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Abstract

This paper introduces a tractable, structural model of subjective beliefs. Since agents that plan for the future care about expected future utility flows, current felicity can be increased by believing that better outcomes are more likely. On the other hand, expectations that are biased towards optimism worsen decision making, leading to poorer realized outcomes on average. Optimal expectations balance these forces by maximizing the total well-being of an agent over time. We apply our framework of optimal expectations to three different economic settings. In a portfolio choice problem, agents overestimate the return of their invesment and underdiversify. In general equilibrium, agents' prior beliefs are endogenously heterogeneous, leading to gambling. Second, in a consumption-saving problem with stochastic income, agents are both overconfident and overoptimistic, and consume more than implied by rational beliefs early in life. Third, in choosing when to undertake a single task with an uncertain cost, agents exhibit several features of procrastination, including regret, intertemporal preference reversal, and a greater readiness to accept commitment.

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1 Introduction

Modern psychology views human behavior as a complex interaction of cognitive and emotional responses to external stimuli that sometimes gives dysfunctional outcomes. Modern economics takes a relatively simple view of human behavior as governed by unlimited cognitive ability applied to a small number of concrete goals and unencumbered by emotion. The central models of economics allow coherent analysis of behavior and of economic policy, but eliminate "dysfunctional" outcomes, and in particular the possibility that households might persistently err in attaining their goals. One area in which there is substantial evidence that households do consistently err is in the assessment of probabilities. In particular, agents often overestimate the probability of good outcomes, such as their success (Alpert and Raiffa (1982), Weinstein (1980), and Buehler, Griffin, and Ross (1994)).

We provide a structural model of subjective beliefs in which agents hold incorrect but optimal beliefs. These optimal beliefs differ from objective beliefs in ways that match many of the claims in the psychology literature about "irrational" behavior. Further, in the canonical economic models that we study, these beliefs lead to economic behaviors that match observed outcomes that have puzzled the economics literature based on rational behavior and common priors. Our approach has three main elements.

First, at any instant, people care about current utility flow and expected future utility flows. We also allow, although it is not central to our main points, that past utility flows also influence current felicity. While it is standard that agents that care about expected future utility plan for the future, forward-looking agents have higher current felicity today if they are optimistic about the future. Phrases like "anticipation exceeds realization"[1] are consistent with this idea. Agents that care about expected future utility flows are happier with distorted beliefs about the payoff of their investments and/or the stochastic process for future labor income.

The second crucial element of our model is that such optimism affects actual decisions. Distorted beliefs distort actions. In this sense, agents are not schizophrenic. For example, an agent cannot derive utility by optimistically believing that she will be rich tomorrow, while also

[1] The common German phrase "Vorfreude ist die schönste Freude" translates roughly to "anticipation exceeds realization."

basing today's consumption-savings decision on rational beliefs about future income.

How are these forces balanced? We assume that subjective beliefs maximize the agent's expected total well-being, the expected discounted sum of felicity across periods. This third key element leads to a balance between the first two – the benefits of optimism and the costs of basing actions on distorted expectations.

There are several reasons why beliefs might maximize overall well-being. First, in the long tradition of economics, we want to explore what is optimal. The results of this paper point to when and how beliefs might stray from rational beliefs. They stray when biases have little cost in realized outcomes. They stray in the direction of optimism – states with greater utility flows are perceived as more likely. Second, if parents care about the happiness of their children, they would choose to endow their children with optimal expectations. This is consistent with children being raised believing they can "do or be anything," "they are special," etc. Third, evolutionary arguments need not favor rational expectations. For example, happiness is linked to better health (Taylor and Brown (1988)), and a happy individual may be favored in the marriage market. Finally, the scientific test of the theory is its performance, not its assumptions. So far, our approach helps to explain heterogenous beliefs, gambling, overconfidence, procrastination, and intertemporal preference reversal, all within one coherent and tractable model.

More specifically, we illustrate our theory of optimal expectations using three examples. First, in a portfolio choice problem, agents overestimate the return of their investment and underdiversify. In general equilibrium, agents' prior beliefs are endogenously heterogeneous and agents gamble against each other. The price of the risky asset may differ from that in an economy populated by agents with rational beliefs. Second, in a consumption-saving problem with quadratic utility and stochastic income, agents are overconfident and overoptimistic; agents consume more than implied by rational beliefs early in life. Third, in choosing when to undertake a single task with an uncertain cost, agents exhibit several features of procrastination, including intertemporal preference reversal, a greater readiness to accept commitment, regret, and a context effect in which non-chosen actions can affect utility.

Our model of beliefs differs markedly from treatments of risk in economics. While early models in macroeconomics specify beliefs exogenously as naive, myopic, or partially updated (e.g. Nerlove (1958)), since Muth (1960, 1961), Lucas (1976) and the rational expectations revolution, nearly all research has proceeded under the maintained assumption that subjective and objective beliefs coincide. There are two main arguments for this. First, the alternatives to rationality lack discipline. But our model provides exactly such discipline for subjective beliefs by specifying an objective for beliefs, that they maximize lifetime well-being. The second argument is that rational expectations is an optimal "as if" – agents have the incentive to hold rational beliefs (or act as if they do) because these expectations make the agent as well off as they can be. However, this rationale for rational expectations relies on an inconsistency: agents care about the future but at the same time expectations about the future do not affect current felicity. Our approach of optimal expectations is the outcome of an optimal "as if" argument that takes into account the fact that agents care in the present about utility flows that are expected in the future.

Most microeconomic models assume that agents share common prior beliefs. This "Harsanyi doctrine" is weaker than the assumption of rational expectations that all agents' prior beliefs are equal to the objective probabilities governing equilibrium dynamics. But like rational expectations, the common priors assumption is quite restrictive and does not allow agents to "agree to disagree" (Aumann (1976)). Savage (1954) provides axiomatic foundations for a more general theory in which agents hold arbitrary prior beliefs, so agents can agree to disagree. But if beliefs can be arbitrary, theory provides little structure or predictive power. Optimal expectations provides discipline to the study of subjective beliefs and heterogeneous priors. Framed in this way, optimal expectations is a theory of prior beliefs for Bayesian rational agents.

The key assumption that agents derive current utility from expectations of future pleasures has its roots in the origins of utilitarianism. Detailed expositions on anticipatory utility can be found in the work of Bentham, Hume, Böhm-Barwerk and other early economists. More recently, the temporal elements of the utility concept have re-emerged in research at the juncture of psychology and economics (Lowenstein (1987), Kahneman, Wakker, and Sarin (1997), Kahneman (2000)), and have been incorporated formally into economic models in the form of belief-dependent utility by Geanakoplos, Pearce, and Stacchetti (1989), Caplin and Leahy (2001), and Yariv (2001). In particular, Caplin and Leahy (2000) shows that competitive equilibria are generically intertemporally sub-optimal and so opens the door for belief distortion to increase well-being.

Several papers in economics study related models in which forward-looking agents distort beliefs.[2] In particular, Akerloff and Dickens (1982) models agents as choosing beliefs to minimize their discomfort from fear of bad outcomes. In a two-period model, agents with rational beliefs choose an industry to work in, understanding that in the second period they will distort their beliefs about the hazards of their work and perhaps not invest in safety technology. Landier (2000) studies a two-period game in which agents choose a prior before receiving a signal and subsequently taking an action based on their updated beliefs. Unlike our approach, belief dynamics are not Bayesian; common to our approach, agents tend to save less and be optimistic about portfolio returns. Finally, Bénabou and Tirole (2002) and Harbaugh (2002) analyze belief biases as resulting from conflicting multiple selves that play intra-personal games constrained by self-reputation. While our approach is not directly related to such settings, models of intra-personal games, bounded rationality, and incomplete memory suggest mechanisms for how households achieve optimal expectations in the face of possibly contradictory data.[3]

The structure of the paper is as follows. In Section 2, we introduce the general optimal expectations framework. Subsequently, we use the optimal expectations framework to study behavior in three different canonical economic settings. Section 3 studies a two-period two-asset portfolio choice problem and shows that agents hold beliefs that are biased towards the belief that their investments will pay off well. Section 4 shows that in a two-agent economy of this type with no aggregate risk, optimal expectations are heterogeneous and agents gamble against one another. Section 5 analyzes a classical consumption-savings problem of an agent with quadratic utility receiving stochastic labor income over time and shows that the agent is biased towards optimism and is overconfident, and so saves less than a rational agent. Section 6 analyzes the

[2] There is also a large body of research, begnining with Simon (1982), in which non-rational beliefs arise from the bounded abilities of agents. More recent examples of this modelling approach include Sims (2001) in which agents have limited information processing capacity and Rubinstein (1998) in which agents have limited memory.

[3] For example, in Mullainathan (2002) individuals forget and remember events based on associativeness and rehearsal. Agents then optimize not taking into account the fact that their memories and therefore beliefs are inaccurate. Our model shows that it is optimal for agents to ignore the biases in their beliefs in some contexts.

decision of when to undertake a single costly task for which the costs of action are stochastic over time. Section 7 concludes.

2 The optimal expectations framework

To solve for optimal beliefs and the resulting actions of agents, we proceed in two steps. First, we describe the problem of the agent given an arbitrary set of beliefs. At any point in time, agents maximize felicity, the present discounted value of expected flow utilities. Second, optimal expectations are the set of beliefs that maximize lifetime well-being in the initial period. Lifetime well-being is the sum of the agent's felicities at each point in time and is a function of the agent's beliefs and the actions these beliefs induce.

2.1 Optimization given beliefs

Consider a wide and canonical class of optimization problems. In each period from 1 to T, agents take their beliefs as given and choose a vector of control variables, c_t, and the implied evolution of a vector of state variables, x_t, to maximize their happiness. We consider first a world where the uncertainty can be described by a finite number of states, S.[4] Let $\pi\left(s_t|\underline{s}_{t-1}\right)$ denote the true probability that state $s_t \in S$ is realized after state history $\underline{s}_{t-1} \in \underline{S}_{t-1}$. We depart from the canonical model in allowing agents to believe subjective probabilities that are not necessarily the same as objective probabilities. Conditional and unconditional subjective probabilities are denoted by $\hat{\pi}\left(s_t|\underline{s}_{t-1}\right)$ and $\hat{\pi}\left(\underline{s}_t\right)$ respectively, and satisfy the basic properties of probabilities (precisely specified subsequently).

The happiness/felicity of an agent at time t depends on current utility flows and subjective expected future utility flows,

$$\hat{E}_t \sum_{\tau=0}^{T-t} \beta^{\tau} u\left(x_{t+\tau}, c_{t+\tau}\right), \tag{1}$$

where $\hat{E}_t$ is the subjective expectations operator, $0 < \beta < 1$, and $u\left(x_{t+\tau}, \cdot\right)$ is the flow utility function which is increasing and concave in $c_{t+\tau}$. Crucially, felicity depends on the complete set of subjective conditional beliefs, denoted $\{\hat{\pi}\}$. If β were zero, there would be no forward-looking

[4] Appendix A defines optimal expectations for the situation with a continuous state space.

behavior. While it is not crucial for our analysis, we allow the possibility that felicity also depends on past utility flows, which we capture by the function $W_t\left(\underline{x}_{t-1}, \underline{c}_{t-1}\right) = \sum_{\tau=1}^{t-1} \delta^{t-\tau} u\left(x_{t-\tau}, c_{t-\tau}\right)$ where $0 \leq \delta < 1$ and $\underline{x}_{t-1}$ and $\underline{c}_{t-1}$ denote the histories of x and c respectively. In the special case with $\delta = 0$ the agent has no experienced utility and current felicity collapses to equation (1).

The agent's problem is standard: at each time t, the agent chooses control variables to maximize felicity subject to the evolution of the state variables, the initial level of x_t and terminal conditions on x_{T+1}. Formally,

$$V_t\left(x_t; \underline{s}_t, \{\hat{\pi}\}\right) = \max_{\{c_t\}} \ \hat{E}_t \sum_{\tau=0}^{T-t} \beta^{\tau} u\left(x_{t+\tau}, c_{t+\tau}\right)$$

subject to

$$x_{t+1} = g\left(x_t, c_t, s_{t+1}\right) \quad (2)$$

$$h\left(x_{T+1}\right) \geq 0 \quad (3)$$

where the agent takes x_t as given, and $g\left(\cdot\right)$ is continuous and differentiable in x and c.

Denoting the optimal choices of the control as $\{c_t^*\left(\underline{s}_t, \{\hat{\pi}\}\right)\}$, the induced path of the state variable as $\{x_t^*\left(\underline{s}_t, \{\hat{\pi}\}\right)\}$, and the corresponding histories as $\underline{c}^*\left(\underline{s}_t, \{\hat{\pi}\}\right)$ and $\underline{x}^*\left(\underline{s}_t, \{\hat{\pi}\}\right)$, we can write V and W in a recursive formulation as:

$$V_t\left(x_t; \underline{s}_t, \{\hat{\pi}\}\right) = \max_{c_t} \ u\left(x_t, c_t\right) + \beta \sum_{s_{t+1} \in S} \hat{\pi}\left(s_{t+1} | \underline{s}_t\right) V_{t+1}\left(g\left(x_t, c_t, s_{t+1}\right); s_{t+1}, \underline{s}_t, \{\hat{\pi}\}\right) \quad (4)$$

$$\text{subject to (2) and (3)} \quad (5)$$

$$W_t\left(\underline{x}_{t-1}^*, \underline{c}_{t-1}^*\right) = \delta u\left(x_{t-1}^*, c_{t-1}^*\right) + \delta W_{t-1}\left(\underline{x}_{t-2}^*, \underline{c}_{t-2}^*\right) \quad (6)$$

$$W_1 = 0, \quad V_{T+1} = 0 \quad (7)$$

So far we have focused on the optimization problem of a single agent. In a competitive economy, each agent faces this maximization problem taking as given his beliefs and the stochastic process of payoff-relevant aggregate variables, and markets clear. Beliefs may differ across agents. Specifically, x_t^i includes the payoff-relevant variables that the agent takes as given, and so reflects the actions of all other agents in the economy. The equilibrium choice of control variables for each agent implies an equilibrium allocation $\{\underline{x}_T, \underline{c}_T\}$, where $\{\underline{c}_T\}$ without a superscript i denotes *all agents'* control variable *along any possible path* of the event tree. In sum, the problem remains standard, with the exception that agents' prior beliefs may be heterogeneous.

2.2 Optimal beliefs

Subjective beliefs are a complete set of conditional probabilities for each branch after any history of the event tree, $\{\hat{\pi}(s_t|\underline{s}_{t-1})\}$. We require that subjective probabilities satisfy four properties.

Assumption 1 (Restrictions on probabilities)

(i) $\sum_{s_t \in S} \hat{\pi}(s_t|\underline{s}_{t-1}) = 1$

(ii) $\hat{\pi}(s_t|\underline{s}_{t-1}) \geq 0$

(iii) $\hat{\pi}(\underline{s}'_t) = \hat{\pi}(s'_t|\underline{s}'_{t-1})\,\hat{\pi}(s'_{t-1}|\underline{s}'_{t-2}) \cdots \hat{\pi}(s'_1)$

(iv) $\hat{\pi}(s_t|\underline{s}_{t-1}) = 0$ if $\pi(s_t|\underline{s}_{t-1}) = 0$.

Assumption 1(i) is simply that probabilities sum to one. Assumptions 1(i) – (iii) imply that the law of iterated expectations holds for subjective probabilities. Assumption 1(iv) implies that in order to believe that something is possible, it must be possible. You cannot believe you will win the lottery unless you buy a lottery ticket.

We further consider the class of problems for which a solution exists and for which $V_t(x_t^*; \underline{s}_t, \{\hat{\pi}\})$ and $W_t(\underline{x}_{t-1}^*, \underline{c}_{t-1}^*)$ are less than infinite. While the conditions to ensure this are standard, we require that these properties hold for all possible subjective beliefs.

Assumption 2 (Conditions on agent's problem)

(i) $E[V_t(x^*(\underline{s}_t, \{\hat{\pi}\}); \underline{s}_t, \{\hat{\pi}\})] < \infty$ for all $\underline{s}_t$ and for all $\{\hat{\pi}\}$ satisfying Assumption 1

(ii) $E[W_t(x^*(\underline{s}_{t-1}, \{\hat{\pi}\}), c^*(\underline{s}_{t-1}, \{\hat{\pi}\}))] < \infty$ for all $\underline{s}_t$ and for all $\{\hat{\pi}\}$ satisfying Assumption 1.

