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Hanno Lustig
Chris Sleet
Şevin Yeltekin

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ABSTRACT

We identify a novel, fiscal hedging motive that helps to explain why governments issue more expensive, long-term debt. We analyze optimal fiscal policy in an economy with distortionary labor income taxes, nominal rigidities and nominal debt of various maturities. The government in our model can smooth labor tax rates by changing the real return it pays on its outstanding liabilities. These changes require state contingent inflation or adjustments in the nominal term structure. In the presence of nominal pricing rigidities and a cash in advance constraint, these changes are themselves distortionary. We show that long term nominal debt can help a government hedge fiscal shocks by spreading out and delaying the distortions associated with increases in nominal interest rates over the maturity of the outstanding long-term debt. After a positive spending shock, the government raises the yield curve and steepens it.

Hanno Lustig
Department of Economics
UCLA
Box 951477
Los Angeles, CA 90095-1477
and NBER
hlustig@econ.ucla.edu

Christopher Sleet
Tepper School
Carnegie Mellon University
Pittsburgh, PA 15213
csleet@andrew.cmu.edu

Şevin Yeltekin
Tepper School
Carnegie Mellon University
Pittsburgh, PA 15213
sevin@andrew.cmu.edu

I. Introduction

Governments have traditionally financed their spending by raising taxes and selling nominal bonds of varied maturities. An old debate concerns the best way for a government to manage the maturity structure of these liabilities. Barro (2002) and Campbell (1995) argue that the government should issue short rather than long term nominal debt, because the persistent nature of inflation makes the real holding return on the former less volatile than the holding return on the latter. Hence, a government that issues short term nominal debt reduces its own risk exposure as well as that of bond investors. In the process, it economizes on the risk premium it pays to investors. These arguments treat inflation and nominal interest rates exogenously.

In this paper, we explore optimal maturity management in a fully specified general equilibrium model in which the only exogenous elements are shocks (to government spending) and all prices are endogenously determined. We find that the optimal policy entails the exclusive use of the longest term debt available coupled with an appropriate management of nominal bond prices and interest rates. The nominal price of outstanding bonds is reduced after unexpectedly large shocks to government spending, by the anticipation of higher, future short rates. Consequently, optimal policy departs from the Friedman rule. Consistent with Barro and Campbell's observations, this policy implies a more volatile ex post real holding return on long term debt relative to short term. However, this volatility is deliberate and managed so as to hedge the fiscal risk the government faces. We argue that the risk premium paid to investors is more like an insurance premium from the government's perspective.

In our model the government finances its activities by selling nominal debt of various maturities, altering the real return on this debt, printing money and levying taxes on income. These taxes distort the labor-consumption margin and, as in many models, the government would like to smooth their path across dates and states via the trading of debt and via state-contingent alterations in the real return that it pays on its debt. Since we assume that the government can only sell debt with a non-contingent *nominal* return, it cannot deliver state-contingent variations in the *real* return directly. Instead, it must do so indirectly through contemporaneous inflation or through variations in the nominal term structure. In particular, the government may hedge a positive government spending shock by devaluing its nominal liabilities through an immediate inflation or through higher expected future inflation and higher future short run nominal interest rates[1], which in turn reduce the price of outstanding longer term debt today.

We introduce two nominal rigidities that render variations in inflation and nominal debt prices costly. First, we assume that some firms set their prices before the realization of the current state. This rigidity implies that contemporaneous state-contingent inflations are associated with a misallocation of production, and labor supply, across sticky and flexible price firms. Absent this rigidity, optimal policy would fully hedge fiscal shocks through contemporaneous inflations that appropriately adjust the real value of the government's nominal liabilities. With the rigidity, the government must trade the benefits from hedging against the production distortion that state-contingent inflations induce. Second, we also assume that households face a cash-in-advance constraint applied to some goods (cash goods), but not others (credit goods). Consequently, deviations in nominal interest rates from zero distort consumption bundles as households substitute credit for cash goods in an effort to economize on cash. The benefits of hedging achieved through variations in nominal debt prices must be traded off against the costs stemming from this second distortion. In summary, variable income tax rates, contemporaneous state-contingent inflations and deviations in nominal interest rates from zero are all costly. The task of the government is to implement a fiscal-monetary policy that minimizes the joint distortions from each of these sources.

In our model, the optimal policy prescribes the use of the longest maturity debt available. While all nominal debt helps with the smoothing of income tax rates and the labor-consumption wedge, *long term* nominal debt helps with the management of the other wedges as well. Reductions in the nominal value of the government's liabilities in the aftermath of a fiscal shock involve positive short term interest rates and corresponding distortions to the cash-credit good margin over the term of these liabilities. Long run debt is useful because it allows the government to postpone these positive interest rates and costly distortions until further into the future. After an unanticipated increase in government spending, the short term nominal interest rate is gradually increased until the initially outstanding debt reaches maturity. In other words, the yield curve is shifted upward and it becomes steeper, up until the largest outstanding maturity.

Our paper follows the Ramsey tradition of placing exogenous restrictions on the set of policy instruments available to the government. Our central restriction, in addition to the standard assumption of a linear income tax, is that the government can only trade debt with a non-contingent nominal return. This assumption has, of course, been widely made in the literature, because, historically, these are the securities that governments have issued. [2] We augment it with a second restriction: that the government cannot lend to households.[3] We discuss the role and theoretical basis for this second assumption in Sections II and VI of the paper. Both assumptions are motivated by empirical considerations. Governments *have* historically sold non-contingent nominal debt of various maturities. Thus, these assumptions allow our model to make closer contact with the data.[4]

Two recent contributions provide empirical evidence that a hedging motive may influence the conduct of nominal interest rate policy. In Lustig, Sleet and Yeltekin (2004), we find evidence that exogenous fiscal shocks do predict higher future nominal yields in post-war US data. In particular, we focus on shocks to the present discounted value of US defense spending. Further corroboration is also provided by Dai and Philippon (2004). Their VAR analysis indicates that shocks to the government deficit lead to subsequent increases in long term interest rates.[5] Both sets of facts are consistent with the optimal policy prescribed by our model.

The plan for the remainder of the paper is as follows. After a brief review of the literature, we outline the model in Section II. In Section III, we characterize competitive allocations. Sections IV and V give Ramsey problems for our environment and for a benchmark complete markets environment. In Section VI we discuss the role of the no lending constraints. We provide a simple example in Section VII that provides intuition and for which a partial analytical solution is available. Section VIII provides a recursive formulation for the general problem, while Section IX implements this formulation numerically and computes optimal policy in several parameterized examples. Finally, Section X concludes.

A. Related literature

The literature on optimal fiscal and monetary policy has made various assumptions about the asset market structure confronting the government. In the benchmark complete markets model, it is optimal for the government to use state-contingent claims to hedge fiscal shocks (see Lucas and Stokey, 1983). This hedging can be achieved via many different portfolios of claims. Angeletos (2002), Barro (1995) and Buera and Nicolini (2004) consider governments restricted to trading non-contingent *real* debt of different maturities. In these models the government can use state-contingent variations in the price of longer term *real* debt to hedge shocks. Angeletos shows that if the set of traded debt maturities is large enough, the government can (almost) achieve the optimal complete markets allocation; when it is not too large, the optimal maturity structure, if it exists, is unique. However, Buera and Nicolini show that it may entail very large asset market positions. We contrast our analysis with these papers in Section VI.

