ASSET PRICING WHEN RISK SHARING IS LIMITED BY DEFAULT

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ABSTRACT

We study the asset pricing implications of a multi-agent endowment economy where agents can default on contracts that would leave them otherwise worse off. We specialize and extend the environment studied by Kocherlakota (1995) and Kehoe and Levine (1993) to make it comparable to standard studies of asset pricing. We make contributions along two fronts. First, we extend the characterization of efficient allocations. Second, we present an equilibrium concept with complete markets and with endogenous solvency constraints. These solvency constraints are such as to prevent default–at the cost of reduced risk sharing. We show a version of the classical welfare theorems for this equilibrium definition. We characterize the pricing kernel, and compare it to the one for economies without participation constraints: interest rates are lower and risk premia can be bigger depending on the covariance of the idiosyncratic and aggregate shocks. We show that those agents whose endowment is very similar to the aggregate endowment are irrelevant for asset pricing. In a quantitative example, for reasonable parameter values, the relevant marginal rates of substitution fall within the Hansen-Jagannathan bounds.

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1. Introduction

Standard equilibrium asset pricing models have problems reproducing some of the basic facts in the data.[1] A promising direction for improvement along this dimension has been to maintain standard preferences and to allow for incomplete risk sharing across agents. Generally, in this class of models, financial markets are exogenously considered as incomplete. Compared to the representative agent model, these models have the attractive feature that agents cannot eliminate all idionsynchratic risk. For this reason, pricing kernels are generally more volatile than those of representative agent economies with the same aggregate consumption. One drawback of studies following this approach, when compared with complete markets economies, is that they require more or less arbitrary assumptions about which set of securities is available. Model conclusions may in turn crucially depend on these assumptions. Another possible problem is tractability. Because finding equilibria of these models involves solving a complicated fixed point problem, it has been very difficult to analyze the case of many assets or many agents. Additionally, previous quantitative studies with incomplete market models required highly persistent relative shocks, a feature not necessarily seen in the data.[2] In this paper, we study a class of models whose equilibrium, in

[1]The shortcomings of standard model versions are widely documented in studies such as Mehra and Prescott (1985), Cochrane and Hansen (1992).

[2]See Telmer (1993), and Heaton and Lucas (1996) for numerical simulations with transitory shocks in two-agent economies, and Constantinides and Duffie (1996) for a structure with permanent relative shocks with a continuum of agents. Empirical estimates of the first order serial correlation for relative incomes range from low values found by Heaton and Lucas (1996) of around 0.5 to higher values found by Storesletten, Telmer and Yaron (1997) of around 0.9 .

general, entail limited risk sharing but that do not have some of these potential drawbacks.

Our approach for limiting risk sharing builds on work by Kehoe and Levine (1993) and Kocherlakota (1996).[3] These authors present and study efficient allocations in endowment economies where *participation constraints* ensure that agents would at no time be better off reverting permanently to autarchy. These participation constraints are motivated by an extreme form of limited liability. We show in our paper that these participation constraints can be modelled as portfolio constraints. We imagine a world, where, if agents default on some debt, they can be punished by seizing all the assets that they may own, but they cannot be punished by garnishing their labor income. In such an environment, risk sharing may be effectively reduced because agents with low income realizations can only borrow up to the amount they are willing to pay back in the future. It is assumed, not modelled, that there is commitment to carry out these permanent punishments.

We adapt previous work within this framework in various ways to make it more suitable for quantitative asset pricing evaluation. We make contributions along two fronts. First, we extend some results on the nature of constrained efficient allocations. Second, we introduce an equilibrium notion where portfolio constraints play a key role. We then explore the connections between efficient

[3] Earlier work in the souvereign debt literature by Eaton and Gersovitz (1981) first formalized the main idea of this approach.

allocations and equilibrium allocations and we characterize some properties of security prices.

We start by extending the analysis of Kocherlakota (1996) to environments with serially correlated shocks and many agents. We also extend Kocherlakota's results by characterizing in more detail the efficient allocations. We pay special attention to the conditions under which different degrees of risk sharing are feasible and to the properties of the process for the marginal rates of substitution of the agents. We characterize the range of parameter values that will produce efficient allocations of either of the three following types: full risk sharing, some intermediate degree of risk sharing and autarchy. A major part of the analysis of efficient allocations is the following characterization of the marginal rate of substitution. The agents whose continuation utility is higher than the value of autarchy at some state in the next date have the highest marginal rate of substitution with respect to that state. Additionally, if autarchy is not the only feasible allocation, there is always an agent whose continuation utility is above autarchy. These properties are key for understanding asset pricing, since we show that in equilibrium the highest marginal rate of substitution determines asset prices. Our characterization of the efficient allocations is simple enough as to be implemented by fast and straightforward computational algorithms. In fact, for some simple cases we can completely characterize efficient allocations analytically. Finally, for the 2 agent case, we show the existence and uniqueness of the invariant distribution of the relevant statistics when full risk sharing is not possible.

Our parameterization of the aggregate shocks has two desirable properties. First, it follows the existing asset pricing literature and thus makes results comparable. Second, it highlights the fact that aggregate shocks will have effects on the marginal rates of substitution over and above the effect that they have in economies without participation constraints to the extent that they are correlated with idiosyncratic shocks.

We introduce a new equilibrium concept that emphasizes portfolio constraints: a competitive equilibrium with *solvency constraints.* Specifically, we focus on constraints that are tight enough to prevent default but allow as much risk sharing as possible. Except for the constraints, our equilibrium is identical to a Radner equilibrium with complete markets (*i.e.* a competitive equilibrium with a sequence of budget constraints). These *endogenously* determined solvency constraints are agent and state specific and ensure that the participation constraints are fulfilled. This means that the amount of wealth that agents can carry to any particular date and event will never be small enough to make them choose to default and to revert to autarchy.

We show versions of the classical first and second welfare theorems for our equilibrium concept. Our decentralization differs from the one in Kehoe and Levine (1993) where the participation constraints are modelled as direct restrictions on the consumption possibility sets. We think that our decentralization relates more closely to the existing asset pricing literature that focuses on portfolio restrictions such as the work by He and Modest (1995) and Luttmer (1996) among others. Indeed, our decentralization provides a justification for such solvency constraints. Additionally, with our definition of equilibrium we obtain a simpler and a very intuitive representation of the prices of securities. In any event, both our equilibrium notion and that of Kehoe and Levine are closely related. We show the circumstances under which the equilibrium allocations are identical and the exact mapping between the two. This last point turns out to be important, since some results are easier to prove in one framework than in the other.

We present some properties of the pricing kernel. One-period contingent claims (Arrow securities) are priced by the agent with the highest marginal rate of substitution, which is the agent that is not constrained with respect to his holding of this asset. Thus the price of a contingent claim (an Arrow security) is equal to the highest marginal valuation across agents. Pricing of an arbitrary asset is accomplished by adding up the prices of the corresponding contingent claims (which, interestingly, does not need to coincide with the highest marginal valuation of the security across agents). As we mentioned before, this framework has two advantages for the purpose of asset pricing over the standard incomplete markets specifications. First, allocations do not depend on a particular arbitrary set of assets that is considered to be available. And second, with markets being complete, any security can be priced. Finally, we compare two properties of asset prices in an equilibrium with solvency constraints with the ones obtained in an identical economy, but without the solvency constraints. The first result is that interest rates are necessarily smaller in an economy with solvency constraints, independently of the precautionary motive (third derivative of the utility function), a feature emphasized in the literature. Second, we examine the following measure of the market price of risk: the excess return of a one-period security whose payments are proportional to the aggregate endowment. We show that unless relative endowment shocks are correlated with the aggregate shocks, this premium is the same that the one in an economy without solvency constraints. Third, we show that asset prices are determined by those agents with substantial individual risk to share. In particular, the presence of agents whose endowment is very similar to the aggregate endowment is irrelevant for asset pricing.

Finally, we engage in a simple numerical exercise for a first quantitative evaluation of this model. In particular, we analyze the variability of the relevant marginal rates of substitution for asset pricing. One motivation for conducting this exercise is that He and Modest (1995) have shown that among the most commonly considered frictions in asset pricing models, solvency constraints could potentially reconcile the variability of aggregate consumption with data on asset prices for reasonable values of risk aversion. They use a version of the Hansen-Jagannathan bounds for complete markets economies with solvency constraints following Luttmer (1996). In our study, we go one step further and present a model with *endogenous* solvency constraints and we find that for reasonably parameterized endowment processes our economy passes the standard Hansen-Jagannathan test for low values of risk aversion. In particular, when individual income processes are calibrated following Heaton and Lucas (1995) based on the

PSID, relative risk aversion is required to be around 2 and we need very low values of the discount factor. With a more persistent processes for the idiosyncratic shocks, motivated by the evidence in Storesletten, Telmer and Yaron (1997), we find similar results for again low risk aversion but higher discount factors. Our companion paper (Alvarez and Jermann (1997)) contains a more detailed analysis of quantitative properties.

The remainder of this paper is organized as follows. In section 2 we present the model environment. Section 3 characterizes the constrained efficient allocations by introducing a recursive representation. We analyze the extent of risk sharing, the long-run dynamics and the case of aggregate uncertainty. In Section 4 we introduce the competitive equilibrium with solvency constraints, show versions of the classical welfare theorems, relate the equilibrium concept to the one by Kehoe and Levine and analyze the pricing kernel. In section 5, we present some quantitative asset pricing implications by solving for efficient (and equilibrium) allocations for a simple case.

2. The Environment

We consider a pure exchange economy with I agents. Agents' endowments follow a finite state markov process, agents' preferences are identical and given by time-separable expected discounted utility. We add to this simple environment participation constraints of the following form: the continuation utility implied by any allocation should be at least as high as the one implied by autarchy at any time and for any history.

Formally, we use the following notation, $i \in \{1, 2, ...I\} \equiv I$, to denote each agent. We use $\{z_t\}$ to denote a markov process with generic elements z, a member of the finite set $Z = \{\mathfrak{z}_1, ..., \mathfrak{z}_N\}$, and transition probabilities given by matrix Π. We use $z^t = (z_0, z_1, z_2, ..., z_t)$ to denote a length t history of z. The matrix Π generates conditional probabilities for histories that we denote as $\pi(z^t | z_0 = z)$. We use the symbol $\succcurlyeq$ for the partial order $z^{t'} \succcurlyeq z^t$ for $t' \geq t$ to indicate that $z^{t'}$ is a possible continuation of z^t, that is, that there exists a history z^s such that $z^{t'} = (z^t, z^s)$ for $s = t' - t$. We use the notation $\{c_i\}$ and $\{e_i\}$ for the stochastic process of consumption and endowment of each agent, hence $\{c_i\} = \{c_{i,t}(z^t) : \forall\, t \geq 0,\ z^t \in Z^t\}$. Individual endowments are given by a function ϵ_i, that depends only on z_t, so that $e_{i,t}(z^t) = \epsilon_i(z_t)$. We assume that $\epsilon_i(z) > 0$ for all i and z. The utility for an agent corresponding to the consumption process $\{c\}$ starting at time t at history z^t is denoted by $U(c)(z^t)$ and is given by:

$$U(c)(z^t) \equiv \sum_{s=0}^{\infty} \sum_{z^{t+s} \in Z^{t+s}} \beta^s u\left(c_{t+s}\left(z^{t+s}\right)\right) \pi(z^{t+s} | z_t).$$

We assume the following properties, $u(\cdot)$ is strictly increasing, strictly concave, C^1 and $u'(0) = +\infty$. We restrict attention to consumption processes, $\{c_i\}$, that satisfy the following Participation Constraints:

$$U(c_i)(z^t) \geq U(e_i)(z^t) \equiv U^i(z_t)\ \forall\, t \geq 0,\ z^t \in Z^t, \tag{2.1}$$

where we define U^i as the utility of consuming the endowment forever starting today when the shock is z_t. Notice that $U(e_i)(z^t)$ depends only on z_t because $e_{i,t}(z^t)$ is a function of z_t.

In many parts of the paper we will further specialize the environment. For that purpose, we provide here the definitions of some of these specializations:

(i) symmetry, the distribution of endowments does not depend on the agent's name, *i.e.*:

$$\begin{aligned} & \Pr\left\{(e_{k_1,t+1}, e_{k_2,t+1}, ...e_{k_I,t+1}) = \widetilde{e}' \,|\, (e_{k_1,t}, e_{k_2,t}, ...e_{k_I,t}) = \widetilde{e}\right\} \\ = & \Pr\left\{(e_{r_1,t+1}, e_{r_2,t+1}, ...e_{r_I,t+1}) = \widetilde{e}' \,|\, (e_{r_1,t}, e_{r_2,t}, ...e_{r_I,t}) = \widetilde{e}\right\} \end{aligned}$$

for any $\widetilde{e}, \widetilde{e}' \in R^I_+$, and two permutations $(k_1, k_2, ..., k_I)$ and $(r_1, r_2, ..., r_I)$ of the set of agents;

(ii) homothetic period utility, $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$;

(iii) no aggregate uncertainty, $\sum_{i=1}^{I} \epsilon_i(z) = e$, for all $z \in Z$;

(iv) ergodicity, Π is such that $\{z_t\}$ has a unique invariant distribution with ergodic set Z.

In section 3.5 we will add assumption (v) bounding the growth rate of output relative to β and γ.

3. Constrained optimal allocations

Constrained optimal allocations are defined as the processes $\{c_i\}$ that maximize period zero expected lifetime utility for agent 1, subject to resource balance and the participation constraints for all agents, given some initial (time zero) promised expected lifetime utility for agents $i = 2, ..., I$. First, we consider the case where the aggregate endowment is constant. At the end of the section we relax this assumption. For simplicity, from now on, we drop the qualification of "constrained" optimal allocations and we refer to them simply as "optimal" or efficient allocations. Optimal allocations solve the following maximization problem:

$$V^*(w_{2,0}w_{3,0}, ...w_{I,0}, z_0) \equiv \max\{U(c_1)(z_0)\} \tag{3.1}$$

$$\sum_i c_{i,t}\left(z^t\right) = e \tag{3.2}$$

$$U\left(c_i\right)\left(z^t\right) \geq U^i\left(z_t\right) \quad \text{all } t, z^t, i \tag{3.3}$$

$$U\left(c_i\right)\left(z_0\right) \geq w_{0,i} \quad i = 2, ..., I.$$

The function $V^*(w_0, z_0)$ can be thought as the time zero utility possibility frontier.