Optimal expectations are the subjective probabilities that maximize expected total or lifetime well-being, $\mathcal{W}$.[5] Following Caplin and Leahy (2000), we define lifetime well-being as the discounted sum of felicity of the agent over its life. Recall that $V_t + W_t$ is felicity, which in turn depends on current and expected future (and past) flow utility, u_t.

[5] A useful alternative for infinite horizon problems is that beliefs maximize average instead of total felicity.

Definition 1 *(i) Optimal expectations (OE) are a set of subjective probabilities* $\left\{\hat{\pi}^{OE}\left(s_\tau|\underline{s}_{\tau-1}\right)\right\}$ *that maximize*

$$\mathcal{W} := E\left[\sum_{t=1}^{T}\beta^{t-1}\left(W_t\left(\underline{x}^*\left(\underline{s}_{t-1},\{\hat{\pi}\}\right),\underline{c}^*\left(\underline{s}_{t-1},\{\hat{\pi}\}\right)\right)+V_t\left(x^*\left(\underline{s}_t,\{\hat{\pi}\}\right);\underline{s}_t,\{\hat{\pi}\}\right)\right)\right] \qquad (8)$$

subject to the four restrictions on subjective probabilities (Assumption 1).

Optimal expectations exist if $c^{OE}\left(\underline{s}_t\right)$ and $x^{OE}\left(\underline{s}_t\right)$ are continuous in probabilities $\hat{\pi}\left(s_t|\underline{s}_{t-1}\right)$ that satisfy Assumption 1 for all t and $\underline{s}_{t-1}$, where for notational simplicity $c^{OE}\left(\underline{s}_t\right) := c^*\left(\underline{s}_t,\left\{\hat{\pi}^{OE}\right\}\right)$ and $x^{OE}\left(\underline{s}_t\right) := x^*\left(\underline{s}_t,\left\{\hat{\pi}^{OE}\right\}\right)$. This follows from the continuity of V_t and W_t in probabilities and controls, Assumption 2, and the compactness of probability spaces. For less regular problems, as for rational expectations equilibria, optimal expectations may or may not exist. As to uniqueness, optimal beliefs need not be unique, as will be clear from the subsequent use of this concept.

In an economy with multiple agents, each agent's beliefs maximize equation (8), where the state variables x^i and control variables c^i are indexed by i, taking the beliefs and actions of the other agents as given.

Definition 2 *A competitive optimal expectations equilibrium (OEE) is a set of beliefs for each agent and an allocation such that*
(i) each agent has optimal expectations, taking as given the stochastic process for aggregate variables;
(ii) each agent solves equation (4) at each t*, taking as given his beliefs and the stochastic process for aggregate variables;*
(iii) markets clear.

Intuitively, an optimal expectations equilibrium (OEE) consists of a set of beliefs for each agent i and the corresponding equilibrium allocations induced by optimization given these beliefs. The optimal beliefs of each agent take as given the aggregate dynamics, and the optimal actions take as given the perceived aggregate dynamics.

2.3 Discussion

Before proceeding to the application of optimal expectations, it is worth emphasizing several points.

First, probabilities, $\hat{\pi}\left(s_t|\text{s}_{t-1}\right)$, are chosen once and forever. Thus, subjective probabilities are time consistent, and the law of iterated expectations holds with respect to the subjective probability measure. Agents learn and update according to Bayes' rule. This allows the use of standard dynamic programming to solve the agent's optimization problem.

Second, optimal subjective probabilities are chosen without any direct relation to reality. Our view is that this frictionless world provides insight into the behaviors generated by the incentive to look forward with optimism when belief distortion is limited only by the costs of poor outcomes. In fact, it may be that beliefs cannot be distorted far from reality. At some cost in terms of tractability, the frictionless model could be extended to include constraints on the choice of probabilities that penalize larger distortions from reality. Beliefs would then bear some relation to reality even in circumstances in which there are no costs associated with behavior caused by distorted beliefs.

Third, optimal expectations are those that maximize lifetime well-being. The argument that is traditionally made for the assumption of rational beliefs – that such beliefs lead agents to the best outcomes – is correct only if one assumes that expected future utility flows do not affect present felicity. This is a somewhat schizophrenic view: one part of the agent makes plans that trade off present and expected future utility flows, while another part of the agent actually enjoys utils but only from present consumption.[6] Under the Jevonian view that an agent who cares about the future has felicity that depends on expectations about the future, optimal expectations give agents the highest lifetime utility levels.

To recast this point, we can ask what objective function for beliefs would make rational expectations optimal. This is the case if the objective function for beliefs ignores the utility from past or future consumption. But this would imply that if we were to decrease future flow utility while keeping present flow utility constant, this would not change current felicity. Because

[6] See Lowenstein (1987) and the discussion of the Samuelsonian and Jevonian views of utility in Caplin and Leahy (2000).

we view this as implausible, we restrict attention to objective functions for beliefs defined over V and W.

Fourth, this discussion also makes clear why lifetime well-being, $\mathcal{W}$, uses the objective expectations operator. Optimal beliefs are not those that maximize the agent's happiness only in the states that the agent views as most likely. Instead, optimal beliefs maximize the happiness of the agent on average, across repeated realizations of uncertainty. The objective expectation captures this since the actual unfolding of uncertainty over the agent's life is determined by objective probabilities.

Fifth, for most of our results we do not need to specify the value of δ. In particular, this allows for $\delta = 0$, in which case the analysis simplifies since all "experience utility" terms W_t vanish. Another special case that is of particular interest is the case in which $\delta = \beta^{-1}$, which we refer to as *preference consistency.* This case lies outside the set assumed by our definitions. However, this parameter configuration has the special feature that the objective function for beliefs evaluated for rational beliefs is identical to the objective function of the agent. Thus, under rational expectations, lifetime well-being is maximized by the decisions of the agent.[7]

Finally, it is also the case that optimal expectations could be derived from generalized objective functions. In particular, an earlier version of this paper considered the possibility that the rate at which future felicity is discounted in the objective function for beliefs differed from the rate at which the agent discounted future utility flows.

3 Portfolio choice: optimism and underdiversification

In this section we consider a two-period investment problem in which an agent chooses between assets in the first period of life and consumes the payoff of the portfolio in the second period of life. The subsequent section places two of these agents into a general equilibrium model with no aggregate risk.

[7] Caplin and Leahy (2000) in the context of a rational expectations model emphasizes that preference consistency is needed for an agent's ranking of utility flows across periods to be time-invariant. Thus it is the only case of time-consistent preferences.

3.1 Portfolio choice given beliefs

There are two periods, zero and one, and two assets. In period zero, the agent allocates his unit endowment between a risk-free asset with gross return R and a risky asset with gross return $R+Z$ (Z is the excess return of the risky asset over the risk-free rate). In period one, the agent consumes the payoff from his period-zero investment.

The agent chooses his portfolio share to invest in the risky asset, w, to maximize expected utility:

$$\begin{aligned} \max_{w} \quad & \beta \hat{E}_0 \left[u\left(c\right) \right] \\ \text{s.t.} \quad & c = R + wZ \\ & c \geq 0 \text{ in all states} \end{aligned}$$

where $u(\cdot)$ is the utility function over consumption, $u' > 0$, $u'' < 0$ and satisfying the Inada conditions. The second constraint is set by the market. Since consumption cannot be negative, the constraint follows from the market requiring the agent to be able to meet his payment obligations in all future states.

Uncertainty is characterized by S states, with ex post excess return Z_s and probabilities $\pi_s > 0$ for $s = 1, \ldots, S$. Let the states be ordered so that the larger the state, the larger the payoff, $Z_{s+1} > Z_s$, and $-R < Z_1 < 0 < Z_S < R$, $Z_s \neq Z_{s'}$ for $s \neq s'$. Beliefs are given by $\{\hat{\pi}_s\}_{s=1}^{S}$ satisfying Assumption 1.

Noting that the second constraint can only bind for the highest or lowest payoff state, the agent problem can be written as a Lagrangian with multipliers λ_1 and λ_S,

$$\max_{w} \beta \sum_{s=1}^{S} \hat{\pi}_s u\left(R + wZ_s\right) - \lambda_1 \left(R + wZ_1\right) - \lambda_S \left(R + wZ_S\right).$$

The necessary conditions for an optimal w are

$$\begin{aligned} 0 &= \sum_{s=1}^{S} \hat{\pi}_s u'\left(R + wZ_s\right) Z_s - \lambda_1 Z_1 - \lambda_S Z_S, \\ 0 &= \lambda_1 \left(R + wZ_1\right), \\ 0 &= \lambda_S \left(R + wZ_S\right). \end{aligned}$$

By complementary slackness, if $\lambda_s \neq 0$ for some s, then $R + wZ_s = 0$. By the Inada condition, this can only occur when $\hat{\pi}_s = 0$ since otherwise $\hat{\pi}_s u'(R + wZ_s) = \infty$. We show subsequently that optimal expectations entail $\hat{\pi}_1 > 0$ and $\hat{\pi}_S > 0$, so that the optimal portfolio is uniquely determined by

$$0 = \sum_{s=1}^{S} \hat{\pi}_s u'(R + wZ_s) Z_s. \tag{9}$$

3.2 Optimal beliefs

Optimal beliefs are a set of $\hat{\pi}_s$ for $s = 1, \ldots, S-1$ with $\hat{\pi}_S = 1 - \sum_{s=1}^{S-1} \hat{\pi}_s$ that maximize total well-being, the expected discounted sum of felicities in periods 0 and 1. In period 0, the agent's felicity is the subjectively expected (anticipated) utility flow in the future period, discounted by β; in period 1, the agent's felicity is the utility flow from actual consumption.

$$\max_{\hat{\pi}} \quad E\left[\beta \hat{E}_0 [u(c)] + \beta u(c)\right]$$

$$\max_{\hat{\pi}} \quad \beta \sum_{s=1}^{S} \hat{\pi}_s u(R + w^* Z_s) + \beta \sum_{s=1}^{S} \pi_s u(R + w^* Z_s)$$

where w^* is given implicitly by equation (9). The first-order conditions for the choice of $\hat{\pi}_s$ are

$$0 = \beta u_{s'} - \beta u_S + \beta \sum_{s=1}^{S} \hat{\pi}_s u'(R + w^* Z_s) Z_s \frac{dw^*}{d\hat{\pi}_{s'}} + \beta \sum_{s=1}^{S} \pi_s u'(R + w^* Z_s) Z_s \frac{dw^*}{d\hat{\pi}_{s'}}.$$

By the envelope condition, small changes in portfolio choice from the optimum caused by small changes in subjective probabilities lead to no change in expected utility, so that this condition simplifies to

$$\beta (u_S - u_{s'}) = \beta \sum_{s=1}^{S} \pi_s u'(R + w^* Z_s) Z_s \frac{dw^*}{d\hat{\pi}_{s'}}. \tag{10}$$

The left-hand side is the marginal gain in 'dream utility' at t from increasing $\hat{\pi}_S$ at the expense of $\hat{\pi}_{s'}$ and is always positive; the right-hand side is the marginal loss in expected utility in $t+1$ from the resultant change in the portfolio share of the risky asset. In equilibrium, the gain in dream utility balances the costs of distorting actual behavior.

Let w^{RE} denote the optimal portfolio choice for rational beliefs. The following proposition, proven in the appendix, states that the agent with optimal expectations is optimistic about the payout of his portfolio. Further, the agent with optimal expectations either takes an opposite position relative to the agent with rational beliefs or is more aggressive – investing even more if the rational agent invests, or shorting more if the rational agent shorts.

Proposition 1 *(Excess risk taking due to optimism)*
(i) if $w^* > 0$, $\sum_{s=1}^{S} (\hat{\pi}_s - \pi_s) u' (R + w^* Z_s) Z_s > 0$*; if* $w^* < 0$, $\sum_{s=1}^{S} (\hat{\pi}_s - \pi_s) u' (R + w^* Z_s) Z_s < 0$.
(ii) if $w^{RE} > 0$*, then* $w^* > w^{RE}$ *or* $w^* < 0$*; if* $w^{RE} < 0$*, then* $w^* < w^{RE}$ *or* $w^* > 0$*; if* $w^{RE} = 0$, $w^* \neq 0$.

The first part of the proposition states that agents with optimal expectations on average hold beliefs that are biased upward for states in which their chosen portfolio payout is high and biased downward for states in which their portfolio payout is low. To see this, note that $u'_s > 0$ for all s, and Z_s is positive for large s and negative for small s. For $w^* > 0$, optimal expectations on average bias up the subjective probability for larger or positive excess return states at the expense of smaller or negative excess return states.

The second part of the proposition states that households either invest more than the rational agent in the risky asset or they short the risky asset. The agent shorts the risky asset if this allows sufficient dream utility, which happens when the asset has the features of a negative lottery ticket. An example clarifies. Let there be only two states and let $R = 1$, $E[Z] = \varepsilon$, $Z_1 = 2\varepsilon$, so that $Z_2 = -\frac{2\pi_1 - 1}{\pi_2}\varepsilon$ and consider large π_1. State 1 is likely and involves little gain; state 2 is unlikely and involves large loss (as $\pi_1 \to 1$, $Z_2 \to -\infty$). $w^{RE} > 0$ since the asset has positive expected value. Assuming log utility, equation (9) implies

$$\begin{aligned} w^{RE} &= \frac{1}{4\varepsilon} \frac{(1 - \pi_1)}{\left(\pi_1 - \frac{1}{2}\right)}, \\ w^{\hat{\pi}} &= w^{RE} + \frac{1}{4\varepsilon} \frac{(\hat{\pi}_1 - \pi_1)}{\left(\pi_1 - \frac{1}{2}\right)}, \end{aligned}$$

for rational beliefs and arbitrary subjective beliefs, respectively. For π_1 near unity, w^{RE} is near zero since the asset's variance is large. As long as $\hat{\pi}_1 \geq \pi_1$, then the agent with subjective beliefs invests more in the risky asset. For π_1 near unity, $\hat{\pi}_1 - \pi_1$ is near zero – subjective beliefs are necessarily near rational beliefs – and $w^{\hat{\pi}}$ is near w^{RE}. However, in this case, if the agent instead is pessimistic about the payoff of the risky asset, $\hat{\pi}_1 < \pi_1$, then he can short the asset and dream about the asset paying off poorly. In fact, for π_1 near unity, $\hat{\pi}_1^{OE} < \pi_1$ and w^{OE} is negative. For our example, when $\pi_1 = 0.90$, the agent is optimistic about the risky asset, $\hat{\pi}_1^{OE} = 0.96$, and invests more in the risky asset. When $\pi_1 = 0.95$, the agent is pessimistic about the risky asset, $\hat{\pi}_1^{OE} = 0.31$, and he shorts the risky asset.

Returning to a general utility function and payout, suppose $E[Z] = 0$. The rational agent chooses $w^{RE} = 0$ since the expected return on the risky asset is the same as the risk-free asset and the risky asset is risky. The agent with optimal expectations holds a nonzero amount of the risky asset. To see this, consider the rational expectations equilibrium and consider a deviation of beliefs from rational beliefs. This deviation leads the agent to choose a nonzero holding of the risky asset which implies that consumption is no longer perfectly smoothed across future states. The associated welfare cost is of second order. This is dominated by the gain in dream utility. The agent can now believe that he is holding (shorting) an asset with positive (negative) expected payoff.

This result implies that agents with optimal expectations are underdiversified in that they are not on the efficient frontier. That is, a portfolio with the same return and less risk is available.

Corollary 1 *(Underdiversification)*
For $E[Z] = 0$, $E[R + w^* Z_s] = E\left[R + w^{RE} Z_s\right]$ *but* $Var[R + w^* Z_s] > Var\left[R + w^{RE} Z_s\right]$.

4 General equilibrium: endogenous heterogenous beliefs and gambling

In this section, we consider an exchange economy with no aggregate risk and derive endogenous heterogeneous beliefs. Agents choose to hold idiosyncratic risk and gamble against one another in equilibrium when perfect consumption insurance is possible. The price of the risky asset may differ from that in an economy populated by agents with rational beliefs.