Bohn (1988) points out that a government can hedge fiscal shocks by combining one period nominal debt with a state-contingent monetary policy that induces appropriate variations in the price level. He assumes that variations in inflation are costless. In contrast, Schmitt-Grohé and Uribe (SU) (2004) and Siu (2004) introduce frictions that render state-contingent inflations distortionary. Consequently, a government must trade the costs of state-contingent inflations off against their hedging benefits. Both SU and Siu show that at moderate levels of debt the inflation costs dominate; it is desirable to have a fairly stable inflation rate with little variation in the real return paid on debt.

Our model is closest to that of SU and, especially, Siu. Like them we assume that state-contingent inflations are costly. Unlike them, we allow the government to trade nominal debt of more than one period maturity. Thus, we are able to consider the optimal nominal maturity structure. Additionally, in our model, the government can influence the price of outstanding nominal bonds via current and future nominal interest rate policy. This opens up a second channel for hedging fiscal shocks that is absent in the Siu and SU models.

II. A model with sticky prices

The economy is inhabited by a population of infinitely-lived households, firms and a government. Let $s_t \in S \subset \mathbb{R}^N$ denote a period t shock and assume that S is a finite set with elements $\{\widehat{s}_1, \ldots, \widehat{s}_N\}$. Let $s^t = \{s_0, \ldots s_t\} \in S^{t+1}$ denote a t-period history of shocks. We assume that s_0 is distributed according to π^0 and that subsequently shocks evolve according to a Markov process with transition matrix π. π is assumed to be monotone: if $s > s'$, then $\sum_{s'' \geq \widetilde{s}} \pi(s''|s) \geq \sum_{s'' \geq \widetilde{s}} \pi(s''|s')$ for each $\widetilde{s} \in S$. Finally, we denote the implied probability distribution over shock histories s^t by π^t.

A. Households

Households have preferences over stochastic sequences of cash goods $\{c_{1t}\}_{t=0}^{\infty}$, credit goods $\{c_{2t}\}_{t=0}^{\infty}$ and labor $\{l_t\}_{t=0}^{\infty}$ of the form:

$$E\left[\sum_{t=0}^{\infty} \beta^t U(c_{1t}, c_{2t}, l_t)\right], \tag{1}$$

where $U : \mathbb{R}_+^2 \times [0, T] \to \mathbb{R}$ is twice continuously differentiable on the interior of its domain, strictly concave, strictly increasing in its first two arguments and decreasing in its third argument. We also assume that U is homothetic in (c_1, c_2) and weakly separable in l. We use the notation U_{jt}, $j = 1, 2, l$ to denote the derivatives of U with respect to each of its arguments at date t. We let U_{jkt}, $j, k = 1, 2, l$ denote its second derivatives at t.

Trading Each household enters period 0 with a portfolio of money $M_0 \geq 0$ and nominal (zero coupon) bonds $\{B_0^k\}_{k=1}^K \in \mathbb{R}_+^K$, where the superscript k denotes the maturity of the bond and K is the maximal maturity traded. The period 0 shock, s_0, is then realized and asset markets open. In equilibrium, households trade bonds and money with the government. We let $\widetilde{M}_0(s^0)$ and $\{\widetilde{B}_0^k(s^0)\}_{k=1}^K$ denote the portfolio of bonds and money purchased by households.

In subsequent periods, asset market trading is assumed to occur in two stages. The first stage takes place before the current period shock is realized, the second stage after this realization. Households receive wages and dividend payments at the end of the period. These are paid in cash (or claims to cash).

First Round of Trading Households invest cash payments received in the previous period in bonds in advance of the current period's shock, and the government sells a portfolio of bonds to households prior to the shock, to better insure itself.

Second Round of Trading Next, the households can liquidate their bond holdings in light of their state-contingent period t cash needs.[67] Let $M_t(s^{t-1})$ and $\{B_t^k(s^{t-1})\}_{k=1}^K$ denote the portfolio of money and bonds purchased by households during the first trading round. Let $\widetilde{M}_t(s^t)$ and $\{\widetilde{B}_t^k(s^t)\}_{k=1}^K$ denote the portfolio purchased in the second round.

Spot markets After asset trading is complete, households split into shoppers and workers. The shopper takes the household's money to the goods market, where she purchases cash and credit goods. The shopper must use money to buy cash goods. Consequently, she faces the cash-in-advance constraints, $\forall t \geq 0,\ s^t \in S^{t+1}$:

$$P_t(s^t)c_{1t}(s^t) \leq \widetilde{M}_t(s^t), \tag{2}$$

where $P_t(s^t)$ is the price of both cash and credit goods. Meanwhile, the worker exerts effort in production $l_t(s^t)$. At the end of the period the household receives nominal income $I_t(s^t)$, where

$$I_t(s^t) = W_t(s^t)l_t(s^t) + \int_0^1 \Pi_t^i(s^t)di.$$

Here $W_t(s^t)$ denotes the period t nominal wage, while $\Pi_t^i(s^t)$ is the nominal profits of intermediate goods firm i at this date. The household then pays its taxes, settles any outstanding credit balances and takes its portfolio into the subsequent period.

Entering into period t, the household has a quantity of money and claims to period t money given by:

$$\begin{aligned}\widetilde{A}_{t-1}(s^{t-1}) &\equiv \widetilde{B}^1_{t-1}(s^{t-1}) + \{\widetilde{M}_{t-1}(s^{t-1}) - P_{t-1}(s^{t-1})c_{1t-1}(s^{t-1})\} \\ &\quad -P_{t-1}(s^{t-1})c_{2t-1}(s^{t-1}) + (1-\tau_{t-1}(s^{t-1}))I_{t-1}(s^{t-1}),\end{aligned}$$

where $\tau_{t-1}(s^{t-1})$ is the tax rate levied on household income. Thus, $\widetilde{A}_{t-1}(s^{t-1})$ includes nominal bonds maturing at date t, money that was not spent on cash or credit goods, after-tax income, less any debts accrued through the purchase of consumption goods on credit. Analogously, define $A_t(s^{t-1}) = M_t(s^{t-1}) + B^1_t(s^{t-1})$ to be the quantity of one period bonds and cash purchased in the first round of asset trading. $\forall t-1 \geq 1$, $s^{t-1} \in S^{t-1}$, the household faces the first round budget constraint:

$$\widetilde{A}_{t-1}(s^{t-1}) + \sum_{k=2}^{K} \widetilde{Q}^k_t(s^{t-1})\widetilde{B}^k_{t-1}(s^{t-1}) \geq A_t(s^{t-1}) + \sum_{k=2}^{K} \widetilde{Q}^k_t(s^{t-1})B^k_t(s^{t-1}), \tag{3}$$

where $\widetilde{Q}^k_t(s^{t-1})$, $k \in \{2, \ldots, K\}$, is the nominal price of the k-th maturity bond and $\widetilde{Q}^1_t(s^{t-1}) = 1$.[8] The household's budget constraint from the second trading round is, $\forall t \geq 0$, $s^t \in S^t$

$$A_t(s^{t-1}) + \sum_{k=1}^{K-1} Q^k_t(s^t)B^{k+1}_t(s^{t-1}) \geq \widetilde{M}_t(s^t) + \sum_{k=1}^{K} Q^k_t(s^t)\widetilde{B}^k_t(s^t), \tag{4}$$

where $Q^k_t(s^t)$, $k \in \{1, \ldots, K\}$, is the nominal price of the k-th maturity bond in this round. We normalize Q^0_t to 1.

