example 2.3** *Consider the risk capital allocation problem $R$, which is defined by $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $N = \{1, 2, 3\}$,*

$$\pi = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}, X_1 = \begin{bmatrix} -5 \\ 25 \\ -5 \end{bmatrix}, X_2 = \begin{bmatrix} 10 \\ 10 \\ -5 \end{bmatrix}, X_3 = \begin{bmatrix} 0 \\ 10 \\ 60 \end{bmatrix},$$

*and $\rho(Y) = \rho^{ES}_{0.10}(Y)$. It is easily verified that the aggregate risk capital is given by $\rho(X_N) = 50$, and that $\beta$ =0.3. Therefore, it follows from (3) that the Aumann-Shapley value is given by*

$$A^{AS}(R) = (-5, -5, 60).$$

*Portfolios 1 and 2 both are allocated negative risk capital. This occurs because the Aumann-Shapley value is based only on the state where portfolio 3 has the highest losses, which in this case is state $\omega_3$. In that state, however, portfolios 1 and 2 make gains.*

The above example shows that when potential losses are negatively correlated and differ in magnitude, the risk capital is determined exclusively by those states of the world where the portfolios for which the losses are largest in magnitude incur losses, leading to negative risk capital for those portfolios that incur gains in those states. Consider for example an insurance company that holds a portfolio of life annuities and a portfolio of death benefit insurance policies. Because the present value of the liabilities of life annuities increases when death rates decrease, and the opposite holds for the present value of the liabilities of death benefit insurance, the two types of liabilities are typically negatively correlated (see, e.g., Cox and Lin, 2007; Wang et al., 2010; and Tsai et al., 2010). The above example suggests that using the CTE rule to allocate risk capital in this case could result in negative risk capital allocated to one of the two portfolios.

Clearly, negative risk capital is not conceptually "wrong"; it is an indication that the corresponding portfolio provides some hedge potential. However, as argued by Denault (2001), if the capital allocation is to be used in a ratio of return over allocated risk capital-type performance measure, a slightly negative risk capital would yield a largely negative risk-adjusted performance measure.

### 2.3 Excess Based Allocation

The examples in the previous section illustrate some drawbacks of allocating risk capital based on the Aumann-Shapley value. Because part of the probability distribution of the individual losses is ignored, it can lead to allocations in which the higher risk portfolio receives the lowest risk capital. This in turn implies that the expected losses of portfolios in excess of the amount of risk capital allocated to them can differ substantially across portfolios. When capital allocation is used to assess the performance of the different portfolios, this is clearly undesirable. We therefore introduce an alternative allocation rule, which we refer to as *Excess Based Allocation* (EBA). The goal is to find capital allocations in which the excess risks are minimized in lexicographical sense.

This section is organized as follows. We first introduce a set of *feasible* risk capital allocations. Next, we define the *excess* of a set of portfolios with respect to a feasible allocation as the expectation of the loss of that set of portfolios in excess of the risk capital allocated to them. We then formally define the allocation rule. We conclude this section by showing that for each risk capital allocation problem $R \in \mathcal{R}$, the allocation rule yields a unique capital allocation, which we refer to as $EBA(R)$.

The allocation rule that we propose determines the allocation that lexicographically minimizes the portfolios' excesses among a set of allocations that satisfies two basic properties. First, no portfolio is allocated more risk capital than the amount of risk capital that it would need to withhold if it were on its own. Second, a portfolio is not allocated less than the minimum loss it can incur. We refer to allocations that satisfy these properties as *feasible* allocations. Formally, we have the following definition.

**Definition 2.4**

**(i)** *The set of all feasible risk capital allocations for a risk capital allocation problem $R \in \mathcal{R}$ is given by*

$$F(R) = \left\{ a \in \mathbb{R}^N \,\middle|\, a_N = \rho\left(X_N\right), \min_{\omega \in \Omega} X_i(\omega) \leq a_i \leq \rho\left(X_i\right) \text{ for all } i \in N \right\}. \quad (6)$$

**(ii)** *A risk capital allocation rule $A$ on $\mathcal{R}' \subseteq \mathcal{R}$ for which it holds that $A(R) \in F(R)$ for all $R \in \mathcal{R}'$ is referred to as a* feasible risk capital allocation rule on $\mathcal{R}'$.

The following example illustrates how the set $F(R)$ is determined for a given capital allocation problem $R \in \mathcal{R}$.

**Example 2.5** *Consider again Example 2.2 with* $\gamma = -15$, *and recall that* $\rho(X_1 + X_2) = 64$. *Moreover, it can be verified that* $\rho(X_1) = 50$ *and* $\rho(X_2) = 50$. *So, the set of feasible allocations is given by*

$$F(R) = \left\{ a \in \mathbb{R}^2 \,|\, a_1 + a_2 = 64, -15 \leq a_1 \leq 50, -15 \leq a_2 \leq 50 \right\}.$$

*Observe that portfolios* 1 *and* 2 *face the same constraints in* $F(R)$. *Now, recall from Example 2.2 that the Aumann-Shapley value in this case equals*

$$A^{AS}(R) = (40, 24).$$

*It is seen immediately that* $A^{AS}(R)$ *is an element of* $F(R)$.

In the case considered in Example 2.5, we find that the Aumann-Shapley value is an element of the feasible set $F(R)$. Recall that the Aumann-Shapley value is the unique allocation that satisfies the no undercut property for fractional portfolios. This raises the question whether more generally allocations that satisfy the no undercut property (i.e., all (fuzzy) core elements) are in the feasible set. The following theorem shows that coherence of the risk measure ensures that this is indeed the case.

**Theorem 2.6** *For all risk capital allocation problems* $R \in \mathcal{R}$, *the set of feasible capital allocations* $F(R)$ *is non-empty and compact. Moreover,*

$$FCore(R) \subseteq Core(R) \subseteq F(R).$$

**Proof.** Clearly the set $F(R)$ is a closed and bounded, hence compact, subset of $\mathbb{R}^N$. Moreover, Denault (2001) showed that coherence of the risk measure implies that $Core(R)$ is non-empty. Because $FCore(R) \subseteq Core(R)$, it suffices to show that $Core(R) \subseteq F(R)$. This was shown in Boonen (2009). We provide an alternative proof. First, $y \in Core(R)$ implies that

$$\begin{aligned} y_i &\leq \rho(X_i), \text{ for all } i \in N, \\ y_N &= \rho(X_N). \end{aligned}$$

Thus, it suffices to show that $y_i \geq \min_{\omega \in \Omega} X_i(\omega)$ for all $i \in N$. Let $i \in N$. First, $y \in Core(R)$ implies that

$$y_{N \setminus \{i\}} \leq \rho(X_{N \setminus \{i\}}).$$

Moreover, *Monotonicity* of the coherent risk measure implies that

$$\rho(X_N) = \rho(X_{N \setminus \{i\}} + X_i) \geq \rho(X_{N \setminus \{i\}} + [\min_{\omega \in \Omega} X_i(\omega)] \cdot \mathbf{1}).$$

Therefore,

$$\begin{aligned} y_i = y_N - y_{N \setminus \{i\}} &= \rho(X_N) - y_{N \setminus \{i\}} \geq \rho(X_N) - \rho(X_{N \setminus \{i\}}) \\ &\geq \rho(X_{N \setminus \{i\}} + [\min_{\omega \in \Omega} X_i(\omega)] \cdot \mathbf{1}) - \rho(X_{N \setminus \{i\}}) \\ &= \rho(X_{N \setminus \{i\}}) + \min_{\omega \in \Omega} X_i(\omega) - \rho(X_{N \setminus \{i\}}) \\ &= \min_{\omega \in \Omega} X_i(\omega). \end{aligned}$$

Hence, $y \in F(R)$. ■

As argued above, our goal is to determine feasible capital allocations for which the expected excess losses of sets of portfolios are minimized in lexicographical sense. To do so, we define the *excess* of (a set of) portfolios as the expectation of the loss of this set of portfolios in excess of the risk capital allocated to them. Later on, these excesses will be used to evaluate the fairness of the risk capital allocation; if a set of portfolios has a higher excess than another set of portfolios, this set of portfolios has been allocated relatively little risk capital.

**Definition 2.7** *For any risk capital allocation $a \in \mathbb{R}^N$ for a risk capital allocation problem $R = (X, \pi, \rho) \in \mathcal{R}$, and for all $S \subseteq N$ the* excess *of $S$ with respect to $a$ is defined as*

$$e(S, a, R) := E\left[(X_S - a_S)^+\right]. \tag{7}$$

Note that for the special case where $S$ consists of a single portfolio, the excess defined above is equal to the expected risk residual considered in Laeven and Goovaerts (2004), Goovaerts, Van den Borre, and Laeven (2005), and Dhaene et al. (2009). In the following example, we show that allocating risk capital on the basis of the Aumann-Shapley rule may lead to large differences in the excesses of sets of portfolios with respect to the amount of risk capital allocated to them.

**Example 2.8** *Consider again the risk capital problem $R = (X, \pi, \rho)$ from Example 2.2 with $\gamma = -15$. Recall that the Aumann-Shapley value is given by $a = (40, 24)$. The excess of a set of portfolios $S$ with respect to the capital allocated to it, $a_S$, can be depicted as the area under the graph of the decumulative distribution function of the aggregate loss, from the allocation point onwards, i.e., $e(S, a, R) = \int_{a_S}^{\infty} P(X_S \geq x)dx$. The graphs for portfolio 1 and 2 (i.e., for $S = \{1\}$, and $S = \{2\}$) are displayed in Figure 1 and Figure 2, respectively.*
*The fact that portfolio 2 is allocated less risk capital even though it is strictly more risky implies that it has a relatively large excess. Specifically, we see that*

$$\begin{aligned} e(\{1\}, a, R) &= E\left[(X_1 - a_1)^+\right] = (60 - 40) \cdot \frac{1}{10} = 2, \\ e(\{2\}, a, R) &= E\left[(X_2 - a_2)^+\right] = (60 - 24) \cdot \frac{1}{10} + (30 - 24) \cdot \frac{2}{5} = 6, \\ e(\{1,2\}, a, R) &= E\left[(X_1 + X_2 - a_1 - a_2)^+\right] = (66 - 64) \cdot \frac{1}{10} = \frac{1}{5}. \end{aligned}$$

In the remainder of this section, we introduce a risk capital allocation rule that aims at minimizing the excesses in lexicographical sense. For any given capital allocation problem $R \in \mathcal{R}$, and any given allocation $a \in F(R)$, we denote

$$\overline{e}(a, R) = (e(S, a, R))_{S \subseteq N} \in \mathbb{R}^{2^N},$$

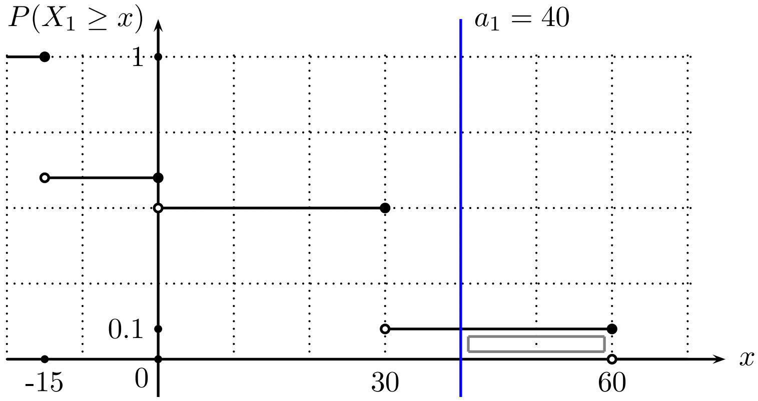


Figure 1: Decumulative distribution function and excess of portfolio 1.

