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PRIVATE INFORMATION, TRANSFERABLE UTILITY, AND THE CORE

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Abstract. Considered here are transferable-utility, coalitional production or market games, featuring differently informed players. It is assumed that personalized contracts must comply with idiosyncratic information. The setting may create two sorts of shadow prices: one for material endowments, the other for knowledge. Focus is on specific, computable solutions that are generated by such prices and belong to the private core. Solutions of that sort obtain under standard regularity assumptions.
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1. Introduction

Economics deals with various ways to handle scarcity. Prominent problems, and corresponding institutions, concern production, valuation or allocation of limited *material* items. Equally important issues revolve though, around acquisition, distribution and sharing of *information*. The latter object is, however, just like other more tangible commodities, often unevenly distributed, scarce, or quite simply lacking.

Efficient instruments that handle lacking but *symmetric* information come as contracts offered say, by insurers or financial bodies. In contrast, presence of *asymmetric* information frequently impedes efficiency, eliminating maybe good opportunities for concerted actions, bilateral exchange, or mutual insurance.

That observation has inspired many studies on contracts under differential knowledge about the state of the world. Main concerns were always with efficiency, incentive compatibility, and existence of appropriate solutions. In particular, the appropriateness and properties of various core versions have been scrutinized in this context.[1] This paper pursues that tack, placing the *private core* at center stage and specializing to transferable utility.

Motivation stems from instances where all parties worship maximization of quasi-linear utility or monetary payoff. For the argument, we construe these as profit-maximizing producers, each willing to accept side-payments. Technologies, endowments, and informations differ across agents. Everyone acts, more or less, in three

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[1] Studies include [1], [2], [11], [19], [22], [25], [26], [27], [38], [39], [40] and references therein.

intertwined roles: as producer, resource owner, and "informer." It appears natural therefore, that contracts pay each party in three corresponding capacities.

We inquire whether there exist acceptable and feasible payment schemes of such sort. A leading maxim says that scarcity commands a price. Another guideline tells that prices of private, perfectly divisible, material commodities typically emerge as shadow items, brought to the fore by differential calculus. There is however, no direct counterpart concerning marginal amounts of information. A rich theory notwithstanding [8], [24], to measure information content still seems difficult - and to divide it even harder.

These pessimistic observations seemingly preclude differentiation, classical or not, as a chief vehicle. Closer scrutiny shows however, that *Lagrangian duality*, already known to furnish standard shadow prices, may help to evaluate information as well.[2] Instrumental to this end are multipliers that relax information constraints.

The prospect of such relaxation motivates our inquiry on several grounds. First, since dual problems often come more tractable than the primal version, one may more easily use them to compute or display *explicit* core solutions. Another bonus of duality is that questions about existence of equilibrium *prices* can be divorced from those concerning *allocations*. Further, to test intuition, it's worthwhile to have handy some simple or practical instances. In particular, one may want to detect information rent if any. Such rent could accrue to totally unproductive, quite poor, but complementary informed parties. Finally, but admittedly on a more technical note, it's interesting to see precisely where, how, and to what degree the availability of price-generated imputations depends on convex preferences.

As always in game theory, whether cooperative or not, it matters much who is informed about what and when. Equally crucial is the protocol that prescribes how play should proceed. Since received models differ on these points, several solution concepts have come up [19], [27]. Our setting is particularly simple. It comprises merely two stages. Everybody commits plans ex ante, and private information obtains only ex post. The absence of an interim stage, and the necessity of maintaining material balance, ensures that actions comply with plans and information. Unlike [18], [20] incentive compatibility will cause no concern here.

The paper addresses several groups of readers. One comprises economists and game theorists who wish to analyze, compute or display some "quantifiable" effects of differential information. Another group include actuaries and finance theorists dealing with differentially informed agents. Also addressed are mathematicians and operations researchers interested in how convex analysis applies to parts of game theory.

Sections 2&3 formalize the setting and the game. Section 4 considers core solutions generated by shadow prices - as illustrated in Section 5. Section 6 records some properties of solutions. Sections 7&8 deal with variational stability and non-transferable utility. Section 9 collects examples, and Section 10 concludes.

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[2] This observation has long been central in stochastic programming; see e.g. [13], [14], [16].

2. Formulation

The subsequent model requires several sorts of data, presented next. Some readers might contend with perusing this section, and return to details when needed.

Players form a finite, fixed set I of economic agents, each construed as a producer who aims at maximal expected profit.

Uncertainty prevails as to which scenario will materialize next. These constitute a comprehensive set S of mutually exclusive states. All parties understand that *one* $s \in S$ will come about in a while. To simplify some technical and mathematical issues assume S finite.[3]

The occurrence of the state separates time in two periods, called ex ante and ex post. Ex ante, decisions are committed in face of non-negligible uncertainty. Ex post, when a state has occurred, players receive private information, honor contracts, and collect proceeds. The realized state need never be identified, and information can remain private.

Information ex post, about the realized state, may differ in degree or nature among players. For example, when s is a vector, various agents may get to see different components. Formally, at the second stage, individual i can only ascertain to which part $P_i(s)$ in a prescribed *partition* $\mathbb{P}_i$ of S the true state s belongs.

For the subsequent analysis let $\mathcal{F}_i$ denote the field formed by taking unions of parts $P_i \in \mathbb{P}_i$. More generally, a non-empty family $\mathcal{F}$ of subsets in S is declared a *field* if stable under complements and unions. Minimal members of $\mathcal{F}$ are referred to as *atoms*. A field $\mathcal{F}$ embodies coarser information than the (finer) field $\hat{\mathcal{F}}$ iff $\mathcal{F} \subsetneq \hat{\mathcal{F}}$.

The polar instance of symmetric information has all fields $\mathcal{F}_i$ equal. Partitions then coincide across players, and ex post everybody knows that merely one and the same part of the state space will be worth caring about. This case is covered below but not especially considered - except as a good case for mutual insurance.

Commodity bundles are codified as vectors in a standard Euclidean space $\mathcal{X}$ with coordinates indexed by the goods in question. A *contingent commodity bundle* $x(\cdot)$ is a mapping $s \in S \mapsto x(s) \in \mathcal{X}$. When confusion cannot result, we write simply x instead of $x(\cdot)$. Let $\mathbb{X} := \mathcal{X}^S$ denote the space of all contingent commodity bundles. $x \in \mathbb{X}$ is declared *adapted* - or *measurable* with respect - to a field $\mathcal{F}$, and we write $x \in \mathcal{F}$, iff x is constant on each atom of $\mathcal{F}$.

Agent i has *endowment* $e_i \in \mathcal{F}_i$. Construe $e_i(s) \in \mathcal{X}$ as the resource bundle owned by him in state s. If e_i, as conceived ex ante, were not adapted to $\mathcal{F}_i$, the latter should be refined.

Given any function f defined on S, its "level sets" constitute a partition that generates a minimal field $\mathcal{F}(f)$ with respect to which f is adapted. Thus we re-

[3] We take care though, to state things so as facilitate generalizations.

quire that $\mathcal{F}(e_i) \subseteq \mathcal{F}_i$. A strict inclusion is acceptable. It would mean that i has at hand more information than imbedded in e_i. We shall not suppose that he observes $e_I(s) := \sum_{i \in I} e_i(s)$ ex post. Clearly, if he does, then $\mathcal{F}(e_I) \subseteq \mathcal{F}_i$.

The objective of player i is to maximize a proper monetary payoff $\pi_i(x_i)$ when $x_i \in \mathcal{F}_i$. We allow π_i to take the value $-\infty$. This device accounts for constraint violation by means of an infinite penalty. It serves as Occam's razor, allowing us to focus on essential objectives - and to shy away from particular features. We refrain therefore, from spelling out what feasibility might mean in full and quite varied detail. Emphasized though, is that $\pi_i(\cdot)$ incorporates all constraints but $x_i \in \mathcal{F}_i$. The latter is singled out for two reasons. First, the only treaties agent i can credibly commit to, are constant across contingencies he cannot discriminate. Second, only such treaties are enforceable. In short, imperfect information makes for incomplete contracts or partial commitments.[4]

Accommodated as a prominent instance is expected payoff

$$\pi_i(x_i) = \sum_{s \in S} \Pi_i(s, x_i(s))\mu(s) \tag{1}$$

featuring a state-dependent "integrand" $\Pi_i(s, \chi)$ and a positive probability measure μ. Because $x_i \in \mathcal{F}_i$, we may replace $\Pi_i(\cdot, \chi)$ with its adapted version $E\left[\Pi_i(\cdot, \chi) \,|\mathcal{F}_i\right]$. Also, if necessary, one may modify Π_i to have a measure μ_i that better mirrors agent i's beliefs.

Exchange and sharing of commodities is presumed frictionless and free of restrictions. That is, all goods are perfectly divisible and transferable. So, ex ante *coalition* $C \subseteq I$ might allocate any $x_i \in \mathcal{F}_i$ to $i \in C$ provided $\pi_i(x_i) > -\infty$ and

$$\sum_{i \in C} x_i = e_C := \sum_{i \in C} e_i. \tag{2}$$

If coalition C were indeed to form, we envisage that this sort of agreement comes as an ensemble of *contracts*, one for each member $i \in C$, specifying, *in terms verifiable by him*, precisely what bundle $x_i(s)$ he is entitled to in state s.