The economy consists of two agents with the same characteristics and facing the same investment problem as in the previous section. There are two states in the second period and two assets (or "trees"), denoted b (bonds) and e (equity). Asset b pays 1 unit of consumption in both states; we normalize the return on this tree to 1 (R equals unity). Asset e returns

$1 + Z$ and the ex post return on asset e is $1 + Z_s = \frac{1+\varepsilon_s}{P_e}$ where s indexes states. We assume $-1 < \varepsilon_1 < \varepsilon_2$. Agent i is initially endowed with X_b^i bonds and X_e^i equity shares. There is no aggregate uncertainty so that asset e is in zero net supply. Aggregate consumption in each state is thus the same, $\sum_{i=1}^{2} X_b^i = X_b = C_1 = C_2$.

Agent i's problem is to take his beliefs, $\{\pi_s^i\}$, and the price of equity, P_e, as given and choose her portfolio to maximize expected utility,

$$\max_w \beta \sum_s \hat{\pi}_{i,s} u\left(A^i\left(1 + w^i Z\right)\right)$$

given initial wealth

$$A^i = X_b^i + P_e X_e^i.$$

The first-order conditions for portfolio choice are

$$0 = \sum_s \hat{\pi}_s^i u'\left(c_s^i\right)\left((1 + \varepsilon_s) - P_e\right).$$

In aggregate, beliefs maximize the lifetime well-being for each agent

$$\max_{\hat{\pi}} \sum_s \left[\beta \hat{\pi}_s^i u\left(c_s^i\right) + \beta \pi_s u\left(c_s^i\right)\right]$$

subject to the restrictions on probabilities (Assumption 2.2), the budget constraint (the definition of consumption), and the agent's first-order conditions for portfolio choice. The Lagrangian for each agent is

$$\begin{aligned}
\mathcal{L} &= \sum_s \left[\beta \hat{\pi}_s^i u\left(\left(X_b^i + P_e X_e^i\right)\left(1 + w^i\left(\tfrac{1+\varepsilon_s}{P_e} - 1\right)\right)\right) + \beta \pi_s u\left(\left(X_b^i + P_e X_e^i\right)\left(1 + w^i\left(\tfrac{1+\varepsilon_s}{P_e} - 1\right)\right)\right)\right] \\
&\quad -\lambda^i\left[\sum_s \hat{\pi}_s^i - 1\right] - \mu^i\left[\sum_s \hat{\pi}_s^i u'\left(\left(X_b^i + P_e X_e^i\right)\left(1 + w^i\left(\frac{1+\varepsilon_s}{P_e} - 1\right)\right)\right)\left((1 + \varepsilon_s) - P_e\right)\right]
\end{aligned}$$

The equilibrium for the economy as a whole is characterized by the first-order conditions for each agent

$$w^i \;:\; 0=\sum_s\left[\left(\beta\hat{\pi}_s^i+\beta\pi_s\right)u'\left(c_s^i\right)\left(\frac{1+\varepsilon_s}{P_e}-1\right)\right] \tag{11a}$$

$$-\mu^i\left[\sum_s\hat{\pi}_s^i u''\left(c_s^i\right)\left(\frac{1+\varepsilon_s}{P_e}-1\right)^2 P_e\right] \text{ for } i=1,2$$

$$\hat{\pi}_s^i \;:\; 0=\beta u\left(c_s^i\right)-\lambda^i-\mu^i\left[u'\left(c_s^i\right)\left(\frac{1+\varepsilon_s}{P_e}-1\right)P_e\right] \text{ for } i=1,2,\; s=1,2 \tag{11b}$$

$$\lambda^{i:} \;:\; 0=\lambda^i\left[\sum_s\hat{\pi}_s^i-1\right] \text{ for } i=1,2$$

$$\mu^i \;:\; 0=\mu^i\left[\sum_s\hat{\pi}_s^i u'\left(c_s^i\right)\left((1+\varepsilon_s)-P_e\right)\right] \text{ for } i=1,2 \tag{11c}$$

and the aggregate resource constraint

$$-P_e\sum_i X_e^i w^i=\sum_i X_b^i w^i, \tag{12}$$

where $c_s^i=\left(X_b^i+P_eX_e^i\right)\left(1+w^i\left(\frac{1+\varepsilon_s}{P_e}-1\right)\right)$.

Before characterizing the equilibrium, we define gambling. Let x_e^i be agent i's equilibrium holding of equity, $x_e^i=w^i\frac{\left(X_b^i+P_eX_e^i\right)}{P_e}$. Agents are gambling against each other in an equilibrium $(*)$ if

$$x_e^{*,i}\neq x_e^{RE,i} \text{ for } i=1,\,2.$$

This says that the amount of equity held by each agent is different in the $(*)$ equilibrium than in the rational expectations equilibrium.

Note that gambling is not implied by a difference between the equilibrium price of equity under rational and optimal expectations, $P_e^{OE}\neq P_e^{RE}$. Gambling results from disagreement about probabilities. Put differently, at the optimal expectations equilibrium we have for each agent

$$\sum_s\hat{\pi}_s u'\left(c_s^{OE,i}\right)\left(\frac{1+\varepsilon_s}{P_e^{OE}}-1\right)=0.$$

If agents are not gambling, then it is also true that

$$\sum_s\pi_s u'\left(c_s^{OE,i}\right)\left(\frac{1+\varepsilon_s}{P_e^{RE}}-1\right)=0.$$

Thus if optimal beliefs differ from objective beliefs, the price of equity will differ in the rational expectations and optimal expectations equilibria even without gambling. Mispricing does not imply gambling. We now state our proposition.

Proposition 2 *(Gambling without aggregate risk)*
(i) $w^{RE,i} = 0$, $x_e^{RE,i} = 0$
(ii) Agents gamble, $x_e^{OE,1} \neq x_e^{RE,i} \neq x_e^{OE,2}$
(iii) In any equilibrium $\hat{\pi}_1^i > \pi_1$, $\hat{\pi}_2^i < \pi_2$, $w^i < 0$, $c_1^i > c_2^i$, *and* $\hat{\pi}_2^{-i} > \pi_2$, $\hat{\pi}_1^{-i} < \pi_1$, $w^{-i} > 0$, $c_2^{-i} > c_1^{-i}$.

The first point states that agents in the rational expectations equilibrium perfectly insure their consumption by trading so that neither agent holds the risky asset. The second point follows from the same logic as in the partial equilibrium section. Agents have a loss from a small amount of betting against one another, which is dominated by the gain in dream utility from gambling. The final point emphasizes the source of the gambling. Each agent believes a different state is more likely than it really is, and they trade so that each of them realizes higher consumption in the state they view as unrealistically likely. Thus, a feature of the optimal expectations equilibrium is endogenous heterogeneity in beliefs.

5 Consumption and saving over time: optimism and overconfidence

This section considers the behavior of an agent with optimal expectations in a multi-period consumption-saving problem with stochastic income. We show that the agent with quadratic utility overestimates the mean of future income and underestimates the uncertainty associated with future income. That is, the agent is both unrealistically optimistic and overconfident. This is consistent with survey evidence that shows that growth rate of expected consumption is greater than that of actual consumption.

5.1 Optimal consumption

In each period $t = 1, \ldots, T$, the agent chooses consumption and saving to maximize the expected present discounted value of utility flows from consumption subject to a budget constraint.

$$\begin{aligned} V_t(A_t) &= \max_{\{c_t\}} \hat{E}\left[\sum_{\tau=0}^{T-t} \beta^{\tau} u(c_{t+\tau}) \,|\underline{\mathrm{y}}_t\right] \\ \text{s.t.} \quad & \sum_{\tau=0}^{T-t} R^{-\tau}(c_{t+\tau} - y_{t+\tau}) = A_t \qquad \text{(BC)} \\ u(c_{t+\tau}) &= ac_{t+\tau} - \frac{b}{2} c_{t+\tau}^2 \end{aligned}$$

where initial wealth $A_1 = 0$, $a, b > 0$ and $\beta R = 1$. The only uncertainty is over income, y_t. y_t is continuously distributed, has support $[\underline{y}, \bar{y}] = Y$ where $0 < \underline{y} < \bar{y} < \frac{a}{bT}$, is independent over time so that $\Pi\left(y_t|\underline{\mathrm{y}}_{t-1}\right) = \Pi(y_t)$, and $d\Pi(y_t) > 0$ for all $y \in Y$. Subjective probabilities are denoted by $\hat{\Pi}\left(y_t|\underline{\mathrm{y}}_{t-1}\right)$ and do not have to be independently distributed over time. Appendix A states the restrictions on probabilities and objective functions for a continuous state space.

Assuming an interior solution, the necessary conditions for an stage-two optimum are, for $t = 1$ to $T - 1$,

$$\begin{aligned} 0 &= u'(c_t) - \beta R \int_{y_{t+1} \in Y} \frac{dV_{t+1}\left(g(A_t, c_t, y_{t+1}), \underline{\mathrm{y}}_t\right)}{dA_{t+1}} d\hat{\Pi}\left(y_{t+1}|\underline{\mathrm{y}}_t\right) \\ \frac{dV_t(A_t)}{dA_t} &= \beta R \int_{y_{t+1} \in Y} \frac{dV_{t+1}\left(g(A_t, c_t, y_{t+1}), \underline{\mathrm{y}}_t\right)}{dA_{t+1}} d\hat{\Pi}\left(y_{t+1}|\underline{\mathrm{y}}_t\right). \end{aligned}$$

Combining these conditions and the assumption of quadratic utility gives the Hall random walk result for consumption but for subjective beliefs

$$c_t\left(A_t, \underline{\mathrm{y}}_t\right) = \hat{E}\left[c_{t+1}\left(A_{t+1}, \underline{\mathrm{y}}_{t+1}\right) |\underline{\mathrm{y}}_t\right]. \qquad (13)$$

Substituting back into the budget constraint gives the optimal consumption rule

$$c_t^*\left(\underline{\mathrm{y}}_t\right) = \frac{1-R^{-1}}{1-R^{-(T-t)}}\left(A_t + y_t + \sum_{\tau=1}^{T-t} R^{-\tau} \hat{E}\left[y_{t+\tau}|\underline{\mathrm{y}}_t\right]\right). \qquad (14)$$

Optimal consumption depends on subjective expectations of future income and the history of income realizations through A_t. Because quadratic utility exhibits certainty equivalence from the perspective of the agent, the problem simplifies significantly. Given the subjective expectation of future income, the subjective variance (and higher moments) of the income process are irrelevant for the optimal consumption-saving choices of the agent.

5.2 Optimal beliefs

We want to choose $\hat{\Pi}\left(y_t|\underline{y}_{t-1}\right)$ for all $\underline{y}_{t-1}$to maximize (weighted) lifetime happiness subject to the probability conditions and the agent's optimal behavior given beliefs. With quadratic utility, the elements of the objective function for beliefs are

$$\begin{aligned} W_t\left(\underline{c}_{t-1}\right) &= \sum_{r=1}^{t-1}\delta^r\left(ac^*_{t-r}-\frac{b}{2}\left(c^*_{t-r}\right)^2\right) \\ V_t\left(A_t;\underline{y}_t\right) &= \sum_{r=0}^{T-t}\beta^r\hat{E}\left[ac^*_{t+r}-\frac{b}{2}\left(c^*_{t+r}\right)^2|\underline{y}_t\right]. \end{aligned}$$

Since the objective is concave in future consumption and since the agent's behavior depends only on the subjective certainty-equivalent of future income, optimal beliefs minimize subjective uncertainty. Thus, future income is optimally perceived as certain, which is an extreme form of overconfidence.

We incorporate optimal behavior directly into the value functions and characterize consumption choices implied by optimal beliefs, $\left\{c^*_t\left(\underline{y}_t\right)\right\}$. Optimal beliefs, $\left\{\hat{\Pi}\left(\underline{y}_t\right)\right\}$, implement these consumption choices given optimal behavior on the part of the agent. In taking this approach, we are assuming that the optimal choice of $c^*_t\left(\underline{y}_t\right)$ and thus $\hat{E}\left[y_{t+\tau}|\underline{y}_t\right]$ does not require violation of the assumptions on probability, which can be checked.[8]

To proceed, optimal behavior is summarized by the agent's Euler equation. Using the Euler equation and the fact that subjective certainty implies $\hat{E}\left[u\left(c^*_{t+\tau}\right)|\underline{y}_t\right]=u\left(\hat{E}\left[c^*_{t+\tau}|\underline{y}_t\right]\right)$, the

[8] If the support of y_t is small, belief distortion may be constrained by the range of possible income realizations. To incorporate these constraints directly, one instead replaces $c^*_t\left(\underline{y}_t\right)$ using equation (14) and searches for optimal $\hat{E}\left[y_{t+\tau}|\underline{y}_t\right]$ while imposing the constraints imposed by Assumption 1 (iv).

value functions over future consumption become

$$\begin{aligned} V_t\left(A_t;\underline{\mathrm{y}}_t\right) &= \sum_{r=0}^{T-t}\beta^r u\left(\hat{E}\left[c^*_{t+r}|\underline{\mathrm{y}}_t\right]\right) \\ &= u\left(c^*_t\left(\underline{\mathrm{y}}_t\right)\right)\sum_{r=0}^{T-t}\beta^r. \end{aligned}$$

Subjective expectations are chosen to yield the path of $\left\{c^*_t\left(\underline{\mathrm{y}}_t\right)\right\}$ that maximizes

$$\begin{aligned} & E\left[\begin{array}{r} \underbrace{u\left(c^*_1\right)\sum_{\tau=1}^{T}\beta^{\tau-1}}_{V_1}+\beta\underbrace{u\left(c^*_1\right)\delta}_{W_2}+\beta\underbrace{u\left(c^*_2\right)\sum_{\tau=1}^{T-1}\beta^{\tau-1}}_{V_2}+\beta^2\underbrace{\left(u\left(c^*_1\right)\delta^2+u\left(c^*_2\right)\delta\right)}_{W_3}+ \\ \beta^2\underbrace{u\left(c^*_3\right)\sum_{\tau=1}^{T-2}\beta^{\tau-1}}_{V_3}+...+\beta^{T-1}\underbrace{\sum_{\tau=1}^{T-1}\delta^{T-\tau}u\left(c^*_\tau\right)}_{W_T}+\beta^{T-1}\underbrace{u\left(c^*_T\right)}_{V_T} \end{array}\right] \\ = \; & E\left[\sum_{t=1}^{T}\beta^{t-1}\left(1+\sum_{\tau=1}^{T-t}\left(\beta^\tau+\left(\beta\delta\right)^\tau\right)\right)u\left(c^*_t\left(\underline{\mathrm{y}}_t\right)\right)\right] \\ = \; & \sum_{t=1}^{T}\psi_t\int_{\underline{\mathrm{y}}_t\in\times_t Y}u\left(c^*_t\left(\underline{\mathrm{y}}_t\right)\right)d\Pi\left(\underline{\mathrm{y}}_t\right) \end{aligned}$$

subject to the budget constraint and where $\psi_t = \beta^{t-1}\left(1+\sum_{\tau=1}^{T-t}\left(\beta^\tau+\left(\beta\delta\right)^\tau\right)\right)$. Expected consumption growth is given by the first-order conditions

$$u'\left(c^*_t\left(\underline{\mathrm{y}}_t\right)\right)=\frac{\psi_{t+\tau}}{\psi_t}R^\tau\int_{\underline{\mathrm{y}}_{t+\tau}|\underline{\mathrm{y}}_t\in\times_\tau Y}u'\left(c^*_{t+\tau}\left(\underline{\mathrm{y}}_{t+\tau}\right)\right)d\Pi\left(\underline{\mathrm{y}}_{t+\tau}|\underline{\mathrm{y}}_t\right),$$

which implies that the actual path of consumption obeys

$$c^*_t\left(\underline{\mathrm{y}}_t\right)=\frac{a}{b}-\frac{\psi_{t+\tau}}{\psi_t}R^\tau\left(\frac{a}{b}-E\left[c^*_{t+\tau}\left(\underline{\mathrm{y}}_{t+\tau}\right)|\underline{\mathrm{y}}_t\right]\right). \tag{15}$$

Level consumption is recovered by substituting into the budget constraint after taking objective expectations.

Given this characterization of optimal behavior, agents are optimistic at every time and state.