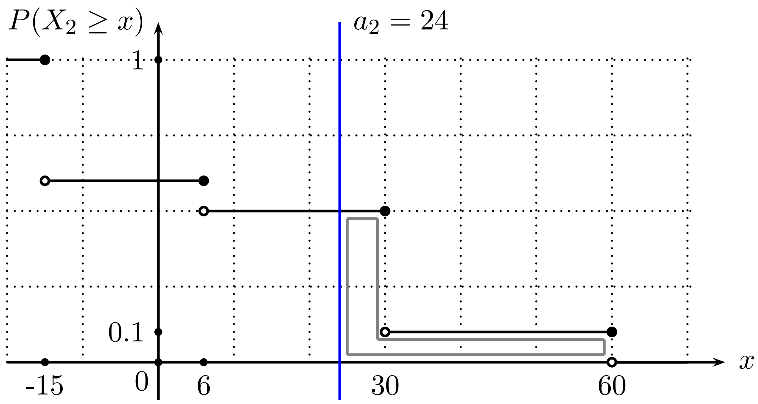


Figure 2: Decumulative distribution function and excess of portfolio 2.

for the vector that contains the excesses of all subsets of portfolios $S \subseteq N$ with respect to the allocation $a$. Our aim is to select feasible risk capital allocations $a$ for which the excesses that arise are lexicographically smaller than those of any other feasible risk capital allocation. For completeness, we recall the definition of lexicographic ordering (see, e.g., Fishburn, 1974)

**Definition 2.9** *For $k \in \mathbb{N}$, and any two vectors $x, y \in \mathbb{R}^k$, $x$ is lexicographically strictly smaller than $y$, denoted as $x <_{lex} y$, if there exists an $i \leq k$ such that $x_i < y_i$, and for all $j < i$ it holds that $x_j = y_j$. Moreover, $x$ is lexicographically smaller than $y$, denoted by $x \leq_{lex} y$, if $x = y$ or $x <_{lex} y$.*

To determine the feasible allocation that minimizes the excesses in lexicographical sense, we first select from the set of feasible allocations those allocations for which the highest excess is minimized. Within the set of allocations for which the highest excess is minimized, we then determine those allocations for which the second highest excess is minimized, and so on.[5]

To formalize this idea, for any $x \in \mathbb{R}^{2^N}$ we let $\theta[x] \in \mathbb{R}^{2^N}$ be the vector that arises from $x$ by arranging the coordinates of $x$ in a non-increasing fashion. Formally, we define $\theta : \mathbb{R}^{2^N} \to \mathbb{R}^{2^N}$ by $\theta_i(x) = x_{\sigma(i)}$ for all $i$, where $\sigma$ is a permutation of $\{1, \ldots, 2^N\}$, such that $x_{\sigma(1)} \geq x_{\sigma(2)} \geq \ldots \geq x_{\sigma(2^N)}$. Now, the allocation rule we propose amounts to finding the allocation that lexicographically minimizes the ordered excesses $\theta[\overline{e}(a, R)]$. Formally, we have the following definition.

**Definition 2.10** *The* Excess Based Allocation *for a risk capital allocation problem $R \in \mathcal{R}$ is given by*

$$EBA(R) = \{a \in F(R) \,|\theta[\overline{e}(a, R)] \leq_{lex} \theta[\overline{e}(y, R)], \forall y \in F(R)\} .$$

Now consider a capital allocation problem $R$. Then, any $a \in EBA(R)$ yields "lower" excesses than any other feasible allocation $y$ in the sense that there exists an $i$ such that the $i-1$ biggest excesses with respect to $a$ are equal to the corresponding $i-1$ biggest excesses with respect to $y$, and the $i$-th biggest excesses with respect to $a$ is strictly smaller than the $i$-th biggest excesses with respect to $y$. For the remaining $k-i$ smallest excesses, no specific relation is required

Note that, by definition, EBA is set-valued. We will show, however, that for any given capital allocation problem $R \in \mathcal{R}$, there is a unique allocation that yields the lexicographical minimum of the excesses, i.e., $EBA(R)$ contains only one allocation. To do so, we first show some basic properties of the excess functions $e(S, \cdot, R)$.

**Lemma 2.11** *Let $R \in \mathcal{R}$. Then,*

**(i)** *for any two allocations $a, b \in F(R)$ and any $S \subseteq N$ it holds that*

- **(a)** *if $e(S, a, R) > e(S, b, R)$, then $a_S < b_S$,*
- **(b)** *if $e(S, a, R) = e(S, b, R) > 0$, then $a_S = b_S$,*
- **(c)** *$e(N, a, R) = e(N, b, R)$.*

**(ii)** *for all $a \in F(R)$ and $i \in N$ it holds that $e(\{i\}, a, R) = 0$ implies $a_i = \rho(X_i)$.*

**(iii)** *for any $S \subseteq N$, it holds that $e(S, \cdot, R)$ is convex on $F(R)$.*

[5] This approach is similar to that in Schmeidler (1969), who introduces this method to find the nucleolus of a cooperative game. Because Schmeidler (1969) considers a different setting (a cooperative game), his definition of excess is different from the one used in this paper.

**Proof.** (a)(i) Let $R \in \mathcal{R}$ be given, and let $a, b \in F(R)$ and $S \subseteq N$.
Suppose $e(S, a, R) > e(S, b, R)$. It then follows from Definition 2.7 that

$$\sum_{\omega \in \Omega} (X_S(\omega) - a_S)^+ \cdot \pi(\omega) = e(S, a, R) > e(S, b, R) = \sum_{\omega \in \Omega} (X_S(\omega) - b_S)^+ \cdot \pi(\omega).$$

This implies that $a_S < b_S$.

(i)(b) Let $a, b \in F(R)$ and $S \subseteq N$, such that $e(S, a, R) = e(S, b, R) > 0$. Then, there exists an $\omega \in \Omega$ such that $X_S(\omega) - a_S > 0$. Therefore, $b_S > a_S$ implies $e(S, a, R) > e(S, b, R)$, and $b_S < a_S$ implies $e(S, a, R) < e(S, b, R)$. This contradicts $e(S, a, R) = e(S, b, R)$, so we can conclude that $a_S = b_S$.

(i)(c) Follows immediately from the fact that $a, b \in F(R)$ implies that $a_N = b_N$.

(ii) Let $i \in N$, $a \in F(R)$ and $e(\{i\}, a, R) = 0$. First, $a \in F(R)$ implies that $a_i \leq \rho(X_i)$.
Next,

$$e(\{i\}, a, R) = \sum_{\omega \in \Omega} (X_i(\omega) - a_i)^+ \cdot \pi(\omega) = 0$$

implies that $X_i(\omega) \leq a_i$ for all $\omega \in \Omega$. Because $\rho$ is a coherent risk measure, $\rho(X_i) \leq \max_{\omega \in \Omega} X_i(\omega)$, which in turn implies $\rho(X_i) \leq a_i$. Therefore, we can conclude that $\rho(X_i) = a_i$.[6]

(iii) Let $a \in F(R)$ and $S \subseteq N$. The results follows immediately from $e(S, a, R) = \sum_{\omega \in \Omega} (X_S(\omega) - a_S)^+ \cdot \pi(\omega)$, and the fact that for any given constant $c \in \mathbb{R}$, the function $f(x) = (c - x)^+$ is convex. ■

We can now show that EBA is a risk capital allocation rule on $\mathcal{R}$, i.e., for every capital allocation problem $R \in \mathcal{R}$, there is a unique allocation that yields the lexicographic minimum.

**Theorem 2.12** *For all $R \in \mathcal{R}$, it holds that $EBA(R)$ is single-valued.*

**Proof.** First, we show that for all $R \in \mathcal{R}$, it holds that $EBA(R) \neq \emptyset$. Let $R = (X, \pi, \rho) \in \mathcal{R}$ be a risk capital allocation problem. Let us denote

$$\begin{aligned} M_1 &= F(R) \\ M_{i+1} &= \{a \in M_i \,|\, \theta_i[\overline{e}(a, R)] \leq \theta_i[\overline{e}(y, R)], \forall y \in M_i\}, \qquad \text{for } i = 1, \cdots, 2^N. \end{aligned}$$

Then, by construction, it follows that

$$EBA(R) = M_{2^N+1}.$$

[6]This inequality follows from *Monotonicity* of the coherent risk measure.

It therefore suffices to show that $M_{2^N+1}$ is non empty. We show by induction that $M_i$ is compact and non-empty for all $i = 1, \cdots, 2^N$. For $i = 1$, it follows immediately from Theorem 2.6 that $M_1 = F(R)$ is non-empty and compact. Now suppose $M_i$ is compact and non-empty for some $i \in \{1, \cdots, 2^N\}$. Because for all $S \subseteq N$, $a \mapsto e(S, a, R)$ is continuous, $\theta_i[\overline{e}(a, R)]$ is also continuous in $a$. Because $M_{i+1} = \arg\max\{\theta_i[\overline{e}(a, R)] : a \in M_i\} \subset M_i$, this immediately implies that $M_{i+1}$ is compact and non-empty. Thus, by induction, we conclude that $M_{2^N+1} \neq \emptyset$.

Next, we show that for all $a, b \in EBA(R)$ and all $S \subseteq N$, it holds that

$$e(S, a, R) = e(S, b, R).$$

Let $a, b \in EBA(R)$ and let

$$\mathcal{M} = \{T \subseteq N : e(T, a, R) \neq e(T, b, R)\}.$$

Suppose that $\mathcal{M} \neq \emptyset$. We will show that that leads to a contradiction. Let $S \in \mathcal{M}$ be such that

$$\max\{e(S, a, R), e(S, b, R)\} \geq \max\{e(T, a, R), e(T, b, R)\}, \text{ for all } T \in \mathcal{M}. \tag{8}$$

We assume, without loss of generality, that $e(S, a, R) > e(S, b, R)$, and let

$$\widetilde{a} := \frac{1}{2}(a + b) \in F(R).$$

We now show that $\theta[\overline{e}(\widetilde{a}, R)] <_{lex} \theta[\overline{e}(a, R)]$, which contradicts $a \in EBA(R)$. Specifically, we show that for those sets of portfolios $T$ for which the excess with respect to $a$ is strictly larger than $e(S, a, R)$, the excess with respect to $\widetilde{a}$ is equal to the excess with respect to $a$, for those sets of portfolios $T$ for which the excess with respect to $a$ is equal to $e(S, a, R)$, the excess with respect to $\widetilde{a}$ is lower than the excess with respect to $a$, with a strict inequality for $T = S$, and, finally, for those sets of portfolios $T$ for which the excess with respect to $a$ is strictly lower than $e(S, a, R)$, the excess with respect to $\widetilde{a}$ is also strictly lower than $e(S, a, R)$. Formally, we distinguish the following two cases:

**(i)** $T \subseteq N$ with $e(T, a, R) > e(S, a, R)$: It follows from (8) that $T \notin \mathcal{M}$, so $e(T, a, R) = e(T, b, R)$. It then follows from Lemma 2.11 that $a_T = b_T = \widetilde{a}_T$. Therefore,

$$e(T, \widetilde{a}, R) = e(T, a, R) > e(S, a, R). \tag{9}$$

**(ii)** $T \subseteq N$ with $e(T, a, R) \leq e(S, a, R)$: It follows from (8) that $e(T, b, R) \leq e(S, a, R)$. Because $e(T, \cdot, R)$ is convex, this implies that

$$e(T, \widetilde{a}, R) \leq \frac{1}{2}\left[e(T, a, R) + e(T, b, R)\right] \leq e(S, a, R). \tag{10}$$

The inequality is strict when $e(T, a, R) < e(S, a, R)$. It is also strict when $T = S$ because $e(S, b, R) < e(S, a, R)$. Thus, we have

$$e(T, \widetilde{a}, R) \begin{array}{ll} \leq e(S, a, R), & \text{if } e(T, a, R) = e(S, a, R), T \neq S, \\ < e(S, a, R), & \text{if } T = S, \\ < e(S, a, R), & \text{if } e(T, a, R) < e(S, a, R). \end{array} \tag{11}$$

Now let $k = \#\{T : e(T, a, R) > e(S, a, R)\}$. It follows from (9) and (10) that the $k$ largest excesses under $\widetilde{a}$ coincide with the $k$ largest excesses under $a$, i.e., $\theta_i[\overline{e}(\widetilde{a}, R)] = \theta_i[\overline{e}(a, R)]$ for all $i \leq k$. Next, let $l = \#\{T : e(T, a, R) = e(S, a, R)\}$, i.e., the next $l$ largest excesses under $a$ are all equal to $e(S, a, R)$. It follows from (11) that $\theta_i[\overline{e}(\widetilde{a}, R)] \leq e(S, a, R) = \theta_i[\overline{e}(a, R)]$ for all $i \in [k+1, k+l]$. Moreover, there is at least one strict inequality because $\#\{T : e(T, \widetilde{a}, R) = e(S, a, R)\} \leq l - 1 < l$. Thus, we conclude that $\theta[\overline{e}(\widetilde{a}, R)] <_{lex} \theta[\overline{e}(a, R)]$, which contradicts $a \in EBA(R)$. Therefore, we have $e(S, a, R) = e(S, b, R)$ for all $S \subseteq N$.

Now we can show that the set $EBA(R)$ contains only one element. Let $R \in \mathcal{R}$. Let $a, b \in EBA(R)$. Then for all $S \subseteq N$ it must hold that $e(S, a, R) = e(S, b, R)$. In particular this holds for all individual portfolios. Let $i \in N$. In order to show that $a_i = b_i$, we can distinguish two cases. If $e(\{i\}, a, R) = e(\{i\}, b, R) > 0$, then according to ii) in Lemma 2.11 we have $a_i = b_i$. If $e(\{i\}, a, R) = e(\{i\}, b, R) = 0$, then according to b) in Lemma 2.11 we have $a_i = b_i = \rho(X_i)$. So, we have $a = b$. ■

From now on, with slight abuse of notation, we denote the unique element of the singleton set $EBA(R)$ by $EBA(R)$ as well.

As argued before, the allocation rule that we propose is inspired by the fact that allocating risk capital based on the Aumann-Shapley value can yield undesirable results in the sense that some portfolios have relatively large expected excess losses. EBA mitigates that problem by determining the allocation for which the largest excess is as small as possible. In addition, it has in common with other existing optimization approaches (e.g., Dhaene, Goovaerts, and Kaas 2003, Laeven and Goovaerts 2004, Goovaerts, Van den Borre, and Laeven 2005, and Dhaene, Tsanakas, Valdez, and Vanduffel 2009) that, in contrast to the Aumann-Shapley value, it does not require differentiability of the risk measure. It is therefore well-defined on a broader class of capital allocation problems. In the next section, we show that EBA satisfies some additional desirable properties

# 3 Properties of EBA

In this section we define some desirable properties of allocation rules, and show that EBA satisfies these properties.

Let $A : \mathcal{R} \to \mathbb{R}^N$ be a risk capital allocation rule. We consider the following properties:

**(i)** *No Diversification*: for all $R \in \mathcal{R}$ it holds that if $\sum_{i=1}^{n} \rho(X_i) = \rho(X_N)$, then

$$A_i(R) = \rho(X_i), \text{ for all } i \in N.$$

**(ii)** *Riskless Portfolio*: for all $R \in \mathcal{R}$ and for all $i \in N$ it holds that if $X_i = c \cdot \mathbf{1}$, then

$$A_i(R) = c.$$

**(iii)** *Symmetry*: for all $R \in \mathcal{R}$ and for all $i, j \in N$ such that $X_i = X_j$, it holds that

$$A_i(R) = A_j(R).$$

**(iv)** *Translation Invariance*: for all $R \in \mathcal{R}$ and for all $i \in N$ it holds that if $\widehat{R} = (\widehat{X}, \pi, \rho)$, where $\widehat{X} = (X_1, \dots, X_i + c \cdot \mathbf{1}, \dots, X_n)$, then

$$A(\widehat{R}) = A(R) + c \cdot e^i,$$

where $e^i$ is the unit vector for portfolio $i$.

**(v)** *Scale Invariance*: for all $R \in \mathcal{R}$ it holds that if $\widehat{R} = (\widehat{X}, \pi, \rho)$, where $\widehat{X} = (c \cdot X_1, \dots, c \cdot X_n)$ for some $c \in \mathbb{R}_+$, then

$$A(\widehat{R}) = c \cdot A(R).$$

*No Diversification* means that if there are no gains for the portfolios in terms of total risk capital required whether they are treated separately or as one entity, then allocation of risk capital is trivial, that is, every portfolio gets allocated exactly what the risk measure yields for that portfolio in isolation. If the *Riskless Portfolio* property holds, then a portfolio with riskless payoffs (i.e., with a deterministic loss) gets allocated as risk capital the loss it will incur for sure. The *Symmetry* property ensures that the allocation depends only on the distribution of the losses of the portfolio. *Translation Invariance* ensures that if in all states of the world an equal amount of losses is added or subtracted to the losses of a portfolio, then that portfolio is allocated an additional risk capital equal to this amount. Finally, *Scale Invariance* ensures, for example, that the allocation of risk capital does not change when another currency is used.

We first show that any feasible risk capital allocation rule, and thus EBA in particular, satisfies the *No Diversification* property and the *Riskless Portfolio* property.

**Theorem 3.1** *Any feasible risk capital allocation rule $A$ on $\mathcal{R}' \subseteq \mathcal{R}$ satisfies* No Diversification *and the* Riskless Portfolio *property.*

**Proof.** Let $A$ be a feasible risk capital allocation rule. First, we prove the *No Diversification* property. Let $R = (X, \pi, \rho) \in \mathcal{R}$ be such that $\sum_{i=1}^{n} \rho(X_i) = \rho(X_N)$. Then, it follows from (6) that $F(R) = \{a \in \mathbb{R}^N | a_i = \rho(X_i), \text{ for all } i \in N\}$. Because $A$ is a feasible risk capital allocation rule, this implies that $A_i(R) = \rho(X_i)$ for all $i \in N$.

Next, we prove the *Riskless Portfolio* property. Let $R = (X, \pi, \rho) \in \mathcal{R}$ be such that $X_i = c \cdot \mathbf{1}$ for some $c \in \mathbb{R}$ and $i \in N$ . Since $\rho$ is a coherent risk measure, it holds that

$$c = \min_{\omega \in \Omega} X_i(\omega) \leq A_i(R) \leq \rho(X_i) \leq \max_{\omega \in \Omega} X_i(\omega) = c,$$

so $A_i(R) = c$. ■

The next theorem shows that EBA also satisfies the *Symmetry*, *Translation Invariance* and *Scale Invariance* properties.

**Theorem 3.2** EBA *satisfies* Symmetry, Translation Invariance *and* Scale Invariance.

**Proof.** Let $R = (X, \pi, \rho) \in \mathcal{R}$ be given.
We start by proving the *Symmetry* property. Let $X_i = X_j$ for some $i, j \in N$. Let $a = EBA(R)$, and define $b \in \mathbb{R}^N$ by $b_k = a_k$ for all $k \in N \backslash \{i, j\}$, $b_i = a_j$ and $b_j = a_i$. It follows immediately from (2.7) that

$$\begin{array}{lll} e(S,a,R) & = e(S,b,R) & \text{if } i,j \notin S \text{ or if } i,j \in S, \\ e(S,a,R) & = e(S \cup \{j\} \backslash \{i\}, b, R) & \text{if } i \in S, j \notin S, \\ e(S,a,R) & = e(S \cup \{i\} \backslash \{j\}, b, R) & \text{if } j \in S, i \notin S. \end{array}$$

Hence, for every $S \subset N$, there exists a $T \subset N$ so that $e(S, a, R) = e(T, b, R)$, i.e., it holds that $\theta[\overline{e}(a, R)] = \theta[\overline{e}(b, R)]$. Because $EBA(R)$ is unique (Theorem 2.12), this implies $a = b$, and we can conclude that $a_j = b_i = a_i$.

Next, we prove the *Translation Invariance* property. Let $i \in N$ and $c \in \mathbb{R}$. Define a second risk capital problem $\widehat{R} = (\widehat{X}, \pi, \rho) \in \mathcal{R}$, where $\widehat{X} = (\widehat{X}_1, \ldots, \widehat{X}_n) = (X_1, \ldots, X_i + c \cdot \mathbf{1}, \ldots, X_n)$. First, note that $\min_{\omega \in \Omega}\{\widehat{X}_j(\omega)\} = \min_{\omega \in \Omega}\{X_j(\omega)\} + c \cdot 1_{(j=i)}$. Second, note that coherence of the risk measure $\rho$ implies that $\rho(\widehat{X}_N) = \rho(X_N) + c$ and $\rho(\widehat{X}_j) = \rho(X_j) + c \cdot 1_{(j=i)}$. Therefore, we know that

$$F(\widehat{R}) = \{a + c \cdot e^i | a \in F(R)\}.$$

Now let $a \in F(R)$ and $\widehat{a} = a + c \cdot e^i$. We now show that $e(S, a, R) = e(S, \widehat{a}, \widehat{R})$ for all $S \subseteq N$. To do so, we distinguish two cases. For all $S \subseteq N, S \not\ni i$, it is clear that $e(S, a, R) = e(S, \widehat{a}, \widehat{R})$. For all $S \subseteq N, S \ni i$ we have

$$\begin{aligned} e(S, \widehat{a}, \widehat{R}) &= E\left[\left(\widehat{X}_S - \widehat{a}_S\right)^+\right] \\ &= E\left[\left(\left(\widehat{X}_S - c \cdot \mathbf{1}\right) - (\widehat{a}_S - c \cdot \mathbf{1})\right)^+\right] = E\left[(X_S - a_S)^+\right] \\ &= e(S, a, R). \end{aligned}$$

So, $\theta[\overline{e}(\widehat{a}, \widehat{R})] = \theta[\overline{e}(a, R)]$. Therefore, it follows that

$$\begin{aligned} EBA(\widehat{R}) &= \arg\min_{\widehat{a}}\ {}_{lex} \left\{\theta[\overline{e}(\widehat{a}, \widehat{R})] : \widehat{a} \in F(\widehat{R})\right\} \\ &= \arg\min_{a}\ {}_{lex} \left\{\theta[\overline{e}(a + c \cdot e^i, \widehat{R})] : a \in F(R)\right\} + c \cdot e^i \\ &= \arg\min_{a}\ {}_{lex} \{\theta[\overline{e}(a, R)] : a \in F(R)\} + c \cdot e^i \\ &= EBA(R) + c \cdot e^i \end{aligned}$$