Now we restate the previous problem recursively. We introduce a functional equation and relate its fixed points to the function V^*. The functional equation is not completely standard. We introduce two operators, T , that maps functions into functions and H, that maps functions into sets. Given a function $V : \prod_z D\left(z\right) \rightarrow R$, with domain described by the sets $D\left(z\right) \subset R^{I-1}$ for each z, the operator $T[V]$ generates a function and $H\ [V]$ generates its domain as follows:

$$T\ [V](w,z) = \max_{\{c_i\}_{i\in I}\ ,\{w_i'(z')\}_{i\geq 2, z'\in Z}} \left\{ u(c_1) + \beta \sum_{z'\in Z} V(w'(z'),z')\ \pi(z'|z) \right\} \quad (3.4)$$

$$\sum_i c_i \leq e \quad (3.5)$$

$$u(c_i) + \beta \sum_{z'\in Z} w_i'(z')\ \pi(z'|z) \geq w_i \quad \text{all } i \geq 2 \quad (3.6)$$

$$w_i'(z') \geq U^i(z') \quad \text{all } z' \in Z\ , i \geq 2 \quad (3.7)$$

$$V(w(z'),z') \geq U^1(z') \quad \text{all } z' \in Z. \quad (3.8)$$

The domain of $T[V]$ is defined for each $z \in Z$ as follows

$$H\ [V](z) = \left\{ w : T\ [V](w,z) \geq U^1(z) \text{ and } w \geq \left(U^2(z), U^3(z), ..., U^I(z)\right)\right\} \quad (3.9)$$

where we used the notation

$$\begin{aligned} w &\equiv (w_2, w_3, ...w_I), \\ w'(z) &\equiv \left(w_2'(z), w_3'(z), ...w_I'(z)\right). \end{aligned}$$

In later parts of the paper, when there is no ambiguity, we will sometimes use the notation: $w_1 \equiv V(w,z)$, and $w_1(z') \equiv V(w(z'),z')$.

It is straightforward to show that the function V^*, defined previously, and its associated domain, are a fixed point of T and H. However, the functional equation (3.4) has more than one fixed point, and hence it cannot be a contraction.[4] In particular, "autarchy" is always a fixed point, since it is immediate to verify that the trivial function v defined on the domain given by singleton $\{U^2(z), ..., U^I(z)\}$ and equal to $v(w) = U^1(z)$ is a fixed point of T. Nevertheless for many parameter values there are other solutions, namely V^*. This observation will "reappear" when we define the decentralized equilibrium and find that autarchy is always an equilibrium.

Even though the T operator is not a contraction it can be useful in computing V^*. Its usefulness comes from the following result. Considering the operator $\tilde{T}$ defined exactly like T in 3.4 except that the participations constraints for both agents 3.8 and 3.7 are removed, and denote its unique fixed point by $\tilde{V}$. The function $\tilde{V}(\cdot, z)$ is the full risk sharing frontier when $z_t = z$. The following proposition states the sense in which T is useful for computational purposes.

Proposition 3.1. $\lim_{n\to\infty} T^n \tilde{V} = V^*$ *pointwise.*

Proof. The proof uses standard monotonicity arguments. It follows from a direct extension of Theorem 4.14 in Stokey and Lucas with Prescott (1989). Alternatively, it follows from the results of Abreu (1988) and Abreu, Pierce and

[4] It is easy to check that T does not satisfy one of the Blackwell sufficient conditions for a contraction, namely *discounting*. That is $T(V + a)$ could be bigger than $TV + \beta a$ for a constant a, since the feasible set of choices for $(w(z'))_{z' \in Z}$ is bigger for $V + a$ than for V.

Stacchetti (1990).[5] ■

Because $\tilde{V}$ is easily computed, the previous proposition makes T useful for computing the fixed point V^* and its associated policies.

3.1. Risk sharing regimes

Now we turn to the analysis of the stochastic process for $\{w_{i,t+1}\}$ and $\{c_{i,t}\}$. Depending on parameter values there are three possible "regimes" for the process for $\{w_{i,t+1}\}$. Independently of the initial condition (w, z) :

1. There is full risk sharing for ever,

2. Only limited risk sharing is possible,

3. Only autarchy is possible.

By full risk sharing we mean that the allocation is Pareto efficient in the standard sense, ignoring the participation constraints, for some initial condition (w_0, z_0). Parameter values that produce the first case are not interesting for us since, for the purposes of asset pricing, its implications are the same as the representative agent economy.

We discuss briefly how one can check for each case. Kocherlakota (1996) presents sufficient conditions for each case when the shocks are i.i.d. and symmetric and $I = 2$. We consider a slightly different case in the following proposition.

[5] An Appendix that contains more details about the proofs in this paper is available on http://finance.wharton.upenn.edu/~jermann/research.html

Proposition 3.2. *If there is no aggregate uncertainty and the endowments are symmetric, then full risk sharing is possible if and only if*

$$\min_{i\in I}\left\{\frac{u(c_i)}{1-\beta}:\sum_i c_i=e\right\}=\frac{u(e/I)}{1-\beta}\geq\max_{i\in I,\ z\in Z}U^i(z)\ . \tag{3.10}$$

Proof. Full risk sharing is characterized by resource feasibility and constancy of the ratio of the marginal utilities across agents. If condition (3.10) is satisfied, then the participation constraints are satisfied for the allocation $c_i = e/I$. Conversely, if full risk sharing is feasible clearly condition (3.10) is satisfied. ■

Figure 1 illustrates this for $I = 2$ for the case when full risk sharing is possible for a range of w. When full risk sharing is not possible there are two cases, one case is where autarchy is the only feasible allocation that satisfies the participation constraints and the other is the one in which some other allocations satisfy the participation constraints. These are the cases that we are interested in, since they are not equivalent to a representative agent economy. Figure 2 illustrates the case when full risk sharing is not possible.

Which case applies depends on how attractive autarchy is relative to some form of risk sharing, which in turn depends on the parameter values, which is detailed in the following remark.

Remark 1. *Consider a case without aggregate uncertainty, and with symmetry in the endowments; let*

$$\Pi_\delta \equiv \delta I + (1-\delta)\,\Pi$$

for $\delta \in (0,1)$, then full risk sharing is not possible in any of the following cases:

(a) The time preference parameter, β is sufficiently small;

(b) The relative risk aversion γ, is sufficiently small;

(c) The persistence of Π_δ, δ is sufficiently close to one;

(d) The variance of $\epsilon_i(z)$ is sufficiently close to zero.

These conditions can be shown to hold by using the relationship from the previous proposition (3.2) by each time taking the appropriate limit. Later in the paper, at proposition (4.14), we proof these conditions implicitly, by showing a stronger version for each of them. In particular, we will show that, not only is full risk sharing not possible, but as the parameters approach the limit values mentioned in each of the four cases, autarchy is the only feasible allocation.

3.2. Marginal rates of substitution with limited risk sharing

The main results of this subsection are two basic properties that will become key for understanding the behavior of prices and solvency constraints once we introduce our equilibrium concept. Namely, there is always at least one agent that is unconstrained by its participation constraint, and an unconstrained agent has the highest marginal rate of substitution—and hence, all unconstrained agents equalize their marginal rates of substitution.

To start, we state without proof some intermediate results about properties of V and its associated policies that are useful for the subsequent analysis. Let $W_{z'}(w,z)$ and $C_i(w,z)$ denote the optimal decision rules for continuation utility

$w(z')$ and current consumption c_i respectively when the state is (w, z).

Proposition 3.3. *V is strictly decreasing and strictly concave on w; furthermore, it is differentiable in its interior. The optimal policy rules are single-valued and continuous.*

The proof follows immediately from the strict monotonicity, strict concavity and differentiability of the period utility function and convexity of the feasible set.

In the next proposition we show that if some risk sharing is feasible between the agents, then at least one agent is always unconstrained.

Proposition 3.4. *Assume that Π is such that it has a unique invariant distribution with ergodic set Z, one of the following is true :*

$$V\left(U^2(z), U^3(z), ..., U^I(z), z\right) = U^1(z) \quad \text{for all } z \in Z, \tag{3.11}$$

which we refer to as saying that "autarchy is the only feasible allocation" or else

$$V\left(U^2(z), U^3(z), ..., U^I(z), z\right) > U^1(z) \quad \text{for all } z \in Z, \tag{3.12}$$

which we refer to as saying that "some risk sharing is feasible". In this case, at least one agent is always unconstrained.

Proof. The proof follows immediately using the strict monotonicity of V and the following observation. If in the future there is a state where some agent's continuation utility is strictly higher than the value of autarchy, then, by the ergodicity of $\{z_t\}$, the current continuation utility of this agent has to be strictly higher than the autarchy value. ■

Notice that this proposition implies that the domains of $V(\cdot, z)$ have either empty interiors for all z or non-empty interiors for all z.

Corollary 3.5. *If some risk sharing is feasible (i.e. 3.12) then for all t and for all $z^t \in Z^t$ there is a feasible allocation that satisfies the participation constraint such that :*

$$c_{i,t+s}(z^{t+s}) > \epsilon_i(z_{t+s})$$

for some i and some $z^{t+s} \succeq z^t$.

To proof this corollary, simply recognize that in order to have lifetime utility above autarchy, consumption will need to be superior to autarchy at some point. The next proposition contains the main result of this section: an unconstrained agent has the highest the marginal rate of substitution, and thus all unconstrained agents equalize their marginal rates of substitution.

Proposition 3.6. *If some risk sharing is feasible, for any (w, z) and z', let j be an agent for which*

$$w_j(z') > U^j(z'), \tag{3.13}$$

then

$$\beta\frac{u'\left(C_j\left(w\left(z'\right),z'\right)\right)}{u'\left(C_j\left(w,z\right)\right)}\pi\left(z'|z\right)=\max_{i\in I}\beta\frac{u'\left(C_i\left(w\left(z'\right),z'\right)\right)}{u'\left(C_i\left(w,z\right)\right)}\pi\left(z'|z\right). \tag{3.14}$$

Proof. The proof is based on a simple variational argument, so we only sketch it here. Let $i \neq j$ be the agent with the highest marginal rate of substitution. One can increase current consumption and decrease future consumption of agent j so as to keep w_j constant. This is feasible since there is slack in (3.13). By decreasing current consumption and increasing future consumption of agent i while keeping material balance, current w_i increases since $i's$ marginal rate of substitution is the higher of the two, which is in contradiction to constrained efficiency. ■

3.3. Decision rules for the 2-agent case

In this subsection and the following we present some further results for the case with only 2 agents. In particular, we analyze some properties of the decision rules $C_i(\cdot)$ and agent two's continuation utility $W_{z'}(\cdot)$. Recall that the consumption decision rule C_2 (as well as the value function V) have different domains $D(z)$ for each value of z. Similarly, the continuation utility decision rules, $W_{z'}(\cdot)$, have different domains for each z and different ranges for each z'. Specifically,

$$\begin{aligned} C_2(\cdot,z) &: [L(z), H(z)] \rightarrow [0,e] \qquad \forall z \\ W_{z'}(\cdot,z) &: [L(z), H(z)] \rightarrow \left[L\left(z'\right), H\left(z'\right)\right] \quad \forall z,\ \forall z', \end{aligned}$$

where the lower and upper bounds of the ranges are given by the values that make agent 2 and agent 1 continuation utility, respectively, equal to the autarchy values, *i.e.*:

$$L(z) \equiv U^2(z) \text{ and } H(z) \equiv V^{-1}(\cdot, z)(U^1(z)) \text{ for } z \in Z.$$

We can show that the time separability of the utility function implies that consumption is increasing in the current continuation utility.

Proposition 3.7. *Consumption is strictly monotone on z, i.e., $\forall z$, $C_2(w,z)$ ($C_1(w,z)$ resp.) is strictly increasing (strictly decreasing resp.) in w.*

Proof. By using Benveniste and Scheinkman, one shows that

$$\frac{\partial V(w,z)}{\partial w} = -\frac{u'(e - C_2(w,z))}{u'(C_2(w,z))} \tag{3.15}$$

which is strictly decreasing in w since the value function is strictly concave. Thus $C_2(w,z)$ is increasing. ■

Next we analyze the decision rules for future continuation utility as a function of current continuation utility. We find that the decision rules are weakly increasing. Specifically, we consider two cases. First, if the shocks z_t is repeated then consumption and continuation utility are the same. If the shock is different then the decision rule is weakly increasing in w. Furthermore, the decision rule is flat only if the assigned continuation utility is such that either agent 1 or 2 is constrained in the next period.

Proposition 3.8. *Let's assume symmetry in the endowments. [I] If $z' = z$, then we have $W_{z'}(w, z) = w$ (the "45^0–rule").*

[II] If $z' \neq z$ and $(\tilde{w}, w, z, z')$ are such that (i) $w < \tilde{w}$ and (ii) both $W_{z'}(\tilde{w}, z)$ and $W_{z'}(w, z)$ are in the interior of the range of $W_{z'}(\cdot, z)$ i.e.,

$$W_{z'}(\tilde{w}, z), W_{z'}(w, z) \in \left(L\left(z'\right), H\left(z'\right)\right)$$

Then $W_{z'}(\tilde{w}, z) > W_{z'}(w, z)$.

Proof. The proof of [I] follows from examining the case where neither agent is constrained in the future, then by Proposition (3.6)

$$\frac{\partial V(w, z)}{\partial w} = -\frac{u'(e - C_2(w, z))}{u'(C_2(w, z))} = -\frac{u'(e - C_2(W_z(w, z), z))}{u'(C_2(W_z(w, z), z))} = \frac{\partial V(W_z(w, z), z)}{\partial w}.$$

Finally, using (3.7) [I] is obtained. [II] follows from a variation of the previous argument. ■

3.4. Long run dynamics for the 2-agent case

Now we consider the invariant distribution of the $\{w_{t+1}, z_{t+1}\}$ process. To this end we use the notation and results from the previous section on the decision rules and we add to the previous assumptions of no aggregate uncertainty and symmetric endowments a requirement that will be satisfied, roughly speaking, if the process for individual endowments is not negatively autocorrelated. Under these circumstances we show that if there is only limited risk sharing, then there is a unique ergodic distribution for the $\{w_{t+1}, z_{t+1}\}$ process.

A similar result was shown by Kocherlakota (1996). A dimension where our results extend Kocherlakota's is that we allow individual endowments to be autocorrelated.[6] We argue that this extension is relevant for our interest. First, recent studies such as Heaton and Lucas (1996) and Storesletten, Telmer and Yaron (1997) find autocorrelations of individual earnings between 0.5 and 0.9. Second, as we will show in proposition (4.14) the persistence of the individual endowment is key in determining whether participation constraints bind or not.

On the other hand, Kocherlakota's treatment is more general, since he allows the level of the aggregate endowment to be *i.i.d.* However, in the next section, we show that all the previous results hold for a specification of aggregate uncertainty different from the one in Kocherlakota (1996). Namely, when the growth rate of the aggregate endowment follows a markov process. This specification simplifies the analysis of aggregate shocks enormously and is also the most commonly used in quantitative asset pricing studies, which is the final objective of our approach.