Proposition 3 *(Overconsumption due to optimism)*

For all $t\in\{1,\dots,T-1\}$,

(i) $\hat{E}\left[\sum_{\tau=0}^{T-t-1}R^{-\tau}y_{t+1+\tau}|\underline{y}_t\right]>E\left[\hat{E}\left[\sum_{\tau=0}^{T-t-1}R^{-\tau}y_{t+1+\tau}|\underline{y}_{t+1}\right]|\underline{y}_t\right]$;

(ii) $c^*_t\left(\underline{y}_t\right)>E\left[c^*_{t+1}\left(\underline{y}_{t+1}\right)|\underline{y}_t\right]$;

(iii) $\hat{E}\left[c^*_{t+1}\left(\underline{y}_{t+1}\right)|\underline{y}_t\right]>E\left[c^*_{t+1}\left(\underline{y}_{t+1}\right)|\underline{y}_t\right]$.

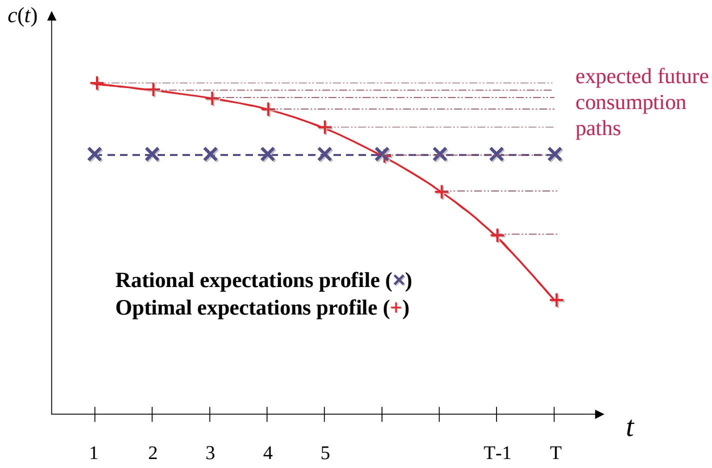

Figure 1: Average Life-cycle Consumption Profiles

The first point of the proposition states that agents overestimate their present discounted value of labor income and on average revise their beliefs downward between t and $t+1$. This downward revision of expected lifetime wealth can come about directly due to y_{t+1} being on average less than expected, or due to news that the realized y_{t+1} brings on average about expectations of future income. The second point states that consumption on average falls between t and $t+1$. Because on average the agent revises down expected future income, on average consumption falls over time. The proof follows directly from the expected change in consumption given by equation (15) and noting that $\frac{a}{b} - c_t^*\left(\underline{\mathrm{y}}_t\right) > 0$ and $\frac{\psi_{t+1}}{\psi_t}R < 1$. Finally, the optimal subjective expectation of future consumption exceeds the rational expectation of future consumption. This is optimism. Part (iii) follows from part (ii) and equation (13). In sum, households are unrealistically optimistic, and, in each period, are on average surprised that their incomes are lower than they expected, and so, on average, household consumption declines over time.

Figure 1 summarizes these results. The agent starts life optimistic about future income. At each point in time the agent expects that on average consumption will remain at the same level. Over time, the agent learns on average that income is less than he expected, and consumption typically declines over the life.

This optimism matches survey evidence on desired and actual life-cycle consumption profiles. Barsky, Juster, Kimball, and Shapiro (1997) finds that households would choose upward sloping consumption profiles. But survey datasets on actual consumption reveal that households have downward sloping or flat consumption profiles (Gourinchas and Parker (2002), Attanasio (1999)). In our model, households expect and plan to have constant marginal utility since $\beta R - 1 = 0$. However, on average marginal utility rises at the age-specific rate $\frac{\psi_t}{\psi_{t+1}} - 1 > 0$. Thus, in the model, the desired rate of increase of consumption exceeds the average rate of increase, as in the real world. Also matching observed household consumption behavior, the model produces average life-cycle consumption profiles that are concave – consumption falls faster (or rises more slowly) later in life.

In general, in consumption-saving problems, the relative curvature of utility and marginal utility determine what beliefs are optimal. Uncertainty about the future enters the objective for beliefs both through the expected future *level* of utility and through the agent's behavior which depends on expected future *marginal* utility. For utility functions with decreasing absolute risk aversion, greater subjective uncertainty leads to greater precautionary saving through the curvature in marginal utility. This has some benefit in terms of less distortion of consumption. In such cases optimal beliefs may consist of a large positive bias for both expected income and its variance.

6 The timing of a task: procrastination, regret, and a context effect

One of the most common forms of overoptimism is the planning fallacy (Buehler, Griffin, and Ross (1994)): people underestimate the time it takes to complete a task. For example, students and teachers repeatedly carry home briefcases stuffed with heavy books and papers in the hope of studying them over the weekend, even though they should know from past experience that their current forecast is not very realistic.

In this section, we consider a problem in which an agent chooses when to undertake a single action with an uncertain effort cost, denoted e_t, that is different in each period. The agent chooses in each period to undertake or postpone the task with the goal of minimizing disutility. We measure the cost in utility terms and its value in each period is realized at the start of that period. Let e_t be independent over time and $e_t \sim U[0,1]$. Each period that the agent postpones the task, she suffers a cost, measured in utility, of ξ, and we allow the action to be postponed indefinitely. To ensure that the solution is nontrivial, we assume that $0 < \xi < \frac{1}{2}$. As an example, consider the choice of when to complete one's income tax forms, given that one expects to receive a refund. Each weekend one has a set of other alternative activities and chooses to do the taxes or postpone them until another weekend.

6.1 Optimal postponement with rational beliefs

The agent's Bellman equation is

$$\begin{aligned} V_t &= \max\left\{-e_t, -\xi + \beta\hat{E}\left[V_{t+1}|\underline{e}_t\right]\right\} \\ \hat{E}\left[V_t\left|\underline{e}_{t-1}\right|\right] &= \max_{e_t^*} \hat{\Pi}\left(e_t^*|\underline{e}_{t-1}\right)\hat{E}\left[-e_t\left|e_t \le e_t^*, \underline{e}_{t-1}\right|\right] + \left(1-\hat{\Pi}\left(e_t^*|\underline{e}_{t-1}\right)\right)\left(-\xi + \beta\hat{E}\left[V_{t+1}\left|\underline{e}_{t-1}\right|\right]\right), \end{aligned}$$

where e_t^* is the optimal cutoff for the effort cost – the agent undertakes the action in period t when $e_t \le e_t^*$. The agent chooses the cutoff $e_t^*\left(\underline{e}_{t-1}\right)$ to maximize the Bellman equation at each possible state. This optimal sequence can be calculated given any set of beliefs.

To set the stage, consider the case in which beliefs are rational. Let $\bar{e}_t$ be the threshold chosen by the agent with rational beliefs. The problem is stationary and the Bellman equation becomes

$$\begin{aligned} E\left[V_t\right] &= \max_{\bar{e}_t} \Pi\left(\bar{e}_t\right)E\left[-e_t|e_t \le \bar{e}_t\right] + \left(1-\Pi\left(\bar{e}_t\right)\right)\left(-\xi + \beta E\left[V_{t+1}\right]\right) \\ E\left[V\right] &= \max_{\bar{e}_t} -\bar{e}\frac{\bar{e}}{2} + \left(1-\bar{e}\right)\left(-\xi + \tfrac{1}{1+\beta}E\left[V\right]\right). \end{aligned}$$

The necessary conditions for an optimum imply that $\frac{dE[V^*]}{d\bar{e}} = 0$ so that the first-order condition yields

$$\bar{e} = \xi - \beta E\left[V\right].$$

Substituting back into the Bellman equation and using the quadratic formula gives the optimal cutoff for an agent with rational expectations

$$\bar{e} = -\frac{1-\beta}{\beta} + \frac{1}{\beta}\sqrt{\left(1-\beta\right)^2 + 2\beta\xi}.$$

6.2 Optimal beliefs

We search for the optimal beliefs and associated actions together. The value functions are

$$\begin{aligned} V_t &= \hat{E}\left[-\sum_{s=0}^{\infty}\beta^s\left\{\mathbf{1}_{[e_{t+s}\le e^*_{t+s}]}e_{t+s}+\left(1-\mathbf{1}_{[e_{t+s}\le e^*_{t+s}]}\right)\xi\right\}\mathbf{1}_{[\underline{e}_{t+s-1}>\underline{e}^*_{t+s-1}]}|\underline{e}_t\right] \\ W_t &= -\sum_{s=1}^{t-1}\delta^{t-s}\left\{\mathbf{1}_{[e_{t-s}\le e^*_{t-s}]}e_{t-s}+\left(1-\mathbf{1}_{[e_{t-s}\le e^*_{t-s}]}\right)\xi\right\}\mathbf{1}_{[\underline{e}_{t-s-1}>\underline{e}^*_{t-s-1}]} \end{aligned}$$

where $\mathbf{1}_{[\underline{e}_{t+s-1}>\underline{e}^*_{t+s-1}]}=\mathbf{1}_{[e_{t+s-1}>e^*_{t+s-1}]}\cdots\mathbf{1}_{[e_1>e^*_1]}$.

We can summarize expectations about the future at time t by the value of the $\hat{E}\left[V_{t+1}|\underline{e}_t\right]$. This also completely determines the decision rule of the agent, which is to undertake the action if $e_t\le\xi-\beta\hat{E}\left[V_{t+1}|\underline{e}_t\right]=:e^*_t$. We can then write the entire objective in a recursive form for general t as a function of e^*_t

$$\max_{\{e^*_t(\underline{e}_t)\}} E\left[J_1\right]$$

where

$$J_t=-\mathbf{1}_{[e_t\le e^*_t]}\tfrac{1}{1-\beta\delta}e_t+\left(1-\mathbf{1}_{[e_t\le e^*_t]}\right)\left\{-\tfrac{1}{1-\beta\delta}\xi+\beta\hat{E}\left[V_{t+1}|\underline{e}_t\right]+\beta J_{t+1}\right\}. \quad (16)$$

The first term on the right is the the indicator function that the agent chooses to undertake the action in t times the total utility cost (effect on V_t and W_{t+s} for all $s\ge 1$) of undertaking the action in period t. The terms in curly brackets are all conditional on postponing, so are pre-multiplied by the indicator function for postponing the action in period t. The first term in curly brackets is the total utility cost (effect on V_t and W_{t+s} for all $s\ge 1$) of the postponement cost suffered in t. The second term in curly brackets, $\beta\hat{E}\left[V_{t+1}|\underline{e}_t\right]$, is the disutility suffered at time t from expecting to undertake the action in the future. The final term in curly brackets, J_{t+1}, captures the future costs of action, of postponing action, and of anticipating undertaking the action.

Inspecting equation (16), whenever the agent's beliefs are such that the action is done today, beliefs about the future are irrelevant for utility. Whenever the agent's beliefs are such that the action is postponed ($e_t>e^*_t$), the agent is best off assuming that the action will be costless to undertake in the next period, $\beta\hat{E}\left[V_{t+1}|\underline{e}_t\right]=0$. Imposing this feature of optimal beliefs simplifies J_t to

$$J_t=-\mathbf{1}_{[e_t\le e^*_t]}\tfrac{1}{1-\beta\delta}e_t+\left(1-\mathbf{1}_{[e_t\le e^*_t]}\right)\left\{-\tfrac{1}{1-\beta\delta}\xi+\beta J_{t+1}\right\}.$$

Taking expectations and using the uniformity of the true distribution leads to

$$\begin{aligned} E\left[J_t\right] &= -\mathbf{P}\left[e_t \leq e_t^*\right] \tfrac{1}{1-\beta\delta} E\left[e_t | e_t \leq e_t^*\right] + \left(1 - \mathbf{P}\left[e_t \leq e_t^*\right]\right) \left\{-\tfrac{1}{1-\beta\delta}\xi + \beta E\left[J_{t+1}\right]\right\} \\ E\left[J\right] &= -e^* \tfrac{1}{1-\beta\delta} \frac{e^*}{2} + \left(1 - e^*\right) \left\{-\tfrac{1}{1-\beta\delta}\xi + \beta E\left[J\right]\right\} \end{aligned}$$

so that

$$E\left[J\right] = -\frac{\frac{1}{1-\beta\delta}\left[\frac{1}{2}\left(e^*\right)^2 + \left(1-e^*\right)\xi\right]}{1-\left(1-e^*\right)\beta}$$

and the first-order condition gives as the optimal cutoff

$$e^* = -\frac{1-\beta}{\beta} + \frac{1}{\beta}\left(\sqrt{1 - 2\beta\left(1+\xi\right) + \beta^2}\right) = \bar{e}.$$

In this case, optimal beliefs do not distort actions.

Proposition 4 *(Optimal timing)*

The optimal expectations equilibrium is described by

$$\left\{ \begin{array}{lll} \text{if } e_t \leq e^*, & \text{then } \hat{E}\left[V_{t+1} | e_t\right] \geq -\xi - e^* & \text{and undertake action} \\ \text{if } e_t > e^*, & \text{then } \hat{E}\left[V_{t+1} | e_t\right] = 0 & \text{and postpone action} \end{array} \right\}$$

While the agent with optimal expectations has the same behavior (decision rule) as the rational agent, this equilibrium has many features of procrastination, also studied in the hyperbolic discounting literature (Strotz (1955-1956), Akerloff (1991), Ainslie (1991), Lowenstein and Prelec (1992), Laibson (1997), Harris and Laibson (2001), O'Donoghue and Rabin (2001)), and as a revealed preference phenomenon by Gul and Pesendorfer (2002). Any agent postponing the action believes that future costs of undertaking the action will be minimal. Thus, the agent is disappointed upon reaching the next period and discovering that the action has costs. An agent with rational beliefs will sometimes regret not taking an action because later costs of action turn out to be high. But the agent with optimal expectations always thinks that she will do the task in the next period and is always disappointed with the subsequent costs of action.[9] Finally, note that in our setting, agents preferences are time-consistent and the law of iterated expectations

[9] Upon undertaking the action, the agent with rational beliefs and the agent with optimal beliefs each have the same probability of wishing ex post that they had undertaken the action earlier. That is, since both types of agent have the same rule for undertaking the action and know past costs, they can each objectively determine if a different strategy would have done better given the realized sequence of costs.

applies to subjective beliefs.[10] This is in contrast to the approach of hyperbolic discounting, where procrastination occurs for reasons of time inconsistency in preferences.

6.3 Intertemporal preference reversal

The most cited experimental evidence for procrastination is intertemporal preference reversal. An agent given the choice between an unpleasant action today and tomorrow is more likely to postpone the action than if the agent were presented with the choice between the same two actions several periods earlier. Optimal expectations agent exhibit this phenomenon. If postponing the action, the agent with optimal expectations intends to undertake the action in the next period with certainty. But in fact the agent may postpone the action again. In contrast, the rational agent correctly assesses the probability of undertaking the action in the future.

For concreteness, suppose that we compare the agent's choices in two situations: a choice at time 1 between taking the action in period τ or in period $\tau + 1$, and the same choice at time τ. Let $a(t)$ denote undertaking the action at t. Let $a(\tau) \succ_{\mathbf{t}} a(\tau')$ denote preference for $a(\tau)$ over $a(\tau')$, given a choice at time t.

Proposition 5 *(Intertemporal preference reversal)*
(i) $a(\tau) \succ_{\mathbf{1}} a(\tau+1)$*, for* $\tau > 1$
(ii) $a(\tau) \prec_{\tau} a(\tau+1)$ *with probability* $\frac{1}{2} - \xi$

The proof is delegated to the appendix. The agent considering the choice of an unpleasant task between two future periods expects the task to be relatively easy, so as to avoid dream disutility or dread as the event approaches. Given this, there is no sense in planning to postpone and suffer the costs of postponing the task, so the agent plans to do the task earlier (in period τ). When the agent faces the choice of undertaking the task in the present, or when the time to undertake the task actually arises, the agent learns the cost of undertaking the action in τ and, if this cost is high, she chooses to postpone the action. Thus, with some probability, the agent

[10] The law of iterated expectations holds somewhat trivially here since the outcome in subsequent periods are zero probability events. But the law holds in general in optimal expectations equilibria and is so extreme in this example because there is only one action to be taken and it is discrete.

chooses to undertake the task later.[11]

6.4 Commitment and a context effect

While agents with optimal expectations exhibit preference reversal, this is not due to problems with self-control. In this model, the agent with optimal expectations cannot gain from an option to commit. But agents with optimal expectations react to the possibility of commitment quite differently than agents with rational expectations. In this section we show that the agent with optimal expectations values the ability to postpone less than the agent with rational expectations. Further, the agent with optimal expectations can be made worse off by an offer of a commitment in exchange for a payment. This is the case because when it is optimal for the agent to reject the offer of commitment, she rejects the offer because she is more pessimistic about future outcomes and perceives a need for flexibility. In other words, when the agent rejects the offer of payment and commitment she also assesses a higher probability on costly future outcomes and so is less happy. The idea that the presentation of a non-chosen alternative affects payoffs is called a "context effect" in the psychology literature.