Finally, we prove the *Scale Invariance* property. Let $c > 0$. We define a second risk capital problem $\widehat{R} = (\widehat{X}, \pi, \rho) \in \mathcal{R}$, where $\widehat{X} = c \cdot X$. Coherence of the risk measure

$\rho$ implies that $\rho(\widehat{X}_N) = c \cdot \rho(X_N)$ and $\rho(\widehat{X}_j) = c \cdot \rho(X_j)$ for all $j \in N$. Moreover $\min_{\omega\in\Omega}\{\widehat{X}_j(\omega)\} = c \cdot \min_{\omega\in\Omega}\{X_j(\omega)\}$. Therefore, it follows that $F(\widehat{R}) = \{c \cdot a | a \in F(R)\}$. Moreover, it is verified easily that $c \cdot e(S, a, R) = e(S, c \cdot a, \widehat{R})$ for every $a \in F(R)$. This implies that

$$\begin{aligned} EBA(\widehat{R}) &= \arg\min_{\widehat{a}}\ {}_{lex} \left\{\theta[\overline{e}(\widehat{a}, \widehat{R})] : \widehat{a} \in F(\widehat{R})\right\} \\ &= c \cdot \arg\min_{a}\ {}_{lex} \left\{\theta[\overline{e}(c \cdot a, \widehat{R})] : a \in F(R)\right\} \\ &= c \cdot \arg\min_{a}\ {}_{lex} \{\theta[c \cdot \overline{e}(a, R)] : a \in F(R)\} \\ &= c \cdot EBA(R) \end{aligned}$$

Therefore, we can conclude that EBA satisfies the *Scale Invariance* property. ■

We know from Theorem 2.6 that the Aumann-Shapley value, when it exists, yields a feasible allocation. Therefore, it follows from Theorem 3.1 that it satisfies the *No Diversification* property and the *Riskless Portfolio* property. Moreover, it follows from Denault (2001) that it also satisfies the *Translation Invariance* and *Scale Invariance* properties (this follows from the *Aggregation Invariance* property of Denault (2001)). Thus, EBA and the Aumann-Shapley both satisfy the set of properties defined above. Another desirable property of a risk capital allocation rule, however, is that "small" changes in the probability distributions of the losses should not cause a major change in the allocation of risk capital. Formally, we introduce the following definition of continuity on the class of risk capital allocation problems.

**Definition 3.3** *A capital allocation rule $A : \mathcal{R} \to \mathbb{R}^N$ is* continuous *on $\mathcal{R}$ if for every sequence $\left\{(X^k, \pi, \rho) : k \in \mathbb{N}\right\} \subset \mathcal{R}$ that satisfies*

**(i)** $\lim_{k\to\infty} X_i^k(\omega) = X_i(\omega)$ *for all $\omega \in \Omega$ and for all $i \in N$,*

**(ii)** $\lim_{k\to\infty} \rho(X_i^k) = \rho(X_i)$ *for all $i \in N$,*

**(iii)** $\lim_{k\to\infty} \rho(X_N^k) = \rho(X_N)$,

*it holds that*

$$\lim_{k\to\infty} A((X^k, \pi, \rho)) = A((X, \pi, \rho)).$$

Because one cannot expect a feasible risk capital allocation rule to be continuous if small changes in the probability distributions of the losses can induce large changes in the feasible set, we add conditions (ii) and (iii). Condition (ii) ensures that the aggregate risk capital, $\rho(X_N)$, converges. Condition (iii) guarantees that the upper bounds of the feasible set (i.e., $\rho(X_i)$ for all $i \in N$) converge. Convergence of the lower bounds (i.e., $\min\{X_i : \omega \in \Omega\}$ for all $i \in N$) is guaranteed by (i). Clearly, when the risk measure $\rho(\cdot)$ is continuous in the sense that (i) implies (ii), then (ii) and (iii) follow from (i).

Recall that Example 2.2 shows that for some risk capital allocation problems, the Aumann-Shapley value does not exist. The following example shows that in the case considered in that example, there does not exist a continuous extension of the Aumann-Shapley value to the set of all risk capital allocation problems.

**Example 3.4** *Consider again Example 2.2. It is verified easily that the aggregate risk capital, $\rho(X_N)$, as well as the amounts of risk capital for each portfolio separately, $\rho(X_1)$ and $\rho(X_2)$, are continuous in $\gamma$. Recall that the Aumann-Shapley value does not exist for $\gamma \in \{30, 36\}$. Moreover, it jumps from $(40, 24)$ to $(50, 14)$ when $\gamma$ increases from just below $30$ to just above $30$. Therefore, there does not exist a continuous extension of the Aumann-Shapley value to the set of all risk capital allocation problems. In contrast, it can be verified that:*

$$
\begin{aligned}
EBA(R) &= (32, 32) && \textit{if } \gamma \leqslant 30 \\
&= (27 + \gamma/6, 27 + \gamma/6) && \textit{if } 30 < \gamma \leqslant 32.4 \\
&= (45 - 7\gamma/18, 9 + 13\gamma/18) && \textit{if } 32.4 < \gamma \leqslant 36 \\
&= (25 + \gamma/6, 5 + 5\gamma/6) && \textit{if } 36 < \gamma \leqslant 66 \\
&= (36, \gamma - 6) && \textit{if } \gamma > 66.
\end{aligned}
$$

*It is verified easily that $EBA(R)$ is continuous in $\gamma$.*

In the above example, we see that $EBA(R)$ is continuous in $\gamma$. The following theorem shows that, more generally, EBA is a continuous allocation rule on the set of all risk capital allocation problems.

**Theorem 3.5** EBA *satisfies* Continuity *on $\mathcal{R}$.*

The proof can be found in Appendix A. It is partly inspired by Kohlberg (1971), who proves continuity for the nucleolus of a cooperative game.

# 4 Linear programming approach

In this section we show how for a given allocation problem $R$, $EBA(R)$ can be found by solving a sequence of linear programming problems. Throughout this section, we let $R$ be given, and denote $EBA(R)$ by $EBA$.

The approach we propose is inspired by an approach to determine the *nucleolus* of a cooperative game (see, e.g., Kohlberg 1971). However, there are important differences. First, for the nucleolus, the "value" of a coalition (in our case a set of portfolios) is a real number; in our case it is a random variable. Second, whereas the objective in case of the nucleolus is the difference between the coalition value and the allocated number, the objective in our case is the expected loss *in excess of* the allocated risk capital, i.e., negative differences are cut off at zero. Third, the feasible set in our case consists of efficient allocations that are coordinatewise bounded both above and below. The nucleolus considers efficient allocations that are coordinatewise bounded below only.[7]

[7] In a cost game, the nucleolus considers efficient allocations that are coordinatewise bounded above only.

The sequence of linear programming problems we propose determines the sets $\mathbf{b}_1, \cdots, \mathbf{b_p}$ and corresponding values of the excesses $\mathbf{e}_1, \ldots, \mathbf{e_p}$ in an inductive manner.

**Definition 4.1** *For any $R \in \mathcal{R}$ and for all $a \in F(R)$ the coalition array $b_{01}, b_{02}, b_1, \ldots, b_p$* belonging *to $(R, a)$ is defined as*

$$\begin{cases} b_{01} & = \{i | a_i = \min_{\omega\in\Omega} X_i(\omega)\}, \\ b_{02} & = \{i | a_i = \rho(X_i)\}, \\ b_1 & = \{S \subseteq N | e(S,a,R) = \max_{T\subseteq N}\{e(T,a,R) | T \subseteq N\}\}, \\ b_j & = \{S \subseteq N | e(S,a,R) = \max_{T\subseteq N}\{e(T,a,R) | T \subseteq N, T \notin b_1 \cup \ldots \cup b_{j-1}\}\}, \end{cases}$$

*for every $j \in \mathbb{N}$, $j \geq 2$, such that $b_1 \cup \cdots \cup b_{j-1} \neq 2^N$. The number $p \in \mathbb{N}$ is such that $b_1 \cup \cdots \cup b_p = 2^N$ and for all $1 \leq k \leq p$ it holds $b_k \neq \emptyset$.*

For every $j \in \mathbb{N}$, the set $b_j$ consists of those sets of portfolios $S$ for which the excess loss with respect to the allocated risk capital is the $j$-th largest among all the excesses, and $e_j$ is the corresponding value of the excess. Furthermore, $b_{01}$ contains all portfolios that are allocated their minimal loss (i.e., for which the lower bound in the feasible set is binding), and $b_{02}$ all portfolios that are allocated the risk capital of their own portfolio (i.e., for which the upper bound in the feasible set is binding). This is illustrated in the following example.

**Example 4.2** *Consider the risk capital allocation problem $R$ given by $\Omega = \{\omega_1, \omega_2\}$, $N = \{1, 2, 3\}$,*

$$\pi = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, X_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, X_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

*and $\rho(Y) = \rho^{ES}_{0.10}(Y)$. Then, since $\rho(X_i) = 1$ and $\min_{\omega\in\Omega} X_i(\omega) = 0$ for all $i \in N$, and $\rho(X_N) = 2$, it holds that*

$$F(R) = \{a \in [0,1]^N | a_1 + a_2 + a_3 = 2\}.$$

*Consider the allocation $a = (\frac{1}{2}, \frac{3}{4}, \frac{3}{4})$. The coalition array belonging to $(R, a)$ is*

$$\begin{aligned} b_{01} &= \emptyset, \\ b_{02} &= \emptyset, \\ b_1 &= \{\{1\}, \{2,3\}\}, \\ b_2 &= \{\{2\}, \{3\}\}, \\ b_3 &= \{\{1,2\}, \{1,3\}, N\}, \end{aligned}$$

*and corresponding values of the excesses are $e_1 = \frac{1}{4}$, $e_2 = \frac{1}{8}$ and $e_3 = 0$.*

We will denote $\mathbf{b}_{01}, \mathbf{b}_{02}, \mathbf{b}_1, \cdots, \mathbf{b_p}$ for the coalition array that belongs to $(R, EBA)$, and $\mathbf{e}_1, \ldots, \mathbf{e_p}$ for the corresponding excesses. Thus, for all $j \leq \mathbf{p}$,

- $\mathbf{b}_j \subset 2^N$ consists of those sets of portfolios $S$ for which the excess loss with respect to $EBA$ is the $j$-th largest among all the excesses;
- $\mathbf{e}_j \in \mathbb{R}$ denotes the value of the excess of each set of portfolios $S \in \mathbf{b}_j$ with respect to $EBA$, i.e., $\mathbf{e}_j = E\left[(X_S - EBA_S)^+\right]$;

Note that, by construction, $\mathbf{p}$ is such that

$$\begin{aligned} \mathbf{b}_1 \cup \cdots \cup \mathbf{b}_k &\subsetneq 2^N, \text{ for all } k < \mathbf{p}, \\ \mathbf{b}_1 \cup \cdots \cup \mathbf{b}_{\mathbf{p}} &= 2^N. \end{aligned}$$

The sequence of linear programming problems we propose determines the sets $\mathbf{b}_1, \cdots, \mathbf{b}_{\mathbf{p}}$ and corresponding values of the excesses $\mathbf{e}_1, \ldots, \mathbf{e}_{\mathbf{p}}$ in an inductive manner.