Denote by $\vee_{i \in C} \mathcal{F}_i$ the smallest field that contains all $\mathcal{F}_i, i \in C$. Evidently, both sides of (2) are adapted to $\vee_{i \in C} \mathcal{F}_i$. It may well happen though, that $\mathcal{F}(e_C)$ is strictly coarser than $\vee_{i \in C} \mathcal{F}_i$. Indeed, it is interesting, and not precluded, that $\mathcal{F}(e_C)$ be totally uninformative, meaning that e_C is a constant.

Anyway, pooling mechanism (2) has two economic advantages. First, it allows resource transfers across C. Second, diversity of information permits greater flexibility

[4] Two competing models deviate at this point. In one, all contracts are written on *common information* $\wedge_{i \in I} \mathcal{F}_i = \cap_{i \in I} \mathcal{F}_i$, this leaving fairly few or slim possibilities for mutual agreements. In the other setting, all information is pooled into $\vee_{i \in I} \mathcal{F}_i$. But then, quite likely, some parties must commit ex ante to terms they hardly can verify ex post.

in adapting pro-actively to various contingencies.

Prices on contingent commodity bundles are linear functionals, mapping $\mathbb{X}$ into $\mathbb{R}$. These functionals constitute a vector space $\mathbb{X}^*$ called *dual* to $\mathbb{X}$. Presence of a *star* henceforth signals that the object in point is a price - or an operator on such items.

It's convenient to have an explicit representation of members $x^* \in \mathbb{X}^*$. For that purpose fix hereafter a probability measure μ on S with $\mu(s) > 0$ for all $s \in S$. In particular, if some positive μ reflects prior and common probabilistic beliefs, then that μ becomes a most natural candidate to use. μ generates a positive definite, bilinear form

$$\langle x^*, x \rangle := \sum_{s \in S} x^*(s) \cdot x(s)\mu(s)$$

on $\mathbb{X}$, the dot indicating the standard (or another) inner product on the underlying commodity space $\mathcal{X}$. By the Riez representation theorem a dual vector corresponds to a unique linear form $\langle x^*, \cdot \rangle$. With this sort of identification the space at hand becomes self-dual; that is: $\mathbb{X} = \mathbb{X}^*$.

Expectations and conditional versions of these are essential below. The positive probability measure μ, just mentioned, gives rise to an *unconditional expectation* $E : \mathbb{X} \to \mathcal{X}$ by $Ex := \sum_{s \in S} x(s)\mu(s)$. Further, for each field $\mathcal{F}$ in S, generated by a partition $\mathbb{P}$, there is a *conditional expectation* operator $E[\cdot |\mathcal{F}] : \mathbb{X} \to \mathbb{X}$, defined by

$$\mu(P)E[x|\mathcal{F}](s) := E[\mathbf{1}_P x] \quad \text{for each state } s \in P \text{ and every part } P \in \mathbb{P}.$$

Here the indicator $\mathbf{1}_P$ equals 1 on part P and 0 elsewhere. Since by assumption $\mu(P) > 0$, the customary formula applies:

$$E[x|\mathcal{F}](s) = \frac{E[\mathbf{1}_P x]}{\mu(P)} = \sum_{s' \in P} x(s')\frac{\mu(s')}{\mu(P)} \quad \text{when } s \in P \in \mathbb{P}.$$

Because $E[x|\mathcal{F}]$ is piecewise constant, it may be construed as a mapping

$$P \in \mathbb{P} \mapsto E[x|\mathcal{F}](P) := \sum_{s' \in P} x(s')\mu(s')/\mu(P).$$

Most important, x is $\mathcal{F}$-measurable iff $x = E[x|\mathcal{F}]$. In particular, writing $E_i := E[\cdot |\mathcal{F}_i]$, we see that $x_i \in \mathcal{F}_i$ amounts to

$$x_i = E_i x_i. \tag{3}$$

Note that $E_\mathcal{F} := E[\cdot |\mathcal{F}]$, when seen as a linear operator from $\mathbb{X}$ to $\mathbb{X}$, has a standard $S \times S$ real matrix representation with $\frac{\mu(s')}{\mu(P)}$ in entry $(s, s') \in S \times S$ when $s, s' \in P$, and 0 otherwise.

To operator $E_\mathcal{F} : \mathbb{X} \to \mathbb{X}$ is associated a *transpose* $E^*_\mathcal{F} : \mathbb{X}^* \to \mathbb{X}^*$, implicitly defined by $\langle E^*_\mathcal{F} x^*, x \rangle = \langle x^*, E_\mathcal{F} x \rangle$ for all $x^* \in \mathbb{X}^*, x \in \mathbb{X}$. To identify $E^*_\mathcal{F}$ explicitly, let B be a basis for the linear space $\mathcal{X}$ and pick any $b \in B$. Denote by $x_{bs} \in \mathbb{X}$ the vector that has b in component s, and 0 elsewhere. That is, $x_{sb} : S \to \mathcal{X}$ equals $\delta_{s's}b$ where $\delta_{s's} = 1$ when $s' = s$, and 0 otherwise. Let $P(s)$ be the part of $\mathbb{P}$ that contains s. Note that

$$E_{\mathcal{F}} x_{sb}(s') = \begin{cases} b\mu(s)/\mu(P) & \text{when } s' \in P(s) \\ 0 & \text{otherwise.} \end{cases}$$

So, for any $x^* \in \mathbb{X}^*$ it holds $E^*_{\mathcal{F}} x^*(s) \cdot b\mu(s) =$

$$\langle E^*_{\mathcal{F}} x^*, x_{sb} \rangle = \langle x^*, E_{\mathcal{F}} x_{sb} \rangle = \sum_{s' \in P(s)} x^*(s') \cdot b \frac{\mu(s)}{\mu(P)} \mu(s') = E_{\mathcal{F}} x^*(s) \cdot b\mu(s),$$

from which it follows that $E^*_{\mathcal{F}} = E_{\mathcal{F}}$. Thus $E_{\mathcal{F}}$ is symmetric. Also,

$$\langle E_{\mathcal{F}} x^*, x \rangle = \langle x^*, E_{\mathcal{F}} x \rangle = \sum_{P \in \mathbb{P}} (E_{\mathcal{F}} x^*)(P) \cdot (E_{\mathcal{F}} x)(P) \mu(P).$$

3. THE GAME AND CORE SOLUTIONS

Every party knows all triples $(\pi_i, \mathbb{P}_i, e_i), i \in I$, ex ante.[5] Since payoffs and resources are transferable, the prescribed data generates a *transferable-utility, cooperative game* in which coalition $C \subseteq I$ can aim at getting value $\geq$

$$v_C := \sup \left\{ \sum_{i \in C} \pi_i(x_i) : \sum_{i \in C} x_i = e_C \text{ and } x_i = E_i x_i \text{ for all } i \in C \right\}. \tag{4}$$

Here $v_\emptyset = 0$, and, as before, $e_C := \sum_{i \in C} e_i$ is shorthand for the aggregate endowment held by coalition C. Note that "excess demand" $x_i - e_i$ of any agent i is adapted to his information. Also note that problem (4) is linearly constrained. This feature is most convenient for theoretical analysis and practical computation. In particular, the Kuhn-Tucker optimality conditions come without any constraint qualification.

The economic attractions of pooling objectives and endowments, as done in (4), are evident: The most efficient producers can utilize scarce resources, and complementary production factors can be brought together. Formally, the advantages of coordination reflect in superadditive values:

$$v_{C_1 \cup C_2} \geq v_{C_1} + v_{C_2} \quad \textit{whenever } C_1, C_2 \subset I \quad \textit{are disjoint.}$$

Remark. When each $\pi_i(e_i) \geq 0$, the set function $C \mapsto v_C$ becomes monotone whence a *capacity* [10], [29]. A capacity is called *convex* iff $v_{C_1 \cup C_2} + v_{C_1 \cap C_2} \geq v_{C_1} + v_{C_2}$. The marginal value $v_{C \cup i} - v_C$ of an outside player i joining coalition C then increases with C. Instance (4) is however, not generally convex. To see this, follow [31], let $\mathcal{F}_i = \{\varnothing, S\}$, and posit

$$\pi_i(x_i) := \sup \{ \langle \bar{y}, y_i \rangle : Ay_i = x_i, y_i \geq 0 \}$$

[5] In particular, players cannot offhand redistribute property or discard endowments; see [5], [21].

where A maps an ordered Hilbert space $\mathbb{Y}$ linearly into $\mathbb{X}$, and $\bar{y} \in \mathbb{Y}^*$. Then

$$v_C = v(e_C) := \sup\{\langle \bar{y}, y\rangle : Ay = e_C,\ y \geq 0\} \tag{5}$$

with $v_C = -\infty$ whenever linear program (5) is infeasible. Since the reduced function $e \mapsto v(e)$ so defined is concave, its generalized differential $\partial v(\cdot)$ is monotone decreasing [7]. This points to possible disadvantages of joining a coalition last. □

Anyway, whenever somebody joins a coalition he may bring *three* benefits. First, if endowed, he adds to the aggregate holding. Second, if efficient, he expands the joint production capacity. Third, if additionally informed, he makes for more flexible exchanges.

Given the characteristic function $C \mapsto v_C$, defined in (4), we want to "solve" the game, using the *core* as solution concept. Specifically, a payment pattern $(u_i) \in \mathbb{R}^I$ is said to reside in the *private core* iff

Pareto efficient:	$\sum_{i\in I} u_i = v_I,$	and
stable against blocking:	$\sum_{i\in C} u_i \geq v_C$	for all $C \subset I$.