The situation we consider is that the agent has the option, exercisable at time τ, to take the action at τ or at $\tau+1$. The agent is also offered a choice at time 1 to accept a payment u in return for giving up the option to postpone and being committed to take the action at time τ. We assume for simplicity that the costs ξ are zero prior to τ.

The agent with rational expectations (given the above preferences) has expected value of postponement of $-\beta^{\tau-1}\left(\xi+\beta E\left[e_{\tau+1}\right]\right)$ and value of taking the action in τ of $-\beta^{\tau-1}E\left[e_{\tau}\right]$. So the agent postpones if $e_{\tau}>\xi+\beta E\left[e_{\tau+1}\right]$. Therefore, the value of the problem with the option to postpone is

$$-\beta^{\tau-1}\Pi\left(\xi+\beta E\left[e_{\tau+1}\right]\right)E\left[e_{\tau}|e_{\tau}<\xi+\beta E\left[e_{\tau+1}\right]\right]-\beta^{\tau-1}\left(1-\Pi\left(\xi+\beta E\left[e_{\tau+1}\right]\right)\right)\left(\xi+\beta E\left[e_{\tau+1}\right]\right).$$

[11] Note that any costs of postponement from 1 to $\tau-1$ are irrelevant. That is, if we suppose that the agent was told in period 1 that she would be given the choice in period τ, then the agent would have additional dream disutility for periods 1 to τ that could be minimized by planning to choose to do the action in period τ. But upon reaching period τ and seeing, surprisingly, that $e_{\tau}>0$, the agent would still choose to postpone the action for the reasons just stated.

The value of the problem without the option is $u-\beta^{\tau-1}E\left[e_{\tau}\right]$. So the agent chooses commitment at time 1 if $E\left[V_1\right|$ commitment$] \geq E\left[V_1\right|$ keep option to postpone$]$ or if

$$u \geq \bar{u} := \beta^{\tau-1}\left(\frac{1}{2}\left(\xi+\beta\frac{1}{2}\right)^2-\left(\xi+\beta\frac{1}{2}\right)+\frac{1}{2}\right) \tag{17}$$

Turning to optimal beliefs, suppose that the agent expects the cost in time τ to be minimal, as above. Then she is indifferent at period 1 between commitment and no commitment if no payment is offered, $u=0$, and a small payment would induce commitment. But then the agent may end up paying a significant cost to undertake the action in period τ. Thus, this choice of expectations, optimal when not faced with the ability to commit, may be suboptimal here. It may be optimal for the agent to maintain the belief that there is some risk of a high cost associated with taking the action in period τ. Whether this is optimal or not depends on whether it is optimal to accept the commitment.

To solve the problem for a given payment, we solve for the optimal expectations and action for the agent accepting commitment, subject to the constraint that the agent, given these expectations, chooses to accept commitment. And we solve for the optimal expectations and action for the agent rejecting commitment, subject to the constraint that the agent, given these expectations, chooses to reject commitment. Equilibrium optimal expectations are the expectations that yield the greater lifetime utility. The maximum payment for which the agent still chooses to reject commitment defines the cost of the commitment.

To begin, what beliefs maximize the utility of accepting commitment subject to preferring to accept commitment? From Proposition 5, the agent believes $e_{\tau}=0$ with certainty and

$$\begin{aligned}\hat{E}\left[V_1|\text{commitment}\right] &= -\beta^{\tau-1}\hat{E}\left[e_{\tau}\right]+u \\ &= u\end{aligned}$$

The incentive compatibility constraint is satisfied since turning commitment down leads to a loss of u and the agent believes she will never exercise the option to postpone.

Second, what beliefs maximize the utility of rejecting commitment subject to the constraint that the agent prefers to reject commitment? The constraint is

$$\begin{aligned}\hat{E}\left[V_1|\text{keep option to postpone}\right] &\geq \hat{E}\left[V_1|\text{commitment}\right] \\ \hat{\mathbf{P}}\left(e_{\tau}<e_{\tau}^*\right)\hat{E}\left[e_{\tau}|e_{\tau}<e_{\tau}^*\right]+\left(1-\hat{\mathbf{P}}\left(e_{\tau}<e_{\tau}^*\right)\right)\left(\xi+\beta\hat{E}\left[e_{\tau+1}|e_{\tau}\geq e_{\tau}^*\right]\right) &\geq \hat{E}\left[e_{\tau}\right]-\beta^{1-\tau}u.\end{aligned}$$

The objective and constraint simplify through the following logic (shown formally in Appendix B.5). It is optimal to believe the action is costless whenever the agent chooses to take the action in period τ. This does not affect the constraint since the constraint only affects behavior when $e_\tau \geq e^*_\tau$. Thus $\hat{E}\left[e_t | e_\tau < e^*_\tau\right] = 0$. To make the agent prefer to reject commitment requires that she believe that the costs in τ could be large. Since this expectation does not affect the agent's utility given that she rejects commitment, it is optimal to set $\hat{E}\left[e_\tau | e_\tau > e^*_\tau\right] = 1$. Turning to expectations about $\tau + 1$, the constraint is also loosened by believing that if the action is postponed the costs in $\tau + 1$ are low. This also maximizes the objective, so $\hat{E}\left[e_{\tau+1} | e_\tau \geq e^*_\tau\right] = 0$. Given that expected values are set, $\hat{\mathbf{P}}\left(e_\tau < e^*_\tau\right)$ is given by the constraint that always binds

$$\left(1 - \hat{\mathbf{P}}\left(e_\tau < e^*_\tau\right)\right) = \frac{u}{\beta^{\tau-1}\left(1-\xi\right)}.$$

Finally, given these expectations and subjective probability, the optimal cutoff, e^*_τ, is given by the first-order condition of the objective. As shown in the Appendix B.5, comparing this equilibrium and the equilibrium conditional on accepting the commitment yields the following proposition.

Proposition 6 *(Cost of Commitment)*
Let $u^* := \min\left\{\beta^{\tau-1}\left(1-\xi\right), \beta^{\tau-1}\frac{\frac{1}{2}-\left(\beta\frac{1}{2}+\xi\right)\left[1-\frac{1}{2}\left(\frac{1}{2}\beta+\xi\right)\right]}{1+(1-\beta\delta)(\tau-1)\frac{\xi}{(1-\xi)}}\right\}$.
(i) For $u < u^*$*, optimal beliefs, prior to* τ *are given by*

$$e_\tau = \left\{\begin{array}{ll} 0 & \textit{with probability } 1 - \frac{u}{\beta^{\tau-1}(1-\xi)} \\ 1 & \textit{with probability } \frac{u}{\beta^{\tau-1}(1-\xi)} \end{array}\right\}$$

$$e_{\tau+1} = \left\{\begin{array}{ll} 0 & \textit{if } e_\tau \geq e^*_\tau, \\ \eta_{\tau+1} & \textit{if } e_\tau < e^*_\tau, \end{array}\right\},$$

where $\eta_{\tau+1}$ *is distributed between zero and one such that* $\hat{E}\left[\eta_{\tau+1} | e_\tau < e^*_\tau\right] \geq \frac{1}{2\beta}$ *and the agent rejects the commitment and sets* $e^*_\tau = \frac{\beta}{2} + \xi$.
(ii) For $u \geq u^*$*, optimal beliefs prior to* τ *are given by* $e_\tau = 0$ *and the agent accepts the commitment and undertakes the action in period* τ.

If the agent sells her option and commits to do the task at τ, her expected overall happiness is $\frac{1}{1-\beta\delta}\left[u-\beta^{\tau-1}\frac{1}{2}\right]$. To see this, note that she immediately receives the payment u at $t=1$ and incurs expected effort costs $E\left[e_{\tau}\right]=\frac{1}{2}$ at τ, which has to be discounted by the factor $\beta^{\tau-1}$. (Since subjective beliefs are such that $\hat{E}\left[e_{\tau}\right]=0$, she suffers no dread.) As these flow utilities also affect future happiness through "memory utility", the term is multiplied by the factor $\frac{1}{1-\beta\delta}$.

If, on the other hand, the agent keeps the option, her total well-being is

$$-\beta^{\tau-1}\left(\tau-1\right)\hat{\mathbf{P}}\left(e_{\tau}>e_{\tau}^{*}\right)\xi-\frac{\beta^{\tau-1}}{1-\beta\delta}\left\{\mathbf{P}\left(e_{\tau}<e_{\tau}^{*}\right)E\left[e_{\tau}|e_{\tau}<e_{\tau}^{*}\right]+\mathbf{P}\left(e_{\tau}\geq e_{\tau}^{*}\right)\left[\beta E\left[e_{\tau+1}\right]+\xi\right]\right\}$$

which, substituting for probabilities, becomes

$$-\beta^{\tau-1}\left(\tau-1\right)\frac{u}{\beta^{\tau-1}(1-\xi)}\xi-\frac{\beta^{\tau-1}}{1-\beta\delta}\left\{e_{\tau}^{*}\frac{1}{2}e_{\tau}^{*}+\left(1-e_{\tau}^{*}\right)\left[\beta\frac{1}{2}+\xi\right]\right\}$$

where $e_{\tau}^{*}=\frac{\beta}{2}+\xi$. Keeping the option makes sense only if the agent (subjectively) believes that she might postpone the task in period τ, so we must have $\hat{\mathbf{P}}\left(e_{\tau}>e_{\tau}^{*}\right)>0$. For $e_{\tau}>e_{\tau}^{*}$, she does so, and hence she incurs postponement costs ξ at τ. But even in the $\tau-1$ periods prior to τ the (subjective) expectation of these costs cause additional dread, which is reflected in the first term above. The second term reflects the actual disutility from undertaking the task at τ or $\tau+1$. If the task is carried out only at $\tau+1$, the effort costs are discounted by β and the agent suffers the postponement costs ξ. All these flow utilities are discounted by $\beta^{\tau-1}$ and divided by $(1-\beta\delta)$ to capture the fact that they also affect all future flow utilities through the memory effect.

Our first main result is shown in Figure 1, which plots the lifetime happiness of accepting the commitment device (increasing function) and the lifetime happiness of retaining the option (decreasing function) for different u. For all u such that $u<u^{*}$, the agent does not accept the commitment payment u and the agent's well-being is lower than if this commitment device had not been presented. This occurs because the presence of the offer of commitment changes the agent's optimal beliefs. These beliefs are less optimistic and so the offer of commitment makes the agent worse off even though this commitment is not taken. This is an example of a context effect.

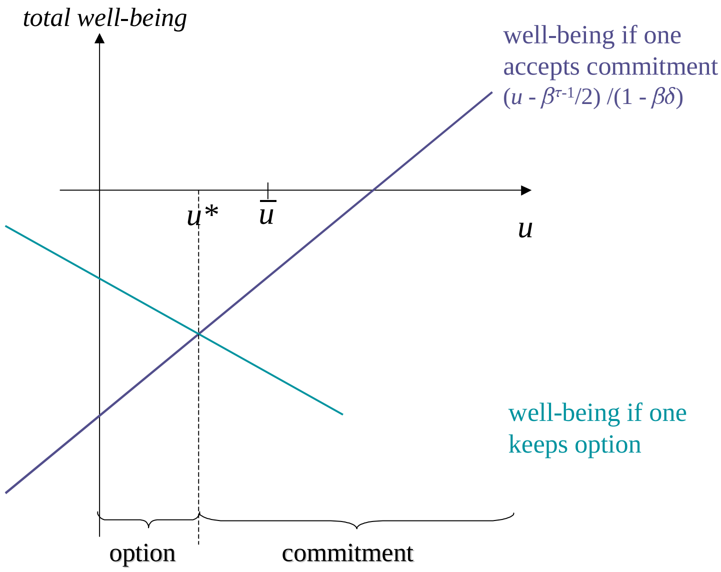

Figure 2: Commitment and Context Effect

Our second main result is that the agent with optimal expectations values the option to postpone less than an agent with rational expectations. To see this, consider the limiting behavior as $\delta \to \beta^{-1}$ from below, that is as we approach the case of preference consistency. The critical option value for the agent with optimal expectations, u^*, converges to the critical value of the agent with rational expectations, $\bar{u} = \beta^{\tau-1}\left[\frac{1}{2}\left(\frac{1}{2}\beta+\xi\right)^2 - \left(\frac{1}{2}\beta+\xi\right)+\frac{1}{2}\right]$. Since u^* is increasing in δ, there is a range of payments u for all $\delta < \beta^{-1}$ for which an agent with optimal expectations accepts the commitment while an agent with rational expectations retains the option, $u^* < \bar{u}$. This occurs despite the fact that, absent commitment, the agent with optimal expectations has the same propensity to postpone the action in τ as the agent with rational expectations. Intuitively this occurs because when the agent rejects the option she also suffers no negative expected future disutility. This "benefit" is not available to the agent with rational beliefs. The only benefit of commitment is the payment u. For the agent with optimal beliefs, there is the additional benefit of avoiding dread.

The coincidence of decisions as $\delta \to \beta^{-1}$ illustrates the fact that when poor actions have larger consequences, optimal expectations are closer to rational expectations. In this setting, the larger δ, the more important experienced utility relative to expected future utility, and the more actual outcomes matter for lifetime well-being relative to expected future utility flows. Thus, optimal beliefs and the actions they induce are closer to those of the rational agent. As $\delta \to \beta^{-1}$, all that matters is experienced utility (which becomes non-finite), and optimal beliefs become rational.

7 Conclusion

This paper introduces a model of utility-serving biases in beliefs. Optimal expectations provides a structural model of non-rational but optimal beliefs. While our applications highlight many of the implications of our theory, many remain to be explored.

First, the specification of possible events is more important for specifying a model with optimal expectations than it is when specifying a model with rational expectations. For example, an optimal expectations equilibrium in a world with only certain outcomes is different from the equilibrium in the same world with an available sunspot or public randomization device. With the randomization device agents can gamble against one another.

Second, agents with optimal expectations can be optimistic about uncertain environments, and therefore can be better off with the later resolution of uncertainty. For instance, you tell someone that they are going to receive gifts on their birthday but not what those gifts are until their birthday.[12] More generally, because more information can change the ability to distort beliefs, agents can be better off not receiving information despite the benefits of better decision making. It is, however, not the case that agents would ever choose that uncertainty be resolved later because agents take their beliefs as given. Bayesian agents never prefer the later resolution of uncertainty (Eliaz and Spiegler (2002)).

Third, we conjecture that the agent who faces the same problem again and again, and so faces the possibility of large losses from an incorrect specification of probabilities, will naturally have a better assessment of probabilities. Thus, optimal expectations agents are not easy to turn into “money pumps,” although they may exhibit behavior far from that generated by rational

[12] A surprise party for an agent raises the possibility in the agent’s mind that he might get more surprise parties in the future and he enjoys looking forward to this possibility.

expectations in one-shot games.

Fourth, and closely related, to what extent are optimal beliefs evolutionarily advantaged or disadvantaged relative to rational beliefs? On the one hand, agents with optimal expectations make poorer decisions. On the other hand, agents with optimal expectations may take on more risk, which can lead to an evolutionary advantage.

Finally, optimal expectations has promising applications in strategic environments. In a strategic setting, each agent's beliefs are set taking as given the reaction functions of other agents.