Finally, following Chari and Kehoe (1993), we assume that households participate anonymously in the bond market. This assumption makes any bonds issued by households unenforceable and ensures that no one is willing to buy such a bond.[9] It clearly prevents households from running Ponzi games. Formally, we assume, for all t, s^t and k,

$$B^k_t(s^{t-1}) \geq 0, \qquad \widetilde{B}^k_t(s^t) \geq 0. \tag{5}$$

This precludes equilibrium lending by the government to households. Consequently, we will refer to (5) as a *no lending constraint.* Ultimately, the repayment of a government loan and the payment of a tax represent transfers to the government from households. Typically, Ramsey models assume that one (the tax) is a linear function of a household's income or consumption, while the other (the repayment) is lump sum. This treatment is somewhat arbitrary. In richer models with heterogeneity and private household information, it may well be desirable, and, perhaps, necessary to allow transfers between governments and households, regardless of whether they are labeled a tax or a repayment, to depend on the households observed income or consumption. If loan repayments depend on such observables, then they will typically be distortionary, just as taxes are. We do not explicitly model such costs, rather we simply rule government loans out. As we discuss in Section VI, absent restrictions on government lending, the government can attain an allocation arbitrarily close to that in a complete markets economy by lending arbitrarily large amounts at shorter maturities and borrowing arbitrarily large amounts at longer ones. Thus, this model forces one to take a position on whether these extreme asset market positions are reasonable. Our no lending constraints preclude them.[10]

Households maximize (1) subject to the constraints (2), (3), (4) and (5).

B. Final goods firms

Final goods firms produce output Y_t for household and government consumption from intermediate goods Y_{it}, using a CES technology:

$$Y_t = \left[\int_0^1 Y_{it}^{\frac{1}{\mu}} di\right]^{\mu}, \ \mu > 1. \tag{6}$$

Intermediate goods are produced by sticky and flexible price firms. The former set their t period price, P_{st}, before s_t is revealed, the latter set their price, $P_{f,t}$, after s_t is revealed. Letting ρ denote the fraction of sticky price firms and assuming symmetry across each type of intermediate good producer, the total output of the final goods firm is given by: $Y_t = \left[(1-\rho)Y_{f,t}^{\frac{1}{\mu}} + \rho Y_{s,t}^{\frac{1}{\mu}}\right]^{\mu}$, where $Y_{f,t}$ and $Y_{s,t}$ are, respectively, the amount of flexible and sticky price intermediate good used. Final goods firms behave competitively. Taking prices as given, they choose quantities of intermediate goods to maximize their profits:

$$\sup_{Y_{f,t}(s^t),Y_{s,t}(s^t)} P_t(s^t)\left[(1-\rho)Y_{f,t}(s^t)^{\frac{1}{\mu}}+\rho Y_{s,t}(s^t)^{\frac{1}{\mu}}\right]^{\mu}-(1-\rho)P_{f,t}(s^t)Y_{f,t}(s^t)-\rho P_{s,t}(s^{t-1})Y_{s,t}(s^t). \quad (7)$$

C. Intermediate goods

Intermediate goods are produced with labor according to the technology: $Y_{it}=L_{it}^{\alpha}$. Substituting this and the demand curves stemming from (7) into its objective, a flexible price intermediate goods firm chooses its price $P_{f,t}(s^t)$ to solve:

$$\Pi_f(s^t)=\sup_{P_{f,t}(s^t)} P_{f,t}(s^t)\left(\frac{P_{f,t}(s^t)}{P_t(s^t)}\right)^{\frac{-\mu}{\mu-1}}Y_t(s^t)-W_t(s^t)\left\{\left(\frac{P_{f,t}(s^t)}{P_t(s^t)}\right)^{\frac{-\mu}{\mu-1}}Y_t(s^t)\right\}^{\frac{1}{\alpha}}.$$

In contrast, a sticky price firm chooses its price $P_{s,t}(s^{t-1})$ before s_t is determined, so as to solve:

$$\sup_{P_{s,t}(s^{t-1})}\sum_{s^t|s^{t-1}}\pi_t(s^t|s^{t-1})(1-\tau_t(s^t))\frac{U_{2t}(s^t)}{P_t(s^t)}\Bigg[P_{s,t}(s^{t-1})\left(\frac{P_{s,t}(s^{t-1})}{P_t(s^t)}\right)^{\frac{-\mu}{\mu-1}}Y_t(s^t)$$
$$-W_t(s^t)\left\{\left(\frac{P_{s,t}(s^{t-1})}{P_t(s^t)}\right)^{\frac{-\mu}{\mu-1}}Y_t(s^t)\right\}^{\frac{1}{\alpha}}\Bigg].$$

D. Government

The government faces a stochastic process for government spending $\{G_t\}_{t=0}^{\infty}$ of the form $G_t(s^t)=G(s_t)$, where $G:S\to\mathbb{R}_+$ is exogenously given and strictly increasing. The government finances its spending by levying taxes on labor and trading non-contingent nominal bonds of maturity $k\leq K$. Like the households, from period $t=1$ onwards, the government engages in two rounds of asset market trading. Its first round budget constraint at t is:

$$\widetilde{A}_{g,t-1}(s^{t-1})+\sum_{k=2}^{K}\widetilde{Q}_t^k(s^{t-1})\widetilde{B}_{g,t-1}^k(s^{t-1})\leq A_{g,t}(s^{t-1})+\sum_{k=2}^{K}\widetilde{Q}_t^k(s^{t-1})B_{g,t}^k(s^{t-1}),$$

where $\widetilde{A}_{g,t-1}(s^{t-1}) = \widetilde{M}_{t-1}(s^{t-1}) + \widetilde{B}^1_{g,t-1}(s^{t-1}) - \tau_{t-1}(s^{t-1})I_{t-1}(s^{t-1}) + P_{t-1}(s^{t-1})G(s_{t-1})$, $A_{g,t}(s^{t-1}) = M_t(s^{t-1}) + B^1_{g,t}(s^{t-1})$, the g subscript is used to denote the government's outstanding debt, and the same notational conventions are used to denote portfolios before and after each round of trading. The government's second round budget constraint at t is:

$$A_{g,t}(s^{t-1}) + \sum_{k=2}^{K} Q^k_t(s^t)B^k_{g,t}(s^{t-1}) \leq \widetilde{M}_t(s^t) + \sum_{k=0}^{K-1} Q^k_t(s^t)\widetilde{B}^{k+1}_{g,t}(s^t). \tag{8}$$

E. Competitive equilibria and allocations

We define a competitive equilibrium as follows.