First, for any given $b_1, \cdots, b_{k-1} \subseteq 2^N$, and $e_1, \cdots, e_{k-1} \in \mathbb{R}$, we define the linear optimization problem $\mathrm{P}_k(b_1, \cdots, b_{k-1}, e_1, \cdots, e_{k-1})$ as follows

$$\min v_k$$

$$a_i^k \leq \rho(X_i), \quad \text{for all } i \in N, \tag{12}$$

$$a_i^k \geq \min_{\omega\in\Omega} X_i(\omega), \quad \text{for all } i \in N, \tag{13}$$

$$a_N^k = \rho(X_N), \tag{14}$$

$$\sum_{\omega\in\Omega} \pi(w) \cdot \lambda_k(S,\omega) = e_j, \quad \text{for all } S \in b_j,\ j < k, \tag{15}$$

$$\sum_{\omega\in\Omega} \pi(w) \cdot \lambda_k(S,\omega) \leq v_k, \quad \text{for all } S \in 2^N \setminus (b_1 \cup \ldots \cup b_{k-1}), \tag{16}$$

$$\lambda_k(S,\omega) \geq X_S(\omega) - a_S^k, \quad \text{for all } S \in 2^N,\ \omega \in \Omega, \tag{17}$$

$$\lambda_k(S,\omega) \geq 0, \quad \text{for all } S \in 2^N,\ \omega \in \Omega, \tag{18}$$

We denote a feasible solution to the linear program by the tuple $(a^k, v_k, \lambda_k) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^{m\times 2^N}$.

The following lemma shows that solving $\mathrm{P}_k(\mathbf{b}_1, \cdots, \mathbf{b}_{k-1}, \mathbf{e}_1, \cdots, \mathbf{e}_{k-1})$ either immediately yields $EBA$, or it yields the $k$-th largest excess with respect to $EBA$, as well as a superset of the corresponding element of the coalition array. Specifically, for any given optimal solution $(\overline{a}^k, \overline{v}_k, \overline{\lambda}_k)$ to $\mathrm{P}_k(\mathbf{b}_1, \cdots, \mathbf{b}_{k-1}, \mathbf{e}_1, \cdots, \mathbf{e}_{k-1})$, we denote

$$\overline{b}_k = \{S \in 2^N | e(S, \overline{a}^k, R) = \overline{v}_k\}, \tag{19}$$

$$\overline{b}_{01}^k = \{i \in N | \overline{a}_i^k = \min_{\omega\in\Omega} X_i(\omega)\}, \text{ and } \overline{b}_{02}^k = \{i \in N | \overline{a}_i^k = \rho(X_i)\}.$$

Then, we have the following result.

**Lemma 4.3** *Let $k \leq \mathbf{p}$. There exists an optimal solution to $P_k(\mathbf{b}_1, \cdots, \mathbf{b}_{k-1}, \mathbf{e}_1, \cdots, \mathbf{e}_{k-1})$. Moreover, for any optimal solution $(\overline{a}^k, \overline{v}_k, \overline{\lambda}_k)$, it holds that $\overline{v}_k = \mathbf{e}_k$, and*

**(i)** *if* $\overline{v}_k = 0$*, then* $\overline{a}^k = EBA$*;*

**(ii)** *if* $\overline{v}_k > 0$*, then:*

> **(a)** $\overline{a}^k_S = EBA_S$ *for all* $S \in \mathbf{b}_1 \cup \cdots \cup \mathbf{b}_k$*;*
>
> **(b)** $\overline{b}_k \supseteq \mathbf{b}_k$.

The proof can be found in Appendix B.

The above lemma shows that solving $\mathrm{P}_k(\mathbf{b}_1, \cdots, \mathbf{b}_{k-1}, \mathbf{e}_1, \cdots, \mathbf{e}_{k-1})$ yields $\mathbf{e}_k$, the $k$-th largest excess with respect to $EBA$. Moreover, if $\overline{v}_k = 0$, then $EBA$ is found. If $\overline{v}_k > 0$, then the solution yields a superset $\overline{b}_k$ of the coalition array $\mathbf{b}_k$ corresponding to the $k$-th largest excess with respect to $EBA$. In order to be able to solve $\mathrm{P}_k(\mathbf{b}_1, \cdots, \mathbf{b}_k, \mathbf{e}_1, \cdots, \mathbf{e}_k)$, we need to reduce the superset $\overline{b}_k$ to $\mathbf{b}_k$, i.e., we need to identify those elements of $\overline{b}_k$ that are also elements of $\mathbf{b}_k$. To do so, for any given $b_1, \cdots, b_{k-1}, d \subseteq 2^N$, and $b_{01}, b_{02} \subseteq 2^N$, we define the optimization problem $\mathrm{Q}_k(b_{01}, b_{02}, b_1, \cdots, b_{k-1}, d)$, with decision variables $y_i$ for all $i \in N$, as follows

$$
\begin{aligned}
f_{\mathrm{Q}_k(b_{01}, b_{02}, b_1, \cdots, b_{k-1}, d)} = \max \sum_{S \in d} & y_S \\
& y_i \leq 1, && \text{for all } i \in N, \\
& y_i \geq 0, && \text{for all } i \in b_{01}, \\
& y_i \leq 0, && \text{for all } i \in b_{02}, \\
& y_S \geq 0, && \text{for all } S \in (b_1 \cup \ldots \cup b_{k-1}) \cup d, \\
& y_N = 0.
\end{aligned}
$$

The following lemma shows that repeatedly solving $\mathrm{Q}_k(\overline{b}^k_{01}, \overline{b}^k_{02}, \mathbf{b}_1, \cdots, \mathbf{b}_{k-1}, d)$ while appropriately updating the set $d$ yields $d = \mathbf{b}_k$.

**Lemma 4.4** *Let* $k \leq \mathbf{p}$*, and let* $(\overline{a}^k, \overline{v}_k, \overline{\lambda}_k)$ *be an optimal solution to* $P_k(\mathbf{b}_1, \cdots, \mathbf{b}_{k-1}, \mathbf{e}_1, \cdots, \mathbf{e}_{k-1})$*. If* $\overline{v}_k > 0$*, then for any* $d \subseteq 2^N$ *such that* $\mathbf{b}_k \subseteq d \subseteq \overline{b}_k$*,*

**(i)** *there exists an optimal solution of* $Q_k(\overline{b}^k_{01}, \overline{b}^k_{02}, \mathbf{b}_1, \cdots, \mathbf{b}_{k-1}, d)$*;*

**(ii)** *for any optimal solution* $\overline{y}$ *of* $Q_k(\overline{b}^k_{01}, \overline{b}^k_{02}, \mathbf{b}_1, \cdots, \mathbf{b}_{k-1}, d)$*, it holds that:*

> **(a)** $\mathbf{b}_k \subseteq \{S \in d | \overline{y}_S = 0\}$*;*
>
> **(b)** $d = \mathbf{b}_k$ *iff* $f_{Q_k(\overline{b}^k_{01}, \overline{b}^k_{02}, \mathbf{b}_1, \cdots, \mathbf{b}_{k-1}, d)} = 0$.

The proof can be found in Appendix B.

The above lemma implies that, starting from $d = \overline{b}_k$, one can can indeed refine the set $d$ to obtain $\mathbf{b}_k$. Specifically, for any given $\mathbf{b}_k \subseteq d$, we determine an optimal solution $\overline{y}$

for $\mathrm{Q}_k(\overline{b}_{01}^k, \overline{b}_{02}^k, \mathbf{b}_1, \cdots, \mathbf{b}_{k-1}, d)$. If it holds that $f_{\mathrm{Q}_k(\overline{b}_{01}^k, \overline{b}_{02}^k, \mathbf{b}_1, \cdots, \mathbf{b}_{k-1}, d)} > 0$, then there exist sets of portfolios $S \in d$ for which $\overline{y}_S > 0$, and it follows from (i)(a) that these sets of portfolios $S$ do not belong to $\mathbf{b}_k$. Thus, we replace $d$ by $\{S \in d | \overline{y}_S = 0\}$, and solve $\mathrm{Q}_k(\overline{b}_{01}^k, \overline{b}_{02}^k, \mathbf{b}_1, \cdots, \mathbf{b}_{k-1}, d)$ again, until $f_{\mathrm{Q}_k(\overline{b}_{01}^k, \overline{b}_{02}^k, \mathbf{b}_1, \cdots, \mathbf{b}_{k-1}, d)} = 0$. It then follows from (ii)(b) that $d = \mathbf{b}_k$.

Combining the above two lemmas yields the following result.

**Theorem 4.5** *Let $R \in \mathcal{R}$ be given, and consider the following algorithm:*

1. *Set $k = 1$.*
2. *Obtain an optimal solution $(\overline{a}^k, \overline{v}_k, \overline{\lambda}_k)$ to $P_k(\overline{b}_1, \cdots, \overline{b}_{k-1}, \overline{v}_1, \cdots, \overline{v}_{k-1})$.*
3. *If $\overline{v}_k > 0$, go to Step 4, else stop.*
4. *Determine $\overline{b}_k$, $\overline{b}_{01}^k$ and $\overline{b}_{02}^k$.*
5. *Obtain an optimal solution $\overline{y}$ to $Q_k(\overline{b}_{01}^k, \overline{b}_{02}^k, \overline{b}_1, \cdots, \overline{b}_{k-1}, \overline{b}_k)$.*
6. *If $f_{Q_k(\overline{b}_{01}^k, \overline{b}_{02}^k, \overline{b}_1, \cdots, \overline{b}_{k-1}, \overline{b}_k)} > 0$, set $\overline{b}_k = \{S \in \overline{b}_k | \overline{y}_S = 0\}$ and return to Step 5. Else, go to Step 7.*
7. *If $\overline{b}_1 \cup \cdots \cup \overline{b}_k \neq 2^N$, set $k = k + 1$, and return to Step 2, else stop.*

*The algorithm terminates after $\mathbf{p}$ iterations, and yields $\overline{a}^{\mathbf{p}} = EBA$.*

**Proof.** First, we show that for all $k < \mathbf{p}$, it holds that after the $k$-the iteration,

$$\left(\overline{b}_1, \cdots, \overline{b}_k, \overline{v}_1, \cdots, \overline{v}_k\right) = \left(\mathbf{b}_1, \cdots, \mathbf{b}_k, \mathbf{e}_1, \cdots, \mathbf{e}_k\right). \tag{20}$$

Let $k = 1$. Lemma 4.3 implies that $\overline{v}_1 = \mathbf{e}_1$, and because excesses are nonnegative and $\mathbf{p} > \mathbf{1}$, we know that $\overline{v}_1 > 0$. Therefore, Lemma 4.4 implies that repeating Step 5 and Step 6 until $f_{\mathrm{Q}_1(\overline{b}_{01}^1, \overline{b}_{02}^1, \overline{b}_1)} = 0$ yields $\overline{b}_1 = \mathbf{b}_1$. Next, suppose (20) holds true for some $k = j < \mathbf{p} - 1$. Then, Lemma 4.3 implies that $\overline{v}_{j+1} = \mathbf{e}_{j+1}$, and $j + 1 < \mathbf{p}$ implies that $\overline{v}_{j+1} > 0$, and it follows from Lemma 4.4 that $\overline{b}_{j+1} = \mathbf{b}_{j+1}$. Therefore, (20) holds true for $k = j + 1$, and it follows by induction that $\left(\overline{b}_1, \cdots, \overline{b}_{\mathbf{p}-1}, \overline{v}_1, \cdots, \overline{v}_{\mathbf{p}-1}\right) = \left(\mathbf{b}_1, \cdots, \mathbf{b}_{\mathbf{p}-1}, \mathbf{e}_1, \cdots, \mathbf{e}_{\mathbf{p}-1}\right)$.
Next, consider $k = \mathbf{p}$. In a similar way, it can be shown that

$$\overline{v}_{\mathbf{p}} = \mathbf{e}_{\mathbf{p}}, \tag{21}$$

$$\text{if } \overline{v}_{\mathbf{p}} > 0, \text{ then } \overline{b}_{\mathbf{p}} = \mathbf{b}_{\mathbf{p}}. \tag{22}$$