Appendixes

A Optimal expectations when the state space is continuous

In the main text, we describe optimal expectations when the state space is finite and discrete. This appendix defines equilibrium when uncertainty is continuous. Let $\Pi\left(s_t|\underline{s}_{t-1}\right)$ denote the conditional cumulative probability distribution function of the vector $s_t \in S$ and $\hat{\Pi}\left(s_t|\underline{s}_{t-1}\right)$ the subjective version. When the state space is continuous, equation (4) becomes

$$V_t\left(x_t;\underline{s}_t,\left\{\hat{\Pi}\right\}\right) = \max_{c_t}\ u\left(x_t,c_t\right) + \beta\int V_{t+1}\left(g\left(x_t,c_t,s_{t+1}\right);s_{t+1},\underline{s}_t,\left\{\hat{\Pi}\right\}\right)d\hat{\Pi}\left(s_{t+1}\right).$$

Assumption 1 is replaced by

Assumption 1´ (Restrictions on probabilities, continuous state space)

(i) $\displaystyle\int_{s_t\in S} d\hat{\Pi}\left(s_t|\underline{s}_{t-1}\right) = 1$

(ii) $d\hat{\Pi}\left(s_t|\underline{s}_{t-1}\right) \geq 0$

(iii) $\hat{\Pi}\left(\underline{s}'_t\right) = \hat{\Pi}\left(s'_t|\underline{s}'_{t-1}\right)\hat{\Pi}\left(s'_{t-1}|\underline{s}'_{t-2}\right)\cdots\hat{\Pi}\left(s'_1\right)$

(iv) $d\hat{\Pi}\left(s_t|\underline{s}_{t-1}\right) = 0$ if $d\Pi\left(s_t|\underline{s}_{t-1}\right) = 0$.

Finally, optimal expectations are determined by choosing continuous probability functions to maximize the functional objective

$$\max_{\left\{\hat{\Pi}\left(s_\tau|\underline{s}_{\tau-1}\right)\right\}}\ E\left[\sum_{t=1}^{T}\beta^{t-1}\left(W_t\left(\underline{x}^*\left(\underline{s}_{t-1},\left\{\hat{\Pi}\right\}\right),\underline{c}^*\left(\underline{s}_{t-1},\left\{\hat{\Pi}\right\}\right)\right)+V_t\left(x^*\left(\underline{s}_t,\left\{\hat{\Pi}\right\}\right);\underline{s}_t,\left\{\hat{\Pi}\right\}\right)\right)\right].$$

B Proofs of propositions

B.1 Proof of Proposition 1

Proof: (i) We prove the case for $w^* > 0$; the case for $w^* < 0$ is analogous. If $w^* > w^{RE}$, then

$$\begin{aligned} u'\left(R+w^*Z_s\right) &\geq u'\left(R+w^{RE}Z_s\right) \text{ for } s \ni Z_s \leq 0 \\ u'\left(R+w^*Z_s\right) &< u'\left(R+w^{RE}Z_s\right) \text{ for } s \ni Z_s > 0 \end{aligned} \tag{B.1}$$

When the asset pays off poorly, marginal utility is higher for the agent with the higher share invested in the risky asset. The agent with rational expectations has first order condition

$$\sum_{s\ni Z_s\leq 0}\pi_s u'\left(R+w^{RE}Z_s\right)Z_s + \sum_{s\ni Z_s>0}\pi_s u'\left(R+w^{RE}Z_s\right)Z_s = 0$$

which combined with equation (B.1) implies

$$\sum_{s \ni Z_s \leq 0} \pi_s u' \left(R + w^* Z_s\right) Z_s + \sum_{s \ni Z_s > 0} \pi_s u' \left(R + w^* Z_s\right) Z_s < 0$$

Subtracting this from the first order condition of the agent with optimal expectations yields the desired inequality

$$\sum_{s=1}^{S} \left(\hat{\pi}_s - \pi_s\right) u' \left(R + w^* Z_s\right) Z_s > 0 \tag{B.2}$$

Thus if we can show that $w^* > 0$ implies $w^* > w^{RE}$ we complete the proof of (i). This follows from the second point of the proposition, which we now prove.

(ii) We first treat the case $w^{RE} > 0$, the case $w^{RE} < 0$ is analogous and we treat $w^{RE} = 0$ subsequently. We first show that an agent invests less in the risky asset as the probability of the worst state is increased ($\frac{dw^*}{d\hat{\pi}_1} < 0$). Examine the agent's first-order condition for w^* and consider a small increase in $\hat{\pi}_{s'}$ at the expense of $\hat{\pi}_S$, so for all $s' \leq S$:

$$\begin{aligned} 0 &= \Big(u' \left(R + w^* Z_{s'}\right) Z_{s'} - u' \left(R + w^* Z_S\right) Z_S\Big) d\hat{\pi}_{s'} + \sum_{s=1}^{S} \hat{\pi}_s u'' \left(R + w^* Z_s\right) Z_s^2 dw^* = 0 \\ \frac{dw^*}{d\hat{\pi}_{s'}} &= \frac{u' \left(R + w^* Z_S\right) Z_S - u' \left(R + w^* Z_{s'}\right) Z_{s'}}{\sum_{s=1}^{S} \hat{\pi}_s u'' \left(R + w^* Z_s\right) Z_s^2} \end{aligned}$$

Note that this equals zero for $s' = S$ and that the denominator is always negative. When the numerator is positive, as for example when $Z_{s'} < 0$, then when the probability of state s' increases at the expense of the best state, the optimal portfolio share in the risky asset decreases. In particular,

$$\frac{dw^*}{d\hat{\pi}_1} < 0$$

When subjective probability moves from the best to the worst state, the optimal portfolio share declines (short position increases).

Suppose for purposes of contradiction, that $0 < w^* \leq w^{RE}$. Since $w^* \leq w^{RE}$, we can follow the same steps as in (i) to show that

$$\sum_{s=1}^{S} \left(\hat{\pi}_s - \pi_s\right) u' \left(R + w^* Z_s\right) Z_s \leq 0 \tag{B.3}$$

We also have predictions about the sign of the left hand side of equation (B.3) from equation (10) – the optimal choice of $\hat{\pi}_1$, which implies

$$\begin{aligned}\text{sign}\left[\beta\left(u_S-u_1\right)\right] &= \text{sign}\left[\frac{dw^*}{d\hat{\pi}_1}\sum_{s=1}^{S}\pi_s u'\left(R+w^*Z_s\right)Z_s\right] \\ &= \text{sign}\left[-\sum_{s=1}^{S}\pi_s u'\left(R+w^*Z_s\right)Z_s\right] \\ &= \text{sign}\left[\sum_{s=1}^{S}\left(\hat{\pi}_s-\pi_s\right)u'\left(R+w^*Z_s\right)Z_s\right]\end{aligned}$$

where the second equality uses $\frac{dw^*}{d\hat{\pi}_1} < 0$ and the third makes use of the agent's first order condition (equation (9)). When the agent is investing in the risky asset $u_S - u_1 > 0$, and when shorting, $u_S - u_1 < 0$ so that

$$\text{sign}\left[w^*\right] = \text{sign}\left[\sum_{s=1}^{S}\left(\hat{\pi}_s-\pi_s\right)u'\left(R+w^*Z_s\right)Z_s\right] \tag{B.4}$$

which with (B.3) implies $w^* \leq 0$. Thus we have a contradiction and we know for $w^{RE} > 0$, $w^* > w^{RE}$ or $w^* \leq 0$. The final step is to rule out $w^* = 0$. If $w^* = 0$ then $\beta\left(u_S - u_1\right) = 0$ so that by the first-order condition for beliefs

$$0 = \sum_{s=1}^{S}\pi_s u'\left(R+w^*Z_s\right)Z_s. \tag{B.5}$$

But since the first-order condition of the rational agent is equation (B.5) with w^{RE} instead of w^* and that this yields a unique $w^{RE} > 0$ we know that $w^* = 0$ cannot solve equation (B.5).

Finally, consider the case $w^{RE} = 0$. This occurs if and only if $E\left[Z\right] = 0$. Suppose $w^* = 0$; then we are supposing that $\hat{\pi}$ is such that $\hat{E}\left[Z\right] = 0$. In fact, such beliefs solve the first-order conditions for optimal beliefs because when $w^* = 0$, $u_S - u_{s'} = 0$ and $\sum_{s=1}^{S}\hat{\pi}_s u'\left(R\right)Z_s = u'\left(R\right)\hat{E}\left[Z_s\right] = 0$. But first order conditions are not sufficient here; the second-order condition is violated. Let $\hat{\pi}^*$ be a candidate optimum with $\hat{E}\left[Z\right] = 0$. Consider the deviation from beliefs that moves probability from the best state to the worst, $\hat{\pi}_1 = \hat{\pi}_1^* + d\hat{\pi}$ and $\hat{\pi}_S = \hat{\pi}_S^* - d\hat{\pi}$. We know $\frac{dw^*}{d\hat{\pi}_1}$ from (ii); the agent responds by reducing his investment in the risky asset. Consider total lifetime well-being as a function of $\hat{\pi}_1$, $F(w^*\left(\hat{\pi}_1\right), \hat{\pi}_1)$. The change in lifetime well-being for our $d\hat{\pi}$ expanded around $\hat{\pi}^*$ (and so $w^*\left(\hat{\pi}^*\right) = 0$) is

$$dF(w^*\left(\hat{\pi}_1\right), \hat{\pi}_1) = F_1\frac{dw^*}{d\hat{\pi}}d\hat{\pi} + F_2 d\hat{\pi} + F_{12}\frac{dw^*}{d\hat{\pi}}\left(d\hat{\pi}\right)^2 + F_1\frac{d^2w^*}{d\hat{\pi}^2}\left(d\hat{\pi}\right)^2 + F_{22}\left(d\hat{\pi}\right)^2.$$

where F_i represents a partial derivative. The first-order terms are zero (since the first-order condition is satisfied), $F_{22} = 0$ since the problem is linear is probabilities, and $F_1 = \beta \sum_{s=1}^{S} \hat{\pi}_s^* u'(R) Z_s + \beta \sum_{s=1}^{S} \pi_s u'(R) Z_s = 0$ since $\hat{E}[Z] = 0$ and $E[Z] = 0$. Thus we have

$$\begin{aligned} dF(w^*(\hat{\pi}_1), \hat{\pi}_1) &= -\beta u'(R)(Z_S - Z_1) \frac{dw^*}{d\hat{\pi}} (d\hat{\pi})^2 \\ &= -\frac{\beta (u'(R)(Z_S - Z_1))^2}{u''(R) \sum_{s=1}^{S} \hat{\pi}_s Z_s^2} (d\hat{\pi})^2 > 0 \end{aligned}$$

which implies that the change in lifetime well-being is positive so that $w^* \neq 0$. QED.

B.2 Proof of Proposition 2

(i) For rational expectations, each agent's first order condition is

$$\sum_s \pi_s \left(\frac{1+\varepsilon_s}{P_e} - 1 \right) u' \left(\left(X_b^i + P_e X_e^i \right) \left(1 + w^i \left(\frac{1+\varepsilon_s}{P_e} - 1 \right) \right) \right) = 0.$$

We guess that the market-clearing price is

$$P_e = 1 + \sum_s \pi_s \varepsilon_s.$$

Substituting into the above first order condition shows after some algebra

$$u'\left(c_1^i\right) - u'\left(c_2^i\right) = 0$$

which implies that consumption is perfectly smoothed across states since $u'(\cdot)$ is monotonic. Writing $c_1^i = c_2^i$ out yields

$$\begin{aligned} \left(X_b^i + P_e X_e^i \right) \left(1 + w^i \left(\frac{1+\varepsilon_1}{P_e} - 1 \right) \right) &= \left(X_b^i + P_e X_e^i \right) \left(1 + w^i \left(\frac{1+\varepsilon_2}{P_e} - 1 \right) \right) \\ w^i \left(\frac{1+\varepsilon_1}{1+\sum_s \pi_s \varepsilon_s} - 1 \right) &= w^i \left(\frac{1+\varepsilon_2}{1+\sum_s \pi_s \varepsilon_s} - 1 \right). \end{aligned}$$

This implies $w^i = 0$ unless the return on the stochastic asset is the same across states, $\frac{1+\varepsilon_1}{1+\sum_s \pi_s \varepsilon_s} = \frac{1+\varepsilon_2}{1+\sum_s \pi_s \varepsilon_s}$. It is straightforward that $x_e^{REi} = 0$ and that these allocations satisfy the aggregate resource constraint.

(ii) Proof by contradiction that $w^i \neq 0$ for both agents, using equations 11c. Suppose $w^i = 0$, then consumption is perfectly smoothed across states, $c_s^i = c$. Some algebra shows that this allocation can satisfy the first order conditions and the resource constraint for rational beliefs and equilibrium price.

But these conditions are not sufficient and this allocation is not an optimal allocation. We show that each agent can change his beliefs and allocations slightly within the feasible set and increase his lifetime well-being. Consider the deviation from objective beliefs, $\hat{\pi}_s^i = \pi_s^i + (-1)^{i+s} d\hat{\pi}$. In this case agents respond by increasing (decreasing) their consumption in the state they perceive as more (less) likely by dc so as to satisfy their first-order condition.

$$0 = \sum_s \left[(-1)^{i+s} u'(c)(1+\varepsilon_s - P_e)\, d\hat{\pi} + \pi_s u''(c) \frac{dc_s^{i*}}{dc_1^{i*}} \frac{dc_1^{i*}}{d\hat{\pi}} d\hat{\pi} (1+\varepsilon_s - P_e) \right]$$

The symmetry of the deviation maintains P_e at the rational expectations price, so that it is clear that the aggregate resource constraint is met. By the budget constraint, for either agent

$$\begin{aligned} \frac{dc_1}{dc_2} &= \frac{1+\varepsilon_1 - P_e}{1+\varepsilon_2 - P_e} \\ &= \frac{\varepsilon_1 - \pi_1\varepsilon_1 - \pi_2\varepsilon_2}{\varepsilon_2 - \pi_1\varepsilon_1 - \pi_2\varepsilon_2} \\ &= -\frac{\pi_2}{\pi_1} \end{aligned}$$

Thus, after some algebra

$$\frac{dc_1^{i*}}{d\hat{\pi}} = (-1)^i \frac{u'(c)}{u''(c)\,\pi_1}.$$

Let lifetime well-being be $B^i\left(c_1^{i*}(\hat{\pi}_1), \hat{\pi}_1^i\right)$ in which we are treating c_2^{i*} as a function of c_1^{i*}. The change in lifetime well-being for agent i and the change $d\hat{\pi}$ has zero first-order effects, as in the partial-equilibrium case. The second order terms are

$$B_{12}^i \frac{dc_1^{i*}}{d\hat{\pi}} (d\hat{\pi})^2 + B_1^i \left(\frac{dc_1^{i*}}{d\hat{\pi}}\right)^2 (d\hat{\pi})^2 + B_{22}^i (d\hat{\pi})^2.$$

$B_{22} = 0$ since B is linear in probabilities.

$$B_1^i = \sum_s \left[\beta\hat{\pi}_s^i u'\left(c_s^i\right) \frac{dc_s^{i*}}{dc_1^{i*}} + \beta\pi_s u'\left(c_s^i\right) \frac{dc_s^{i*}}{dc_1^{i*}} \right] = 0$$

where the last equality comes from evaluating at $c_s^i = c$ and substituting $\frac{dc_1}{dc_2}$. Thus the total welfare gain simplifies to

$$\begin{aligned} B_{12}^i \frac{dc_1^{i*}}{d\hat{\pi}} (d\hat{\pi})^2 &= 2\beta(-1)^{i+1} u'(c) \left(1 + \frac{\pi_1}{\pi_2}\right) (-1)^i \frac{u'(c)}{u''(c)\,\pi_1} (d\hat{\pi})^2 \\ &= 2\beta \frac{(u'(c))^2 \left(\frac{1}{\pi_1} + \frac{1}{\pi_2}\right)}{-u''(c)} (d\hat{\pi})^2 > 0. \end{aligned}$$

Intuitively, in this alternative allocation each agent suffers a second-order welfare loss in the second period for standard arguments about perfect consumption smoothing. However, the overall welfare gain is positive because the agent can invest in the state that he believes more likely and anticipate a utility gain. This yields a contradiction so that we have $w^1 \neq w^2$. Since at least one agent has $w \neq 0$, both agents have $w \neq 0$ by the aggregate resource constraint and noting that each agent has a positive endowment. So agents are gambling.

(iii) We proceed in 5 steps.

Step 1 $\mu^i \neq 0$:

Suppose $\mu^i = 0$, then $\beta u_s\left(c_s^i\right) = \lambda^i$ for both s by the first order condition $\left(\hat{\pi}_s^i\right)$ (equation 11b). Since λ^i does not depend on s, this implies that the agent would perfectly smooth consumption across states, $u\left(c_1^i\right) = u\left(c_2^i\right)$. This in turn, implies $w^i = 0$, as shown above, which by the aggregate resource constraint implies the other agent's portfolio share in the risky asset is also zero. A contradiction arises and hence, we must have $\mu^i \neq 0$ at the optimum.