Definition 1. *$\{c_{1t},\ c_{2t},\ l_t,\ L_{f,t},\ L_{s,t},\ \tau_t,\ W_t,\ P_{s,t+1},\ P_{f,t},\ P_t,\ \{Q^k_t\}^K_{k=1},\ \{\widetilde{Q}^k_{t+1}\}^K_{k=1},\ \{B^k_t\}^K_{k=1},\ \{B^k_{g,t}\}^K_{k=1},\ M_t,\ \{\widetilde{B}^k_t\}^K_{k=1},\ \{\widetilde{B}^k_{g,t}\}^K_{k=1},\ \widetilde{M}_t\}^\infty_{t=0}$ is a* ***competitive equilibrium*** *at $\{P_{s0},\ M_0,\ \{B^k_0\}^K_{k=1}\}$ if:*

1. *$\{c_{1t}, c_{2t}, l_t, \{B^k_t\}^K_{k=1}, M_t, \{\widetilde{B}^k_t\}^K_{k=1}, \widetilde{M}_t\}^\infty_{t=0}$ solves the household's problem given the household's initial portfolio M_0, $\{B^k_0\}^K_{k=1}$ and the tax and price sequence $\{\tau_t,\ W_t,\ P_t,\ \{Q^k_t\}^K_{k=1}, \{\widetilde{Q}^k_{t+1}\}^K_{k=1}\}^\infty_{t=0}$;*

2. *the sequence of input amounts $\{L^\alpha_{f,t}\}^\infty_{t=0}$ and $\{L^\alpha_{s,t}\}^\infty_{t=0}$ solve the final goods firm's problem given the price sequence $\{P_{s,t},\ P_{f,t}\}^\infty_{t=0}$;*

3. *the price sequences $\{P_{f,t}\}^\infty_{t=0}$ and $\{P_{s,t+1}\}^\infty_{t=0}$ solve the intermediate goods firms' problems;*

4. *the government's budget constraints hold at each date;*

5. *the labor market clears: $\forall t,\ s^t,\ l_t(s^t) = (1-\rho)L_{f,t}(s^t) + \rho L_{s,t}(s^t)$;*

6. *the bond markets clear: $\forall t,\ s^t,\ k,\ B^k_t(s^{t-1}) = B^k_{g,t}(s^{t-1}),\ \widetilde{B}^k_t(s^t) = \widetilde{B}^k_{g,t}(s^t)$.*

7. *the no lending constraints hold: $\forall t,\ s^t$, and $\forall\ k \geq 1,\ B^k_{g,t}(s^{t-1}) \geq 0,\ \widetilde{B}^k_{g,t}(s^t) \geq 0$.*

We will call a sequence $e^\infty = \{c_{1t},\ c_{2t},\ L_{f,t},\ L_{s,t}\}^\infty_{t=0}$ an *allocation* and a sequence $e^\infty(s^{t-1}) = \{c_{1t+r}(s^{t-1},\cdot),\ c_{2t+r}(s^{t-1},\cdot),\ L_{ft+r}(s^{t-1},\cdot),\ L_{st+r}(s^{t-1},\cdot)\}^\infty_{r=0}$ a *continuation allocation.* We will say that e^∞ is a *competitive allocation* if it is part of a competitive equilibrium.

III. Characterizing competitive allocations

We take a primal approach to the government's problem. To this end we provide a set of conditions that characterize competitive allocations. We begin by informally listing and discussing these conditions; we then establish their necessity and sufficiency for a competitive allocation. First, we list three more or less standard conditions.

A. Standard Constraints

To begin with, we have a sequence of *no arbitrage conditions*: for all t, s^t,

$$\frac{U_{1t}(s^t)}{U_{2t}(s^t)} \geq 1. \tag{9}$$

These ensure that the nominal interest rate is always non-negative and that there is no opportunity for arbitrage between money and nominal debt. Next, we have a set of *resource constraints* that ensure total consumption equals total final goods output, for all t, s^t,

$$G(s_t) + c_{1t}(s^t) + c_{2t}(s^t) = Y_t(s^t) \equiv \left[(1-\rho)L_{f,t}^{\frac{\alpha}{\mu}}(s^t) + \rho L_{s,t}^{\frac{\alpha}{\mu}}(s^t)\right]^{\mu}. \tag{10}$$

Third, we have a set of constraints that guarantee consistency of the allocation with profit maximizing behavior on the part of the sticky price intermediate goods firms:

$$\sum_{s^t|s^{t-1}} \pi(s^t|s^{t-1}) U_{lt}(s^t)\Phi(s^t) = 0, \tag{11}$$

where $\Phi(s_t) \equiv L_{f,t}(s^t)^{1-\frac{\alpha}{\mu}} L_{s,t}(s^t)^{\frac{\alpha}{\mu}} - L_{s,t}(s^t)$. These will be referred to as *sticky price firm optimality conditions* in what follows.

B. Implementability Constraints

Our fourth set of constraints are the counterparts of the implementability constraint, found in most analyses of dynamic optimal taxation, and the measurability constraints found in the non-contingent debt model of Aiyagari, Marcet, Sargent and Seppälä (2002).[11] To state them we introduce, and give economic interpretations for, the following pieces of notation.

Primary Surplus Value First, we define the *primary surplus value*:

$$\xi_t(s^t) \equiv E_{s^t}\left[\sum_{j=0}^{\infty} \beta^{t+j} \Lambda_{t+j}(s^{t+j})\right], \tag{12}$$

where $\Lambda_{t+j} = U_{1t+j} c_{1t+j} + U_{2t+j} c_{2t+j} + U_{lt+j} \Upsilon_{t+j}$ and $\Upsilon_{t+j} \equiv \frac{\mu}{\alpha}[(1-\rho)L_{ft+j} + \rho L_{ft+j}^{1-\frac{\alpha}{\mu}} L_{st+j}^{\frac{a}{\mu}}]$. $\xi_t(s^t)$ gives the present discounted value of future primary surpluses accruing to the government after s^t. To see this, note that the household's first order conditions and the expression for profits from an intermediate goods firms, gives $\Lambda_t = \mu_t M_t + \beta E_t[\lambda_{t+1}]\ \{P_t c_{1t} + P_t c_{2t} - (1-\tau_t) I_t\}$. Also, we can rewrite the income term as: $I_t = (1-\rho)[\Pi_{ft} + W_t L_{f,t}] + \rho[\Pi_{st} + W_t L_{s,t}] = (1-\rho) P_{f,t} Y_{f,t} + \rho P_{s,t} Y_{s,t}$ $= P_t Y_t$. Combining the expression for Λ_t and I_t, the resource constraint, and the household's first order conditions gives $\Lambda_t = U_{2t}\ \{i_t^1 \frac{M_t}{P_t} + [\tau_t \frac{I_t}{P_t} - G_t]\}$, where $i_t^1 = \frac{1}{Q_t^1} - 1$ is the nominal interest rate. It then follows from the (12) that $\xi_t(s^t)$ does indeed give the present discounted value of future primary surpluses.

Unanticipated Inflation Next, we derive an expression for the ratio of the sticky price level to the general price level, $\frac{P_{s,t}(s^{t-1})}{P_t(s^t)}$. Define:

$$N_t(s^t) \equiv \left(\frac{\left[(1-\rho)L_{f,t}^{\frac{\alpha}{\mu}}(s^{t-1}, s_t) + \rho L_{s,t}^{\frac{\alpha}{\mu}}(s^{t-1}, s_t)\right]^{\mu}}{L_{s,t}^{\alpha}(s^{t-1}, s_t)}\right)^{\frac{\mu-1}{\mu}}.$$

In Proposition 2 below we show that in a competitive equilibrium, $\frac{P_{s,t}(s^{t-1})}{P_t(s^t)} = N_t(s^t)$. Absent uncertainty, the prices of sticky and flexible price firms are equal. Thus, $N_t(s^t)$ can be interpreted as an "unanticipated inflation" term, that departs from one when shocks to the environment prompt flexible price firms to set their prices to values different from those set by sticky price firms.