Let's denote the transition function for the process $\{w_{t+1}, z_{t+1}\}$ generated by

[6] The reader familiar with the literature on savings under uncertainty will recognize that the uniqueness of the invariant distribution is not a trivial result. For instance, in their theorethical sections Aiyagari (1994) and Laitner (1992) assume that income shocks are *i.i.d.*. Hugget (1993) shows an uniqueness result for a special case with two shocks. The uniqueness of the invariant distribution is obtained only for fixed prices (and not necessarily the equilibrium ones). These authors show existence of an equilibrium invariant measure, but not its uniqueness.

the probabilities Π and the optimal decision rule $W(\cdot)$ by

$$P(G, z'|w, z) \equiv \Pr\{w_{t+1} \in G,\ z_{t+1} = z' |\ w_t = w,\ z_t = z\,; \Pi, W\}$$

for suitably chosen sets $G \subseteq D \equiv [\min_z L(z), \max_z H(z)]$. We will assume that the matrix Π and the functions $e_i(\cdot)$ are such that $U^i(\cdot)$ is increasing in e_i. Now we outline the results of the rest of this section. With this extra assumption and the previous result that the optimal decision rule $W_{z'}(\cdot, z)$ is increasing and continuous on w, we show that the transition function for $\{w_{t+1}, z_{t+1}\}$ is monotone and satisfies the Feller property. Furthermore, we show that for the no aggregate uncertainty, symmetric endowment case, if there is only limited risk sharing, the domains of the function $V^*(\cdot, z)$ for different z are such that the markov process for $\{w_{t+1}, z_{t+1}\}$ satisfies a mixing condition which is sufficient to ensure the uniqueness of the invariant distribution.

Now we state the definitions and intermediate results to show the existence and uniqueness of the invariant measure.

Definition 3.9. *Π is monotone means that $\pi(e_{2,t+1} \geq e' |\ e_{2,t} = e)$ is increasing in e for all e'.*

The following lemma follows immediately from the monotonicity of the decision rules as shown in Proposition (3.8).

Lemma 3.10. ***Monotonicity and Feller Property:*** *Assume that Π is monotone.*

Then the transition function for the process $\{w_{t+1}, z_{t+1}\}$ is monotone and satisfies the Feller property.

Now we turn to show that the process for (w_{t+1}, z_{t+1}) mixes high and low values enough, so that it has a unique ergodic set. To do so we show a series of lemmas that make the connection between the domain of $V(\cdot, z)$ for different values of z and monotonicity properties. We introduce the following assumption,

Assumption: Monotonicity of U^i: the values of Z ,π, and the function $\epsilon_i(\cdot)$ are such that :

$$\epsilon_2(\mathfrak{z}_i) < \epsilon_2(\mathfrak{z}_{i+1}) \text{ and } U^2(\mathfrak{z}_i) < U^2(\mathfrak{z}_{i+1}) \text{ for } i = 1, 2, ..., N-1$$

Notice that this assumption is implied by monotonicity of Π.

The next lemma, which is merely a restatement of its assumptions, says that we can order the lower and upper bounds of the domains of V for different $z's$.

Lemma 3.11. ***Monotonicity on*** U^i. *If there is monotonicity on U^i then , $L(\mathfrak{z}_i) < L(\mathfrak{z}_{i+1})$ and $H(\mathfrak{z}_i) < H(\mathfrak{z}_{i+1})$ for all $i = 1, 2, ...N-1$*

The next lemma makes the connection between the domains of $V(\cdot, z)$ and the extent of risk sharing. This is the key insight of why there is enough recurrence so that there is a unique ergodic set. The logic of this result is the following. If full risk sharing is possible an allocation with constant $w_t = \tilde{w}$ is feasible for all values of z, hence $\tilde{w} \in [L(z), H(z)]$ for all z, *i.e.* the intersection of the domains should be non-empty. Conversely, if only limited risk sharing is possible, then such a $\tilde{w}$ should not exists, *i.e.* the intersection of the domains should be empty.

Lemma 3.12. *Domains of V and risk sharing.* *Assume that there is monotonicity on U^i, symmetry on the endowments and that there is no aggregate uncertainty, then efficient allocations have only limited risk sharing if and only if* $H(\mathfrak{z}_1) < L(\mathfrak{z}_N)$.

We are ready to state a definition of "mixing" that will ensure the uniqueness of the invariant distribution.

Definition 3.13. *Mixing:* *there is a* $w^0 \in [L(\mathfrak{z}_1),\ H(\mathfrak{z}_N)]$ *, and integer* M *and a number* $\varepsilon > 0$ *such that*

$$P^N\left(\left[w^0, H(\mathfrak{z}_N)\right], \mathfrak{z}_N \mid L(\mathfrak{z}_1), \mathfrak{z}_1\right) \geq \varepsilon\,, \text{ and } P^N\left(\left[L(\mathfrak{z}_1), w^0\right], \mathfrak{z}_1 \mid H(\mathfrak{z}_N), \mathfrak{z}_N\right) \geq \varepsilon.$$

where P^n is the n composition of the transition function.

Finally, the next lemma verifies that this property is satisfied. It uses the result about the empty intersection of the domains of the value function.

Lemma 3.14. *Mixing.* *If there is monotonicity on U^i and there is no aggregate uncertainty then the transition function satisfies mixing.*

Collecting the result from the previous five lemmas we have that by Theorem 12.12 in Stokey and Lucas with Prescott (1989) :

Proposition 3.15. *If there is monotonicity on U^i, monotonicity of Π, symmetry on the endowments and no aggregate uncertainty, then there is a unique invariant distribution for the process $\{w, z\}$.*

3.5. Optimal allocations and growth

We introduce aggregate growth in the same fashion as the specification of the aggregate endowment process in Mehra-Prescott (1985) and in much of the quantitative consumption based asset pricing literature. To simplify the notation we consider the case of $I = 2$, but it will be obvious that the case for $I > 2$ is identical. In particular, the growth rate of aggregate endowment, $\lambda(z_{t+1})$, and the relative endowment of each agent $\epsilon_i(z_{t+1})$ follow a finite state markov process described as :

$$e_{t+1} = e_t \cdot \lambda(z_{t+1}) \text{ and } e_{i,t} = e_t \cdot \epsilon_i(z_t) \text{ for } i = 1, 2.$$

For this case, we write $e' = \lambda(z') \cdot e$ and the state for the corresponding functional equation—and hence value function and policies—is (w, z, e). Also notice that the value of autarchy is expressed as $U^i(e, z_t) \equiv U(e_i)(z^t)$ for $e_{i,t}(z^t) = e$ and $z_t = z$. The corresponding functional equation is obvious and we omit it to save space.

When the period utility function is homothetic of the form $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ for some positive γ (for simplicity lets assume that $\gamma \neq 1$), then the value function

$V(\cdot)$ and the policies $\{C_1(w,z,e), C_2(w,z,e), W_{z'}(w,z,e)\}$ satisfy the following homogeneity property :

Proposition 3.16. *For any $y > 0$ and any (w,z,e) :*

$$\begin{aligned}
V(y^{1-\gamma}\cdot w,\ z,\ y\cdot e) &= y^{1-\gamma}\cdot V(w,z,e)\\
C_i(y^{1-\gamma}\cdot w,\ z,\ y\cdot e) &= y\cdot C_i(w,z,e) \text{ for } i=1,2\\
W_{z'}(y^{1-\gamma}\cdot w,\ z,\ y\cdot e) &= y^{1-\gamma}\cdot W_{z'}(w,z,e).
\end{aligned}$$

The proof of this proposition is standard so we omit it. By using these homogeneity properties the functional equation for the economy with stochastic growth can be rewritten as[7] :

$$T\ V(\widehat{w},z,1) = \max_{\widehat{c}_1,\widehat{c}_2,\{\widehat{w(z')}\}\ z'\in Z} \left\{ u(\widehat{c_1}) + \hat{\beta}(z) \sum_{z'\in Z} V(\widehat{w(z')},z',1)\,\hat{\pi}(z'|z) \right\}$$

$$\widehat{c_1} + \widehat{c_2} \leq 1$$

$$u(\widehat{c_2}) + \hat{\beta}(z) \sum_{z'\in Z} \widehat{w(z')}\ \hat{\pi}(z'|z) \geq \widehat{w}$$

$$\widehat{w(z')} \geq \widehat{U^2(z',1)} \text{ all } z' \in Z$$

$$V(\widehat{w}(z'),z',1) \geq \widehat{U^1(z',1)} \text{ all } z' \in Z$$

[7] We omit the corresponding H operator defining the domain of TV to save space.

where the " hat " variables are defined as follows

$$\begin{aligned}
\hat{c}_i &\equiv \frac{c_i}{e}, \quad \widehat{w} \equiv \frac{w}{e^{1-\gamma}}, \quad \widehat{w}' \equiv \frac{w'}{e'^{1-\gamma}} = \frac{w'}{(\lambda' e)^{1-\gamma}} \\
\hat{\pi}\left(z'|z\right) &\equiv \frac{\pi\left(z'|z\right)\cdot\lambda\left(z'\right)^{1-\gamma}}{\sum_{z'}\pi\left(z'|z\right)\cdot\lambda\left(z'\right)^{1-\gamma}} \text{ and } \hat{\beta}\left(z\right) \equiv \beta\cdot\sum_{z'}\pi\left(z'|z\right)\cdot\lambda\left(z'\right)^{1-\gamma}.
\end{aligned}$$

Hence the functional equation for an economy with stochastic growth is identical to the previous one except for the state dependent discount factor $\hat{\beta}(z)$ and the "growth discounted" probabilities $\hat{\pi}$. The next assumption provides a sufficient condition for the continuation utilities to be well defined:

(v) uniform bound on the growth rates, $\hat{\beta}(z) < 1$ for all $z \in Z$.

All the propositions shown in previous sections, appropriately restated, with the new stochastic discount factor, will hold. Because the proofs are almost identical, except for details in the notation, we will not repeat them. This should not be surprising given that the shocks to the discount factor affect both agents identically. A major benefit from this representation is that we will be able to compute this transformed stationary economy in exactly the same way as the economy with constant aggregate endowment.

4. Equilibrium, efficiency and asset prices

In order to meaningfully analyze asset prices we propose a definition of a decentralized market equilibrium for these economies. We define a competitive equilibrium as a Radner equilibrium with complete markets and with solvency constraints. The securities are all Arrow securities and each agent faces a constraint on the minimum amount of the Arrow security holding. The idea behind these constraints is that by not allowing agents to take large amounts of contingent debt, default will be prevented. In general, these solvency constraints restrict the wealth that is carried to each state z' differently, since the relative value of staying in the economy or going to autarchy should be different.[8]

After defining an equilibrium we show the condition under which efficient allocations can be decentralized as a competitive equilibrium with solvency constraints. Then we analyze the relationship between our equilibrium concept and the one introduced by Kehoe and Levine (1993) and show the conditions under which the first welfare theorem holds. Next, we use these results to characterize the extent of risk sharing. Finally we analyze some properties of the asset pricing kernel.

In this section we keep the maintained assumptions introduced in section 2 and use the additional assumptions (ii), homothetic period utility, and (iii), no-aggregate uncertainty, in the following way. If assumption (iii) does not hold, then

[8]In most of the quantitative asset pricing literature portfolio constraints are usually set exogenously and not as a function of the environment, see for instance Heaton and Lucas (1996) and Telmer (1993). Zhang (1996) computes numerical examples of an equilibrium subjet to a borrowing constraint chosen so that agents are better off than in autarchy. Our model differs from Zhang's in that he exogenoulsy limits agents to trade only one-period riskless bonds and that his borrowing constraint is neither time nor state dependent; additionally, there is no formal analysis of the relationship between equilibrium and efficient allocations. On the theory side, there is a substantial amount of work on how to model borrowing or solvency constraints in the infinite horizon incomplete markets model such as Levine and Zame (1996), Hernandez and Santos (1994), Santos and Woodford (1997) and Magill and Quinzii (1994). This work addresses a different question, namely what is the natural way to extend no Ponzi-game conditions to the *incomplete market* case.

we restrict preferences to satisfy assumption (ii) and preferences and endowments to satisfy assumption (v).[9] Assumptions (i), symmetry, and (iv), ergodicity, are not required.

4.1. Definition of competitive equilibrium with solvency constraints

In this section we introduce a competitive equilibrium with solvency constraints. We also define two important related concepts: a condition on the Arrow-Debreu prices and a condition on solvency constraints. These conditions are relevant, because, as we will show, the welfare theorems hold for competitive equilibria that satisfy them.

We use $q_t\left(z^t, z'\right)$ to denote the period t, state z^t, price of one unit of the consumption good delivered at $t+1$, contingent on the realization of $z_{t+1} = z'$, in terms of period t consumption goods. The corresponding holdings for agent i at t of the Arrow security are denoted by $a_{i,t+1}(z^t, z')$. The lower limit on the holdings of the corresponding Arrow securities, at time t, is denoted by $B_{i,t+1}(z^t, z')$.

Definition 4.1. *An equilibrium with Solvency Constraints* $\{B_{i,t+1}\}$ *for initial conditions* $\{a_{i,0}\}$ *has* $\{c_{i,t}, a_{i,t+1}\}$ *and prices* $\{q_t\}$ *such that:*

a. agents maximize given $\{B_{i,t+1}\}, \{q_t\}$ and $a_{i,0}$; *i.e.* there is a sequence of

[9]This could be relaxed. Assumption (ii) need not be required when the process for aggregate endowment is uniformly bounded above and below.

functions $\{J_{i,t}\}$ such that :

$$\begin{aligned} J_{i,t}\left(a,z^{t}\right) &= \max_{c,\{a(z')\}_{z\in Z}} \left\{ u\left(c\right)+\beta\sum_{z'} J_{i,t+1}\left(a'\left(z'\right),\left(z^{t},z'\right)\right)\pi\left(z'|z_{t}\right)\right\} && (4.1)\\ e_{i,t}\left(z^{t}\right)+a &= \sum_{z'\in Z} a\left(z'\right)q_{t}\left(z^{t},z'\right)+c && (4.2)\\ a\left(z'\right) &\geq B_{i,t+1}\left(z^{t},z'\right) \quad \text{all } z'\in Z, && (4.3)\end{aligned}$$

where we denote by $\{a_{i,t+1}, c_{i,t}\}$ the optimal path for agent i with initial assets $a_{i,0}$;

b. markets clear, *i.e.*, for all $t = 0, 1, ...$ and for all $z^t \in Z^t$,

$$\sum_{i\in I} c_{i,t}\left(z^{t}\right)=e_{t}\left(z^{t}\right),$$

and for all $z' \in Z$

$$\sum_{i\in I} a_{i,t+1}\left(z^{t},z'\right)=0.$$

We introduce a condition on solvency constraints which we describe as *not too tight.* These constraints prevent default by not allowing agents to accumulate more contingent debt than what they are willing to pay back. At the same time they allow as much insurance as possible, in the sense that they bind if and only if the continuation utility is equal to the value of autarchy.