A chief concern is that the core could be empty. Put differently: the question is whether the game is *balanced* or not? In that regard the following result can be established along well known lines; see [30]:

Proposition 3.1. (**Balanced games** [35]) *Suppose all payoff functions $\pi_i(\cdot)$ are concave. Then the core is non-empty in every subgame which involves a player community $C \subseteq I$ that has finite value v_C. In particular, when v_C is finite for all $C \subseteq I$, the entire game becomes totally balanced.* □

4. Price-Supported Core Solutions

Proposition 3.1 isn't quite satisfying. It just deals with existence, and it presumes concave payoffs. Further, one would want *computable* solutions, brought out in constructive or explicit manner. And most important, Proposition 3.1 doesn't indicate how cooperation could come about.

These objections make us envision exchange markets where the agents meet anonymously and sign price-mediated contracts. Accordingly, consider problem (4) from a dual and price-oriented vantage-ground. As usual, associate a multiplier (price) vector $x^* \in \mathbb{X}^*$ to constraint (2) and a similar vector $x_i^* \in \mathbb{X}^*$ to constraint (3). Related to problem (4) is thus a standard *Lagrangian*

$$\begin{aligned} L_C(\vec{x}, \vec{x}^*) :&= \textstyle\sum_{i\in C}\{\pi_i(x_i) + \langle x^*, e_i - x_i\rangle + \langle x_i^*, E_i x_i - x_i\rangle\} \\ &= \textstyle\sum_{i\in C}\{\pi_i(x_i) - \langle x^* + x_i^* - E_i x_i^*, x_i\rangle + \langle x^*, e_i\rangle\}\,. \end{aligned}$$

Here $\vec{x} := (x_i)$, $\vec{x}^* := (x^*, x_i^*, i \in I)$, and $\langle x_i^*, E_i x_i\rangle = \langle E_i^* x_i^*, x_i\rangle = \langle E_i x_i^*, x_i\rangle$. The interpretation of L_C is commonplace but worth recalling all the same. Suppose individual $i \in C$ could add a perturbation $\Delta e_i \in \mathbb{X}$ to his endowment at cost $\langle x^*, \Delta e_i\rangle$.

Upon doing so constraint (2) would take the relaxed form

$$\sum_{i\in C} x_i = \sum_{i\in C}(e_i + \Delta e_i). \tag{6}$$

Further imagine that instead of (3) member $i \in C$ could face the looser constraint

$$x_i = E_i x_i + \Delta x_i, \tag{7}$$

with $\Delta x_i \in \mathbb{X}$ chosen freely but at extra cost $\langle x_i^*, \Delta x_i\rangle$. In that relaxed setting coalition C could achieve overall payoff

$$L_C(\vec{x}, \vec{x}^*) = \sup_{(\Delta e_i, \Delta x_i), i\in C} \left\{ \sum_{i\in C} [\pi_i(x_i) - \langle x^*, \Delta e_i\rangle - \langle x_i^*, \Delta x_i\rangle] : (6) \text{ \& } (7) \text{ hold} \right\}.$$

Plainly, the more freedom in choosing perturbations, the richer in detail the corresponding price regimes. For such reasons we face, right here, a crucial modelling choice, namely: *Should the perturbed version* (6) *of an equation that, in essence, accounts for material balances, also embody extra information*? We choose to block this avenue, our motivation being to divorce payments for tangible endowments from those concerning information. Accordingly, and because the *grand coalition* $C = I$ is of chief interest, we insist from here on that *any endowment price* x^* *be* $\mathcal{F}(e_I)$*-measurable.*

After these considerations declare now $\vec{x}^* = (x^*, x_i^*, i \in I)$ a *shadow price* or *Lagrange multiplier vector* iff, under that price regime, access to a competitive market for

$$\begin{cases} \text{material perturbations:} & \Delta e = E\left[\Delta e \,|\mathcal{F}(e_I)\right] \quad \text{and} \\ \text{informational perturbations:} & \Delta x_i, i \in I, \end{cases}$$

offers the grand coalition no advantage. Formally and more simply, call $\vec{x}^*$ a shadow price iff

$$v_I \geq \sup_{\vec{x}} L_I(\vec{x}, \vec{x}^*). \tag{8}$$

In mathematical terms, $\vec{x}^*$ realizes v_I as the *saddle value* of L_I. To wit, $\vec{x}^*$ qualifies as shadow price iff

$$v_I \geq \sup_{\vec{x}} L_I(\vec{x}, \vec{x}^*) \geq \inf_{\vec{x}^*} \sup_{\vec{x}} L_I(\vec{x}, \vec{x}^*) \geq \sup_{\vec{x}} \inf_{\vec{x}^*} L_I(\vec{x}, \vec{x}^*) \geq v_I.$$

To bring out economic and game-theoretic implications of such objects let

$$f^{(*)}(x^*) := \sup\left\{f(x) - \langle x^*, x\rangle : x \in \mathbb{X}\right\}$$

denote the *conjugate* of a proper function $f : \mathbb{X} \to \mathbb{R} \cup \{-\infty\}$. Clearly, $f^{(*)}(x^*)$ is lower semicontinuous convex, and it records the profit that accrues to a producer who enjoys revenue function $f(\cdot)$ and pays price x^* for inputs. Separable instances $f(x) = \sum_{s\in S} f_s(x(s))\mu(s)$ give

$$f^{(*)}(x^*) = \sum_{s\in S} f_s^{(*)}(x^*(s))\mu(s).$$

In terms of conjugates the additive separability of L_C implies that

$$\sup_{\vec{x}} L_C(\vec{x}, \vec{x}^*) = \sum_{i\in C} \left\{ \pi_i^{(*)}(x^* + x_i^* - E_i x_i^*) + \langle x^*, e_i \rangle \right\}. \quad (9)$$

We can now state a chief result forthwith:

Theorem 4.1. (**Price-supported core solutions**) *Each shadow price* $\vec{x}^* = (x^*, x_i^*, i \in I)$ *generates a solution* $(u_i) \in \mathbb{R}^I$ *in the private core by the formula*

$$u_i = u_i(\vec{x}^*) := \pi_i^{(*)}(x^* + x_i^* - E_i x_i^*) + \langle x^*, e_i \rangle. \quad (10)$$

Proof. For any coalition $C \subseteq I$ and any multiplier vector $\vec{x}^*$ it holds via (9) that

$$\begin{aligned} \sum_{i\in C} u_i &= \sum_{i\in C} \left\{ \pi_i^{(*)}(x^* + x_i^* - E_i x_i^*) + \langle x^*, e_i \rangle \right\} = \sup_{\vec{x}} L_C(\vec{x}, \vec{x}^*) \\ &\geq \inf_{\vec{x}^*} \sup_{\vec{x}} L_C(\vec{x}, \vec{x}^*) \geq \sup_{\vec{x}} \inf_{\vec{x}^*} L_C(\vec{x}, \vec{x}^*) = v_C. \end{aligned}$$

Thus $\sum_{i\in C} u_i \geq v_C$. Since $C \subseteq I$ was arbitrary, this takes care of stability against blocking. Further, because $\sum_{i\in I} u_i \geq v_I$, for Pareto optimality we need now only verify that $\sum_{i\in I} u_i \leq v_I$. But the last inequality follows from (8) and (9). □

Theorem 4.1 begs the question whether Lagrange multipliers exist? To ensure existence, concavity of each $\pi_i(\cdot)$ would be most convenient - as Proposition 3.1 already indicated. That property embodies risk aversion, but is really not required. Instead comes a somewhat weaker assumption about *convoluted preferences*, often assigned a so-called a *representative agent.*

Before regarding the preferences of that fictive fellow, recall that sup-convolution (4) contributes towards concavity of the resulting, reduced function. Broadly, by admitting many and small agents the optimal value $v_I = v(e_I)$ becomes "more concave" in e_I. The linear support of $e \mapsto v(e)$ from above at $e = e_I$ is what decides existence of shadow prices. To emphasize this fact consider the aggregate but perturbed payoff function

$$\pi(\Delta e, \Delta x) := \sup \left\{ \sum_{i\in I} \pi_i(x_i) : \sum_{i\in I} x_i = e_I + \Delta e \ \& \ x_i = E_i x_i + \Delta x_i \ \forall i \in I \right\} \quad (11)$$

where Δe is $\mathcal{F}(e_I)$-measurable. Observe that $\pi(0,0) = v_I$. Since shadow prices bear on differential properties of π, recall that a proper function f, mapping a vector space $\mathbb{Y}$ into $\mathbb{R} \cup \{-\infty\}$, has a *supergradient* $y^* \in \mathbb{Y}^*$ at a point y iff

$$f(\bullet) \leq f(y) + \langle y^*, \bullet - y \rangle.$$

We then write $y^* \in \partial f(y)$ and declare f *superdifferentiable* at y.

Theorem 4.2. (**Characterization and existence of shadow prices**)
• $\vec{x}^* = (x^*, x_i^*, i \in I)$ *is a shadow price iff* $\vec{x}^* \in \partial\pi(0,0)$. *Thus existence of a shadow price is ensured iff the perturbation function* π *is superdifferentiable at* $(0,0)$.
• *Denote by* $\hat{\pi}$ *the smallest concave function* $\geq \pi$*, the latter defined in* (11)*. It suffices for existence of a shadow price, whence of a core solution* (10)*, that* $\hat{\pi}(\cdot,\cdot)$ *be finite-valued near* $(0,0)$ *with* $\hat{\pi}(0,0) = v_I$*. In particular, if all* π_i *are concave, with* $\pi(\cdot,\cdot)$ *finite near* $(0,0)$*, then at least one shadow price regime exists.*
• *No core solution of the sort* (10) *exists if there is a strictly positive duality gap:*

$$d := \inf_{\vec{x}^*} \sup_{\vec{x}} L_I(\vec{x}, \vec{x}^*) - v_I.$$

In this case, any scheme (10) *entails aggregate overpayment* $\geq d$.