Bond Pricing We are now ready to characterize equilibrium bond prices. Define the sequence $\{D_{t+1}^k\}_{k=1}^K$ by

$$D_{t+1}^k(s^t) \equiv \begin{cases} E_{s^t}\left[\prod_{j=0}^{k-2}\left\{\frac{N_{t+j+1}U_{2t+j+1}}{E_{t+j}[N_{t+j+1}U_{1t+j+1}]}\right\}\right] & k > 1 \\ 1 & k = 1. \end{cases}$$

In Proposition 2, we establish that in a competitive equilibrium, the bond price $Q_t^k(s^t)$ equals $\frac{U_{2t}(s^t)}{U_{1t}(s^t)}D_{t+1}^k(s^t)$. Absent uncertainty, this equality reduces to $Q_t^k = \prod_{j=0}^{k-1}\frac{U_{2t+j}}{U_{1t+j}} = \prod_{j=0}^{k-1}Q_{t+j}^1$. In this case, the expectations hypothesis holds and the price of a bond equals the product of credit-cash good wedges over the term of the bond. Notice that D_{t+1}^k describes how the future allocation influences the current price of the k-maturity bond.

Portfolio Weights Finally, we define the bond *portfolio weights* for cash and bonds of different maturities:

$$a_t(s^{t-1}) = \frac{A_t(s^{t-1})}{P_{s,t}(s^{t-1})}, \quad \text{and for each } k,\ b_t^k(s^{t-1}) = \frac{B_t^k(s^{t-1})}{P_{s,t}(s^{t-1})}. \tag{13}$$

Measurability Using this notation, our fourth set of restrictions can be stated as:

$$\underbrace{\frac{\xi_t(s^t)}{U_{1t}(s^t)}}_{\textit{primary surplus}} = \underbrace{N_t(s^t)\left\{a_t(s^{t-1}) + \frac{U_{2t}(s^t)}{U_{1t}(s^t)}\sum_{k=1}^{K-1} b_t^{k+1}(s^{t-1})D_t^k(s^t)\right\}}_{\textit{liabilities}}. \tag{14}$$

The portfolio weights at date 0 are taken to be predetermined and we will refer to the date 0 versions of (14) as the *implementability constraints.* At dates $t > 0$, the portfolio weights will be chosen as part of the competitive equilibrium. However, since the $\{a_t, \{b_t^k\}_{k=2}^K\}$ are measurable with respect to s^{t-1} they place cross-state restrictions on the process for ξ_t. Following Aiyagari, Marcet, Sargent and Seppälä (2002), we will refer to these conditions as *measurability constraints.* They are crucial in what follows. The left hand side of (14) can again be interpreted as the value of the government's primary surpluses (now priced in terms of period t cash good consumption). The right hand side of this equation can be interpreted as the value of the government's liabilities. To see this, notice that the government's real liabilities in the second round of asset trading satisfy:

$$\frac{A_t(s^{t-1})}{P_t(s^t)} + \sum_{k=1}^{K-1} \frac{B_t^{k+1}(s^{t-1})}{P_t(s^t)} Q_t^k(s^t) = \underbrace{\frac{P_{s,t}(s^{t-1})}{P_t(s^t)}}_{unanticipatedinflation} \left\{ a_t(s^{t-1}) + \sum_{k=1}^{K-1} b_t^{k+1}(s^{t-1}) \underbrace{Q_t^k(s^t)}_{termstructure} \right\}. \tag{15}$$

Using $N_t(s^t) = \frac{P_{s,t}(s^{t-1})}{P_t(s^t)}$ and $Q_t^k(s^t) = \frac{U_{2t}(s^t)}{U_{1t}(s^t)} D_{t+1}^k(s^t)$ in (15) gives the right hand side of (14). In any competitive equilibrium, this liability value must equal the value of the government's primary surpluses and (14) must hold.

Two Types of Hedging Although the portfolio weights $\{a_t, \{b_t^{k+1}\}_{k=1}^{K-1}\}$ are predetermined at t, the values of the different liabilities in the government's portfolio are not. These can be altered in two ways: (1) by unanticipated changes to the price level and (2) by changes to the nominal term structure. As the formulas above indicate, unanticipated increases in the price level are associated with reductions in $N_t(s^t)$, while decreases in nominal bond prices are associated with falls in the $\frac{U_{2t}(s^t)}{U_{1t}(s^t)} D_t^k(s^t)$ values. Both changes cause the value of the government's liabilities on the right hand side of (14) to fall. Such a reduction is desirable if the government is trying to attain a lower primary surplus value $\frac{\xi_t(s^t)}{U_{1t}(s^t)}$ in the aftermath of a fiscal shock.

On the other hand, the $N_t(s^t)$ and $\frac{U_{2t}(s^t)}{U_{1t}(s^t)} D_{t+1}^k(s^t)$ also capture the costly distortions to the pattern of production across firms and consumption across goods introduced by unanticipated price level and term structure changes. Specifically, if events occur at t that induce firms to alter their prices and to which only flexible price firms can react, an inefficient allocation of production across firm types will occur. A wedge is driven between the marginal products of labor at sticky and flexible price firms. Reductions in the price of outstanding bonds with maturity greater than 1, and of the $\{\frac{U_{2t}(s^t)}{U_{1t}(s^t)} D_t^k(s^t)\}_{k=1}^K$ terms, are associated with nominal interest rates in excess of zero.[12] These result in a misallocation of consumption across cash and credit goods as households seek to economize on their use of cash. A wedge is driven between the marginal utilities of these two goods.

C. Comparison with Existing Models

The key difference between our model and others becomes apparent in the measurability constraints. It is worth contrasting our version of these conditions (14) with those in the more restrictive environments of Siu (2004) and Aiyagari, Marcet, Sargent and Seppälä (2002), and the less restrictive environment of Lucas and Stokey (1983).

In the model of Lucas and Stokey, the government can trade real state contingent debt, and so the analogue of (14) is:[13]

$$\frac{\xi_t(s^t)}{U_t(s^t)} = a_t(s^t). \tag{16}$$

The portfolio weight a_t is s^t-measurable, so that unlike (14) or (17), (16) does not represent a collection of cross state restrictions. Except at date 0, when $a_0(s^0)$ is fixed, the constraints in (16) are redundant.

On the other hand, in Siu (2004), nominal debt of only one period is traded and (14) reduces to:

$$\frac{\xi_t(s^t)}{U_{1t}(s^t)} = N_t(s^t) a_t(s^{t-1}). \tag{17}$$

Thus, $\frac{\xi_t(s^t)}{U_{1t}(s^t)}$ can be varied across states s_t only by creating unanticipated inflation or changing $N_t(s^t)$. Since there is no long term debt, there is clearly no opportunity to devalue this debt through increases in future nominal interest rates.

Finally, in Aiyagari, Marcet, Sargent and Seppälä (2002), only real debt of one period is traded, which implies:

$$\frac{\xi_t(s^t)}{U_{1t}(s^t)} = a_t(s^{t-1}). \tag{18}$$

In this most limiting case, there is not even the opportunity to devalue debt through unexpected inflation. To summarize the discussion in the preceding paragraphs, (14) incorporates the fact that when the government can only trade nominal non-contingent debt implementing stochastic variations in the value of the government's outstanding liabilities is costly. These costs constrain the evolution of the primary surplus value process $\{\xi_t\}_{t=0}^{\infty}$.