For future reference we reproduce the sufficient Euler and Transversality conditions for the individual maximization (4.1). Given prices $\{q_t\}$ and solvency constraints $\{B_{i,t+1}\}$ if $\{c_{i,t}, a_{i,t+1}\}$ satisfy the following Euler equations, for all z^t, and z'

$$-u'\left(c_{i,t}\left(z^t\right)\right) q_t\left(z^t, z'\right) + \beta\pi\left(z'|z\right) u'\left(c_{i,t+1}\left(z^t, z'\right)\right) \leq 0 \quad (4.4)$$

with equality if $a_{i,t+1} > B_{i,t+1}\left(z^t, z'\right)$and Transversality condition :

$$\lim_{t\to\infty} \sum_{z^t\in Z^t} \beta^t u'\left(c_{i,t}\left(z^t\right)\right) \cdot \left[a_{i,t}\left(z^t\right) - B_{i,t}\left(z^t\right)\right] \cdot \pi\left(z^t|z_0\right) = 0, \quad (4.5)$$

then $\{c_{i,t}, a_{i,t+1}\}$ solves problem (4.1). This result follows from the concavity of the objective function and the convexity of the feasible set.

Definition 4.2. *An equilibrium with Solvency constraints that are not too tight is such that the solvency constraints satisfy*

$$J_{i,t+1}\left(B_{i,t+1}\left(z^{t+1}\right), z^{t+1}\right) = U\left(e_i\right)\left(z^{t+1}\right),$$

for all $t = 0, 1, \ldots$ *and for all* $z^{t+1} \in Z^{t+1}$.

We argue that this is a natural condition to require for the solvency constraints. If the solvency constraint binds and the continuation utility is strictly higher than the value of autarchy, the solvency constraint could be relaxed a bit, without "causing" the agent to default. Having relaxed the solvency constraint, then the agent will take a bit more debt and be better off. As an additional motivation, consider an interpretation where competitive financial intermediaries are price takers but are able to set the solvency constraints to attract agents.

On one hand, intermediaries that were to lend an extra amount to a solvency constrained agent would never be reimbursed. On the other hand, if some agents' continuation utility is strictly above autarchy and their solvency constraints bind, an intermediary could attract all of them by offering to lend some extra amount. Consequently, in an equilibrium with competition among intermediaries solvency constraints should not be too tight.

Notice that given the previous definition, and the fact that $J_{i,t}$ is strictly increasing, an equilibrium allocation $\{a_{i,t+1}, c_{i,t}\}$ with solvency constraints that are not too tight implies that for all z^t and for all i

$$\begin{aligned} U(c_i)(z^t) &\geq U(e_i)(z^t) \quad \text{and} \\ U(c_i)(z^t) &= U(e_i)(z^t) \Leftrightarrow a_{i,t}(z^t) = B_{i,t}(z^t). \end{aligned} \tag{4.6}$$

Finally, we introduce the concept of *high implied interest rates.* This is a condition that ensures finiteness of the value of endowment implied by a given allocation. Given an allocation $\left\{c_{i,t}^*\right\}$, for $i = 1, 2..., I$, we start by defining candidate Arrow prices $\{q_t^*\}$ and Arrow-Debreu prices $\{Q_t^*\}$.

Definition 4.3. *Given an allocation* $\left\{c_{i,t}^*\right\}$, *the period* t, *event* z^t, *candidate Arrow prices are defined as*

$$q_t^*(z^t, z') \equiv \max_{i \in I} \left\{ \beta \frac{u'\left(c_{i,t+1}^*(z^t, z')\right)}{u'\left(c_{i,t}^*(z^t)\right)} \pi(z'|z) \right\}, \tag{4.7}$$

for all t, $z^t \in Z^t$ *and* $z' \in Z$.

These Arrow prices imply the following Arrow-Debreu prices:

Definition 4.4. *Given Arrow prices* $\{q_t^*\}$ *the candidate time* 0 *Arrow-Debreu price of a unit of consumption contingent on the realization of* z^t *at time* t *is defined as follows,*

$$Q_0^*\left(z^t|z_0\right) = q_0^*(z_0, z_1) \cdot q_1^*\left(z_0, z_1, z_2\right) \cdots q_{t-1}^*\left(z^{t-1}, z_t\right). \tag{4.8}$$

Now we are ready to define high implied interest rates.

Definition 4.5. *We say that the implied interest rates for the allocation* $\left\{c_{i,t}^*\right\}$ *are high if the value of the aggregate consumption computed using the Arrow-Debreu candidate prices for the allocation* $\left\{c_{i,t}^*\right\}$ *is finite:*

$$\sum_{t\geq 0} \sum_{z^t \in Z^t} Q_0^*\left(z^t|z_0\right)\left(\sum_{i\in I} c_{i,t}^*\left(z^t\right)\right) < +\infty. \tag{4.9}$$

4.2. Decentralizing optimal allocations: Second welfare theorem

In this section we characterize the allocations that can be supported as a competitive equilibrium with solvency constraints. In particular, we decentralize (constrained) optimal allocations as an equilibrium with some solvency constraints, *i.e.* we state a version of the second welfare theorem.

Proposition 4.6. *Given any allocation $\left\{c_{i,t}^*\right\}$ that satisfies : (a) resource feasibility at any time and event, (b) the participation constraints 2.1 for each agent at each time and event, (c) if the participation constraint of an agent does not bind then this agent has the higher marginal rate of substitution, i.e., for all $t \geq 0$ and $z^t \in Z^t$, $z' \in Z$ then if for agent i*

$$\begin{aligned} U\left(c_i^*\right)\left(z', z^t\right) &> U\left(e_i\right)\left(z', z^t\right) \Longrightarrow \\ \beta \frac{u'\left(c_{i,t+1}^*\left(z^t, z'\right)\right)}{u'\left(c_{i,t}^*\left(z^t\right)\right)} \pi\left(z'|z_t\right) &= \max_{j \in I}\left\{\beta \frac{u'\left(c_{j,t+1}^*\left(z^t, z'\right)\right)}{u'\left(c_{j,t}^*\left(z^t\right)\right)} \pi\left(z'|z_t\right)\right\}, \end{aligned}$$

and, (d) the allocation has high implied interest rates, there exists (i) a process $\{B_{i,t}\}$, initial wealth $a_{i,0}$ and an asset holding process $\left\{a_{i,t}^*\right\}$ such that the plan $\left\{a_{i,t}^*, c_i^*\right\}$ is a competitive equilibrium for the solvency constraints $\{B_{i,t}\}$ and the initial wealth $a_{i,0}$. Moreover, (ii) the process for the solvency constraint $\{B_{i,t}\}$ can be chosen so that the solvency constraints for both agents satisfy 4.6 (are not too tight).

Proof. The proof proceeds by construction. Given the allocation we use 4.7 as candidates for the equilibrium Arrow prices and 4.8 for the implied Arrow-Debreu prices. Since by assumption the implied interest rates are high, we use the Arrow-Debreu budget constraint to construct asset holdings and solvency constraints at each time and event. For a given agent we choose the solvency constraint equal to the holding of the corresponding Arrow security, when this agent's marginal rate of substitution is not the highest one. Then, we check the sufficient Euler and

Transversality conditions. The Euler conditions (4.4) follows from the definition of the Arrow prices (4.7) and the assumption that the unconstrained agent has the highest marginal rate of substitution. The Transversality condition (4.5) follows from our definition of candidate Arrow prices (4.7) and the assumption that the implied interest rates are high. Finally, we construct the function $J_{i,t}$ and use it to find the values of the solvency constraints for the cases where they do not bind. ■

As a corollary of the previous proposition we get the second welfare theorem, since efficient allocations satisfy properties (a), (b) and (c) of proposition (4.6).

Corollary 4.7. *Any constrained optimal allocation that has high implied interest rates can be decentralized as a competitive equilibrium with solvency constraints where the constraints are not too tight.*

The connection between equilibrium allocations and efficient allocations is clear. In the equilibrium, solvency constraints that are not too tight take the place of the participation constraints of the efficient allocations. In equilibrium allocations, an agent whose solvency constraint is not binding has the highest marginal rate of substitution. In efficient allocations, an agent whose participation constraint does not bind has the highest marginal rate of substitution.

In the previous corollary we have restricted ourselves to the case where the optimal allocation satisfied the assumption that the implied interest rates are high. Later on we will develop some results that will allow us to show that if the allocation $\left\{c^*_{i,t}\right\}$ is a constrained optimal allocation where there is some risk sharing, then 4.9 has to hold (the implied interest rate are high).

We finally establish a result about autarchy. Autarchy can always be decentralized as an equilibrium with solvency constraints that satisfy 4.6 (that are not too tight).

Lemma 4.8. *Define $\{c_{a,i,t}, q_{at}, a_{a,i,t+1}, B_{a,it}\}$ as follows: for all $t \geq 0$, $z^t Z^t, z' \in Z$*

$$\begin{aligned} c_{a,i,t}\left(z^t\right) &= e_{i,t}\left(z^t\right), \\ a_{a,i,t+1}\left(z^t, z'\right) &= B_{a,it}\left(z^t\right) = 0, \\ q_{at}\left(z^t, z'\right) &\equiv \max_{i \in I} \beta \frac{u'\left(e_{i,t+1}\left(z^t, z'\right)\right)}{u'\left(e_{i,t}\left(z^t\right)\right)} \pi\left(z'|z_t\right) \end{aligned} \tag{4.10}$$

and $\{Q_{a,t}\}$ are the A-D prices implied by $\{q_{at}\}$ as defined by 4.8. Then $\{c_{a,i,t}, q_{at}, a_{a,it}\}$ is a competitive equilibrium with solvency constraints given $\{B_{a,i,t+1}\}$ and the initial conditions $a_{i,0} = 0$, $i \in I$. Moreover $\{B_{a,i,t+1}\}$ are not too tight.

Proof. The proof follows by verifying the sufficient conditions of Proposition (4.6). ■

Notice that even though autarchy allocations can always be decentralized, in general, they are not (constrained) efficient allocations. This result is analogous to the fact stated above about multiple solutions of the functional equation. That is, that the value of autarchy was always a solution of the functional equation characterizing optimal allocations. Also notice that the implied interest rate in

$\{Q_{a,t}\}$ can be very low and hence equation (4.9) may be violated.

4.3. 1^{st} Welfare theorem and comparison with Kehoe-Levine equilibrium

In this section we address two related issues, the comparison of our equilibrium concept with the one proposed by Kehoe and Levine (1993), henceforth K-L, and the conditions under which the first welfare theorem holds for an equilibrium with solvency constraints.

Let us start setting up some notation. The problem for agent i in the K-L decentralization is

$$\max_{c_i} U(c_i)(z_0)$$

$$s.t. \quad : \quad \mathfrak{P}_0(c_i - e_i) \leq a_{i,0} \tag{4.11}$$

$$U(c_i)(z^t) \geq U(e_i)(z^t) \text{ for all } t \geq 0 \text{ and } z^t \in Z^t \tag{4.12}$$

where $\mathfrak{P}_0$ is a non-negative linear function. Thus, except for the extra inequalities in the consumption possibility set, an equilibrium is a standard Arrow-Debreu equilibrium, henceforth A-D .

Given $\mathfrak{P}_0$, the dot product representation of the A-D prices is defined as follows. For any t and $z^t \in Z^t$ define

$$Q_0\left(z^t|z_0\right) \equiv \mathfrak{P}_0\left(\widetilde{c}\right)$$

where $\widetilde{c}_s\left(z^s\right) = 0$ if $s \neq t$ and $z^s \neq z^t$ and otherwise $\widetilde{c}_t\left(z^t\right) = 1$.

Definition 4.9. *The A-D prices are said to have a dot product representation if*

$$\mathfrak{P}_0\left(c\right) = \sum_{t \geq 0} \sum_{z^t \in Z^t} c_t\left(z^t\right) Q_0\left(z^t|z_0\right).$$

With this notation we can write the budget constraint of agent i as

$$\sum_{t \geq 0} \sum_{z^t \in Z^t} \left(c_{i,t}\left(z^t\right) - e_{i,t}\left(z^t\right)\right) Q_0\left(z^t|z_0\right) \leq a_{i,0}.$$

The corresponding Arrow prices are defined as

$$q_{0,t}\left(z^t, z'\right) = \frac{Q_0\left(z^t, z'|z_0\right)}{Q_0\left(z^t|z_0\right)}.$$

Compared to our equilibrium concept the K-L equilibrium differs along two dimensions. First, they include the participation constraints in the consumption possibility sets, as opposed to our limits to borrowing (our solvency constraints). We think that our alternative decentralization provides a closer link with the literature on equilibrium asset pricing with borrowing and solvency constraints and that it makes the form of the pricing kernel immediate. In particular, it allows us to use the results of Luttmer (1996) and He and Modest (1995). Second, in the K-L decentralization agents are not given the option to default and walk way from their debts. In particular, lifetime utility of consumption plans are limited within an A-D budget constraint. Whereas our equilibrium with solvency constraints that are not too tight is consistent with agents having the option to default on their debt. For instance, agents can choose to borrow now in order to consume immediately by planning to default on their debt next period. In an equilibrium with solvency constraints that are not too tight agents will prefer not to do so. We find this interpretation of the solvency constraint as enforcing individual rationality an attractive feature of our equilibrium concept.

Next we derive two common points between the K-L decentralization and an equilibrium with solvency constraints that are not too tight.

First, the implied Arrow prices in the K-L decentralization are equal to the highest marginal rate of substitution across agents, like in our equilibrium with solvency constraints that are not too tight. Second, under weak conditions, the allocation of an equilibrium with solvency constraints corresponds to the allocations of a K-L equilibrium. An immediate consequence of these results is that we can use the first welfare theorem shown by K-L. And thus, we obtain a (qualified) version of the first welfare theorem for our equilibrium concept. That is, allocations corresponding to an equilibrium for solvency constraints that satisfy 4.6 (solvency constraints not too tight) and 4.9 (that have high implied interest rates) are constrained efficient.

The next proposition characterizes the Arrow prices in a K-L equilibrium.