Proof. Plainly, $\vec{x}^* = (x^*, x_i^*, i \in I) \in \partial\pi(0,0)$ iff

$$\pi(\Delta e, \Delta x) - \langle x^*, \Delta e\rangle - \sum_{i\in I} \langle x_i^*, \Delta x_i\rangle \leq \pi(0,0)$$

for all $\Delta x = (\Delta x_i)$ and each $\mathcal{F}(e_I)$-measurable Δe. In turn, via substitutions $\Delta e = \sum_{i\in I}(x_i - e_i)$, $\Delta x_i = x_i - E_i x_i$, and $\pi(0,0) = v_I$, this is equivalent to

$$L_I(\vec{x}, \vec{x}^*) = \sum_{i\in I} \{\pi_i(x_i) + \langle x^*, e_i - x_i\rangle + \langle x_i^*, E_i x_i - x_i\rangle\} \leq v_I \quad \text{for all } \vec{x},$$

whence to (8). This takes care of the first bullet. For the second simply note that the "concavification" $\hat{\pi}$ of π has a supergradient at each point near which it is finite-valued, and evidently, $\partial\hat{\pi}(0,0) \subseteq \partial\pi(0,0)$ because $\hat{\pi}(0,0) = \pi(0,0)$. Finally, the assertion after the third bullet is justified by the fact that each instance of (10) yields $\sum_{i\in I} u_i \geq v_I + d$. □

Appendix mentions further properties of shadow prices and discusses existence of optimal allocations.

5. SOME PRODUCTION GAMES

For more concreteness and intuition this section singles out a few instances, all motivated by joint production.

Example 5.1. (Linear Production Games) The computational and expressive power of linear programming, with modern extensions [6], motivates a brief look at cooperative producers who all enjoy linear technologies [31]. A special instance was already considered in (5). Here, more generally posit

$$\pi_i(e_i) := v_i := \sup\{\langle c_i, y_i\rangle : A_i y_i \leq e_i,\ y_i \geq 0\} \qquad (P_i)$$

where the objective

$$\langle c_i, y_i \rangle := E\left[c_i \cdot y_i\right] = \sum_{s \in S} c_i(s) \cdot y_i(s)\mu(s),$$

embodies $\mathcal{F}_i$-adapted vectors $c_i(s)$ and $y_i(s)$ that reside in an Euclidean space Y_i. The constraints in (P_i) mean that $A_i(s)y_i(s) \leq e_i(s)$ and $y_i(s) \geq 0$ for all s. The $\mathcal{F}_i$-adapted operator (or matrix) $A_i(s)$ maps Y_i into $\mathcal{X}$, and both these spaces are ordered. Problem (4) now amounts to the following aggregate linear program:

$$v_C := \sup\left\{\sum_{i \in C} \langle c_i, y_i \rangle : \sum_{i \in C} A_i y_i \leq e_C \text{ with } y_i \geq 0 \text{ and } \mathcal{F}_i\text{-adapted}\right\}. \quad \square \qquad (P_C)$$

π_i as defined in (P_i) is a reduced function: $\pi_i(x_i) := \sup_{y_i} \mathbf{\Pi}_i(x_i, y_i)$. This feature, and the importance of such instances, speaks against presuming π_i smooth.[6] Linear instances, like the one just described, cause few concerns with (primal-dual) existence. Also, as one would expect, no direct information rent accrues because players are risk-neutral:

Proposition 5.1. (**Linear imputations**) *Suppose the aggregate linear problem (P_I) has finite optimal value v_I. Let x^* and $y_i^*, i \in I$, be Lagrange multipliers - alias optimal dual variables - associated to $\sum_{i \in I} A_i y_i \leq e_I$ and $y_i = E_i y_i$, $i \in I$, respectively. Then the payment pattern*

$$i \in I \rightarrow \langle x^*, e_i \rangle$$

belongs to the private core. This happens if x^* and $y_i^*, i \in I$, optimally solve the dual problem

$$\min \langle x^*, e_I \rangle \;\; s.\; t.\; x^* \geq 0 and\; c_i \leq A_i^* x^* + y_i^* - E_i y_i^* \; for\; all \;\; i. \; \square$$

Example 5.2 (**Piecewise linear objectives)** Existence of several production lines often leads to instances

$$\pi_i(x_i) = \begin{cases} \min\left\{A_h(x_i) : h \in H(i)\right\} & \text{when } x_i \in X_i \cap \mathcal{F}_i \\ -\infty & \text{otherwise,} \end{cases}$$

with each A_h affine, the index set $H(i)$ finite, and the constraint set X_i polyhedral. Then (4) amounts to the linear program

$$v_C = \max \sum_{i \in C} t_i \text{ s. t. } t_i \leq A_h(x_i),\; x_i \in X_i \cap \mathcal{F}_i \text{ for each } h \in H(i) \text{ and } i \in C.$$

[6] Linear objectives belong to the wider and most important class of *polyhedral functions*, defined as those whose hypograph equals the intersection of finitely many closed half-spaces [33]. Since the conjugate of such functions are polyhedral as well, formula (10) becomes tractable.

When, as right here, $f(x) := \min\{x_h^* x + r_h : h \in H\}$, one may show that

$$f^{(*)}(x^*) = \inf\left\{\sum_h r_h^* r_h : r_h^* \geq 0, \quad \sum_h r_h^* = 1, \quad \sum_h r_h^* x_h^* = x^*\right\}, \tag{12}$$

with the understanding that $\inf \emptyset = +\infty$. Thus, $f^{(*)}(x^*) = +\infty$ iff $x^* \notin conv\{x_h^* : h \in h\}$.
□

Example 5.3. (A single producer and private resource owners) Producer 0 has endowment $e_0 = 0$ and concave, state-dependent payoff function $\Pi_{0s} : \mathcal{X} \to \mathbb{R} \cup \{-\infty\}$. Resource owner $i \in I_{-0}$ has endowment e_i and gets payoff $\pi_i(x_i) = 0$ when $x_i = 0$, otherwise $-\infty$. Posit $I := \{0\} \cup I_{-0}$. Then

$$v_C := \begin{cases} -\infty & \text{if } 0 \notin C \text{ and } e_C \neq 0, \\ 0 & \text{if } 0 \notin C \text{ and } e_C = 0, \\ \pi_0(e_C) := E\Pi_{0s}(e_C(s)) & \text{otherwise.} \end{cases}$$

Trivially $\pi_i^{(*)} = 0$ for each $i \neq 0$. Consequently, resource owner i receives payment $u_i = \langle x^*, e_i \rangle = \sum_s x^*(s) \cdot e_i(s)\mu(s)$, and the producer gets

$$u_0 = \pi_0^{(*)}(x^* + x_0^* - E_0 x_0^*) = \sum_{s \in S} \left\{\Pi_{0s}^{(*)}(x^*(s) + x_0^*(s) - E_0 x_0^*(s))\right\} \mu(s).$$

Note that, as long as $\mathcal{F}(e_I)$ remains fixed, only the producer's information structure comes into effect. Therefore, if $\mathcal{F}(e_I)$ is replaced by a finer field $\hat{\mathcal{F}}(e_I)$, the resource price $\hat{x}^*$ is likely to vary more across S. In extremis, suppose an atom $P \in \cap_{i \in I} \mathbb{P}_i$ splits into two non-empty parts P^-, P^+ identifiable only by the producer and *one* resource owner i. Further suppose resources are markedly less valuable in P^- than in P^+. Consequently, if e_i is high in P^- and low in P^+, its owner looses. In short, better information improves v_I, but the distributional impacts are not clear. □

6. Some Properties of Price-generated Imputations

The last term in formula (10) reimburses agent i the value $\langle x^*, e_i \rangle$ of his endowment. In case $\mathcal{X} = \mathbb{R}^G$ for a finite set G of goods,

$$\langle x^*, e_i \rangle = \sum_{g \in G} E(x_g^* \cdot e_{ig}) = \sum_{g \in G} \left\{E x_g^* \cdot E e_{ig} + cov(x_g^*, e_{ig})\right\}. \tag{13}$$

As in finance, i receives, besides his risk-free value, a covariance correction for his good g endowment e_{ig}. When e_{ig} is anti-correlated with e_{Ig}, that correction is positive. This feature derives from the monotonicity of the endogenous price curve $e_I = e \mapsto x^*(e)$:

Proposition 6.1. (**A decreasing price curve**) *It holds that*

$$\langle e - e', x^*(e) - x^*(e') \rangle \leq 0 \tag{14}$$

for all aggregate endowments e, e' where the shadow resource prices $x^*(e)$ and $x^*(e')$ exist.