D. No Lending Constraints

Our final set of constraints are the *no lending constraints.* These ensure that the household's bond holdings are non-negative. For maturities $k > 1$, we have:

$$\forall t, s^t, k > 1, \;\; b_t^k(s^{t-1}) \geq 0,$$

while for one period nominal liabilities:

$$a_t(s^{t-1}) \geq 0.$$

The following proposition formally establishes the necessity and sufficiency of the conditions for a competitive allocation.

Proposition 2. *$e^{\infty} = \{c_{1t}, c_{2t}, L_{f,t}, L_{s,t}\}_{t=0}^{\infty}$ is a competitive allocation at $\left\{P_{s0}, M_0, \left\{B_0^k\right\}_{k=1}^{K}\right\}$ if there exists a sequence of portfolio weights $\left\{a_t, \left\{b_t^{k+1}\right\}_{k=1}^{K-1}\right\}_{t=0}^{\infty}$ with $a_0 = \frac{M_0+B_0^1}{P_{s0}}$ and $b_0^{k+1} = \frac{B_0^{k+1}}{P_{s0}}$, $k = 1, \cdots, K-1$, such that the portfolio weight sequence and e^{∞} satisfy:*

1. *for all t, s^t,*

$$\frac{U_{1t}(s^t)}{U_{2t}(s^t)} \geq 1; \tag{19}$$

2. *for all t, s^t,*

$$G(s_t) + c_{1t}(s^t) + c_{2t}(s^t) = \left[(1-\rho)L_{f,t}^{\frac{\alpha}{\mu}}(s^t) + \rho L_{s,t}^{\frac{\alpha}{\mu}}(s^t)\right]^{\mu}; \tag{20}$$

3. for all $t > 0$, s^{t-1},

$$\sum_{s^t|s^{t-1}} \pi(s^t|s^{t-1}) U_{lt}(s^t)\Phi(s^t) = 0; \tag{21}$$

4. for all t, s^t,

$$\xi_t(s^t) = N_t(s^t)\left\{U_{1t}(s^t)a_t(s^{t-1}) + U_{2t}(s^t)\sum_{k=1}^{K-1} b_t^{k+1}(s^{t-1})D_{t+1}^k(s^t)\right\}; \tag{22}$$

5. for all $t-1$, s^{t-1}, $k > 1$,

$$\begin{aligned} a_t(s^{t-1}) &\geq 0; \\ b_t^k(s^{t-1}) &\geq 0. \end{aligned} \tag{23}$$

If $e^\infty = \{c_{1t}, c_{2t}, L_{f,t}, L_{s,t}\}_{t=0}^\infty$ is a competitive allocation at $\left\{P_{s0}, M_0, \{B_0^k\}_{k=1}^K\right\}$ with $c_{jt} > 0$, $j = 1, 2$ and $(1-v)L_{f,t} + vL_{s,t} \in (0, T)$, then e^∞ satisfies conditions (19)-(23)for some sequence of portfolio weights $\{a_t, \{b_t^{k+1}\}_{k=1}^{K-1}\}_{t=0}^\infty$, with $a_0 = \frac{M_0+B_0^1}{P_{s0}}$ and $b_0^{k+1} = \frac{B_0^{k+1}}{P_{s0}}$, $k = 1, \cdots, K-1$.

Proof: See Appendix A. ■

Remark The requirement that the competitive allocation be interior in the second part of the above proposition is satisfied if the household's utility function satisfies appropriate Inada conditions and if the associated competitive equilibrium has tax rates τ_t strictly less than 1.

IV. The Ramsey problem

Given Proposition 2, the optimal policy problem of a government in an economy with initial triple $\left\{P_{s0}, M_0, \{B_0^k\}_{k=1}^K\right\}$ can be formulated as:

$$\textbf{Problem 1:} \quad \sup_{\{c_{1t}, c_{2t}, L_{f,t}, L_{s,t}, a_t, \{b_t^k\}_{k=2}^K\}_{t=0}^\infty} E\left[\sum_{t=0}^\infty \beta^t U(c_{1t}, c_{2t}, (1-\rho)L_{f,t} + \rho L_{s,t})\right] \tag{24}$$

subject to the restrictions $a_0 = \frac{M_0 + B_0^1}{P_{s0}}$ and $b_0^k = \frac{B_0^k}{P_{s0}}$, the no arbitrage (19) and resource (20) constraints, the sticky price firm optimality conditions (21), the measurability and implementability (22), and no lending (23) constraints. We will call this the *Ramsey problem with nominal debt and no government lending.*

The initial period of Problem 1 is somewhat different from later periods in that the portfolio weights $\{a_0, \{b_0^k\}_{k=2}^K\}$ are fixed, whereas in later periods they are chosen. Consequently, it is useful to split the government's choice problem into two pieces. In the first piece, the government picks a period 0 allocation and some state variables. In the second piece, the government selects a continuation allocation $\{c_{1t}, c_{2t}, L_{f,t}, L_{s,t}\}_{t=1}^{\infty}$ for period 1 onwards and a sequence of portfolios $\{a_t, \{b_t^k\}_{k=2}^K\}_{t=1}^{\infty}$ taking the state variables as given. These state variables ensure that the continuation allocation satisfies all of the period 0 constraints. A solution to Problem 1 can then be reconstructed from the solutions to these two sub-problems. More formally, notice that the implementability constraint ((22) with $t = 0$) can be rewritten as:

$$\left[U_{20}(s^0) \left\{ \sum_{k=1}^{K-1} D_1^k(s^0) b_0^{k+1} \right\} + U_{10}(s^0) a_0 \right] N_0(s^0) = \Lambda_0(s^0) + \beta \phi_1(s^0); \tag{25}$$

where

$$\phi_1(s^0) = E_{s^0} \left[\sum_{t=0}^{\infty} \beta^t \Lambda_{t+1} \right]. \tag{26}$$

Thus, if an allocation $\{c_{1t}, c_{2t}, L_{f,t}, L_{s,t}\}_{t=0}^{\infty}$ satisfies the implementability constraint, then there exists a tuple $\{\phi_1, \{D_1^k\}_{k=1}^{K-1}\}$ such that $\{c_{10}, c_{20}, L_{f0}, L_{s0}, \phi_1, \{D_1^k\}_{k=1}^{K-1}\}$ satisfies (25), and the continuation allocations $\{c_{1t}(s_0, \cdot), c_{2t}(s_0, \cdot), L_{f,t}(s_0, \cdot), L_{s,t}(s_0, \cdot)\}_{t=1}^{\infty}$ satisfy (26). The converse is also true. $\{\phi_1, \{D_0^k\}_{k=1}^{K-1}\}$ act as state variables for the government's continuation problem. ϕ_1 describes how the continuation allocation affects the value of the government's stream of primary surpluses, while the $\{D_1^k\}_{k=1}^{K-1}$ variables describe how this allocation affects the value of the government's period 0 liabilities. In Section VIII we develop a fully recursive formulation of the Ramsey problem based on this observation. In Sections VI and VII we consider simpler problems in which $b_0^k = 0$, $k > 2$. This simplification implies that the government's period 0 liability value (the left hand side of (25)) is independent of the $\{D_1^k\}_{k=1}^{K-1}$ values. Thus, ϕ_1 serves as the only state variable. The government's continuation problem can then be written as:

$$\textbf{Cont. Problem 1:} \quad \sup_{\left\{c_{1t}, c_{2t}, L_{f,t}, L_{s,t}, a_t, \{b_t^k\}_{k=2}^{K}\right\}_{t=1}^{\infty}} E_{s^0}\left[\sum_{t=1}^{\infty} \beta^t U(c_{1t}, c_{2t}, (1-\rho)L_{f,t} + \rho L_{s,t})\right] \quad (27)$$

subject to the constraints (19) to (23) from period one onwards and the implementability constraint

$$\phi_1 = E_{s^0}[\xi_1(s^1)]. \quad (28)$$

We will call ϕ_1 an *expected primary surplus value.* We now introduce a benchmark complete markets economy in which the government implements the Friedman rule.