Proposition 4.10. *Let $\{c_i\}, i = 1, ..., I,$ and $\{Q_0\}$ be the allocations and A-D prices corresponding to a K-L equilibrium. Let q_0 be the corresponding Arrow prices. Then*

$$q_{0,t}\left(z^t, z'\right) = \max_{i \in I}\left\{\beta \frac{u'\left(c_{i,t+1}\left(z^t, z'\right)\right)}{u'\left(c_{i,t}\left(z^t\right)\right)} \pi\left(z'|z_t\right)\right\}, \tag{4.13}$$

and if

$$U\left(c_i\right)\left(z^t, z'\right) > U\left(e_i\right)\left(z^t, z'\right)$$

then

$$q_{0,t}\left(z^t, z'\right) = \beta \frac{u'\left(c_{i,t+1}\left(z^t, z'\right)\right)}{u'\left(c_{i,t}\left(z^t\right)\right)} \pi\left(z'|z_t\right).$$

Proof. It follows a variational argument in a modified problem. In particular we find necessary conditions such that the following feasible deviation does not decrease utility. Decrease consumption at some date and increase it in the next at a particular state. We do not have to worry about the participation constraints at the future date since we are increasing consumption in the future. We do not have to worry about the current participation constraints, since current continuation utility cannot decrease, given that no deviation is feasible for the modified problem. ■

Now we show that for any equilibrium with solvency constraints, if the implied interest rate are high and the constraints are not too tight, then the implied A-D prices and consumption allocations constitute a K-L equilibrium.

Proposition 4.11. *Let $\{q_t, c_{it}, a_{it+1}\}$ be an equilibrium given the solvency constraints $\{B_{i,t+1}\}$ and the initial wealth $\{a_{i0}\}$. Assume that the A-D prices $\{Q_t\}$ implied by $\{q_t\}$ satisfy 4.9 (that the implied interest rates are high) and that the solvency constraints satisfy 4.6 (i.e. they are not too tight). Additionally, assume that for each $i \in I$ there is a constant ξ_i such that for all t, z^t,*

$$\left|u\left(c_{i,t}\left(z^t\right)\right)\right| \leq \xi_i \cdot u'\left(c_{i,t}\left(z^t\right)\right) \cdot c_{i,t}\left(z^t\right) \ . \tag{4.14}$$

Then the consumption allocations $\{c_{it}\}$ and the A-D prices $\{Q_t\}$ constitute a K-L equilibrium.

Remark 2. *The condition (4.14) is a joint requirement on the consumption allocation and the utility functions. It is satisfied automatically in several relevant cases. For instance, it holds for agent i if (a) $u(\cdot)$ has relative risk aversion different from one at zero consumption, i.e.:*

$$\lim_{c \to 0} -\frac{c\, u''(c)}{u'(c)} \neq 1$$

which can be verified by repeated application of L'Hôpital's rule, or (b) $u'(0) < +\infty$, or (c) if consumption for agent i is uniformly bounded away from zero.

Proof. In an equilibrium for given solvency constraints the consumption allocation is feasible. If the solvency constraints are not too tight, the consumption allocations satisfy the participation constraints for each agent at all times and states. It only remains to be shown that given the A-D prices implied by $\{q_t\}$ the consumption allocation maximizes utility subject to budget and participation constraints. It will suffice to find non-negative shadow values (multipliers) associated with the budget constraints and participation constraints. The following algorithm defines these multipliers as a function of the consumption allocation, Arrow prices and participation constraints. The multiplier on the A-D budget constraint (4.11) is set to

$$\zeta_i = \frac{u'\left(c_{i,0}\left(z_0\right)\right)}{Q\left(z_0|z_0\right)} = u'\left(c_{i,0}\left(z_0\right)\right).$$

The multiplier for the participation constraints (4.12) at time $t = 0$ is $\eta_{i,0}\left(z_0\right) = 0$. The multipliers for the participation constraints (4.12) for all $t > 0$ and $z^t \in Z^t$ are defined recursively by

$$\beta^t u'\left(c_{i,t}\left(z^t\right)\right)\left[1 + \sum_{z^r \preceq z^t} \eta_{i,r}\left(z^r\right)\right]\pi\left(z^t|z_0\right) = \zeta_i Q\left(z^t|z_0\right).$$

Finally, one verifies that these multipliers together with the consumption allocation are indeed a saddle. This is accomplished by verifying the first order conditions for a saddle. The fact that Arrow prices are defined as in (4.13) is used to show that the multipliers are non-negative. The assumption that the implied interest rates are high is used, together with (4.14) to show that the summations in the Lagrangian are finite. ■

We obtain the first welfare theorem as a corollary of the previous proposition.

Corollary 4.12. *Since a K-L equilibrium is a standard Arrow-Debreu equilibrium, its consumption allocation is efficient. Then by proposition (4.11) under the assumption of equation (4.14), an equilibrium with solvency constraints that are not too tight (i.e. they satisfy (4.6)) and with implied high interest rates (i.e. they satisfy 4.9) is efficient.*

4.4. Solvency constraints and the extent of risk sharing

In this section we use the equivalence between a K-L equilibrium, equilibrium with solvency constraints that are not too tight and efficient allocations to address three issues. The first is to characterize the circumstances under which autarchy is the only feasible allocation. The second is to show that if an efficient allocation is different from autarchy, then it has high implied interest rates. The third is to show that solvency constraints are negative.

Proposition 4.13. *Consider the autarchy allocation, i.e. the one where the consumption process c_i is equal to the endowment process e_i for all i. If the implied interest rates for this allocation are high (i.e. satisfy 4.9) then autarchy is an efficient allocation, and hence is the only feasible allocation.*

Proof. Recall that lemma (4.8) shows that autarchy is an equilibrium with solvency constraints that are not too tight for $B_{i,t} = 0$ and $a_{i,0} = 0$. Then the result follows using the first welfare theorem (4.12) since by assumption the implied interest rates are high. ■

In the next proposition we give sufficient conditions so that autarchy is the only feasible allocation. We assume that condition (v) on the growth rates and preferences is satisfied for the limit values of the parameters considered.

Proposition 4.14. *Autarchy is the only feasible allocation in either of the four cases: (i) the time discount factor is sufficiently small (i.e.e $\beta \downarrow 0$), (ii) risk aversion is sufficiently small (i.e. $\gamma \downarrow 0$), (iii) the transition probability matrix is sufficiently close to identity, (i.e. $\delta \uparrow 1$) and (iv) the variance of the idiosyncratic shock is sufficiently close to zero (i.e. $var(\epsilon_i) \downarrow 0$ all i).*

Proof. We need to show that under the given condition the autarchy-value of endowment converges to a finite limit. In autarchy, Arrow prices depend only on the current and past z, so they are markov. Thus the value of the aggregate endowment satisfies a simple recursion. This recursion satisfies monotonicity and discounts at a rate that depends on the characteristics (i) to (iv). This rate is strictly smaller than one for the limit values in (i) to (iv). ■

The previous proposition has clear implications for asset pricing. In particular, it suggests the type of parameter values needed so that idiosyncratic shocks generate high and volatile pricing kernels. These are the properties found in many empirical studies, such as Hansen and Cochrane (1992). We will further illustrate these effects in section 5.

Next we show that for any (constrained) optimal allocation where some risk sharing is possible, the implied interest rates are high. This result complements our statement of the second welfare theorem, that uses as an assumption that the implied interest rates are high. These two results imply that efficient allocation with some risk sharing can be decentralized as an equilibrium with solvency constraints that are not too tight.

Proposition 4.15. *Let $\{c_{it}\}$ be a (constrained) efficient allocation. Assume that some risk sharing is possible, so that for each t, z^t, there is $z' \in Z$ such that one of the agent $j \in I$:*

$$U(c_j)(z^t, z') > U(e_j)(z^t, z') \ . \tag{4.15}$$

Then 4.9 is satisfied (i.e. the implied interest rates are high).

Proof. A K-L economy is a standard Arrow-Debreu convex economy, hence by the second welfare theorem any efficient allocation can be supported as a quasi-equilibrium by a linear function $\mathfrak{P}_0$. By a straightforward adaptation of the argument in Proposition (4.10) under the assumed condition (4.15) the Arrow prices implicit in the supporting prices $\mathfrak{P}_0$ in a quasi equilibrium satisfy (4.13). Thus the Arrow prices are strictly positive. This establishes that agent j has a cheaper point in his consumption possibility set, and hence a quasi-equilibrium is an equilibrium. In a K-L equilibrium the value of the agent j endowment is finite. Since $\epsilon_i > 0$ is uniformly bounded then the implied interest rates are high. ■

In the last section of the paper we will analyze a simple case with constant aggregate endowment where we show that interest rates are constant. In terms of that example, the last two propositions say that some risk sharing is possible if and only if the interest rate implied by autarchy is negative.

We end this section by showing that the solvency constraints are negative. Indeed, in the definition of an equilibrium with solvency constraints there is no requirement that the constraints are negative.[10] We show that, for the cases of high interest rates, solvency constraints are effectively constraints on contingent borrowing *i.e.* they satisfy $B_{i,t+1} < 0$.

Proposition 4.16. *If* $\{c_{i,t}, a_{i,t+1}, q_t, B_{i,t+1}, a_{i,0}\}$ *is an equilibrium with solvency constraints that are not too tight, where the implied interest rates are high and where there is only limited risk sharing, then*

$$B_{i,t+1}\left(z^t, z'\right) < 0.$$

Proof. The proof uses the following lemma about a K-L equilibrium. If the allocation allows some risk sharing and has high implied interest rates, then, if an agent's utility is equal to the utility value of autarchy, the initial asset holding of this agent has to be negative. In this case, the proof of the proposition is an immediate consequence of the lemma, and the equivalence of the efficient allocations, K-L equilibrium allocations, and allocations in an equilibrium with

[10] A positive $B_{i,t+1}$ means that agent i has to *save* a minimum amount.

solvency constraints that are not too tight. ■

4.5. Properties of asset returns in economies with solvency constraints

In this section we analyze some properties of the pricing kernels in economies with solvency constraints. We start by describing the pricing of securities that are more complex than Arrow securities. We then compare pricing implications in economies with and without participation constraints by considering marginal valuations, interest rates and risk premia. We present a simple irrelevance result.

4.5.1. Pricing complex securities

Instead of allowing agents to trade only Arrow securities (one-period contingent claims) we want to let them trade any security, particularly multiperiod securities such as stocks and bonds. A straightforward extension of our framework allows us to do this. It turns out that the pricing of any security (under certain conditions stated below) can be obtained by pricing the corresponding portfolio of Arrow securities

We assume that at time t the set of securities that could have possibly been bought (or sold) at dates and states priors to t, z^t is given by $K_t\left(z^t\right)$. These securities have prices $q_{k,t}$ and may pay dividends $d_{k,t}$ at multiple dates. Analogously the set $\left\{K_{t+1}\left(z^t, z'\right) : z' \in Z\right\}$ contains the set of securities that can be bought (or sold) at time and state t, z^t. In this case, agents face the following sequence of budget constraints for all $t = 0, 1, ...$ and for all $z^t \in Z^t$:

$$\sum_{z' \; k' \in K_{t+1}(z^t, z')} a_{i,t+1,k'}(z^t) q_{t,k'}\left(z^t\right) + c_{i,t}(z^t)$$
$$\leq \sum_{k \in K_t(z^t)} a_{i,t,k}(z^{t-1}) \left[q_{t,k}(z^t) + d_{t,k}\left(z^t\right)\right] + e_{i,t}(z^t) \; .$$

To have the same budget set as in the case with Arrow securities we need two conditions: first, that this set of securities be rich enough so that markets are dynamically complete, and second, that the portfolio choice be restricted so that the value of an agent's portfolio be high enough in the different states. Specifically, for each period, a set of solvency constraints that limits the minimum value of next period's portfolio for each state is defined as :

$$\sum_{k' \in K_{t+1}(z^t, z')} \left[q_{t+1,k'}\left(z^t, z'\right) + d_{t+1,k'}\left(z^t, z'\right)\right] a_{i,t+1,k'}\left(z^t, z'\right) \geq B_{i,t+1}(z^t, z')$$

for all $t = 0, 1, ...$ and for all $z^t \in Z^t$. Clearly, every portfolio of securities is just a portfolio of Arrow-Debreu securities, and Arrow-Debreu prices are simple products of the sequence of Arrow prices.

4.5.2. Prices and marginal valuations

Given the frictions introduced by solvency constraints, we find that the prices of securities with non-negative payoffs are generally higher than any of the agents' valuation. Superficially, this result may seem to imply an arbitrage opportunity, in the sense that the prices are too high for everyone, but recall that agents can short sell only limited amounts of securities.[11]

Assume that k' is a security available at time t , *i.e.* $k' \in K_{t+1}\left(z^t, z'\right)$ and that

$$d_{t+s,k'}\left(z^{t+s}\right) \geq 0 \text{ for all } s > 0.$$

Let us denote the price of this security for the equilibrium with solvency constraints by $q_{t,k'}$ and the marginal valuation of agent i by $MV_{i,t,k'}$ where the marginal valuation is defined as

$$MV_{i,t,k'}\left(z^t\right) \equiv \sum_{s>0} \beta^s \sum_{z^s \in Z^s} d_{t+s,k'}\left(z^t, z^s\right) \frac{u'\left(c_{i,t+s}\left(z^t, z^s\right)\right)}{u'\left(c_{i,t}\left(z^t\right)\right)} \pi\left(z^s | z_t\right).$$

This quantity measures the marginal change in utility, in terms of time t consumption, produced by an increase in the i^{th} agent's consumption that is proportional to the dividends of security k' at each future date. When agents are never constrained, or equivalently for the case of perfect insurance, we have that for all i,

$$q_{t,k'}\left(z^t\right) = MV_{i,t,k'}\left(z^t\right).$$

In our environment we have the following result:

Lemma 4.17. *In an equilibrium with solvency constraints, when there is only*

[11] We thank George Constantinides for hihglighting this point.

limited risk sharing, then

$$q_{t,k'}\left(z^t\right) \geq \max_{i\in I} MV_{i,t,k'}\left(z^t\right)$$

for all agents i, securities k', time periods $t = 0, 1, ..,$ and states $z^t \in Z^t$. With strict inequality for an agent i if he is constrained at least once between t and $t + s$.

Proof. The proof of this lemma follows directly from the fact that Arrow prices are equal to the larger marginal rate of substitution of the two agents. ■

Notice that since we assume that the dividends are non-negative, increasing the holding of security k' can never lead to default in the future if the original plan did not already contemplate default. One may therefore be tempted to conclude that in equilibrium $q_{t,k'}\left(z^t\right) = \max_{i\in I} MV_{i,t,k'}\left(z^t\right)$. The misleading part of this conjecture is that this equality holds only for Arrow securities and not for general securities. The latter may pay in different states, but the pricing kernel is defined by the *max* across agents of the marginal rate of substitution, state by state. That is, the agent whose marginal rate of substitution is equal to the price of the Arrow security in a given state may not price the Arrow security in a different state.