Proof. Let $\hat{v}_I(e_I)$ denote the smallest concave function $\geq v_I(e_I)$ defined by (4). Since $x^*(e) \in \partial\hat{v}_I(e)$ and $x^*(e') \in \partial\hat{v}_I(e')$, it holds

$$\hat{v}_I(e') \leq \hat{v}_I(e) + \langle x^*(e), e' - e\rangle \quad \text{and} \quad \hat{v}_I(e) \leq \hat{v}_I(e') + \langle x^*(e'), e - e'\rangle \,.$$

The conclusion now obtains by adding the last two inequalities. □

The first component in (10) reflects production profit, calculated at a resource price x^* translated by an idiosyncratic component $x_i^* - E_i x_i^*$ that stems from private information. One might call $p_i := x^* + x_i^* - E_i x_i^*$ an *information-corrected shadow price* for agent i. As one would expect, most often that price benefits him:

Proposition 6.2. (Individual gains) *Agent i strictly benefits from collaboration if $\pi_i^{(*)}(p_i) > \pi_i^{(*)}(x^*)$.*

Proof. This is immediate from $u_i = \pi_i^{(*)}(p_i) + \langle x^*, e_i\rangle > \pi_i^{(*)}(x^*) + \langle x^*, e_i\rangle \geq \pi_i(e_i)$. □

While equal treatment is standard in the customary core, and in Walras equilibrium as well, differential information may overthrow that property; see [1]. Here though, transferable utility restores it:

Proposition 6.3. (Equal treatment) *Agents who have equal endowments, information, and preferences, receive the same price-generated imputation* (10). □

We have stressed the advantages of cooperation. It may happen though, that some agent prefers to play no part:

Proposition 6.4. (**On dummies or outsiders**) *Imputation* (10) *offers agent i autarky payment $u_i = \pi_i(e_i)$ iff the information-corrected shadow price "coincides" with his marginal payoff; that is, iff*

$$p_i := x^* + x_i^* - E_i x_i^* \in \partial\pi_i(e_i). \tag{15}$$

Proof. Since $x_i^* - E_i x_i^* \in \ker E_i$ and e_i is $\mathcal{F}_i$-measurable, $\langle x_i^* - E_i x_i^*, e_i\rangle = 0$. Therefore autarky payment happens iff

$$\pi_i^{(*)}(p_i) + \langle p_i, e_i\rangle = \pi_i(e_i),$$

or equivalently, when

$$\pi_i^{(*)}(p_i) := \sup\left\{\pi_i(x_i) - \langle p_i, x_i\rangle : x_i \in \mathbb{X}\right\} = \pi_i(e_i) - \langle p_i, e_i\rangle \,.$$

Plainly, the function $x_i \mapsto \pi_i(x_i) - \langle p_i, x_i \rangle$ is maximal at $x_i = e_i$ iff (15) holds. □

Presence of players with linear objectives facilitate risk sharing. Likewise, when information is symmetric the prospects of mutual insurance appear good:

Proposition 6.5. (**Symmetric information and mutual insurance**) *Suppose all $\mathcal{F}_i = \mathcal{F}$ are equal and generated by a common partition $\mathbb{P}$. Also suppose π_i is of form* (1) *with $\Pi_i(s, \cdot)$ adapted to the common $\mathcal{F}$. Then coalition C has value $v_C = \sum_{P \in \mathbb{P}} v_C(P)\mu(P)$ where*

$$v_C(P) := \sup \left\{ \sum_{i \in C} \Pi_i(s, \chi_i) : \sum_{i \in C} \chi_i = e_C(s) \right\} \text{ for each } s \in P.$$

Moreover, $u_i = \sum_{P \in \mathbb{P}} u_i(P)\mu(P)$ *with*

$$u_i(P) = \Pi_i^{(*)}(s, x^*(s)) + x^*(s) \cdot e_i(s) \text{ for each } s \in P.$$

Thus, cooperative gains obtain only via contingent transfers.

Proof. With no loss of generality replace S with $\mathbb{P}$. After such replacement everybody has a perfect information structure whence the information constraints can all be ignored. □

An opposite extreme setting deserves notice. Suppose players are *exclusively informed* in that there exists a partition $\{S_i, i \in I\}$ of S such that for any $i \in I$ it holds

$$P_i \in \mathbb{P}_i \Rightarrow \left\{ \begin{array}{ll} P_i = S_i^c & \text{or} \\ P_i \subseteq S_i. & \end{array} \right.$$

Let $\Delta x_i = x_i - e_i$ denote the net demand of player i. Clearly, Δx_i is constant on S_i^c. But Δx_i must be constant on S_i as well. If not, some agent $j \neq i$ would have x_j vary across S_i, a possibility blocked by his measurability constraint. In short, even if e_i is highly variable within S_i, player i can only exchange bundles that are constant across S_i against others that stay constant across S_i^c. Thus, information held by only one player helps nobody.

7. VARIATIONAL STABILITY

This section digresses to inquire briefly about the robustness or stability of core imputations (10). The question is: *how do these items fare under perturbations of endowments, payoffs and information structures?*

The issue can be formalized as follows: Let $\vec{x}^{*n}$ be a shadow price of a game $\Gamma^n := (\pi_i^n, E_i^n, e_i^n)_{i \in I}$. Suppose the latter converges to $\Gamma := (\pi_i, E_i, e_i)_{i \in I}$ in a sense to be made precise. Then, will each cluster point $\vec{x}^*$ of the sequence $(\vec{x}^{*n})$ be a shadow price for Γ? Further, will $u_i^n := u_i^n(\vec{x}^{*n}) \to u_i := u_i(\vec{x}^*)$?

Plainly, in asking these questions, there is no ambiguity or choice as to what $(e_i^n, \vec{x}^{*n}, u_i^n) \to (e_i, \vec{x}^*, u_i)$ should mean. Also, $E_i^n \to E_i$ amounts to have the matrix representation of E_i^n converge in each entry to that of E_i. But some care is needed in defining the appropriate notion of convergence $\pi_i^n \to \pi_i$. We say that a sequence of functions $f^n : \mathbb{X} \to \mathbb{R} \cup \{\pm\infty\}$ *epi-converges* to $f : \mathbb{X} \to \mathbb{R} \cup \{\pm\infty\}$, and write $f^n \to^{epi} f$, iff

$$\begin{cases} \forall x \in \mathbb{X}\ \forall x^n \to x & \text{it holds that} \quad \liminf f^n(x^n) \geq f(x) \text{ and} \\ \forall x \in \mathbb{X}\ \exists x^n \to x & \text{such that} \quad \limsup f^n(x^n) \leq f(x). \end{cases}$$

Proposition 7.1. (**Stability of shadow prices and imputations**) *Suppose*
- $(e_i^n, E_i^n) \to (e_i, E_i)$;
- $\forall i \in I$, $\forall x_i^* \in \mathbb{X}^*, \forall x_i^{n*} \to x_i^*$ *it holds that*

$$\liminf \pi_i^{n(*)}(x_i^{n*}) \geq \pi_i^{(*)}(x_i^*), \quad \textit{and} \quad \pi_i^{n(*)}(x_i^*) \to \pi_i^{(*)}(x_i^*);$$

- *the lower level set* $\left\{\pi_i^{(*)} \leq r\right\}$ *is bounded for every* $r \in \mathbb{R}$ *and every* i.

Let $\vec{x}^{*n}$ *be a shadow price of game* $\Gamma^n = (\pi_i^n, E_i^n, e_i^n)_{i\in I}$. *Then each cluster point* $\vec{x}^*$ *of the sequence* $(\vec{x}^{*n})$ *is a shadow price of the unperturbed game* $\Gamma = (\pi_i, E_i, e_i)_{i\in I}$. *Moreover,* $u_i^n = u_i^n(\vec{x}^{*n}) \to u_i = u_i(\vec{x}^*)$ *for each* i.

Proof. Denote by $L_i^n : \mathbb{X}^* \times \mathbb{X}^* \to \mathbb{X}^*$ the linear mapping defined by $L_i^n(x^*, x_i^*) := p_i^n := x^* + x_i^* - E_i x_i^*$. Clearly, $L_i^n \to L_i$ pointwise for each i. Now define

$$F^n(\vec{x}^*) := \sum_{i\in I} \left\{\pi_i^{n(*)} \circ L_i^n(x^*, x_i^*) + \langle x^*, e_i^n\rangle\right\} \text{ and}$$
$$F(\vec{x}^*) := \sum_{i\in I} \left\{\pi_i^{(*)} \circ L_i(x^*, x_i^*) + \langle x^*, e_i\rangle\right\}.$$

Observe that $F^n \to^{epi} F$. Since $\vec{x}^{*n} \in \arg\min F^n$, the conclusion follows from Theorem 7.33 in [34]. □

8. Non-Transferable Utility

So far arguments hinged upon utility being transferable. This section drops that assumption at the cost of a less constructive approach to core solutions.