Next, we show that the algorithm terminates after $\mathbf{p}$ iterations. The algorithm terminates after the first iteration for which either $\overline{v}_k = 0$ or $\overline{b}_1 \cup \cdots \cup \overline{b}_k = 2^N$. Now (20), (21), (22), and the fact that by definition, $\mathbf{b}_1 \cup \cdots \cup \mathbf{b}_\mathbf{p} = 2^N$, imply that $k \leq \mathbf{p}$. Now if $\overline{v}_k = 0$, then $\overline{v}_j > 0$ for all $j < k$, and it follows from (20) and (21) that $\mathbf{e}_j = \overline{v}_j > 0$ for all $j < k$, and $\mathbf{e}_k = \overline{v}_k = 0$. Because excesses are nonnegative, this implies that $k = \mathbf{p}$. If $\overline{b}_1 \cup \cdots \cup \overline{b}_k = 2^N$ and $\overline{v}_k > 0$, then $\overline{b}_1 \cup \cdots \cup \overline{b}_j \subsetneq 2^N$ for all $j < k$ and it follows from (20) and (22) that $\mathbf{b}_1 \cup \cdots \cup \mathbf{b}_j \subsetneq 2^N$ for all $j < k$, and $\mathbf{b}_1 \cup \cdots \cup \mathbf{b}_k = 2^N$. This implies that $k = \mathbf{p}$.

Finally, we show that the algorithm yields $\overline{a}^\mathbf{p} = EBA$. We know the algorithm terminates after the $\mathbf{p}$-th iteration. Then either $\overline{v}_\mathbf{p} = 0$, or $\overline{b}_1 \cup \cdots \cup \overline{b}_\mathbf{p} = \mathbf{b}_1 \cup \cdots \cup \mathbf{b}_\mathbf{p} = 2^N$ and $\overline{v}_\mathbf{p} > 0$. It then follows from Lemma 4.3(i) and (ii)(c), respectively, that $\overline{a}^\mathbf{p} = EBA$. ■

# 5 Conclusions

This paper considers the allocation problem that arises when the total risk capital withheld by a firm needs to be divided over several portfolios within the firm. We propose an alternative allocation rule that amounts to determining the unique allocation that lexicographically minimizes the expected losses of all sets of portfolios in excess of the risk capital allocated to them. The approach is inspired by the fact that an allocation rule that has received considerable attention in the literature, the Aumann-Shapley value, can yield undesirable results in the sense that some portfolios have large expected excess losses. The alternative allocation rule that we propose mitigates that problem by determining the allocation for which the largest excess is as small as possible. As compared to the Aumann-Shapley value, it has some additional advantages in common with existing optimization approaches, e.g., it does not require differentiability of the risk measure, and it satisfies continuity. Finally, we show that the allocation that we propose can be determined by solving a series of linear programming problems.

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## A Proof of Theorem 3.5

Before we give the actual proof of the continuity of EBA, we first introduce several notions.

**Definition A.1** *A coalition array $b_{01}, b_{02}, b_1, \cdots, b_p$ satisfies the* No Improvement Property[8] *if for all $k = 1, 2, \ldots, p$, it holds that if $y \in \mathbb{R}^N$ satisfies*

$$y_i \geq 0, \qquad \textit{for all } i \in b_{01}, \tag{23}$$

$$y_i \leq 0, \qquad \textit{for all } i \in b_{02}, \tag{24}$$

$$y_N = 0, \tag{25}$$

$$y_S \geq 0, \qquad \textit{for all } S \in b_1 \cup \cdots \cup b_k, \tag{26}$$

*then it satisfies*

$$y_S = 0, \textit{for all } S \in b_1 \cup \cdots \cup b_k. \tag{27}$$

This property means for an allowed risk capital allocation that there is no lexicographic improvement of the allocation possible, without either being outside the set $F(R)$, or worsening the excess of a set of portfolios that already has a high excess.

**Lemma A.2** *Let $R \in \mathcal{R}$, $a \in F(R)$ and $b_{01}, b_{02}, b_1, \cdots, b_p$ is the coalition array belonging to $(R, a)$. Suppose that the implication in the* No Improvement Property *holds for $k = 1, \cdots, p-1$ and all $y \in \mathbb{R}^N$, and that $e(S, a, R) = 0$ for all $S \in b_p$. Then the implication also holds for $k = p$ and all $y \in \mathbb{R}^N$.*

**Proof.** Note that $b_1 \cup \ldots \cup b_p$ contains all sets of portfolios, so in particular all individual portfolios. Let $y \in \mathbb{R}^N$ satisfy conditions (23)-(26) for $k = p$ and let $i \in N$. If $\{i\} \in b_1 \cup \cdots \cup b_{p-1}$, we have $y_i = 0$ since equation (27) holds for $k = p-1$. If $\{i\} \in b_p$, we have that $e(\{i\}, a, R) = 0$. From Lemma 2.11 it follows that $a_i = \rho(X_i)$, so $i \in b_{02}$. However, since equations (24) and (26) for $k = p$ imply that $y_i \leq 0$ and $y_i \geq 0$ respectively, we get that $y_i = 0$. Because $y$ is additive and $y_i = 0$ for all $i \in N$, we find that $y_S = 0$ for all $S \subseteq N$. ■

**Lemma A.3** *Let $R \in \mathcal{R}$. Then the coalition array belonging to $(R, EBA(R))$ satisfies the* No Improvement Property*.*

**Proof.** Let $EBA = EBA(R)$ and $b_{01}, b_{02}, b_1, \cdots, b_p$ the coalition array belonging to $(R, EBA)$. Assume there exists a $y \in \mathbb{R}^N$ and a $k \in \{1, \cdots, p\}$ such that conditions (23)-(26) are satisfied, but for at least one $T \in b_1 \cup \cdots \cup b_k$ it holds $y_T > 0$. Let $k^*$ be the smallest number with this property. Because of this choice of $k^*$, $y_S = 0$ for all $S \in b_1 \cup \cdots \cup b_{k^*-1}$, so $T \in b_{k^*}$. Then for $i \in b_{01}$ we have that $EBA_i = \min_{\omega \in \Omega} X_i(\omega)$ and $y_i \geq 0$, so $EBA_i + t \cdot y_i \geq \min_{\omega \in \Omega} X_i(\omega)$ for positive $t$. For $i \in b_{02}$ we have

[8] The property is called *Property I* in Kohlberg (1971).

that $EBA_i = \rho(X_i)$ and $y_i \leq 0$, so $EBA_i + t \cdot y_i \leq \rho(X_i)$. Furthermore, $y_N = 0$, so $EBA_N + t \cdot y_N = EBA_N = \rho(X_N)$. Because for all $i \notin b_{01} \cup b_{02}$ it holds that $\min_{\omega \in \Omega} X_i(\omega) < EBA_i < \rho(X_i)$, it follows that $EBA + t \cdot y \in F(R)$ for $t$ small enough. For all sets of portfolios $S \in b_1 \cup \cdots \cup b_{k^*}$ we have that

$$\begin{aligned} e(S, EBA + t \cdot y, R) &= E\left[(X_S - (EBA + t \cdot y)_S)^+\right] = E\left[(X_S - EBA_S - t \cdot y_S)^+\right] \\ &\leq E\left[(X_S - EBA_S)^+\right] = e(S, EBA, R), \end{aligned}$$

so no set of portfolios in $b_1 \cup \cdots \cup b_{k^*}$ has a higher excess. We can now distinguish two cases.

**(i)** $e(T, EBA, R) > 0$. Then

$$\begin{aligned} e(T, EBA + t \cdot y, R) &= E\left[(X_T - EBA_T - t \cdot y_T))^+\right] \\ &< E\left[(X_T - EBA_T)^+\right] = e(T, EBA, R), \end{aligned}$$

so $T$ has a lower excess, and thus $EBA + t \cdot y$ is lexicographically better than $EBA$, which is a contradiction.

**(ii)** $e(T, EBA, R) = 0$. Then we have $k^* = p$ since all excesses are at least zero, hence $e(S, EBA, R) = 0$ for all $S \in b_p$. So for $k = 1, \cdots, p-1$ the implications of the *No Improvement Property* hold. But then Lemma A.2 states the implication for $k = p$ also holds, and thus the *No Improvement Property* is satisfied.

■

**Lemma A.4** *If the coalition array that belongs to $(R, a)$ for some $R \in \mathcal{R}$ and $a \in F(R)$ possesses the* No Improvement Property*, then $a = EBA(R)$.*

**Proof.** Let $R \in \mathcal{R}$, $a \in F(R)$. Let $EBA = EBA(R)$. Then $EBA$ is as least as good, lexicographically, as $a$. Let $b_{01}, b_{02}, b_1, \cdots, b_p$ denote the coalition array of $(R, a)$. Obviously we have $EBA_N = \rho(X_N) = a_N$, thus

$$(EBA - a)_N = 0.$$

For all $i \in b_{01}$ we have that $a_i = \min_{\omega \in \Omega} X_i(\omega)$ and $EBA_i \geq \min_{\omega \in \Omega} X_i(\omega)$, so

$$(EBA - a)_i \geq 0.$$

Also, since for all $i \in b_{02}$ we have that $a_i = \rho(X_i)$ and $EBA_i \leq \rho(X_i)$, we have

$$(EBA - a)_i \leq 0.$$

So the inequalities (23), (24) and (25) are satisfied for $(EBA - a) \in \mathbb{R}^N$.
Using induction, we show that $EBA_S = a_S$ for all $S \subset N$. Take $k = 1$. For all $S \in b_1$ we have that $e(S, a, R) \geq e(S, EBA, R)$ because these are by definition the highest excesses.

If $e(S, a, R) = 0$ for all $S \in b_1$, then $e(T, a, R) = e(T, EBA, R) = 0$ for all $T \subset N$ and hence $a = EBA$. If $e(S, a, R) > 0$ for all $S \in b_1$, we use Lemma 2.11 to find

$$(EBA - a)_S \geq 0.$$

According to the *No Improvement Property*, this implies for all $S \in b_1$ that

$$(EBA - a)_S = 0.$$

Now let $1 \leq k < p$. Assume for all $T \in b_1 \cup \ldots \cup b_k$ it holds that $(EBA - a)_T = 0$. Then we have for all $S \in b_{k+1}$ that $e(S, a, R) \geq e(S, EBA, R)$. If $e(S, a, R) = 0$, then $k + 1 = p$ and hence $e(S, EBA, R) = 0$. So, $e(S, a, R) = e(S, EBA, R)$ for all $S$ and hence $a = EBA$. Otherwise, we have that $e(S, a, R) > 0$. Then Lemma 2.11 implies

$$(EBA - a)_S \geq 0,$$

which in turn implies that

$$(EBA - a)_S = 0.$$

So, for all $S \subseteq N$ we find $(EBA - a)_S = 0$, and thus $a = EBA = EBA(R)$. ■
We can now show that EBA satisfies *Continuity*.