Step 2: there are then two cases, case (a) $c_1^1 > c_2^1$ and $c_2^2 > c_1^2$ and case (b) $c_2^1 > c_1^1$ and $c_1^2 > c_2^2$. The μ^i equations imply that $(1+\varepsilon_2) > P_e > (1+\varepsilon_1) > 0$ since $\hat{\pi} \geq 0$ and $u' > 0$. The aggregate resource constraint implies that (a) $w^2 > 0 > w^1$ or (b) $w^1 > 0 > w^2$. These in turn imply by the consumption definition in (a) $c_1^1 = \left(X_b^1 + P_e X_e^1\right)\left(1 + w^1\left(\frac{1+\varepsilon_1}{P_e} - 1\right)\right) > c_2^1 = \left(X_b^1 + P_e X_e^1\right)\left(1 + w^1\left(\frac{1+\varepsilon_2}{P_e} - 1\right)\right)$ and $c_2^2 > c_1^2$; case (b) $c_2^1 > c_1^1$ and $c_1^2 > c_2^2$.

Step 3: in case (a) $\mu^1 > 0 > \mu^2$ and in case (b) $\mu^2 > 0 > \mu^1$.

The next step provides more information about μ^i. From equations $\hat{\pi}_s^i$

$$
\begin{aligned}
0 &= \beta u\left(c_1^i\right) - \lambda^i - \mu^i\left[u'\left(c_1^i\right)\left(\frac{1+\varepsilon_1}{P_e} - 1\right)P_e\right] \\
0 &= \beta u\left(c_2^i\right) - \lambda^i - \mu^i\left[u'\left(c_2^i\right)\left(\frac{1+\varepsilon_2}{P_e} - 1\right)P_e\right]
\end{aligned}
$$

subtract the equations for the same agent

$$
\beta u\left(c_1^i\right) - \beta u\left(c_2^i\right) = \mu^i P_e\left[u'\left(c_1^i\right)\left(\frac{1+\varepsilon_1}{P_e} - 1\right) - u'\left(c_2^i\right)\left(\frac{1+\varepsilon_2}{P_e} - 1\right)\right]
$$

Case (a) $w^2 > 0 > w^1$, $c_1^1 > c_2^1$ and $c_2^2 > c_1^2$ the left hand side is positive for $i = 1$, negative for $i = 2$, so

$$\begin{aligned}
\mu^1 \left[u'\left(c_1^1\right)(1+\varepsilon_1-P_e)-u'\left(c_2^1\right)(1+\varepsilon_2-P_e)\right] &> 0 \\
\mu^2 \left[u'\left(c_1^2\right)(1+\varepsilon_1-P_e)-u'\left(c_2^2\right)(1+\varepsilon_2-P_e)\right] &< 0
\end{aligned}$$

Now the two first order conditions for w^i :

$$0=\sum_s \left[\left(\beta\hat{\pi}_s^i+\beta\pi_s\right)u'\left(c_s^i\right)\left(\frac{1+\varepsilon_s}{P_e}-1\right)\right]+\mu^i\,[+]$$

where [+] is some positive number.

For $i=1$, write out and substitute π_2 and $\hat{\pi}_2$ using the fact the probabilities sum to one, and multiply through by μ^i, giving

$$\begin{aligned}
0 &= \left(\hat{\pi}_1^i+\pi_1\right)u'\left(c_1^i\right)\left(\tfrac{1+\varepsilon_1}{P_e}-1\right)+2u'\left(c_2^i\right)\left(\tfrac{1+\varepsilon_2}{P_e}-1\right)-\left(\hat{\pi}_1^i+\pi_1\right)u'\left(c_2^i\right)\left(\tfrac{1+\varepsilon_2}{P_e}-1\right)+\mu^i\,[+] \\
&= 2\mu^i u'\left(c_2^i\right)(1+\varepsilon_2-P_e)+\left(\hat{\pi}_1^i+\pi_1\right)\mu^i\left\{u'\left(c_1^i\right)(1+\varepsilon_1-P_e)-u'\left(c_2^i\right)(1+\varepsilon_2-P_e)\right\}+\left(\mu^i\right)^2[+].
\end{aligned}$$

The inequality above implies the term $\mu^1\{\}>0$ so that we have

$$0=[+]+\mu^i\,[+]+[+]$$

so that $\mu^1<0$.

Similarly, for $i=2$, we get

$$\begin{aligned}
0 &= 2u'\left(c_1^2\right)\left(\tfrac{1+\varepsilon_1}{P_e}-1\right)-\left(\hat{\pi}_2^2+\pi_2\right)u'\left(c_1^2\right)\left(\tfrac{1+\varepsilon_1}{P_e}-1\right)+\left(\hat{\pi}_2^2+\pi_2\right)u'\left(c_2^2\right)\left(\tfrac{1+\varepsilon_2}{P_e}-1\right)+\mu^i\,[+] \\
&= 2\mu^i u'\left(c_1^2\right)(1+\varepsilon_1-P_e)-\left(\hat{\pi}_2^2+\pi_2\right)\mu^i\left\{u'\left(c_1^2\right)(1+\varepsilon_1-P_e)+u'\left(c_2^2\right)(1+\varepsilon_2-P_e)\right\}+\left(\mu^i\right)^2[+]
\end{aligned}$$

The inequality above implies the term $\mu^i\{\}<0$ so that we have

$$0=\mu^2\,[-]-[-]+[+]$$

so that $\mu^2>0$.

Step 4: Optimism about high consumption state

Turning to probabilities, from the first equation, $0=\sum_s\left(\beta\hat{\pi}_s^i+\beta\pi_s\right)u'\left(c_s^i\right)\left(\frac{1+\varepsilon_s}{P_e}-1\right)-\mu^i\,[-]$ since $u''<0$ so that

$$\operatorname{sign}\left\{\sum_s\left(\beta\hat{\pi}_s^i+\beta\pi_s\right)u'\left(c_s^i\right)\left(\frac{1+\varepsilon_s}{P_e}-1\right)\right\}=\operatorname{sign}\left\{-\mu^i\right\}$$

For $i=1$, $c_1^1 > c_2^1$ so $u'\left(c_1^1\right) < u'\left(c_2^1\right)$

$$\left(\beta\hat{\pi}_1^1 + \beta\pi_1\right) u'\left(c_1^1\right)\left(\frac{1+\varepsilon_1}{P_e} - 1\right) + \left(\beta\hat{\pi}_2^1 + \beta\pi_2\right) u'\left(c_2^1\right)\left(\frac{1+\varepsilon_2}{P_e} - 1\right) > 0$$

From the agent's optimization problem, $\sum_s \hat{\pi}_s^i u'\left(c_s^i\right)\left((1+\varepsilon_s) - P_e\right) = 0$. This implies

$$\pi_1 u'\left(c_1^1\right)\left(\frac{1+\varepsilon_1}{P_e} - 1\right) + \pi_2 u'\left(c_2^1\right)\left(\frac{1+\varepsilon_2}{P_e} - 1\right) > 0$$

and again

$$\left(\pi_1 - \hat{\pi}_1^1\right) u'\left(c_1^1\right)\left(\frac{1+\varepsilon_1}{P_e} - 1\right) + \left(\pi_2 - \hat{\pi}_2^1\right) u'\left(c_2^1\right)\left(\frac{1+\varepsilon_2}{P_e} - 1\right) > 0$$

and using the fact that probabilities sum to unity

$$\begin{aligned}
\left(\pi_1 - \hat{\pi}_1^1\right) u'\left(c_1^1\right)\left(\frac{1+\varepsilon_1}{P_e} - 1\right) + \left(1 - \pi_1 - 1 + \hat{\pi}_1^1\right) u'\left(c_2^1\right)\left(\frac{1+\varepsilon_2}{P_e} - 1\right) &> 0 \\
\left(\pi_1 - \hat{\pi}_1^1\right)\left\{u'\left(c_1^1\right)\left(\frac{1+\varepsilon_1}{P_e} - 1\right) - u'\left(c_2^1\right)\left(\frac{1+\varepsilon_2}{P_e} - 1\right)\right\} &> 0
\end{aligned}$$

Finally, we know $\mu^1\left[u'\left(c_1^1\right)(1+\varepsilon_1 - P_e) - u'\left(c_2^1\right)(1+\varepsilon_2 - P_e)\right] > 0$ and $u'\left(c_1^1\right)(1+\varepsilon_1 - P_e) - u'\left(c_2^1\right)(1+\varepsilon_2 - P_e) < 0$ so we have $\left(\pi_1 - \hat{\pi}_1^1\right) < 0$ or $\hat{\pi}_1^1 > \pi_1$. A symmetric argument shows $\hat{\pi}_1^2 < \pi_1$ and the probabilities of the second state follows.

B.3 Proof of Proposition 3

Part (ii): Subtract $E\left[c_{t+1}^*\left(\underline{y}_{t+1}\right)|\underline{y}_t\right]$ from each side of equation (15) with $\tau = 1$

$$\begin{aligned}
c_t^*\left(\underline{y}_t\right) - E\left[c_{t+1}^*\left(\underline{y}_{t+1}\right)|\underline{y}_t\right] &= \frac{a}{b} - \frac{\psi_{t+1}}{\psi_t}R\left(\frac{a}{b} - E\left[c_{t+1}^*\left(\underline{y}_{t+1}\right)|\underline{y}_t\right]\right) - E\left[c_{t+1}^*\left(\underline{y}_{t+1}\right)|\underline{y}_t\right] \\
&= \left(1 - \frac{\psi_{t+1}}{\psi_t}R\right)\left(\frac{a}{b} - E\left[c_{t+1}^*\left(\underline{y}_{t+1}\right)|\underline{y}_t\right]\right)
\end{aligned}$$

Since the support of the income process does not admit a plan such that $E\left[c_{t+\tau}^*\left(\underline{y}_{t+\tau}\right)|\underline{y}_t\right] > \frac{a}{b}$ the second term is positive. The following demonstrates that the first term is positive.

$$\begin{aligned}
\frac{\psi_{t+1}}{\psi_t}R &= \frac{\beta^t}{\beta^{t-1}}R\frac{1+\sum_{\tau=1}^{T-t-1}\left(\beta^\tau + \delta^\tau\right)}{1+\sum_{\tau=1}^{T-t}\left(\beta^\tau + \delta^\tau\right)} \\
&< \beta R\frac{1+\sum_{\tau=1}^{T-t-1}\left(\beta^\tau + \delta^\tau\right)}{1+\sum_{\tau=1}^{T-t-1}\left(\beta^\tau + \delta^\tau\right) + \left(\beta^{T-t} + \delta^{T-t}\right)} \\
&< 1
\end{aligned}$$

therefore

$$c_t^* \left(\underline{\mathrm{y}}_t\right) - E\left[c_{t+1}^* \left(\underline{\mathrm{y}}_{t+1}\right) |\underline{\mathrm{y}}_t\right] > 0$$

and we have the result.

Part (iii): From the agent's consumption Euler equation

$$\hat{E}\left[c_{t+1}^* \left(A_{t+1}, \underline{\mathrm{y}}_{t+1}\right) |\underline{\mathrm{y}}_t\right] = c_t^* \left(A_t, \underline{\mathrm{y}}_t\right) > E\left[c_{t+1}^* \left(\underline{\mathrm{y}}_{t+1}\right) |\underline{\mathrm{y}}_t\right]$$

where the inequality follows from part (ii).

Part (i): the consumption rule at $t+1$ is

$$c_{t+1}^* \left(\underline{\mathrm{y}}_{t+1}\right) = \tfrac{1-R^{-1}}{1-R^{-(T-t)}} \left(A_{t+1} + \sum_{\tau=0}^{T-t-1} R^{-\tau} \hat{E}\left[y_{t+1+\tau} |\underline{\mathrm{y}}_{t+1}\right]\right)$$

From part (iii) we have

$$\begin{aligned}
&\hat{E}\left[c_{t+1}^* \left(A_{t+1}, \underline{\mathrm{y}}_{t+1}\right) |\underline{\mathrm{y}}_t\right] > E\left[c_{t+1}^* \left(\underline{\mathrm{y}}_{t+1}\right) |\underline{\mathrm{y}}_t\right] \\
&\hat{E}\left[\tfrac{1-R^{-1}}{1-R^{-(T-t)}} \left(A_{t+1} + \sum_{\tau=0}^{T-t-1} R^{-\tau} \hat{E}\left[y_{t+1+\tau} |\underline{\mathrm{y}}_{t+1}\right]\right) |\underline{\mathrm{y}}_t\right] > \\
&E\left[\tfrac{1-R^{-1}}{1-R^{-(T-t)}} \left(A_{t+1} + \sum_{\tau=0}^{T-t-1} R^{-\tau} \hat{E}\left[y_{t+1+\tau} |\underline{\mathrm{y}}_{t+1}\right]\right) |\underline{\mathrm{y}}_t\right]
\end{aligned}$$

and by the law of iterated expectations

$$\hat{E}\left[\sum_{\tau=0}^{T-t-1} R^{-\tau} y_{t+1+\tau} |\underline{\mathrm{y}}_t\right] > E\left[\sum_{\tau=0}^{T-t-1} R^{-\tau} \hat{E}\left[y_{t+1+\tau} |\underline{\mathrm{y}}_{t+1}\right] |\underline{\mathrm{y}}_t\right]$$

QED.

B.4 Proof of Proposition 5

Proof. Consider the utility of the agent who chooses $a(\tau)$ at period 1 where $\tau > 1$

$$
\begin{aligned}
V_1 &= -\left(\xi + \beta\xi + \ldots + \beta^{\tau-2}\xi + \beta^{\tau-1}\hat{E}[x_\tau]\right) \\
W_1 &= 0 \\
V_2 &= -\left(\xi + \beta\xi + \ldots + \beta^{\tau-3}\xi + \beta^{\tau-2}\hat{E}[x_\tau]\right) \\
W_2 &= -\delta\xi \\
&\ldots \\
V_\tau &= -x_\tau \\
W_\tau &= -\left(\delta^{\tau-1} + \ldots + \delta^2 + \delta\right)\xi \\
V_t &= 0 \text{ for } t > \tau \\
W_t &= -\delta^{t-\tau}\left(x_\tau + \left(\delta^{\tau-1} + \ldots + \delta^2 + \delta\right)\xi\right) \text{ for } t > \tau
\end{aligned}
$$

Adding these together and taking unconditional expectations

$$
-(\tau-1)\xi - (\tau-2)\beta\xi - \ldots - \beta^{\tau-2}\xi - \frac{\beta - \beta^\tau}{1-\beta}\hat{E}[x_\tau] - E[x_\tau] - \tfrac{\delta}{1-\delta}\left(E[x_\tau] + (\tau-1)\xi\right)
$$

So optimal expectations given the choice between $a(\tau)$ and $a(\tau+1)$ sets $\hat{E}[x] = 0$ in whichever period is going to be chosen so as to minimize dream disutility. Given that the agent has greater expected disutility from doing the action in $\tau+1$ than τ, $a(\tau) \succ_{\mathbf{1}} a(\tau+1)$. At time τ the agent already knows x_τ. If the plan is to undertake the action at $\tau+1$, the above formula from the perspective of time τ with $\hat{E}[x_{\tau+1}] = 0$ applies

$$
-\xi - E[x_{\tau+1}] - \tfrac{\delta}{1-\delta}\left(E[x_{\tau+1}] + \xi\right)
$$

For choosing to undertake the action today,

$$
\begin{aligned}
V_\tau &= -x_\tau \\
W_\tau &= 0 \\
V_t &= 0 \text{ for } t > \tau \\
W_t &= -\delta^{t-\tau}E[x_\tau] \text{ for } t > \tau
\end{aligned}
$$

Adding these together gives

$$-x_\tau - \tfrac{\delta}{1-\delta}\left(E\left[x_\tau\right] + \xi\right)$$

Then,

$$\begin{aligned}
\Pr\left[-\xi - E\left[x_{\tau+1}\right] - \tfrac{\delta}{1-\delta}\left(E\left[x_{\tau+1}\right] + \xi\right) > -c_\tau - \tfrac{\delta}{1-\delta}\left(E\left[x_\tau\right] + \xi\right)\right] &= \\
&= \Pr\left[\xi + \tfrac{1}{2} < x_\tau\right] \\
&= \tfrac{1}{2} - \xi
\end{aligned}$$

B.5 Proof of Proposition 6

Step 1: Derive overall utility for commitment and beliefs such that agent prefers to commit.