As hitherto, by a *price* system is understood a profile $\vec{x}^* := (x^*, x_i^*, i \in I)$ such that $x^* \in \mathcal{F}(e_I)$ and $x_i^* \in range(\mathbb{I} - E_i)$. For any price system let

$$c_i(\vec{x}^*, x_i) := \langle x^*, x_i\rangle + \langle x_i^*, x_i - E_i x_i\rangle$$

denote the cost incurred by player i when he purchases $x_i \in \mathbb{X}$. Note that $c_i(\vec{x}^*, e_i) = \langle x^*, e_i\rangle$. Recall that $\vec{x} = (x_i)$ is declared a feasible allocation iff $\sum_I x_i = \sum_I e_i$ with

$\pi_i(x_i) > -\infty$ and $x_i = E_i x_i$ for each i. A price $\vec{x}^*$ together with a feasible allocation $\vec{x}$ constitutes a *Walras equilibrium* if for each i

$$c_i(\vec{x}^*, x_i) \leq \langle x^*, e_i \rangle, \quad \text{and} \quad \pi_i(x_i') > \pi_i(x_i) \Rightarrow c_i(\vec{x}^*, x_i') > \langle x^*, e_i \rangle .$$

The pair $(\vec{x}^*, \vec{x})$ is declared a *quasi-equilibrium* if for each i

$$c_i(\vec{x}^*, x_i) = \langle x^*, e_i \rangle, \quad \text{and} \quad \pi_i(x_i') \geq \pi_i(x_i) \Rightarrow c_i(\vec{x}^*, x_i') \geq \langle x^*, e_i \rangle .$$

A feasible allocation $\vec{x}$ is in the *Core* if no coalition $C \subseteq I$ can find another allocation $(x_i')_{i\in C}$, feasible for itself such that $\pi_i(x_i') \geq \pi_i(x_i)$ $\forall i \in C$, with at least one inequality strict.

Proposition 8.1. (**Existence of quasi-equilibrium**) *Assume each π_i is Lipschitz continuous, concave on $dom\pi_i := \{x_i : \pi_i(x_i) > -\infty\}$, and that the latter set is non-empty compact. Then there exists a quasi-equilibrium.*

Proof. We follow [19]. Denote by Δ the standard unit simplex in $\mathbb{R}^I$. That is, $\delta = (\delta_i) \in \Delta$ iff each $\delta_i \geq 0$ and $\sum_{i\in I} \delta_i = 1$. For any $\delta \in \Delta$ let $\mathbf{s}^\delta = (\vec{x}^{*\delta}, \vec{x}^\delta)$ be a min-max saddle-point of the Lagrangian

$$L^\delta(\vec{x}^*, \vec{x}) := \sum_{i\in I} \{\delta_i \pi_i(x_i) - c_i(\vec{x}^*, x_i) + \langle x^*, e_i \rangle\} .$$

Then

$$\delta_i \left\{\pi_i(x_i^\delta) - \pi_i(x_i)\right\} \geq c_i(\vec{x}^{*\delta}, x_i^\delta) - c_i(\vec{x}^{*\delta}, x_i) \text{ for each } x_i. \tag{16}$$

Let $\mathbf{S}^\delta$ equal the set of all saddle points $\mathbf{s}^\delta = (\vec{x}^{*\delta}, \vec{x}^\delta)$ of L^δ, and posit for any $(\vec{x}^*, \vec{x})$,

$$D(\vec{x}^*, \vec{x}) := \{\delta \in \Delta : \delta_i = 0 \text{ if } c_i(\vec{x}^*, x_i) > c_i(\vec{x}^*, e_i)\} .$$

Since each π_i is Lipschitz continuous on its domain, so are all functions $\vec{x} = (x_i) \mapsto \sum_{i\in I} \delta_i \pi_i(x_i)$ on $K := \Pi_{i\in I} dom\pi_i$ with a modulus that doesn't depend on δ. Consequently, the components of the multiplier vectors $\vec{x}^{*\delta}$, having the nature of supergradients

$$p_i^\delta := x^{*\delta} + x_i^{*\delta} - E_i x_i^{*\delta} \in \delta_i \partial \pi_i(x_i^\delta),$$

must be uniformly bounded. This entails that, modulo the transformation $\vec{x}^{*\delta} \mapsto (p_i^\delta) = (x^{*\delta} + x_i^{*\delta} - E_i x_i^{*\delta})$, we can restrict $\vec{x}^{*\delta}$ to belong to a compact convex set K^*. The correspondence $(\vec{x}^*, \vec{x}, \delta) \rightsquigarrow \mathbf{S}^\delta \times D(\vec{x}^*, \vec{x})$ has a fixed point $(\vec{x}^*, \vec{x}, \delta)$ on the set $K^* \times K \times \Delta$.

We claim that $c_i(\vec{x}^*, x_i) = c_i(\vec{x}^*, e_i)$ for all i. Indeed, if some $c_i(\vec{x}^*, x_i) > c_i(\vec{x}^*, e_i)$, then by construction $\delta_i = 0$, and (16) would yield the contradiction $c_i(\vec{x}^*, x_i) \leq c_i(\vec{x}^*, e_i)$. Consequently, $c_i(\vec{x}^*, x_i) \leq c_i(\vec{x}^*, e_i)$ for all i. But, if some such inequality were strict, another contradiction comes up, namely: $\sum_{i\in I} c_i(\vec{x}^*, x_i) < \sum_{i\in I} c_i(\vec{x}^*, e_i)$. This proves the claim.

Similarly, if $\pi_i(x_i') \geq \pi_i(x_i)$ and $c_i(\vec{x}^*, x_i') < c_i(\vec{x}^*, x_i)$ for some $\mathcal{F}_i$-measurable x_i', then $L^\delta(\vec{x}^*, \cdot)$ cannot be maximal at $\vec{x}$. □

Proposition 8.2. (**Walras equilibrium**) *Suppose each π_i is lower semicontinuous on its effective domain $dom\pi_i$ and that this set is starshaped with respect to* 0. *Then each quasi-equilibrium for which all $\langle \vec{x}^*, e_i \rangle > 0$, is a Walras equilibrium.*

Proof. If a quasi-equilibrium $(\vec{x}^*, \vec{x})$ is not a Walras equilibrium, then some agent i has a $\mathcal{F}_i$-measurable x_i' such that $\pi_i(x_i') > \pi_i(x_i)$ and $c_i(\vec{x}^*, x_i') = c_i(\vec{x}^*, e_i)$. Since $dom\pi_i$ is starshaped with respect to 0, we have $rx_i' \in dom\pi_i$ for all $r \in [0,1]$. By the lower semicontinuity of π_i on its effective domain, for $r < 1$ sufficiently close to 1 we still get $\pi_i(rx_i') > \pi_i(x_i)$ but $c_i(\vec{x}^*, rx_i') < c_i(\vec{x}^*, e_i)$ which contradicts the quasi-equilibrium. □

Proposition 8.3. (**Non-empty core**) *Under the hypotheses of Propositions 8.1-2 there exists a core solution.*

Proof. Pick any quasi-equilibrium $(\vec{x}^*, \vec{x})$. If $\vec{x}$ is not in the core, some proper coalition C has an alternative feasible allocation $(x_i')_{i\in C}$ satisfying $\pi_i(x_i') \geq \pi_i(x_i)$ for all $i \in C$, with at least one inequality is strict. By quasi-equilibrium $c_i(\vec{x}^*, x_i') \geq c_i(\vec{x}^*, e_i)$ for all i. By Walras equilibrium, $c_i(\vec{x}^*, x_i') > c_i(\vec{x}^*, e_i)$ for each strictly improving agent. The upshot is the contradiction $\sum_{i\in C} c_i(\vec{x}^*, x_i') > \sum_{i\in C} c_i(\vec{x}^*, e_i)$. □

9. SOME EXAMPLES

Since payment

$$u_i = \pi_i^{(*)}(x^* + x_i^* - E_i x_i^*) + \langle x^*, e_i \rangle$$

is convex in $\vec{x}^*$, impacts of changes in measurability become interesting. For the argument maintain e_i but replace $[x^*, x_i^* - E_i x_i^*, \mathcal{F}(e_I)]$ with a strictly "finer" version $\left[\hat{x}^*, \hat{x}_i^* - \hat{E}_i \hat{x}_i^*, \mathcal{F}(\hat{e}_I)\right]$, satisfying $E\left[\hat{x}^* \left| \mathcal{F}(e_I)\right.\right] = x^*$ and

$$E\left[\hat{x}_i^* - \hat{E}_i \hat{x}_i^* \middle| \mathcal{F}(e_I)\right] = x_i^* - E_i \hat{x}_i^*.$$

Then, if $\pi_i^{(*)}$ is strictly convex, $\hat{u}_i := \pi_i^{(*)}(\hat{x}^* + \hat{x}_i^* - \hat{E}_i^* \hat{x}_i^*) + \langle \hat{x}^*, e_i \rangle > u_i$. In particular, *if player i is propertyless, perfectly informed, and has $\pi_i^{(*)}$ is strictly convex, he is likely to benefit form a refinement of the field $\mathcal{F}(e_I)$.*

If e_i changes, there is, of course, a material effect, but possibly also repercussions via the information structure. To better isolate the latter, let i be a pure resource owner. He has conjugate $\pi_i^{(*)} \equiv 0$ and gets core payment $u_i = \langle x^*, e_i \rangle$. A pair y_1, y_2 of real-valued random variables, defined on the same probability space, is said to exhibit *negative dependence* if

$$\Pr\{y_1 \leq r_1 \,|y_2 \leq r_2\} \quad \leq \quad \Pr\{y_1 \leq r_1\} \cdot \Pr\{y_2 \leq r_2\} \text{ for all real } r_1, r_2,$$

with strict inequality for at least one choice r_1, r_2. With *one* good, $\mathcal{X} = \mathbb{R}$, and (13) gives $u_i = Ex^* \cdot Ee_i + cov(x^*, e_i)$. Thanks to (14) we can posit that the resource price x^* is a decreasing function of e_I. So, if e_I and e_i are negatively dependent, $cov(x^*, e_i) > 0$; see [28] Proposition 16.9. Consequently, when e_I and e_i are anticorrelated, agent i receives a bonus beyond the "average payment" $E(x^*) \cdot E(e_i)$.