**Theorem 3.5** EBA satisfies *Continuity* on $\mathcal{R}$.

**Proof.** Consider the sequence of risk capital allocation problems $R^1, R^2, \ldots$ where $R^k = (X^k, \pi, \rho) \in \mathcal{R}$ for all $k \in \mathbb{N}$, that converges to $R = (X, \pi, \rho) \in \mathcal{R}$. Let $a^k = EBA(R^k)$ for all $k \in \mathbb{N}$ and $a = EBA(R)$. We have to show that $\lim_{k\to\infty} a^k = a$.
Let $i \in N$. Choose $L_i = \inf_{k\in\mathbb{N}}\{\min_{\omega\in\Omega} X_i^k(\omega)\}$ and $U_i = \sup_{k\in\mathbb{N}}\{\rho(X_i^k)\}$. Because the definition of convergence of risk capital allocation problems includes convergence of $X_i^k$ and $\rho(X_i^k)$, this infimum and supremum exist. Define the compact set $H = \{y \in \mathbb{R}^N | L_i \leq y_i \leq U_i, \text{ for all } i \in N\}$. Clearly $F(R^k) \subseteq H$ for all $k \in \mathbb{N}$. Since $H$ is compact, it suffices to show that every converging subsequence $a^{k_1}, a^{k_2}, \ldots$ converges to $a$.
Let $a^{k_1}, a^{k_2}, \ldots$ be a subsequence converging to $g$. Assume without loss of generality that the coalition array belonging to each $(R^k, a^k))$ is $b_{01}, b_{02}, b_1, \cdots, b_p$.[9] According to Lemma A.3, for every element of $a^{k_1}, a^{k_2}, \ldots$ the *No Improvement Property* holds. Denote the coalition array of $g$ by $c_{01}, c_{02}, c_1, \cdots, c_q$. Since for all $S, T \in b_i$, for all $i \leq p$ we have that

$$\begin{aligned} E\left[\left(X_S^{k_i} - a_S^{k_i}\right)^+\right] &= e(S, a^{k_i}, R^{k_i}) \\ &= e(T, a^{k_i}, R^{k_i}) = E\left[\left(X_T^{k_i} - a_T^{k_i}\right)^+\right], \end{aligned}$$

[9]This assumption can be made because the number of sets of portfolios is finite. Thus there are finitely many possibilities for the coalition array of any $(R^k, a^k)$. Since the sequence $a^{k_1}, a^{k_2}, \ldots$ is infinitely long, there exists a subsequence that has the same coalition array $b_{01}, b_{02}, b_1, \ldots, b_p$ for all elements of that sequence, and this subsequence also converges to $g$. Therefore, any results about the convergence of that subsequence can be extended to $a^{k_1}, a^{k_2}, \ldots$.

so because $E[\cdot]$ is a continuous function, we have that $e(S,g,R) = e(T,g,R)$ in the limit. Hence in the coalition array of $g$ we have that all sets of portfolios of $b_i$ are still in the same set, formally $b_i \subseteq c_j$ for some $j \in \{1,\cdots,q\}$. So we have that $c_1,\cdots,c_q$ is a coarsening of $b_1,\cdots,b_p$. Similarly, for $S \in b_i$ and $T \in b_j$ with $i < j$ we have

$$e(S,a^{k_i},R^{k_i}) > e(T,a^{k_i},R^{k_i}),$$

so in the limit we have

$$e(S,g,R) \geq e(T,g,R),$$

meaning this coalition array $c_1,\cdots,c_q$ does not change the order with respect to the sets of portfolios as given by $b_1,\cdots,b_p$. Because the implication of the *No Improvement Property* is weaker for $c_{01},c_{02},c_1,\ldots,c_q$ than for $b_{01},b_{02},b_1,\cdots,b_p$, the *No Improvement Property* also holds for the coalition array of $g$.
It remains to be shown that $g \in F(R)$. For $i \in b_{02}$ we have that $a_i^k = \rho(X_i^k)$ for all $k$ and all $i \in N$, so we see that $g_i = \rho(X_i)$ since both sides of the equality converge. Thus $i \in c_{02}$, so $b_{02} \subseteq c_{02}$. For $i \in b_{01}$ we have that $a_i^k = \min_{\omega\in\Omega} X_i^k(\omega)$, and this equality is preserved if we take the limit, so $i \in c_{01}$ and hence $b_{01} \subseteq c_{01}$.
Because $g \in F(R)$ and the *No Improvement Property* holds, $g = EBA(R)$. Because $a$ is a single point according to Theorem 2.12, we find that $g = a$ and thus any converging subsequence converges to $a$. ■

## B Proofs of Lemma 4.3 and Lemma 4.4

In order to prove Lemmas 4.3 and 4.4, we need an auxiliary lemma.

**Lemma B.1** *Let $k \leq \mathbf{p}$, and let $(\overline{a}^k,\overline{v}_k,\overline{\lambda}_k)$ be an optimal solution to $P_k(\mathbf{b}_1,\cdots,\mathbf{b}_{k-1},\mathbf{e}_1,\cdots,\mathbf{e}_{k-1})$. Then, we have:*

**(i)** $\overline{a}^k \in F(R)$,

**(ii)** $e(S,\overline{a}^k,R) \leq \sum_{\omega\in\Omega}\pi(\omega)\cdot\overline{\lambda}_k(S,\omega)$ *for all* $S \subseteq N$,

**(iii)** $e(S,\overline{a}^k,R) = \mathbf{e}_j$ *for all* $S \in \mathbf{b}_j$, *where* $j < k$.

**Proof.** (i) It follows from (12)-(14) and Definition 2.4 that $\overline{a}^k \in F(R)$.

(ii) Let $S \subseteq N$ and $\omega \in \Omega$. Then from equations (17) and (18) it follows that $\overline{\lambda}_k(S,\omega) \geq \max\{0, X_S(\omega) - \overline{a}_S^k\}$. Thus, we find that

$$\sum_{\omega\in\Omega}\pi(\omega)\cdot\overline{\lambda}_k(S,\omega) \geq \sum_{\omega\in\Omega}\pi(\omega)\cdot\Big(X_S(\omega)-\overline{a}_S^k\Big)^+ = e(S,\overline{a}^k,R). \qquad (28)$$

(iii) Define the vector $y = \overline{a}^k - EBA \in \mathbb{R}^N$. We will show that $y_S = 0$ and thus $\overline{a}^k_S = EBA_S$ for all $S \in \mathbf{b}_1 \cup \cdots \cup \mathbf{b}_{k-1}$. First, because $\overline{a}^k \in F(R)$, it holds that

$$y_i = \overline{a}^k_i - \min_{\omega\in\Omega} X_i(\omega) \geq 0, \qquad \text{for all } i \in \overline{b}^k_{01}, \tag{29}$$

$$y_i = \overline{a}^k_i - \rho(X_i) \leq 0, \qquad \text{for all } i \in \overline{b}^k_{02}, \tag{30}$$

$$y_N = \overline{a}^k_N - EBA_N = 0. \tag{31}$$

Moreover, for all $j < k$ and all $S \in \mathbf{b}_j$ it follows from (ii) and (15) that

$$e(S, \overline{a}^k, R) \leq \sum_{\omega\in\Omega} \pi(\omega) \cdot \overline{\lambda}_k(S,\omega) = \mathbf{e}_j = e(S, EBA, R). \tag{32}$$

Because $k \leq \mathbf{p}$ implies that $\mathbf{b}_1 \cup \cdots \cup \mathbf{b}_j \subseteq \mathbf{b}_1 \cup \cdots \cup \mathbf{b}_{k-1} \neq 2^N$, it holds that $\mathbf{e}_j > 0$. Therefore, it follows from Lemma 2.11 (a)(ii) that $\overline{a}^k_S \geq EBA_S$ for all $S \in \mathbf{b}_1 \cup \cdots \cup \mathbf{b}_{k-1}$, and thus

$$y_S = \overline{a}^k_S - EBA_S \geq 0, \text{for all } S \in \mathbf{b}_1 \cup \cdots \cup \mathbf{b}_{k-1}.$$

Hence, it follows from Lemma A.3 (the *No Improvement Property*) that $y_S = 0$, and thus $\overline{a}^k_S = EBA_S$, for all $S \in \mathbf{b}_1 \cup \ldots \cup \mathbf{b}_{k-1}$. Therefore, we conclude that

$$e(S, \overline{a}^k, R) = e(S, EBA, R) = \mathbf{e}_j, \text{for all } S \in b_j,\ j < k.$$

■

**Proof of Lemma 4.3.** We first show that for any $k \leq \mathbf{p}$, there exists an optimal solution to $\mathrm{P}_k(\mathbf{b}_1, \cdots, \mathbf{b}_{k-1}, \mathbf{e}_1, \cdots, \mathbf{e}_{k-1})$. First, $k \leq \mathbf{p}$ implies $\mathbf{b}_1 \cup \cdots \cup \mathbf{b}_{k-1} \subsetneq 2^N$, and therefore it follows from (16) and (18) that any feasible $(a^k, v_k, \lambda_k)$ satisfies $v_k \geq \sum_{\omega\in\Omega} \pi(\omega) \cdot \lambda_k(S,\omega) \geq 0$. Thus, the objective value is bounded below. It therefore remains to see that the feasible region is non-empty. It can be verified easily that $a^k = EBA$, $v_k = \mathbf{e}_k$, and $\lambda_k(S,\omega) = (X_S(\omega) - EBA_S)^+$, for all $S \in 2^N$, and $\omega \in \Omega$ is a feasible solution.

(i) We know from Lemma B.1 (iii) that

$$e(S, \overline{a}^k, R) = \mathbf{e}_j = e(S, EBA, R), \text{ for all } S \in \mathbf{b}_j, j < k. \tag{33}$$

Because excesses are non-negative by definition, and because $\overline{v}_k = 0$, (16) and Lemma B.1(ii) imply that

$$0 \leq e(S, \overline{a}^k, R) \leq \sum_{\omega\in\Omega} \pi(\omega) \cdot \overline{\lambda}_k(S,\omega) \leq \overline{v}_k = 0, \text{ for all } S \in 2^N \setminus (\mathbf{b}_1 \cup \cdots \cup \mathbf{b}_{k-1}),$$

hence

$$e(S, \overline{a}^k, R) = 0 \leq e(S, EBA, R), \text{ for all } S \in 2^N \setminus (\mathbf{b}_1 \cup \cdots \cup \mathbf{b}_{k-1}). \tag{34}$$

Combining (33) and (34) yields that $e(S, \overline{a}^k, S) \leq e(S, EBA, R)$ for all $S \subseteq N$, and therefore $\overline{a}^k \in F(R)$ is lexicographically at least as good as $EBA$. Because $EBA$ is the unique lexicographical minimum over all risk capital allocations in $F(R)$, this implies that $\overline{a}^k = EBA$.

Finally, $\overline{a}^k = EBA$ and (34) implies that

$$\mathbf{e}_k = e(S, EBA, R) = e(S, \overline{a}^k, R) = 0, \text{for all } S \in \mathbf{b}_k.$$

Because $\overline{v}_k = \mathbf{0}$, this yields $\overline{v}_k = \mathbf{e}_k$.