4.5.3. Interest rates and security prices

As a general property, interest rates are smaller in economies with solvency constraints than in corresponding economies without such constraints. Moreover, as opposed to the findings in some applications of incomplete markets economies, this effect does not rely on the precautionary savings motive (convexity of the marginal utility).

Proposition 4.18. *The price of a one-period bond is higher in an economy with solvency constraints than in the corresponding economy without these constraints. It is strictly so, if one agent at least is constrained in each period. Regardless of this, the unconditional mean of the risk free rate is strictly lower in the solvency constraints economy.*

Proof. The price of a one-period bond in an equilibrium with solvency constraints and with no aggregate shocks, *i.e.* $e_t = e$, is given by

$$P_t^{PC}[1_{t+1}] = \beta \sum_{z'} \max_{i \in I} \left\{ \frac{u'\left(c_{i,t+1}\left(z^t, z'\right)\right)}{u'\left(c_{i,t}\left(z^t\right)\right)} \right\} \pi\left(z'|z_t\right).$$

The price of a one period bond in an economy without solvency constraint and with no aggregate uncertainty is $P_t[1_{t+1}] = \beta$. Finally, $\max_{i \in I} \left\{ \frac{u'\left(c_{i,t+1}\left(z^t, z'\right)\right)}{u'\left(c_{i,t}\left(z^t\right)\right)} \right\} \geq 1$, otherwise we will arrive to a contradiction with u' being strictly decreasing and the allocations being resource feasible. For the case were the aggregate endowment follow a markov process, with the additional assumption of constant relative risk aversion utility functions we have

$$P_t^{PC}[1_{t+1}] = \beta \sum_{z'} [\lambda_{z'}]^{-\gamma} \max_{i \in I} \left\{ \left[\frac{c_{i,t+1}(z^t, z') / e_{t+1}(z^t, z')}{c_{i,t}(z^t) / e_t(z^t)} \right]^{-\gamma} \right\} \cdot \pi(z'|z_t)$$

Clearly, the same argument as without aggregate uncertainty holds. ■

Notice that in the case of constant aggregate endowment we only use that u' is decreasing. Also, it should be clear that more generally any security with non-negative payouts will have a weakly higher price in the solvency constraints economy compared to the corresponding representative agent one.

4.5.4. The premium for aggregate risk with idiosyncratic shocks

A major issue for any "heterogenous-agent" asset pricing framework is the mechanism through which idiosyncratic shocks can generate a risk premium for claims contingent on the aggregate shock, that is, the market risk premium. We provide here a look at this issue by deriving sufficient conditions under which the economy with idiosyncratic shocks generates a risk premium on a one-period risky strip identical to the representative agent economy.

We consider the risk premium for one-period risky strips, defined as

$$\frac{E_t d_{t+1}}{P_t[d_{t+1}]} P_t[1_{t+1}],$$

where d_{t+1} is the payout, (which we restrict to be a function of $t+1$ aggregate output), $P_t[d_{t+1}]$ is the price of this strip and $P_t[1_{t+1}]$ is the price of a one-period risk-free bond. This is sometimes called the "multiplicative excess return of the one-period risky strip" over the risk-free rate. We choose this premium as opposed to the equity premium for its tractability. Indeed, the equity premium is a weighted sum of the entire infinite sequence of strips, one for each period.

In this section we specialize to $I = 2$. Additionally, we specialize $\lambda(z)$ and $\epsilon_2(z)$ to express the idea that they are statistically independent.

Definition 4.19. *We say that the aggregate shock is i.i.d. and independent of the idiosyncratic shock if* λ , ϵ *and* Π *are such that we can write*

$$\begin{aligned} \lambda(z) &= \lambda_r \text{ for } r \in R, \\ \epsilon_2(z) &= \epsilon_j \text{ for } j \in J, \end{aligned}$$

and

$$\Pr\{\lambda(z') = \lambda_{r'}, \epsilon_2(z') = \epsilon_{j'} | \lambda(z) = \lambda_r, \epsilon_2(z) = \epsilon_j\} = \phi_{r'} \psi_{j,j'}$$

for some probabilities ϕ *and* ψ.

Proposition 4.20. *If assumptions (i) on endowments and (ii) on preferences hold and if the aggregate shock is i.i.d. and independent of the idiosyncratic shock. Then the multiplicative premium on a one-period risky strip with payout contingent on the aggregate output is the same in an economy with and without participation constraints.*

Proof. By the first welfare theorem and the representation of Arrow prices it suffices to analyze the properties for the marginal utility of any efficient allocation. Under the stated assumptions we can write $U^i(e,z)/e^{1-\gamma} = \hat{U}_j^i$ for each agent i. Using this, it follows that $\widehat{c}_i(\hat{w},z)$ can be written as $\widehat{c}_i(\hat{w},j)$, a function of $\hat{w}$ and j alone, where $\widehat{c}_i$ denotes $i's$ consumption share. The desired result follows from computing the relevant marginal rate of substitution. ■

The extra assumption that the aggregate shock is *i.i.d.* is due to the fact that the aggregate shock is expressed in rates of change and the idiosyncratic in levels. At any rate, quantitatively, the assumption of *i.i.d.* growth rates of aggregate consumption is not a bad first approximation. From the proof it is clear that the assumption that $I = 2$, is only for notational convenience. The relevance of this proposition is that it indicates that dependence of the cross sectional distribution of earnings on the aggregate is required to obtain interesting results for excess returns. This result complements similar results by Mankiw (1986) and Constantinides and Duffie (1996) that were obtained in environments with exogenous asset market incompleteness..

4.5.5. Irrelevance of the average agent

The next result is important for thinking about quantitative implementation of this equilibrium concept. Consider first an equilibrium with solvency that are not too tight and add one agents to that economy. If the marginal rates of substitution under autarchy of this agent is smaller than the equilibrium prices, then there is an equilibrium for the expanded economy with the same prices. Additionally the extra agent will be constrained all the time.

Proposition 4.21. *Let the processes* $\left\{c_{i,t}^*, a_{i,t+1}^*\right\}_{i=1,\ldots,I}, \{q_t^*\}, \left\{B_{i,t+1}^*\right\}_{i=1,\ldots,I}$ *and the initial conditions* $\left\{a_{i,0}^*\right\}_{i=1,\ldots I}$ *be an equilibrium with solvency constraints that are not too tight. Add to the previous economy one agent with endowment process* $\{e_{I+1,t}\}$. *Assume that for all* z^t *and* z_{t+1}

$$\beta \frac{u'\left(e_{I+1,t}\left(z^t, z_{t+1}\right)\right)}{u'\left(e_{I+1,t}\left(z^t\right)\right)} \pi\left(z_{t+1}|z_t\right) \leq q_t^*\left(z^t, z_{t+1}\right). \tag{4.16}$$

Then $\left\{c_{i,t}^*, a_{i,t+1}^*\right\}_{i=1,\ldots,I+1}, \{q_t^*\}, \left\{B_{i,t+1}^*\right\}_{i=1,\ldots,I+1}$ *and the initial condition* $\left\{a_{i,0}^*\right\}_{i=1,\ldots I+1}$ *is an equilibrium with solvency constraints that are not too tight where* $\left\{c_{I+1,t}^*\right\} = \{e_{I+1,t}\}, \left\{a_{I+1,t+1}^*\right\} = \{0\}$ *and* $\left\{B_{I+1,t+1}^*\right\} = \{0\}$ *and initial condition* $a_{i,0}^* = 0$.

Proof. The allocation is clearly resource feasible and satisfies the participation constraints. It suffices to show that $\left\{c_{i,t}^*, a_{i,t+1}^*\right\}$ solve (4.1) given $\{q_t^*\}$ and $\{B_{i,t+1}\}$and $\left\{a_{i,0}^*\right\}$ for $i = I+1$. It is immediate to verify that given (4.16) the Euler equations (4.4) are satisfied for all t and that since $a_{i,t+1} = B_{i,t+1}$ for all t the Transversality condition (4.5) is satisfied. ■

The example in the next remark says that adding an extra agent whose endowment is perfectly correlated with aggregate endowment will have no effect on equilibrium prices.

Remark 3. *Let $\alpha > 0$ be a constant. Assumption (4.16) is satisfied if*

$$e_{I+1,t}\left(z^t\right) = \alpha \cdot \sum_{i=1}^{I} e_{i,t}\left(z^t\right) .$$

Agent $I + 1$ has no reason to participate in future insurance arrangements with the I agents, and thus he will default if he acquires any debt. By continuity, agents with individual endowment processes that are very similar to the aggregate endowment are not important for the determination of asset prices. Instead, only the agents with substantial idiosyncratic risk are key to determining asset prices.

5. Analytical solution and calibration of a simple example

In order to illustrate model mechanisms at work and in order to get a very first idea about the quantitative potential of our model for explaining asset returns, we present here some calibration results. We consider a model version with two agents and two shocks and no aggregate uncertainty. We completely characterize the optimal allocations and we look at the implications for consumption and Hansen-Jagannathan bounds. The main point here is to illustrate model mechanisms, for a more thorough quantitative analysis of more general model versions see Alvarez and Jermann (1997). This example suggests that powerful limitations on risk sharing occur in a reasonable parameter region.

5.1. Analytical solution with 2 agents and 2 shocks

Consider the case of two symmetric agents and two shocks $Z = \{\mathfrak{z}_1, \mathfrak{z}_2\}$, with $\epsilon_2(\mathfrak{z}_1) < \epsilon_2(\mathfrak{z}_2)$ and no aggregate uncertainty. We completely solve the decision rules analytically. When full risk sharing is not possible, the decision rules imply a unique ergodic set, where consumption (and hence continuation utility w) depend exclusively on the current value of z. We use this example to illustrate several results derived in the previous sections, and to suggest simple algorithms to compute more general cases.

The decision rules are completely described by analyzing two cases. If $z' = z$, then the optimal policy is the 45^o line as we have shown before. For the remaining case of $z' \neq z$ we show that the policies rules are constant, a result that we refer to as saying that they are flat after reversal.

Definition 5.1. *For a given V we say that decision rules that achieve TV are "flat" after a reversal of the shock if for all $w \in D(z) = [L(z), H(z)]$*

$$\begin{aligned} W_{\mathfrak{z}_2}(w, \mathfrak{z}_1) &= L(\mathfrak{z}_2) \equiv \bar{w}(\mathfrak{z}_2) \quad \text{and} \\ W_{\mathfrak{z}_1}(w, \mathfrak{z}_2) &= H(\mathfrak{z}_1) \equiv \bar{w}(\mathfrak{z}_1) \,. \end{aligned}$$

Recall that $L(z) = U^2(z)$ and that $H(z)$ solves $V(H(z), z) = U^1(z)$. In order to demonstrate this property we state the following two lemmas.

Lemma 5.2. *If*

$$\partial V\left(\bar{w}\left(\mathfrak{z}_1\right),\mathfrak{z}_1\right)/\partial w > \partial V\left(\bar{w}\left(\mathfrak{z}_2\right),\mathfrak{z}_2\right)/\partial w$$

then the decision rules that achieve TV are flat after reversal.

Proof. The proof follows immediately from the first order conditions of problem defined by the RHS of (3.4). ■

Lemma 5.3. *If the decision rules after reversal are flat, then*

$$\partial TV(\bar{w}(\mathfrak{z}_1),\mathfrak{z}_1)/\partial w > \partial TV(\bar{w}(\mathfrak{z}_2),\mathfrak{z}_2)/\partial w.$$

Proof. Using the "45^0 line" result from Proposition (3.8) and the assumption that the "decision rules are flat after reversal" in the two promise keeping equations (3.6) evaluated at $\bar{w}(\mathfrak{z}_1)$ and $\bar{w}(\mathfrak{z}_2)$ we obtain

$$\bar{w}(\mathfrak{z}_1) - \bar{w}(\mathfrak{z}_2) = \frac{u\left(e - C_2\left(\bar{w}(\mathfrak{z}_1),\mathfrak{z}_1\right)\right) - u\left(C_2\left(\bar{w}(\mathfrak{z}_2),\mathfrak{z}_2\right)\right)}{1-\beta\pi+\beta(1-\pi)}.$$

The desired result now follows by using the envelop condition. ■

If full risk sharing is not possible it follows that $\partial \tilde{V}\left(\bar{w}\left(\mathfrak{z}_1\right),\mathfrak{z}_1\right)/\partial w > \partial \tilde{V}\left(\bar{w}\left(\mathfrak{z}_2\right),\mathfrak{z}_2\right)/\partial w$. Then the result that the optimal decision rules are flat after reversal for the fixed point V^* follows by the combination of the previous two lemmas with the result that $\lim_{n\to\infty} T^n\tilde{V} = V^*$.

Figure 3 plots the decision rules $W_{z'}(w,z)$. For any initial (w,z), after one reversal of the shock z, continuation utility and consumption will attain the values $\overline{w}(z)$ and $\overline{c}(z)$, and depend only on the given state z. These decision rules illustrate Proposition (3.15), which for a more general case shows that there is a unique invariant distribution of $\{w_t, z_t\}$ if and only full risk sharing is not possible.

The values of consumption under the invariant distribution have to satisfy the following system of the two promise keeping conditions :

$$\bar{w}(\mathfrak{z}_1) = u(\overline{c}(\mathfrak{z}_1)) + \beta\pi\bar{w}(\mathfrak{z}_1) + \beta(1-\pi)\bar{w}(\mathfrak{z}_2), \text{ and}$$

$$\bar{w}(\mathfrak{z}_2) = u(\overline{c}(\mathfrak{z}_2)) + \beta\pi\bar{w}(\mathfrak{z}_2) + \beta(1-\pi)\bar{w}(\mathfrak{z}_1),$$

in addition to the known boundary value $\bar{w}(\mathfrak{z}_2) = U^2(\mathfrak{z}_2)$, and the symmetry condition $\overline{c}(\mathfrak{z}_1) = e - \overline{c}(\mathfrak{z}_2)$. This system can be written as a single equation in the single unknown $u_2 = u(\overline{c}(\mathfrak{z}_2))$:

$$u_2 = h(u_2) \equiv \frac{(1-\beta)(1-2\beta\pi-1)}{(1-\beta\pi)}U^2(\mathfrak{z}_2) - \frac{\beta(1-\pi)}{1-\beta\pi}f(u_2), \tag{5.1}$$

with $f(u_2) \equiv u\left(e - u^{-1}(u_2)\right)$. The function h is strictly increasing and convex, so it has at most two solutions. Autarchy always satisfies (3.6) so (5.1) has at least one solution.