It is noteworthy that the first fundamental theorem of welfare economics no longer holds. The reason is that (rational expectation) Walras equilibria, in so far as ascribing value merely to commodities, need not belong to the private core. For example, an agent i with $e_i = 0$ and perfect information structure $\mathcal{F}_i = \{\{s\}\}$ gets production profit $\pi_i^{(*)}(x^*)$. So, provided $\pi_i^{(*)}(x^*) > 0$, he is left with some purchasing power. The private core is apt to reward him for information that allow risk averters to write more detailed and diversified contracts; see Example 9.6. In contrast, Walras equilibrium gives him zero wealth, nullifies his consumption - irrespective of what information he brings. It also deserves mention that Walras equilibrium may fail to exist in cases where the core is non-empty:

Example 9.1. (An instance with no Walras equilibrium but non-empty core) Let there be two goods $g \in G = \{g_1, g_2\}$, two players $i \in I = \{1, 2\}$, and two equally likely states $s \in S = \{s_1, s_2\}$. Posit $e_1(s) = (1, 0)$, $e_2(s) = (1, 1)$ in each state s to have constant endowments. Define

$$\pi_i(x_i) := \begin{cases} Ex_{i,g_i}(s) & \text{if } x_i(s) \in \mathbb{R}_+^2 \text{ for all } s, \\ -\infty & \text{otherwise.} \end{cases}$$

Player i chooses $x_i = [x_{ig}(s)] \in \mathbb{R}^{G \times S}$ and enjoys merely good g_i. For any price $x^* = \left[x_g^*(s)\right] \in \mathbb{R}_+^{G \times S}$ we get

$$\pi_i^{(*)}(x^*) := \begin{cases} 0 & \text{if } x_{g_i}^*(s) \geq 1 \text{ for each } s, \\ +\infty & \text{otherwise.} \end{cases}$$

Let each $\mathcal{F}_i$ be generated by a perfect partition. Then every allocation that gives player 1 a constant amount $x_{1g_1} \in [1, 2]$ of the first good - and player 2 the rest - belongs to the core. The shadow price $x^* \equiv \mathbf{1}$ supports that outcome. Consequently, $u_1 = 1, u_2 = 2$ is a price-generated core imputation. There is however, no competitive equilibrium. Indeed, an equilibrium price vector $p = [p_g(s)]$ must be constant across S, but cannot have $p_1(s) \equiv p_1 = 0$, leaving agent 1 destitute. Further, if $p_1 > 0$, then agent 2 will demand more of good 2 than available. Changing $\mathcal{F}_1$ to $\{\varnothing, S\}$ would not upset this conclusion. □

As is fairly well known, differential information may impede the writing of good contracts:

Example 9.2. (A case for autarky) Accommodate two agents, one good, and three states as follows:

Agent i	$\mathbb{P}_i$	state s :	s_1	s_2	s_3
1	$\{s_1\}, \{s_2, s_3\}$	endowment $e_1(s)$:	$e_1(s_1)$	0	0
2	$\{s_2\}, \{s_1, s_3\}$	endowment $e_2(s)$:	0	$e_2(s_2)$	0

If $e_1(s_1)$, $e_2(s_2) > 0$ are different, $\mathcal{F}(e_I)$ is strictly finer than $\mathcal{F}_1, \mathcal{F}_2$. Also, when $e_1(s_1) = e_2(s_2) > 0$, $\mathcal{F}(e_I) = \mathcal{F}(\{s_1, s_2\}, \{s_3\}) \neq \mathcal{F}_i$. Posit format (1) with $\Pi_i(s, 0) = 0$ and $x_i(s) \geq 0$, to get $v_i = \Pi_i(s_i, e_i(s_i))\mu(s_i)$ for each i. Both players get 0 is state s_3. Therefore, by measurability $x_1(s_2) = 0$ and $x_2(s_1) = 0$, - to the effect that no contract becomes possible apart from the autarkic one. While both parties might want to write contracts in terms of s_1, s_2, either is unable to disentangle s_3 as a special contingency. □

Example 9.3. (Autarky or arbitrage) For players $i \in \{1, 2\}$ let there be *one* good ($\mathcal{X} = \mathbb{R}$) and posit $\pi_i(x_i) := \sum_{s \in S} a_i(s)x_i(s)$ with $a_i \in \mathcal{F}_i$ and $x_i \geq 0$. Suppose partition $\mathbb{P}_1$ is strictly finer than $\mathbb{P}_2 = \{S\}$. Also suppose $e_1(s) = 0$ for at least one state. If $\sum_{s \in S} a_1(s) \leq \sum_{s \in S} a_2(s)$, then autarky is optimal. In case the last inequality is strict, and the constraint $x_1 \geq 0$ is dropped, arbitrage obtains to the effect that $v_I = +\infty$. □

Example 9.4. (On the advantage of being informed) Accommodated are two agents, two goods, and two states as follows:

Agent i	$\mathbb{P}_i$	state s :	s_1	s_2
1	$\{s_1\}, \{s_2\}$	endowment $e_1(s)$:	$(1, 0)$	$(0, 0)$
2	$\{s_1, s_2\}$	endowment $e_2(s)$:	$(0, 2)$	$(0, 2)$

Note that $\mathcal{F}(e_I) = \mathcal{F}_1$ is strictly finer than $\mathcal{F}_2$. Let the two goods $g \in \{1, 2\}$ be *perfect complements*. Accordingly, posit

$$\pi_i(x_i) = \sum_s \min_g x_{ig}(s)\mu(s)$$

with $x_i = [x_{ig}(s)] \geq 0$, to get $v_i = 0$ for each i. Because player 1 can't transfer any positive amount of good 1 to player 2,

$$v_I = \max\left\{\sum_s \min_g x_{1g}(s)\mu(s) : 0 \leq x_1(s) \leq e_I(s)\right\} = \mu(s_1).$$

The shadow price x^* on resources is the constant vector $(1, 0)$. Here $\pi_1^{(*)}(x^*) = 0$. Thus the price-generated core payments are

$$u_1 = \langle x^*, e_1 \rangle = \mu(s_1) \text{ and } u_2 = \langle x^*, e_2 \rangle = 0.$$

When $\mu(s_1) \leq 0.5$, player 1 has less attractive endowment and technology. But his information advantage allows him to produce the cake - and have it all. □

Example 9.5. **(On syndication)** It is known that players who hold relatively scarce resources may loose by forming a syndicate. It appears though, that price-generated core solutions may mitigate this. To illustrate, accommodate 5 players, only one good, but three states:

Agent i	$\mathbb{P}_i$	state s :	s_1	s_2	s_3
$i = 1, 2$	$\{s_1\}, \{s_2, s_3\}$	endowment $e_i(s)$:	1	0	0
$i = 3, 4, 5$	$\{s_1\}, \{s_2\}, \{s_3\}$	endowment $e_i(s)$:	0	1/2	1

Let $ext\Delta$ denote the set of extreme points in the unit simplex $\Delta \subset \mathbb{R}^S$. For each player i let

$$\pi_i(x_i) := \min_{s \in S} x_i(s)\mu(s) = \min\{\langle \delta, x_i \rangle : \delta \in ext\Delta\} = \min \langle \Delta, x_i \rangle, \qquad (17)$$

to have from (12) that $\pi_i^{(*)}$is nil on Δ and $+\infty$ elsewhere.[7] Like in [32] we get

$$v_S = \frac{1}{3} \min\{|S \cap \{1, 2\}|, |S \cap \{3, 4, 5\}| / 2\}$$

The private core reduces to the single profile $u = (0, 0, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$. The resource price $x^* = [0, 1, 0] \in \mathcal{F}(e_I)$, and $u_i = \langle x^*, e_i \rangle$.

If owners of the scarce resource $e_I(s_2)$ form a syndicate $\{3, 4, 5\}$, the resulting core has extreme points $(u_1, u_2, u_{\{3,4,5\}})$ at the four vectors $\left(0, 0, \frac{1}{2}\right)$, $\left(\frac{1}{6}, 0, \frac{1}{3}\right)$, $(0, \frac{1}{6}, \frac{1}{3})$, and $(\frac{1}{6}, \frac{1}{6}, \frac{1}{6})$. Thus, $u_{\{3,4,5\}} < 1/2$ in all but one point. However, since syndication does not affect x^*, for the price-generated selection we still get $u_{\{3,4,5\}}(x^*) = \left\langle x^*, e_{\{3,4,5\}} \right\rangle = 1/2$. This attests to the competitive nature of formula (10). □

Example 9.6. (On bringing useful information) First admit merely two players, each with payoff (17) and imperfect information:

Player	*Partition*	$e_i(s_1)$	$e_i(s_2)$	$e_i(s_3)$
1	$\{s_1, s_2\}, \{s_3\}$	1	1	0
2	$\{s_1, s_3\}, \{s_2\}$	1	0	1

Then $v_i = 0$, $v_{\{1,2\}} = 0$. Now add a third, totally destitute, but perfectly informed player:

Player	*Partition*	$e_3(s_1)$	$e_3(s_2)$	$e_3(s_3)$
3	$\{s_1\}, \{s_2\}, \{s_3\}$	0	0	0

[7]The utility function (17) equals the *Choquet expected utility* $\int x_i dc$ when the normalized capacity c is strictly positive only on the sure event S; see [10].