V. Complete markets problem

The economy considered above incorporated two sorts of asset market frictions. First, the government could only trade nominally non-contingent debt; second, the government could not lend. The complete markets environment described in this section removes both of these frictions. Households and the government can now trade nominal state-contingent claims $\{F_t\}_{t=0}^{\infty}$.

We assume (wlog) that these claims are of one period maturity only and that there is only one trading round. The household's budget constraint is now:

$$\begin{aligned} F_t(s^{t+1}) + M_t(s^t) \quad \geq \quad & P_t(s^t)c_{1t}(s^t) + P_t(s^t)c_{2t}(s^t) - (1-\tau_t(s^t))I_t(s^t) \\ & + \sum_{s^{t+2}|s^{t+1}} Q_{t+1}^S(s^{t+2})F_{t+1}(s^{t+2}) + M_{t+1}(s^{t+1}), \end{aligned}$$

where Q_{t+1}^S is the pricing kernel for this market structure. The government's budget constraint is similarly altered. By an argument similar to that in Proposition 2, one can show that a sequence $\{c_{1t}, c_{2t}, L_{f,t}, L_{s,t}\}_{t=0}^{\infty}$ is a competitive allocation in this environment at (P_{s0}, F_0), if it satisfies

(19)-(21) and

$$\xi_0(s^0) = U_{10}(s^0)N_0(s^0)f_0(s^0); \tag{29}$$

where $f_0(s^0) = \frac{F_0(s^0)}{P_{s0}}$. The incorporation of state-contingent debt renders the measurability constraints redundant.[14] In addition, the absence of the no lending constraints on the government removes any non-negativity restrictions on the government's debt portfolio.

The Ramsey problem in this environment entails the government maximizing its objective subject to (19)-(21) and (29). To avoid the special nature of the initial period, we again focus on the government's continuation problem. In analogy with Continuation Problem 1 of the previous section, this can be stated as:[15]

$$\textbf{Cont. Problem 2:} \quad \sup_{\{c_{1t}, c_{2t}, L_{f,t}, L_{s,t}\}_{t=1}^{\infty}} E_{s^0}\left[\sum_{t=1}^{\infty} \beta^t U(c_{1t}, c_{2t}, (1-\rho)L_{f,t} + \rho L_{s,t})\right] \tag{30}$$

subject to, for $t \geq 1$, no arbitrage (19), resource (20), sticky price firm optimality (21), and the implementability (28) constraints.

In Appendix B we show the following:

Proposition 3. *Under our assumed preferences, the solution to Continuation Problem 2 is such that:*

1. *The Friedman rule holds and* $U_{1t} = U_{2t}$.

2. *Flexible price firms always set their prices to the same value as sticky price firms and* $N_t = 1$.

VI. The role of the no lending constraint

We use Continuation Problem 1 and Continuation Problem 2 above to explain the role of the no lending constraints in the clearest way. To do this, we first rewrite the measurability constraints in a more compact way. We define a vector of primary surplus values $\Xi_t(s^{t-1}) = (\frac{\xi_t(s^{t-1},\hat{s}_1)}{U_{1t}(s^{t-1},\hat{s}_1)}, \cdots, \frac{\xi_t(s^{t-1},\hat{s}_n)}{U_{1t}(s^{t-1},\hat{s}_1)})'$ and a vector of portfolio weights $\Gamma_t(s^{t-1}) = (a_t(s^{t-1}), \cdots, b_t^K(s^{t-1}))$. We next define a matrix of cash and bond holding returns, by setting $\psi_{1,t}(s^t) = N_t(s^t)$, $\psi_{k,t}(s^t) = N_t(s^t)\frac{U_{2t}(s^t)}{U_{1t}(s^t)}D_{t+1}^{k-1}(s^t)$, $k = 2, \cdots, K$ and letting $\Psi_t(s^{t-1})$ be the $N \times K$-matrix

$$\Psi_t(s^{t-1}) = \begin{pmatrix} \psi_{1,t}(s^{t-1}, \hat{s}^1) & \dots & \psi_{K,t}(s^{t-1}, \hat{s}^1) \\ \vdots & \ddots & \vdots \\ \psi_{1,t}(s^{t-1}, \hat{s}^N) & \dots & \psi_{K,t}(s^{t-1}, \hat{s}^N) \end{pmatrix}. \tag{31}$$

Using this notation, the measurability constraints can be rewritten as, for all t, s^{t-1},

$$\Xi_t(s^{t-1}) \in \text{Span}(\Psi_t(s^{t-1})). \tag{32}$$

In words, the time t primary surplus values need to lie in the space spanned by the cash and bond return matrix. It follows from Proposition 3 that when markets are complete, the optimal continuation allocation equates the marginal utilities of cash and credit goods and the prices of the sticky and flexible-type firms. In that case, $\Psi_t(s^{t-1})$ is reduced to the unit matrix. By (32), implementation of this allocation in the non-contingent debt economy would require that $\frac{\xi_t(s^t)}{U_{1t}(s^t)}$ be s^{t-1}-measurable. If this requirement does not hold, and in general it will not,[16] then the optimal complete markets allocation cannot be implemented with non-contingent nominal debt. The logic is straightforward; this allocation precludes state-contingent variations in interest rates or in inflation, yet it is in precisely these ways that the government introduces state contingency into its liability values in the non-contingent nominal debt economy. However, if we drop the no lending constraints from the non-contingent debt economy, then arbitrarily small deviations from the optimal complete markets allocation can then be used to ensure that $\Xi_t(s^{t-1}) \in \text{Span}(\Psi_t(s^{t-1}))$ and that the measurability constraints are satisfied. Such small deviations will perturb the $\Psi_t(s^{t-1})$ only slightly from the unit matrix. Consequently, the government will need to take extreme positions in asset markets to induce the required variation in primary surplus values. One further implication is that absent additional restrictions on debt trades, there exists no solution to the government's problem since, when $K \geq 2$, the optimal complete markets allocation does not lie in the set of allocations implementable in the non-contingent debt economy, but does lie in its closure. In an example in Appendix C, we make this explicit.

We now reinstate the no lending constraints. In this case, $\Xi_t(s^{t-1}) \in \text{Span}_+(\Psi_t(s^{t-1}))$, where $\text{Span}_+(\Psi_t(s^{t-1})) = \{y \in \mathbb{R}^N : y = \Psi_t(s^{t-1})x, x \in \mathbb{R}^K_+\}$. It follows that the government can no longer take extreme asset positions. Consequently, it will not in general be able to implement the optimal allocation from the complete markets problem.