(ii)(a) Consider the vector $y = \overline{a}^k - EBA \in \mathbb{R}^N$. It follows from the proof of Lemma B.1 that

$$y_i \geq 0, \qquad \text{for all } i \in \overline{b}_{01}^k, \tag{35}$$

$$y_i \leq 0, \qquad \text{for all } i \in \overline{b}_{02}^k, \tag{36}$$

$$y_N = 0, \tag{37}$$

and

$$y_S = 0, \text{ for all } S \in \mathbf{b}_1 \cup \cdots \cup \mathbf{b}_{k-1}. \tag{38}$$

Moreover, because $(EBA, \mathbf{e}_k, ((X_S(\omega) - EBA_S)^+)_{S \subseteq N, \omega \in \Omega})$ is feasible for $P_k(\mathbf{b}_1, \cdots, \mathbf{b}_{k-1}, \mathbf{e}_1, \cdots, \mathbf{e}_{k-1})$, we have

$$\overline{v}_k \leq \mathbf{e}_k. \tag{39}$$

Therefore, it follows from Lemma B.1 and (16) that for all $S \in \mathbf{b}_k$ we have that

$$e(S, \overline{a}^k, R) \leq \sum_{\omega \in \Omega} \pi(\omega) \cdot \overline{\lambda}_k(S, \omega) \leq \overline{v}_k \leq \mathbf{e}_k = e(S, EBA, R), \tag{40}$$

and Lemma 2.11 implies that $\overline{a}_S^k \geq EBA_S$. Therefore,

$$y_S = \overline{a}_S^k - EBA_S \geq 0, \text{for all } S \in \mathbf{b}_k.$$

Then, it follows from Lemma A.3 (the *No Improvement Property* for $EBA$) implies that $y_S = 0$ for all $S \in \mathbf{b}_1 \cup \cdots \cup \mathbf{b}_k$.

(b) We show by contradiction that $\mathbf{b}_k \subseteq \overline{b}_k$. Suppose there exists an $S \in \mathbf{b}_k$ such that $S \notin \overline{b}_k$. Then it follows from (16), (19), and (39) that $e(S, \overline{a}^k, R) < \overline{v}_k \leq \mathbf{e}_k = e(S, EBA, R)$, and Lemma 2.11 implies that $EBA_S < \overline{a}_S^k$. This contradicts $EBA_S = \overline{a}_S^k$. Thus there does not exist an $S \in \mathbf{b}_k$ such that $S \notin \overline{b}_k$, hence $\mathbf{b}_k \subseteq \overline{b}_k$.

Finally, we know from part (a) that $EBA_S = \overline{a}_S^k$ for all $S \in \mathbf{b}_k$. This implies that

$$e(S, \overline{a}^k, R) = e(S, EBA, R) = \mathbf{e}_k > 0, \quad \text{for all } S \in \mathbf{b}_k.$$

Combined with (40), this yields $\overline{v}_k = \mathbf{e}_k$. ∎

In order to prove Lemma 4.2, we need the following lemma.

**Lemma B.2** *If $z \in \mathbb{R}^N$ satisfies*

$$z_i \geq 0, \qquad \textit{for all } i \in \overline{b}^k_{01}, \tag{41}$$
$$z_i \leq 0, \qquad \textit{for all } i \in \overline{b}^k_{02}, \tag{42}$$
$$z_N = 0, \tag{43}$$
$$z_S \geq 0, \qquad \textit{for all } S \in \mathbf{b}_1 \cup \ldots \cup \mathbf{b}_k, \tag{44}$$

*then it satisfies*

$$z_S = 0, \textit{for all } S \in \mathbf{b}_1 \cup \ldots \cup \mathbf{b}_k. \tag{45}$$

**Proof.** We show by contradiction that for $\overline{b}^k_{01}, \overline{b}^k_{02}, \mathbf{b}_1, \cdots, \mathbf{b}_k$ the above implication holds. Suppose there exists a vector $z \in \mathbb{R}^N$ such that conditions (41)-(44) hold, but the implication in equation (45) is violated. Let $j$ be the smallest index such that there is an $S \in \mathbf{b}_j$ with $z_S > 0$, and let $T \in \mathbf{b}_j$ be a set of portfolios such that $z_T > 0$. Observe that, similar to the proof of Lemma A.3, we have that $\overline{a}^k + t \cdot z \in F(R)$, for $t$ small enough.
We show that the vector $y = \overline{a}^k + t \cdot z - EBA \in \mathbb{R}^N$ violates the *No Improvement Property* for $EBA$. Because $\overline{a}^k + t \cdot z \in F(R)$, it holds that

$$y_i = \overline{a}^k_i + t \cdot z_i - \min_{\omega \in \Omega} X_i(\omega) \geq 0, \qquad \text{for all } i \in \overline{b}^k_{01}, \tag{46}$$
$$y_i = \overline{a}^k_i + t \cdot z_i - \rho(X_i) \leq 0, \qquad \text{for all } i \in \overline{b}^k_{02}, \tag{47}$$
$$y_N = \overline{a}^k_N + t \cdot z_N - EBA_N = 0. \tag{48}$$

Because by construction $z_S = 0$ for all $S \in \mathbf{b}_1 \cup \cdots \cup \mathbf{b}_{j-1}$, it follows that $y_S = \overline{a}^k_S - EBA_S$ for all $S \in \mathbf{b}_1 \cup \cdots \cup \mathbf{b}_{j-1}$. Combined with Lemma 4.3(ii)(a), this yields

$$y_S = 0, \text{for all } S \in \mathbf{b}_1 \cup \cdots \cup \mathbf{b}_{j-1}.$$

Moreover, (44) implies that for all $S \in \mathbf{b}_j$ we have

$$y_S = \overline{a}^k_S + t \cdot z_S - EBA_S = t \cdot z_S \geq 0.$$

and, because by construction $z_T$,

$$y_T = t \cdot z_T > 0.$$

This contradicts the fact that $EBA$ satisfies the *No Improvement Property.* ■

**Proof.** We first show that for any given $b_1, \cdots, b_{k-1}, d \subseteq 2^N$, and $b_{01}, b_{02} \subseteq 2^N$, there exists an optimal solution to $\mathrm{Q}_k(b_{01}, b_{02}, b_1, \cdots, b_{k-1}, d)$. Because $y_i = 0$ for all $i \in N$ is feasible, the value of the objective function is always at least zero, i.e., $f_{\mathrm{Q}_k(b_{01}, b_{02}, b_1, \cdots, b_{k-1}, d)} \geq 0$. Thus the feasible region is non-empty, and since also $y_i \leq 1$ for all $i \in N$, the value of the objective function is bounded and thus this problem has a solution.

i) Any feasible solution of $Q_k(\overline{b}_{01}^k, \overline{b}_{02}^k, \mathbf{b}_1, \cdots, \mathbf{b}_{k-1}, d)$ satisfies (41)-(44). Therefore, it follows from Lemma B.2 that if $y \in \mathbb{R}^N$ is a feasible solution to $Q_k(\overline{b}_{01}^k, \overline{b}_{02}^k, \mathbf{b}_1, \cdots, \mathbf{b}_{k-1}, d)$, then $y_S = 0$ for all $S \in \mathbf{b}_k$. Since this holds true for any feasible solution, it holds true for the optimal solution $\overline{y}$.

ii) Suppose $\overline{v}_k > 0$, and let $d$ be such that $\mathbf{b}_k \subset d \subseteq \overline{b}_k$ . We first show that $f_{Q_k(\overline{b}_{01}^k, \overline{b}_{02}^k, \mathbf{b}_1, \cdots, \mathbf{b}_{k-1}, d)} = 0$ implies $\mathbf{b}_k = d$. Suppose $d \neq \mathbf{b}_k$. Let $y = \alpha \cdot (EBA - \overline{a}^k) \in \mathbb{R}^N$, with $\alpha > 0$ such that

$$y_i \leq 1, \text{ for all } i \in N.$$

First, Lemma 4.3 implies that

$$y_S = \alpha \cdot (EBA_S - \overline{a}_S^k) = 0, \text{ for all } S \in \mathbf{b}_1 \cup \cdots \cup \mathbf{b}_k.$$

Moreover, $EBA \in F(R)$ implies

$$y_i = \alpha \cdot \left[ EBA_i - \min_{\omega \in \Omega} X_i(\omega) \right] \geq 0, \qquad \text{for all } i \in \overline{b}_{01}^k, \tag{49}$$

$$y_i = \alpha \cdot [EBA_i - \rho(X_i)] \leq 0, \qquad \text{for all } i \in \overline{b}_{02}^k, \tag{50}$$

$$y_N = \alpha \cdot \left[ EBA_N - \overline{a}_N^k \right] = 0. \tag{51}$$

Next, suppose there exists a $S \in d \backslash \mathbf{b}_k$. Then, because $d \subseteq \overline{b}_k$ and because it follows from Lemma 4.3 that $\overline{v}_k = \mathbf{e}_k$, so that $\overline{b}_k \subseteq 2^N \backslash (\mathbf{b}_1 \cup \cdots \cup \mathbf{b}_{k-1})$, it holds that $S \notin \mathbf{b}_1 \cup \cdots \cup \mathbf{b}_k$, so that

$$e(S, EBA, R) < \mathbf{e}_k. \tag{52}$$

Moreover, we know from Lemma 4.3 that $\overline{v}_k = \mathbf{e}_k$. Then, $S \in d \subseteq \overline{b}_k$ and (19) imply that

$$e(S, \overline{a}^k, R) = \overline{v}_k = \mathbf{e}_k > 0. \tag{53}$$

Combined with (52), this implies $e(S, \overline{a}^k, R) > e(S, EBA, R)$, and thus $\overline{a}_S^k < EBA_S$. Hence

$$y_S > 0, \text{for all } S \in d \backslash \mathbf{b}_k.$$

Thus, $y$ is a feasible solution to $Q_k(\overline{b}_{01}^k, \overline{b}_{02}^k, \mathbf{b}_1, \cdots, \mathbf{b}_{k-1}, d)$, and

$$\sum_{S \in d} y_S = \sum_{S \in d \backslash \mathbf{b}_k} y_S + \sum_{S \in \mathbf{b}_k} y_S = \sum_{S \in d \backslash \mathbf{b}_k} y_S + 0 > 0.$$

Therefore, $f_{Q_k(\overline{b}_{01}^k, \overline{b}_{02}^k, \mathbf{b}_1, \cdots, \mathbf{b}_{k-1}, d)} = \sum_{S \in d} \overline{y}_S \geq \sum_{S \in d} y_S > 0$, which contradicts $f_{Q_k(\overline{b}_{01}^k, \overline{b}_{02}^k, \mathbf{b}_1, \cdots, \mathbf{b}_{k-1}, d)} = 0$. Thus $\mathbf{b}_k = d$.

Next, suppose $\mathbf{b}_k = d$ and let $\overline{y} \in \mathbb{R}^N$ be an optimal solution to $Q_k(\overline{b}_{01}^k, \overline{b}_{02}^k, \mathbf{b}_1, \cdots, \mathbf{b}_{k-1}, d)$. Then,

$$f_{Q_k(\overline{b}_{01}^k, \overline{b}_{02}^k, \mathbf{b}_1, \cdots, \mathbf{b}_{k-1}, d)} = \sum_{S \in d} \overline{y}_S = \sum_{S \in \mathbf{b}_k} \overline{y}_S = 0,$$

where the last equality follows from part (i). ■