As shown in the text $\hat{E}\left[e_\tau\right] = 0$. Hence, the overall utility for commitment is

$$\tfrac{1}{1-\beta\delta}\left[u - \beta^{\tau-1}E\left[e_\tau\right]\right] = \tfrac{1}{1-\beta\delta}\left[u - \beta^{\tau-1}\frac{1}{2}\right].$$

Step 2: Derive overall utility for option and beliefs such that the agent prefers to reject the commitment.

The expected value functions without commitment are

$$\begin{aligned}
E\left[V_1\right] &= -\beta^{\tau-1}\hat{\mathbf{P}}\left(e_\tau < e_\tau^*\right)\hat{E}\left[e_\tau|e_\tau < e_\tau^*\right] - \beta^{\tau-1}\left(1 - \hat{\mathbf{P}}\left(e_\tau < e_\tau^*\right)\right)\left(\xi + \beta\hat{E}\left[e_{\tau+1}|e_\tau \geq e_\tau^*\right]\right) \\
E\left[W_1\right] &= 0 \\
E\left[V_2\right] &= -\beta^{\tau-2}\hat{\mathbf{P}}\left(e_\tau < e_\tau^*\right)\hat{E}\left[e_\tau|e_\tau < e_\tau^*\right] - \beta^{\tau-2}\left(1 - \hat{\mathbf{P}}\left(e_\tau < e_\tau^*\right)\right)\left(\xi + \beta\hat{E}\left[e_{\tau+1}|e_\tau \geq e_\tau^*\right]\right) \\
E\left[W_2\right] &= 0 \\
&\ldots \\
E\left[V_\tau\right] &= -\mathbf{P}\left(e_\tau < e_\tau^*\right)E\left[e_\tau|e_\tau < e_\tau^*\right] - \left(1 - \mathbf{P}\left(e_\tau < e_\tau^*\right)\right)\left(\xi + \beta\hat{E}\left[e_{\tau+1}|e_\tau \geq e_\tau^*\right]\right) \\
E\left[W_\tau\right] &= 0 \\
E\left[V_{\tau+1}\right] &= \left(1 - \mathbf{P}\left(e_\tau < e_\tau^*\right)\right)E\left[e_{\tau+1}\right] \\
E\left[W_{\tau+1}\right] &= -\mathbf{P}\left(e_\tau < e_\tau^*\right)\delta E\left[e_\tau|e_\tau < e_\tau^*\right] - \left(1 - \mathbf{P}\left(e_\tau < e_\tau^*\right)\right)\delta\xi \\
E\left[V_t\right] &= 0 \text{ for } t > \tau + 1 \\
E\left[W_t\right] &= -\delta^{t-\tau-1}\left(\mathbf{P}\left(e_\tau < e_\tau^*\right)\delta E\left[e_\tau|e_\tau < e_\tau^*\right] + \left(1 - \mathbf{P}\left(e_\tau < e_\tau^*\right)\right)\left(\delta\xi + E\left[e_{\tau+1}\right]\right)\right) \text{ for } t > \tau + 1
\end{aligned}$$

After discounting the value functions with the discount factor β, their sum forms the planner's objective function. The constraint that $\hat{E}\left[V_1 | \text{keep option to postpone}\right] \geq \hat{E}\left[V_1 | \text{commitment}\right]$ can be rewritten as

$$\hat{\mathbf{P}}\left(e_\tau < e_\tau^*\right) \hat{E}\left[e_\tau | e_\tau < e_\tau^*\right] + \left(1 - \hat{\mathbf{P}}\left(e_\tau < e_\tau^*\right)\right)\left(\xi + \beta \hat{E}\left[e_{\tau+1} | e_\tau \geq e_\tau^*\right]\right) \geq \hat{E}\left[e_\tau\right] - \beta^{1-\tau} u$$

By inspection, it is optimal for the agent to believe that $e_{\tau+1} = 0$ whenever $e_\tau \geq e_\tau^*$ since this minimizes the disutility of expecting a bad future outcome and also relaxes the constraint. Substituting $\hat{E}\left[e_\tau\right] = \hat{P}\left(e_\tau < e_\tau^*\right) \hat{E}\left[e_\tau | e_\tau < e_\tau^*\right] + \left(1 - \hat{\mathbf{P}}\left(e_\tau < e_\tau^*\right)\right) \hat{E}\left[e_\tau | e_\tau \geq e_\tau^*\right]$ into the constraint gives

$$\left(1 - \hat{\mathbf{P}}\left(e_\tau < e_\tau^*\right)\right) \xi \leq \left(1 - \hat{\mathbf{P}}\left(e_\tau < e_\tau^*\right)\right) \hat{E}\left[e_\tau | e_\tau \geq e_\tau^*\right] - \beta^{1-\tau} u \qquad \text{(B.6)}$$

so that the constraint does not depend on $\hat{E}\left[e_\tau | e_\tau < e_\tau^*\right]$. Therefore, to minimize dread, it is optimal for the agent to believe that $e_\tau = 0$ whenever $e_\tau < e_\tau^*$. Similarly, $\hat{E}\left[e_\tau | e_\tau \geq e_\tau^*\right]$ does not appear in the objective and to loosen the constraint $\hat{E}\left[e_\tau | e_\tau \geq e_\tau^*\right] = 1$. So the problem is to maximize the discounted sum of expected value functions

$$\begin{aligned} & -(\tau-1)\beta^{\tau-1}\left(1 - \hat{\mathbf{P}}\left(e_\tau < e_\tau^*\right)\right)\xi - \tfrac{\beta^{\tau-1}}{1-\beta\delta}\left[\mathbf{P}\left(e_\tau < e_\tau^*\right) E\left[e_\tau | e_\tau < e_\tau^*\right] + \left(1 - \mathbf{P}\left(e_\tau < e_\tau^*\right)\right)\left(\beta E\left[e_{\tau+1}\right] + \xi\right.\right. \\ = \; & -(\tau-1)\beta^{\tau-1}\left(1 - \hat{\mathbf{P}}\left(e_\tau < e_\tau^*\right)\right)\xi - \tfrac{\beta^{\tau-1}}{1-\beta\delta}\left[\tfrac{1}{2}\left(e_\tau^*\right)^2 + \left(1 - e_\tau^*\right)\left(\beta\tfrac{1}{2} + \xi\right)\right] \qquad \text{(B.7} \end{aligned}$$

subject to

$$u \leq \left(1 - \hat{\mathbf{P}}\left(e_\tau < e_\tau^*\right)\right) \beta^{\tau-1}\left(1 - \xi\right). \qquad \text{(B.8)}$$

Note first that if $u > \beta^{\tau-1}\left(1 - \xi\right)$, the constraint always fails so that the agent chooses commitment. Note second that the objective function decreases with $1 - \hat{P}\left(e_\tau < e_\tau^*\right)$, so that unconstrained optimal beliefs would set $\hat{P}\left(e_\tau < e_\tau^*\right) = 1$, so that the agent believes it will take the action in τ with certainty (at no cost). But this would lead to a violation of the constraint (unless $u = 0$). Thus the constraint binds (if $u < \beta^{\tau-1}\left(1 - \xi\right)$) so that

$$\left(1 - \hat{\mathbf{P}}\left(e_\tau < e_\tau^*\right)\right) = \frac{u}{\beta^{\tau-1}\left(1 - \xi\right)}$$

So we seek to maximize

$$-(\tau-1)\beta^{\tau-1}\left(\frac{u}{\beta^{\tau-1}\left(1 - \xi\right)}\right)\xi - \tfrac{\beta^{\tau-1}}{1-\beta\delta}\left[\tfrac{1}{2}\left(e_\tau^*\right)^2 + \left(1 - e_\tau^*\right)\left(\beta\tfrac{1}{2} + \xi\right)\right]$$

which has first order condition

$$e_\tau^* = \tfrac{1}{2}\beta + \xi.$$

Turning to $\hat{E}\left[e_{\tau+1}^*|e_\tau < e_\tau^*\right]$, in order to ensure incentive compatibility, it must be that $-e_\tau^* \geq -\xi - \beta\hat{E}\left[e_{\tau+1}^*|e_\tau < e_\tau^*\right]$ or $1 \geq \hat{E}\left[e_{\tau+1}^*|e_\tau < e_\tau^*\right] \geq \frac{1}{\beta}\left(e_\tau^* - \xi\right)$.

The overall utility is

$$-(\tau-1)\frac{\xi}{(1-\xi)}u - \tfrac{\beta^{\tau-1}}{1-\beta\delta}\left(\beta\tfrac{1}{2}+\xi\right)\left[1-\tfrac{1}{2}\left(\tfrac{1}{2}\beta+\xi\right)\right]$$

Step 3: Solve for the set of u for which the lifetime well-being of commitment is higher than the lifetime well-being of rejecting commitment.

$$\begin{aligned} \frac{1}{1-\beta\delta}\left[u-\beta^{\tau-1}\frac{1}{2}\right] &\geq -(\tau-1)\frac{\xi}{(1-\xi)}u - \tfrac{\beta^{\tau-1}}{1-\beta\delta}\left(\beta\tfrac{1}{2}+\xi\right)\left[1-\tfrac{1}{2}\left(\tfrac{1}{2}\beta+\xi\right)\right] \\ u+(1-\beta\delta)(\tau-1)\frac{\xi}{(1-\xi)}u &\geq \beta^{\tau-1}\frac{1}{2} - \beta^{\tau-1}\left(\beta\tfrac{1}{2}+\xi\right)\left[1-\tfrac{1}{2}\left(\tfrac{1}{2}\beta+\xi\right)\right] \\ u &\geq \beta^{\tau-1}\frac{\frac{1}{2}-\left(\beta\frac{1}{2}+\xi\right)\left[1-\frac{1}{2}\left(\frac{1}{2}\beta+\xi\right)\right]}{1+(1-\beta\delta)(\tau-1)\frac{\xi}{(1-\xi)}} \end{aligned}$$

For all $u > \beta^{\tau-1}(1-\xi)$, the agent always chooses commitment regardless of beliefs.

Hence, choose commitment whenever

$$u \geq \min\left\{\beta^{\tau-1}(1-\xi), \beta^{\tau-1}\frac{\frac{1}{2}-\left(\beta\frac{1}{2}+\xi\right)\left[1-\frac{1}{2}\left(\frac{1}{2}\beta+\xi\right)\right]}{1+(1-\beta\delta)(\tau-1)\frac{\xi}{(1-\xi)}}\right\}.$$

References

AINSLIE, G. (1991): "Derivation of 'Rational' Economic Behavior from Hyperbolic Discount Curves," *American Economic Review*, 81, 334–340.

AKERLOFF, G. (1991): "Procrastination and Obedience," *American Economic Review*, 81, 1–19.

AKERLOFF, G., AND W. T. DICKENS (1982): "The Economic Consequences of Cognitive Dissonance," *American Economic Review*, 72(3), 307–19.

ALPERT, M., AND H. RAIFFA (1982): "A Progress Report on the Training of Probability Assessors," in *Judgement Under Uncertainty: Heuristics and Biases*, ed. by D. Kahneman, P. Slovic, and A. Tversky. Cambridge University Press, Cambridge.

ATTANASIO, O. (1999): "Consumption," in *Handbook of Macroeconomics*, ed. by J. B. Taylor, and M. Woodford, vol. 1, chap. 11, pp. 741–812. Elsevier Press.

AUMANN, R. J. (1976): "Agreeing to Disagree," *Annals of Statistics*, 4, 1236–1239.

BARSKY, R., T. JUSTER, M. KIMBALL, AND M. SHAPIRO (1997): "Preference Parameters and Behavioral Heterogeneity: an Experimental Approach in the Health and Retirement Survey," *Quarterly Journal of Economics*, 112(2), 537–80.

BÉNABOU, R., AND J. TIROLE (2002): "Self-confidence and Personal Motivation," *Quarterly Journal of Economics*, 117(3), 871–915.

BUEHLER, R., D. GRIFFIN, AND M. ROSS (1994): "Exploring the "Planning Fallacy": Why People Underestimate Their Task Completion Times," *Journal of Personality and Social Psychology*, 67(3), 366–381.

CAPLIN, A. J., AND J. LEAHY (2000): "The Social Discount Rate," *NBER Working Paper*, 7983.

——— (2001): "Psychological Expected Utility Theory and Anticipatory Feelings," *Quarterly Journal of Economics*, 106(1), 55–80.

ELIAZ, K., AND R. SPIEGLER (2002): "Anticipatory Feelings and Attitudes Towards Information," Manuscript, NYU.

GEANAKOPLOS, J., D. PEARCE, AND E. STACCHETTI (1989): "Psychological Games and Sequential Rationality," *Games and Economic Behavior*, 1, 60–79.

GOURINCHAS, P.-O., AND J. A. PARKER (2002): "Consumption Over the Lifecycle," *Econometrica*, 70(1), 47–89.

GUL, F., AND W. PESENDORFER (2002): "Temptation and Self-Control," *Econometrica*, 69(6), 1403–35.

HARBAUGH, R. (2002): "Skill Reputation, Prospect Theory, and Regret Theory," *Mimeo, Claremont Colleges*.

HARRIS, C., AND D. LAIBSON (2001): "Dynamic Choices of Hyperbolic Consumers," *Econometrica*, 69(4), 935–57.

KAHNEMAN, D. (2000): "Experienced Utility and Objective Happiness: A Moment-Based Approach," in *Choices, Values and Frames*, ed. by D. Kahneman, and A. Tversky, pp. 637–692. Cambridge University Press, New York.

KAHNEMAN, D., P. P. WAKKER, AND R. SARIN (1997): "Back To Bentham? Explorations of Experienced Utility," *Quarterly Journal of Economics*, 112(2), 375–405.

Laibson, D. (1997): "Golden Eggs and Hyperbolic Discounting," *Quarterly Journal of Economics*, 112(2), 443–77.

Landier, A. (2000): "Wishful Thinking: A Model of Optimal Reality Denial," MIT Manuscript.

Lowenstein, G. (1987): "Anticipation and the Valuation of Delayed Consumption," *Economic Journal*, 47, 666–84.

Lowenstein, G., and D. Prelec (1992): "Anomolies in Intertemporal Choice: Evidence and an Interpretation," *Quarterly Journal of Economics*, 107, 573–97.

Lucas, Jr., R. E. (1976): "Econometric Policy Evaluation: A Critique," in *The Phillips Curve and Labor Markets*, vol. 1 of *Carnegie Rochester Conference Series on Public Policy*, pp. 19–46. North-Holland, Amsterdam.

Mullainathan, S. (2002): "A Memory-Based Model of Bounded Rationality," *Quarterly Journal of Economics*, 117(3), 735–774.

Muth, J. R. (1960): "Optimal Properties of Exponentially Weighted Forecasts," *Journal of the American Statistical Association*, 55, 229–305.

——— (1961): "Rational Expectations and the Theory of Price Movements," *Econometrica*, 24, 315–335.

Nerlove, M. (1958): "Adaptive expectations and cobweb phenomena," *Quarterly Journal of Economics*, 72, 227–240.

O'Donoghue, T., and M. Rabin (2001): "Choice and Procrastination," *Quarterly Journal of Economics*, 116(1), 121–60.

Rubinstein, A. (1998): *Modeling Bounded Rationality*. MIT Press, Cambridge, MA.

Savage, L. J. (1954): *The Foundation of Statistics*. Wiley, New York.

Simon, H. (1982): *Models of Bounded Rationality*. MIT Press.

Sims, C. A. (2001): "Implications of Rational Inattention," Princeton University Manuscript.

Strotz, R. H. (1955-1956): "Myopia and Inconsistency in Dynamic Utility Maximization," *Review of Economic Studies*, 23(3), 165–80.

Taylor, S. E., and J. D. Brown (1988): "Illusion and Well-Being: A Social Psychological Perspective on Mental Health," *Psychological Bulletin*, 103(2), 192–210.

Weinstein, N. D. (1980): "Unrealistic Optimism About Future Life Events," *Journal of Personality and Social Psychology*, 39(5), 806–820.

Yariv, L. (2001): "Believe and Let Believe: Axiomatic Foundations for Belief Dependent Utility," Cowles Foundation Discussion Paper No. 1344, Yale University.