If his payoff also is of the form (17), $v_I = 1/3$, $p_3 = x^* \in \Delta$, and coalition $\{1,2\}$ takes all. Player 3 then merely serves as a nexus for exchange. If however, $\pi_3(x_3) := -\frac{1}{2}\sum_s x_3^2(s)\mu(s)$, we get $\pi_3^{(*)}(x^*) = \frac{1}{2}\sum_s x_3^{*2}(s)\mu(s) > 0$ for all $x^* \neq 0$. Because $x^*(s_2), x^*(s_3) > 0$, players receive positive rewards $u_1 = \langle x^*, e_1\rangle$, $u_2 = \langle x^*, e_2\rangle$, $u_3 = \pi_3^{(*)}(x^*)$. The upshot is that players 1 and 2, while unable to tango, find it best to join the grand coalition. This makes the utterly poor player a right honorable member.

10. Concluding Remarks

The core, a most popular solution concept, occupied center stage here. Moreover, a price-generated selection was made.

Such selection points toward Walras equilibrium and various ways of shrinking the core. Specifically, to have the core non-empty but small, one may invoke replicated agents [9], nonatomic player sets [4], convexified preferences [17], or fuzzy coalitions [23].

None of these remedies were used here. Instead we simply presumed that overall payoff π was superdifferentiable - that is, concave - at the point of reference. More global concavity could come about by aggregating representative agents as follows: Let $I := \{1, ..., |I|\}$ and introduce for each $t \in (i-1, i]$, $i \in I$, a player with endowment $e_t = e_i dt$, upper semicontinuous payoff $\pi_t = \pi_i$, and partition $\mathbb{P}_t = \mathbb{P}_i$. Thus player i becomes a representative for a continuum of identical agents. Introduce next the functions

$$\hat{\pi}_i(x_i) := \sup\left\{\int_{i-1}^{i} \pi_t(x_t)dt : x_t = E_i x_t \text{ and } \int_{i-1}^{i} x_t dt = x_i\right\}.$$

The functions $\hat{\pi}_i$ so constructed are all concave [37], and

$$\sup\left\{\int_0^{|I|} \pi_t(x_t)dt : x_t = E_i x_t \text{ for } t \in (i-1, i]\text{, and } \int_0^{|I|} x_t dt = e_I\right\}$$
$$= \sup\left\{\sum_{i\in I} \hat{\pi}_i(x_i) : x_i = E_i x_i \text{ and } \sum_{i\in I} x_i = e_I\right\}.$$

The resulting, "representative" triples $(\hat{\pi}_i, \mathbb{P}_i, e_i), i \in I$, generates a concave perturbed function $\hat{\pi}$ (11), and the preceding analysis applies.

Appendix

Collected here are some comments and results on shadow prices and optimal allocations.

Uniqueness of a shadow price amounts, of course, to have $\pi(\cdot,\cdot)$, as defined in (11), differentiable at $(0,0)$. We shall not explore this issue.

A non-negative resource price x^* results when the commodity space $\mathcal{X}$ is ordered, and at least one agent has monotone payoff. Then, for material balance it suffices that $\sum_{i\in I} x_i \leq e_I$.

Superdifferentiability of π (11) at $(0,0)$ does not demand that all underlying π_i be concave. If some π_i isn't concave, one may "board up its holes" by employing instead the smallest concave function $\hat{\pi}_i \geq \pi_i$. This done, each price regime $\vec{x}^*$ generates imputations $\hat{u}_i(\vec{x}^*) \geq u_i(\vec{x}^*)$. And any shadow price $\vec{x}^*$ for the concavified game gives

$$\begin{array}{ll} \sum_{i\in I} \hat{u}_i(\vec{x}^*) \leq v_I + d, & \text{and} \\ \sum_{i\in C} \hat{u}_i(\vec{x}^*) \geq v_C & \text{for all } C \subseteq I. \end{array}$$

The Shapley-Folkman theorem [12] asserts that concavification of payoffs affects at most $\dim \mathcal{X} + 2$ agents. For more on this issue, and for estimates of the duality gap (or core deficit) d, see [3], [15], [17], [36]. The upshot is that there is room for agents whose payoffs are non-concave in regions of no economic interest. It is hard however, to accommodate a player whose payoff is globally convex and finite-valued. Indeed, his presence suffices to render the perturbed function π convex. When moreover, that π has a supergradient somewhere, it must be affine. Definitely, such an instance has little of interest or realism.

As customary Lagrange multipliers relate to geometry, and they mirror the willingness to pay. These features are recorded next. For the statement denote by $f'(y;\Delta y)$ the directional derivative of $f : \mathbb{Y} \to \mathbb{R} \cup \{-\infty\}$ at $y \in \mathbb{Y}$ in the direction Δy.

Proposition A.1. (**Properties of shadow prices** $\vec{x}^* = (x^*, x_i^*, i \in I)$)
1) x^* *must be orthogonal on the affine subspace spanned by equation* $\sum_{i\in I} x_i = e_I$. *More precisely,* $\langle x^*, \sum_{i\in I} x_i - e_I \rangle = 0$ *for all* $\sum_{i\in I} x_i \in \mathcal{F}(e_I)$. *Further,* $x_i^* \perp \ker(\mathbb{I} - E_i) = \mathcal{F}_i$ *and* $x_i^* - E_i x_i^* \perp \mathcal{F}_i$ *in that* $\langle x_i^*, x_i - E_i x_i\rangle = \langle x_i^* - E_i x_i^*, x_i\rangle = 0$ for all $x_i \in \mathcal{F}_i$.
2) x^* *is, or can be made,* $\mathcal{F}(e_I)$*-measurable, and one can posit* $x_i^* \in range(\mathbb{I} - E_i)$.
3) *If function* π (11) *is differentiable at* $(0,0)$ *in the direction* $(\Delta e, \Delta x)$, *then*

$$\pi'(0,0;\Delta e,\Delta x) \leq \inf \left\{ \langle x^*, \Delta e\rangle + \sum_{i\in I} \langle x_i^*, \Delta x_i\rangle : \vec{x}^* \textit{ shadow price} \right\}.$$

In case π *is concave and finite near* $(0,0)$ *equality holds here.*

Proof. 1) just expresses standard complementarity. In 2) the first assertion derives from the hypothesis that only $\mathcal{F}(e_I)$-measurable perturbations of the aggregate endowment were accommodated. Plainly, the dual space to $E_{\mathcal{F}(e_I)}\mathbb{X}$ comprises only functionals of corresponding measurability. Because x_i^* is nil on $\ker(\mathbb{I} - E_i)$, it belongs to $range(\mathbb{I} - E_i)^* = range(\mathbb{I} - E_i)$.

In 3), by Theorem 4.2 each shadow price $\vec{x}^*$ is a supergradient of π at $(0,0)$. This implies that

$$\pi(t\Delta e, t\Delta x) - \pi(0,0) \leq t \left\{ \langle x^*, \Delta e\rangle + \sum_{i\in I} \langle x_i^*, \Delta x_i\rangle \right\}$$

for any $t > 0$ and shadow price $\vec{x}^*$. Consequently,

$$\frac{\pi(t\Delta e, t\Delta x) - \pi(0,0)}{t} \leq \inf \left\{ \langle x^*, \Delta e\rangle + \sum_{i\in I} \langle x_i^*, \Delta x_i\rangle : \vec{x}^* \text{ shadow price} \right\}$$

and the first assertion om 3) follows. The last one there is a standard result of convex analysis. □

For completeness, we record a primal existence result:

Proposition A.2. (**Existence of optimal allocations**) *An optimal allocation* $\vec{x} = (x_i)$ *exists, and the value* v_I *is attained, in each the following three cases*:
1) *The upper level set*

$$\left\{ (x_i) : \sum_{i \in I} \pi_i(x_i) \geq r, \ x_i = E_i x_i, \ \sum_{i \in I} x_i = e_I \right\}$$

is non-empty bounded for at least one $r \in \mathbb{R}$ *and closed for all* $r \in \mathbb{R}$.
2) *Each* π_i *is upper semicontinuous and concave on* $rangeE_i$, *and the recession functions*

$$0^- \pi_i(d_i) := \inf_{r>0} \frac{\pi_i(x_i + rd_i) - \pi_i(x_i)}{r}, \quad \pi_i(x_i) \text{ finite},$$

satisfy

$$\sum_{i \in I} 0^- \pi_i(d_i) \geq 0 \ \ \& \ \ \sum_{i \in I} 0^- \pi_i(-d_i) < 0 \ \text{ implies } \sum_{i \in I} d_i \neq 0.$$

3) *Each* π_i *is upper semicontinuous with a conjugate* $\pi_i^{(*)}$ *that is finite-valued continuous at* 0.

Proof. Statement 1) is standard. For 2) see Rockafellar (1970) Corollary 9.2.1. For 3) let

$$f_*(y_*) := \inf_y \left\{ \langle y_*, y \rangle - f(y) \right\}$$

denote the concave conjugate of a proper function f that maps a Hilbert space into $[-\infty, +\infty)$. Then, on the same space, $\hat{f} := (f_*)_*$ equals the smallest concave upper semicontinuous function $\geq f$. The fact that π_{i*} is finite-valued and continuous at 0 implies, by the Moreau-Rockafellar theorem [7], that each upper level set $\{\hat{\pi}_i \geq r_i\}$ is compact. Now consider any maximizing, feasible sequence $x^k = (x_i^k)$. Since v_I is finite there exist real numbers r_i such that $x_i^k \in \{\hat{\pi}_i \geq r_i\}$ for all k and i. Extract a convergent subsequence to conclude. □

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