To verify which of the two solutions of (5.1) corresponds to the efficient allocation we use Proposition (4.13), which shows that autarchy is the only feasible allocation if autarchy has high implied interest rates. For each of the two solutions of (5.1) the implied allocation has a constant interest rate. Hence the corresponding allocation has *high implied interest rates* (*i.e.* the value of aggregate endowment is finite) if and only if the corresponding interest rate is positive. Thus we simply verify that the interest rate implied by autarchy i_a is positive, *i.e.* we verify that

$$\frac{1}{1+i_a} = \beta \left[\pi + (1-\pi) \frac{u'(\epsilon_2(\mathfrak{z}_1))}{u'(\epsilon_2(\mathfrak{z}_2))} \right] < 1 \ .$$

Equivalently, analysis of Equation (5.1) shows that $i_a > 0$ if and only if the solution of (5.1) different from autarchy has more volatile consumption than endowment.

5.2. Calibration: risk sharing and the Hansen-Jagannathan bounds

We calibrate individual income following Heaton and Lucas (1996) based on a large sample from the PSID. In particular, the log of an agent's income, relative to the aggregate, that is $\ln(y_i / \sum y_j)$, is stationary with a first order serial correlation of 0.5 and a standard deviation of 0.29 for annual data. Initially we set $\beta =$ 0.65 and explore the effect of risk aversion for consumption and for asset pricing implications, we will explore the quantitative effects of β below.[12]

[12]The parameters for this case are the following: $\Pi = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$, $\begin{bmatrix} \epsilon_1(\mathfrak{z}_1) \\ \epsilon_1(\mathfrak{z}_2) \end{bmatrix} = \begin{bmatrix} 0.641 \\ 0.359 \end{bmatrix}$, $\begin{bmatrix} \epsilon_2(\mathfrak{z}_1) \\ \epsilon_2(\mathfrak{z}_2) \end{bmatrix} = \begin{bmatrix} 0.359 \\ 0.641 \end{bmatrix}$.

Figure 4 presents consumption of agent 2 as a function of risk aversion. For low risk aversion, agents do not share income risk because autarchy is sufficiently attractive to prevent any borrowing and lending. As risk aversion increases, autarchy becomes relatively less attractive, risk sharing starts to take place and consumption itself becomes less volatile. When risk aversion equals 4.5, income risk is fully shared and consumption is constant.

Hansen and Jagannathan's volatility bounds for stochastic discount factors provide a concise and widely used diagnostic device.[13] The idea is, that, assuming the existence of an arbitrage free pricing kernel, data on asset returns can be used to construct a lower bound on the unconditional standard deviation of this kernel, for a given unconditional mean. Candidate kernels obtained from theoretical model structures can then be compared to this benchmark. Figure 5 presents test results for kernels generated by our model. It can be taken as a clear success that the model is able to generate kernels that fall inside the HJ-bound for risk aversion coefficients around 2, whereas the representative agent economy fails this test for such values of risk aversion.[14] This positive finding also confirms empirical results by He and Modest (1995) that show that solvency constraints can pass a similar type of volatility bound test for aggregate consumption data. A close look at this picture reveals the two-sided role of the risk aversion coefficient. In most of the asset pricing literature, increasing risk aversion increases the volatility of

[13]For a detailled survey of applications of this test see, for example, Cochrane and Hansen (1992).

[14]Introducing aggregate uncertainty would of course generally give further volatility to the pricing kernel and help it pass the test.

the pricing kernel, because for a given consumption process marginal utility is more volatile. In our framework however, the extent of risk sharing and thus the consumption process is endogenous. The highest volatility for the pricing kernel is achieved with very limited risk sharing.

If we were to choose $\beta > 0.65$, as is usually done in the literature, risk sharing would be limited only for even lower coefficients of risk aversion. Consequently the volatility of the pricing kernel would no longer be above the minimum bound imposed by the given HJ-bound. By calibrating to a higher value of serial autocorrelation of the shocks one can obtain pricing kernels that are sufficiently volatile to be inside the admissible HJ-region for much higher values of β. For instance in Figure 6 we set the first order serial correlation at 0.95, which is consistent with the values estimated by Storesletten, Telmer and Yaron (1997). Then we examine the volatility of the pricing kernel for $\beta = 0.90$, a much more standard value.[15] In this case, risk sharing is limited for risk aversion coefficients that are slightly larger than in the previous case. This combination still generates a pricing kernel that is volatile enough to fall within the HJ bounds.

An alternative way of thinking about our model framework is to take it as a two-country open economy, where relative income shocks are rather small. This seems of some interest given that the literature on sovereign debt, in particular the widely cited paper by Eaton and Gersovitz (1981), have studied the same

[15] We keep the unconditional variance of the endowment at the same value as in the previous exercise.

mechanism of lack of commitment of the debtor country.[16] Figure 7 presents consumption for the case where the log of relative income volatility is a mere 3% annually.[17] This can be taken as the order of magnitude of country specific shocks to GDP. The endogenous limitations in risk sharing do appear to be even more powerful in this case given that they occur for a much wider range of risk aversion coefficients. In this last example we can clearly see that the fact that autarchy is less of a punishment with low income volatility ends up reducing risk sharing in equilibrium.

6. Conclusions

We have presented a framework for analyzing asset prices where endogenous solvency constraints may end up limiting risk sharing. After characterizing allocations with participation constraints in a recursive framework we proposed a concept of a market equilibrium with explicit endogenous portfolio constraints. We derived the classical first and second welfare theorems and studied the pricing kernel that emerges in this setup. We derive some general properties of asset prices, for example we show that interest rates in the environment with solvency constraints are lower than in the corresponding representative agent economy, without the usual need for precautionary saving. Finally, a simple quantitative

[16]Eaton and Gersovitz (1981) present a partial equilibrium framework with a given world interest rate where the borrower can default if continuation utility is below the autarchy value.

[17]The parameters for this case are the following: $\Pi = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$, $\beta = 0.65$, $\begin{bmatrix} \epsilon_1(\mathfrak{z}_1) \\ \epsilon_1(\mathfrak{z}_2) \end{bmatrix} = \begin{bmatrix} 0.515 \\ 0.485 \end{bmatrix}$, $\begin{bmatrix} \epsilon_2(\mathfrak{z}_1) \\ \epsilon_2(\mathfrak{z}_2) \end{bmatrix} = \begin{bmatrix} 0.485 \\ 0.515 \end{bmatrix}$.

example shows that powerful limitations to risk sharing occur in the relevant region of the parameter space.

We view this paper as a first step in exploring asset pricing relationships in an economy where the possibility of default limits risk sharing. In a companion paper we have started examining the quantitative side more in depth. Given the thorough characterization of the equilibrium allocations of this paper we are able to design simple and fast algorithms for computing efficient allocations. Applying these allows us to address several quantitative issues. We are interested in the first moments such as the mean risk free rate and the equity premium that have attracted so much attention in the recent literature on equilibrium asset pricing. We are also interested in the business cycle behavior of excess returns and the term structure that have been documented empirically.

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Figure 1

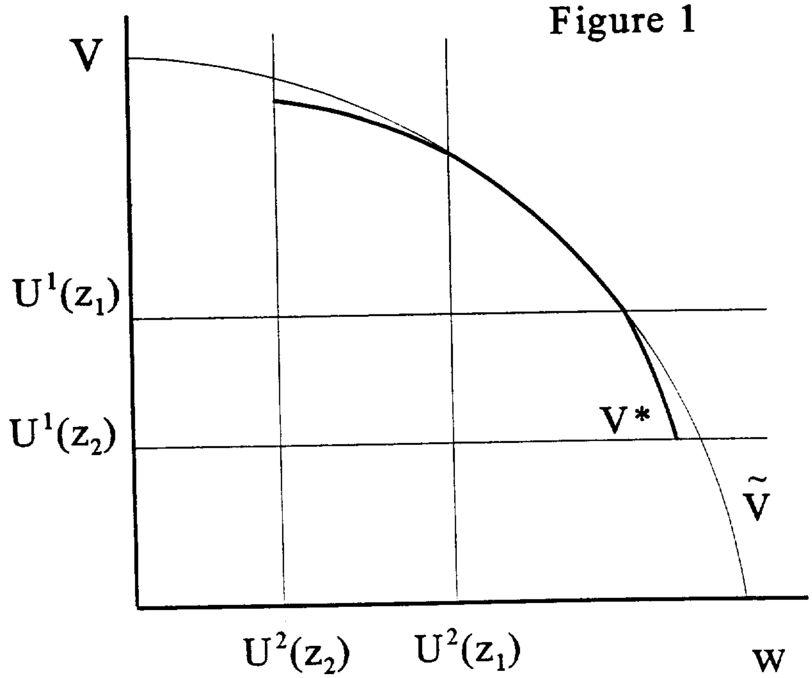


Figure 2

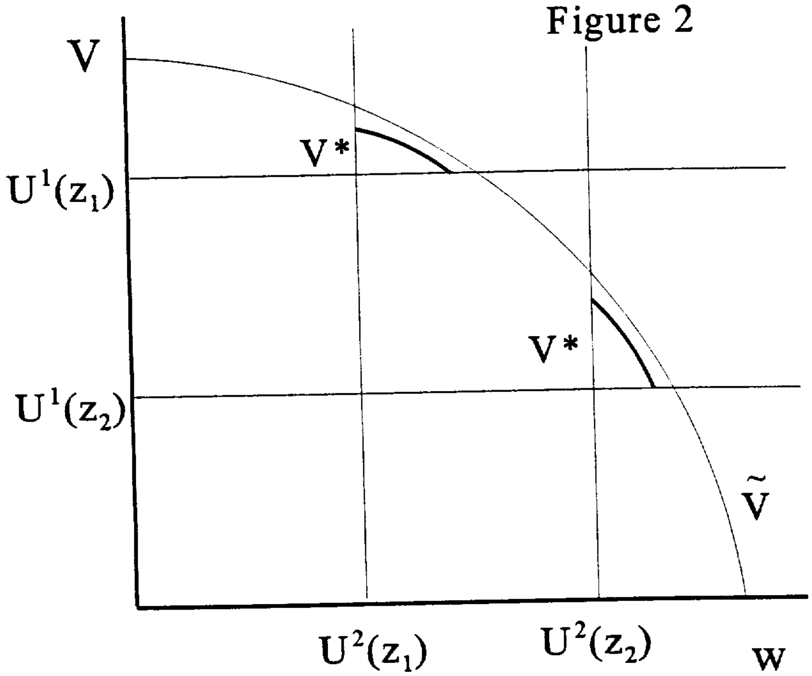


Figure 3

Decision rules for 2 by 2 case

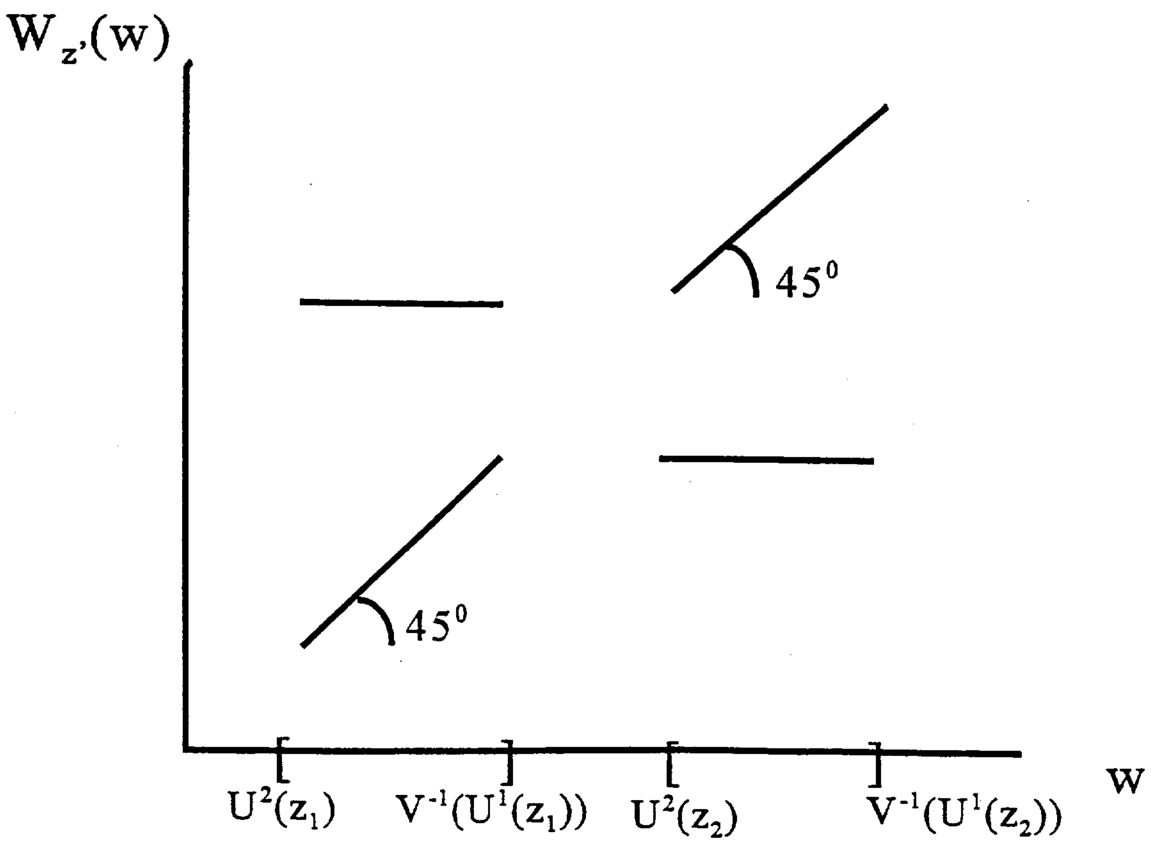
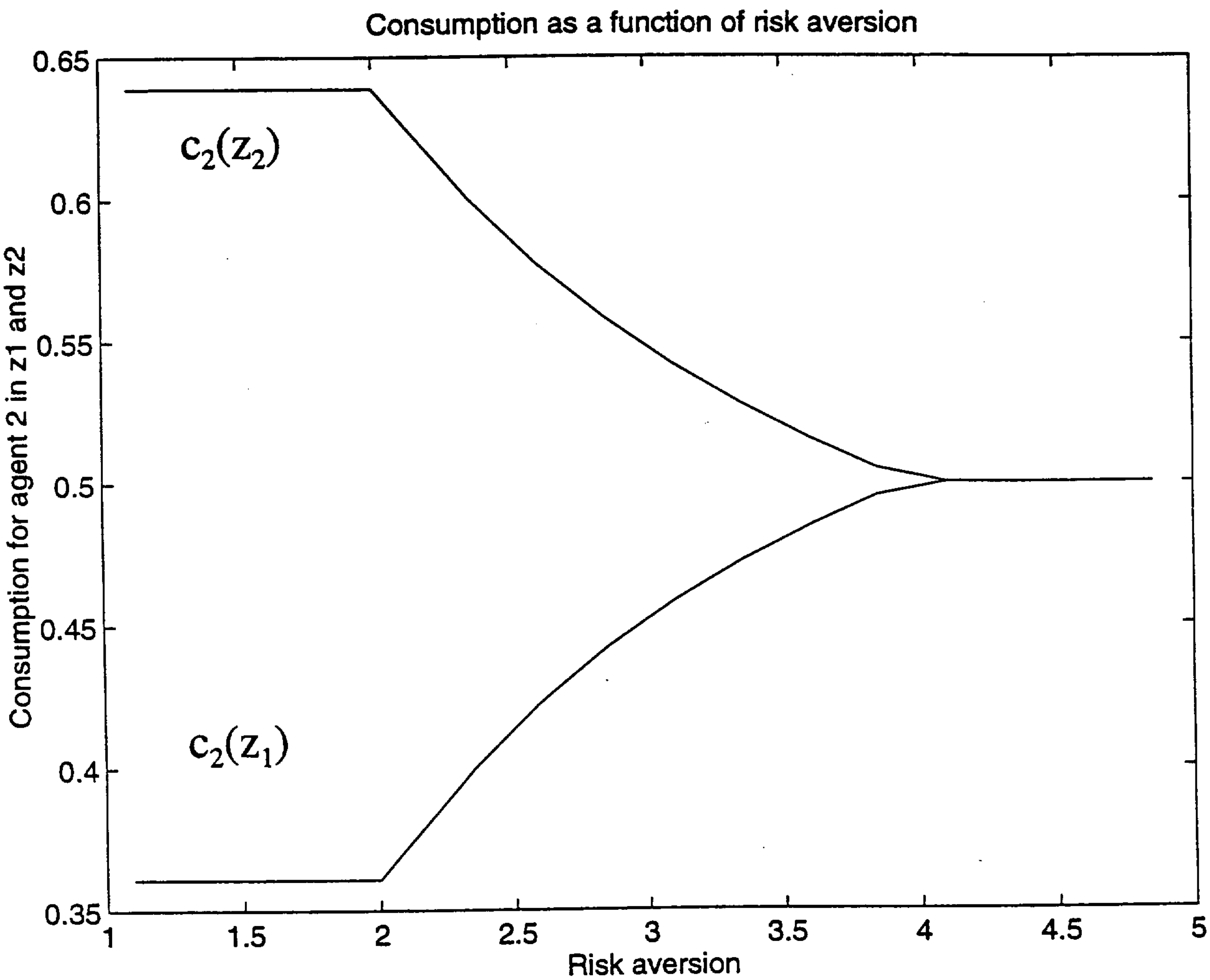
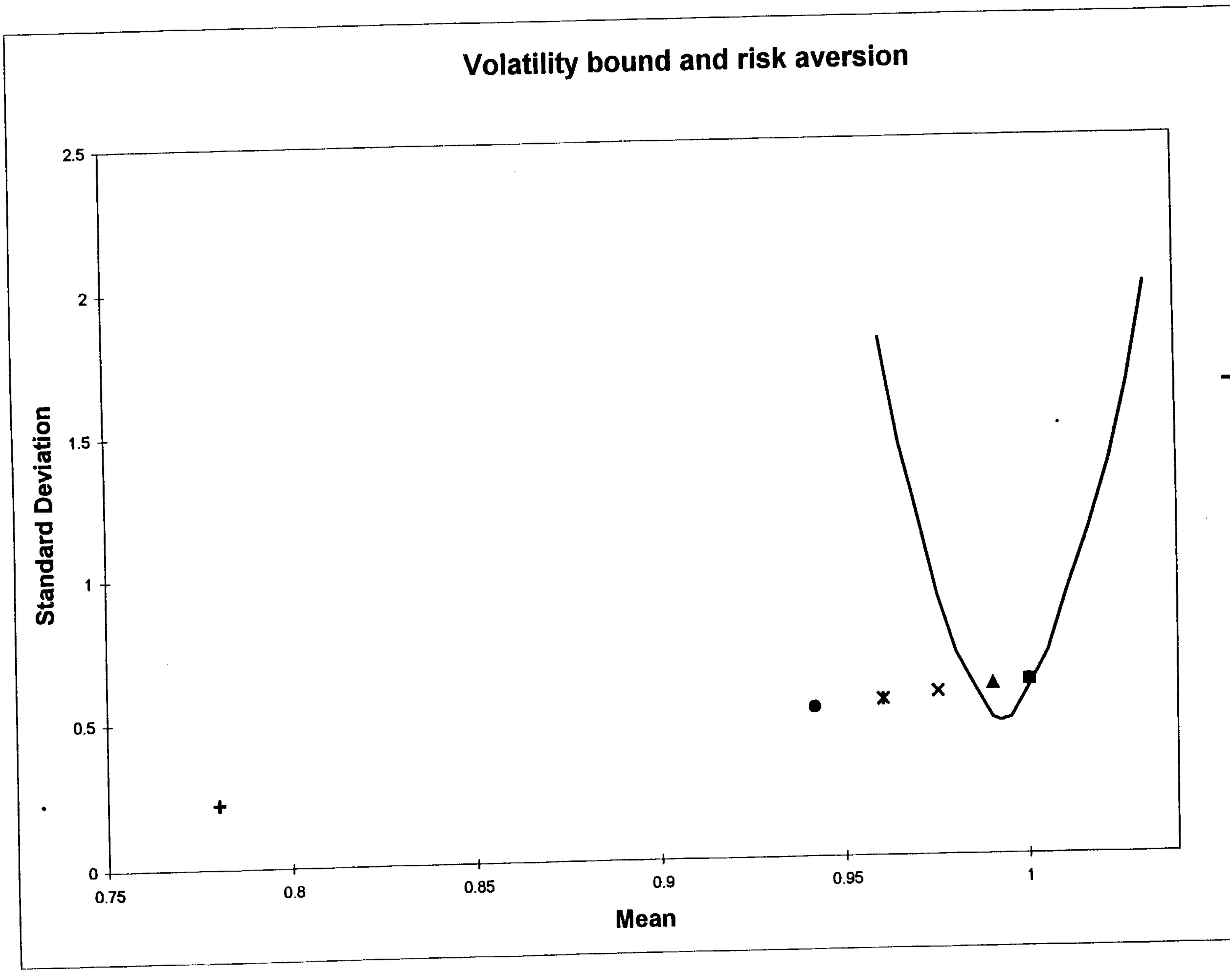


Figure 4



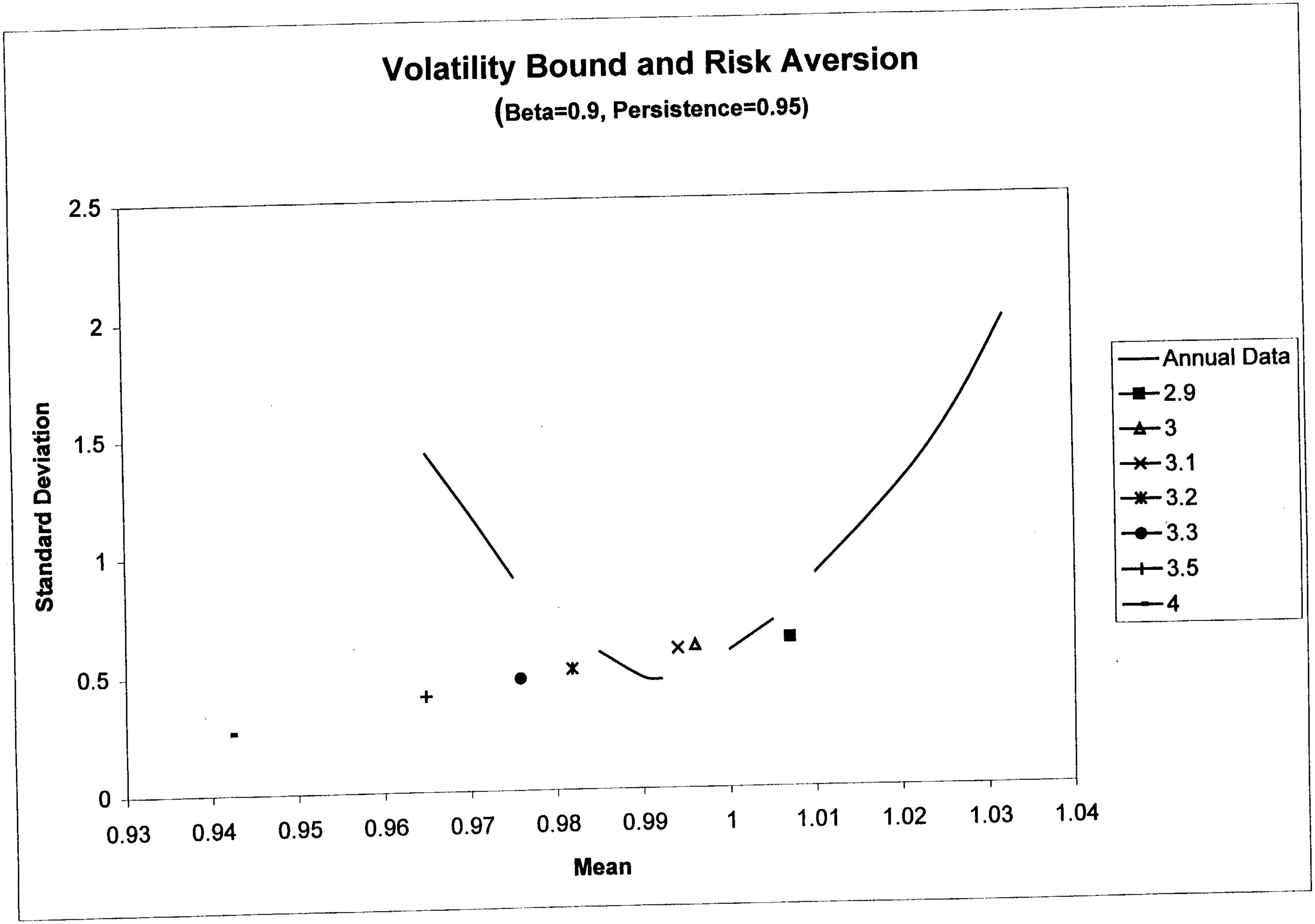
Consumption as a function of risk aversion
0.65
0.6
0.55
0.5
0.45
0.4
0.35
Consumption for agent 2 in z1 and z2
$c_2(z_2)$
$c_2(z_1)$
1
1.5
2
2.5
3
3.5
4
4.5
5
Risk aversion

Figure 5

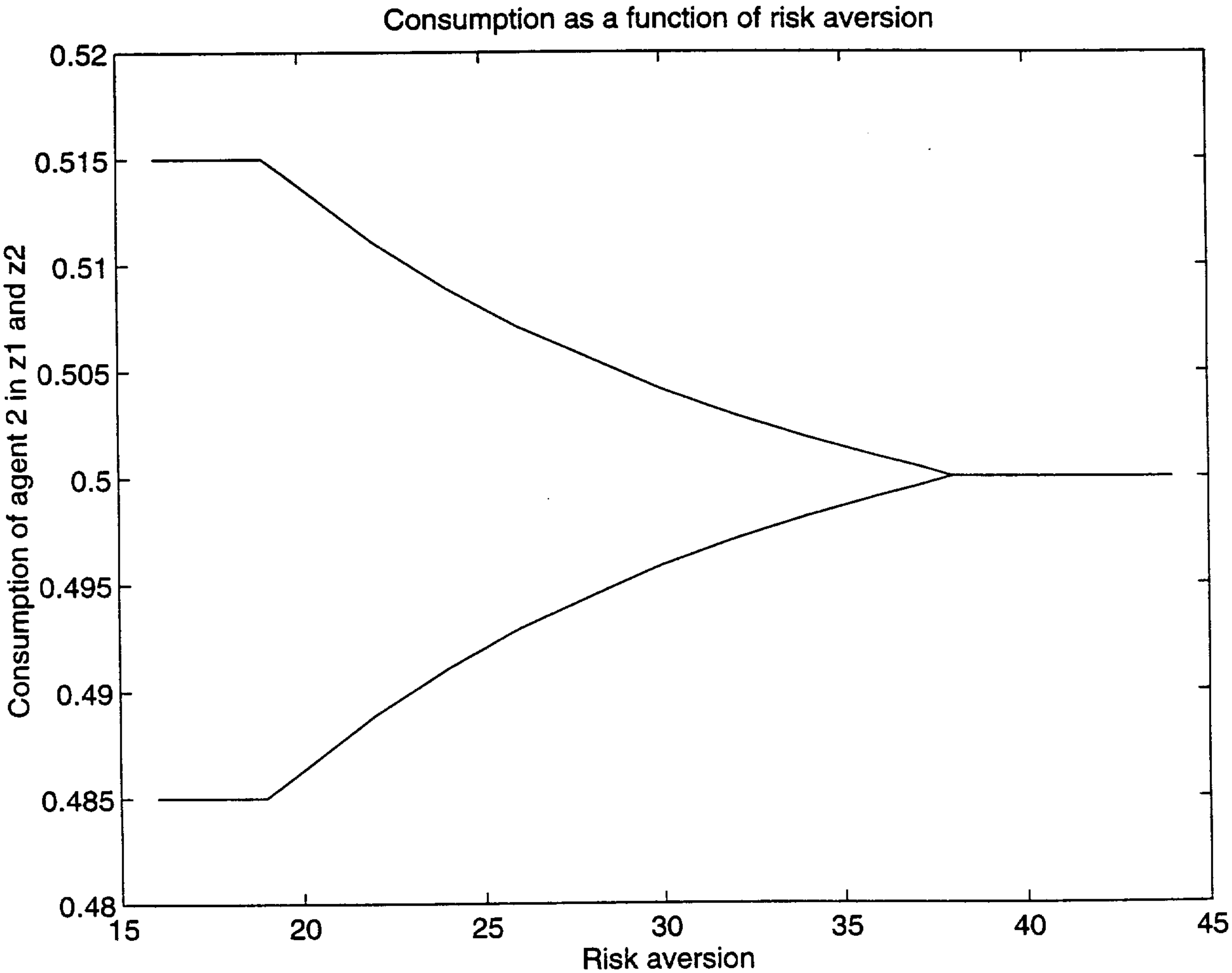


The annual data is from Cochrane and Hansen (1992). The individual income process have standard deviation of 29% and first order se
0.5, this is consistent with estimates from Heaton and Lucas (1994).

Figure 6



The annual data is from Cochrane and Hansen (1992).
The individual income processes have standards deviation of 29% following Heaton and Lucas (1994).


Consumption as a function of risk aversion
Consumption of agent 2 in z1 and z2
Risk aversion
0.52
0.515
0.51
0.505
0.5
0.495
0.49
0.485
0.48
15
20
25
30
35
40
45