Before proceeding, we briefly compare these observations to the results of Buera and Nicolini (2004) and Angeletos (2002). These authors obtain related measurability constraints in economies with non-contingent *real* debt of various maturities and no restrictions on lending. Let $\Psi_t^{\text{real}}(s^{t-1})$ denote the pricing matrix for non-contingent real debt after history s^{t-1}. Angeletos shows that when $K \geq N$, the set of allocations for which $\text{Span}(\Psi_t^{\text{real}}(s^{t-1})) = \mathbb{R}^N$, all t, s^{t-1}, is dense in the set of complete markets allocations. Hence, for this case, the closure of the set of allocations implementable in the non-contingent debt economy equals the set implementable in the complete markets economy. Generically, the optimal complete markets allocation can be implemented with non-contingent real debt when $K \geq N$. However, Buera and Nicolini show in a series of calibrated numerical examples that the government may need to take large debt positions to achieve this implementation. Buera and Nicolini regard this as a problem. In contrast, with non-contingent *nominal* debt, the optimal complete markets allocation can never be implemented, and the implementation of allocations close to it always require extreme debt positions. In this sense the problem identified by Buera and Nicolini is more severe in an economy with nominal debt.

VII. An illustrative example

To obtain insight into optimal nominal debt management, we begin by considering a simplified version of *Continuation Problem 1.* In this version, we show that it is weakly optimal for the government to use only the longest term nominal debt. Specifically, in the first round of asset trading in each period, the government buys the outstanding portfolio of money and bonds and sells only the longest term debt. The households hold this portfolio when the shock occurs. In the case of separable utility, we show that the government uses this greater flexibility to postpone the distortionary positive nominal interest rates needed to undertake a state-contingent debt devaluation. In doing so, our government clearly does not minimize the cost of debt service. The long term debt that it sells is relatively risky for households and they must be compensated accordingly. The risk premium earned by households who hold long run debt is, from the government's point of view, an insurance premium.

Setup We assume the following process for shocks. In period 1, a state s is drawn from the set $S = \{\underline{s}, \overline{s}\}$ according to probability distribution π. If $s = \underline{s}$ then the government faces a low period 1 spending shock, $G_1(\underline{s}) = \underline{G}$, followed by a constant sequence of moderate spending levels in subsequent periods, $G_t(\underline{s}) \in (\underline{G}, \overline{G})$, $t = 2, \cdots$. If $s = \overline{s}$, then the government faces a high period 1 spending shock, $G(\overline{s}) = \overline{G}$ followed by the same sequence of moderate spending levels. All uncertainty is resolved in period 1. Consequently, the date $t > 1$ sticky price, measurability and no lending constraints will not bind and can be dropped.

Government Problem The government's continuation problem then reduces to:

$$\sup_{\left\{\{c_{1t}, c_{2t}, L_{f,t}, L_{s,t}\}_{t=1}^{\infty}, a_1, \{b_1^k\}_{k=2}^{K}\right\}} \sum_{s \in S} \left[\sum_{t=1}^{\infty} \beta^{t-1} U(c_{1t}(s), c_{2t}(s), (1-\rho)L_{f,t}(s) + \rho L_{s,t}(s)) \right] \pi(s) \tag{33}$$

subject to the no arbitrage conditions,

$$\forall t, s \in S, \qquad U_{1t}(s) - U_{2t}(s) \geq 0, \tag{34}$$

the resource constraints,

$$\forall t, s \in S, \qquad G_t(s) + c_{1t}(s) + c_{2t}(s) = \left[(1-\rho) L_{f,t}^{\frac{\alpha}{\mu}}(s) + \rho L_{s,t}^{\frac{\alpha}{\mu}}(s) \right]^{\mu}, \tag{35}$$

the optimality condition for sticky price firms in period 1,

$$\sum_{s \in S} \pi(s) U_{l1}(s) \Phi_1(s) = 0, \tag{36}$$

the continuation implementability constraint,

$$\phi_1 = \sum_{s \in S} \xi_1(s) \pi(s), \tag{37}$$

the first period measurability constraint,

$$\left[\left\{ \sum_{k=1}^{K-1} \prod_{j=1}^{k} \frac{U_{2j}(s)}{U_{1j}(s)} b_1^{k+1} \right\} + a_1 \right] U_{11}(s) N_1(s) = \xi_1(s), \tag{38}$$

and the first period no lending conditions,

$$a_1 \geq 0, \qquad \text{for } k = 2, \cdots, K, \quad b_1^k \geq 0. \tag{39}$$

Notice that since this is a continuation problem, the government inherits an expected primary surplus value, ϕ_1, from period 0. We assume $\phi_1 > 0$. The constraint (37) captures the fact that the government must implement this value.

Let ζ_0 denote the Lagrange multiplier on the implementability constraint (37) and $\zeta_1(s)\pi(s)$ the multiplier on the s-th measurability constraint (38). We assume that $\zeta_0 < 0$. Also define ς_1 to be the multiplier on the sticky price constraint, set $\varsigma_t = 0$, $t > 1$ and for $i = 1, 2$, let

$$H_{it}(s) = [\zeta_0 + \zeta_1(s)\pi(s)]\beta^t \{U_{1it}(s)c_{1t}(s) + U_{it}(s) + U_{2it}(s)c_{2t}(s) + U_{lit}\Upsilon_t(s)\} + \varsigma_t U_{lit}(s)\Phi_t(s)\pi(s).$$

The first order condition for c_{it}, $t = 2, \cdots, K-1$ can then be expressed as:

$$\begin{aligned} 0 &= U_{it} + \eta_t [U_{1it} - U_{2it}] - \chi_t - H_{it} \\ &\quad + \beta^{-(t-1)} \zeta_1 \sum_{k=t}^{K-1} b_1^{k+1} \left\{ \prod_{j=2}^{k} \frac{U_{2j}}{U_{1j}} \left[-\frac{U_{1it}}{U_{1t}} + \frac{U_{2it}}{U_{2t}} \right] \right\} U_{21} N_1 \end{aligned}$$

where $\beta^t \eta_t(s)\pi(s)$ and $\beta^t \chi_t(s)\pi(s)$ are, respectively, the multipliers on the (s,t)-th no arbitrage and resource constraints. For $t = 1$, the corresponding first order condition is:

$$\begin{aligned} 0 &= U_{i1} + \eta_1 [U_{1i1} - U_{2i1}] - \chi_1 - H_{i1} \\ &\quad + \zeta_1 \left[a_1 U_{1i1} + b_1^2 U_{2i1} + \sum_{k=2}^{K-1} b_1^{k+1} \left\{ \prod_{j=2}^{k} \frac{U_{2j}}{U_{1j}} \right\} U_{2i1} \right] N_1. \end{aligned}$$

Additionally, the first order conditions for a_1 and b_1^{k+1} and $k \in \{2, \cdots, K-1\}$ are:

$$0 = \sum_{s \in S} \zeta_1(s) N_1(s) U_{11}(s) \pi(s) + \theta^1, \tag{40}$$

$$0 = \sum_{s \in S} \zeta_1(s) N_1(s) U_{21}(s) \prod_{j=2}^{k} \frac{U_{2j}(s)}{U_{1j}(s)} \pi(s) + \theta^{k+1}, \tag{41}$$

where θ^1 and θ^{k+1} are the Lagrange multipliers on the no lending constraints. We now use these first order conditions to establish (1) that there is a solution to the government's problem in which the government uses only the longest term debt, and, in the case of separable utility, that (2) the spending shock's effect on nominal interest rates persists until the longest term debt matures, and (3) after a high government spending shock, the short run nominal interest rate rises over the term of the initially outstanding debt. The expected holding return increases in the maturity as well.

Lemma 4. *There exists a solution to the government's problem in which* $a_1 = 0$*,* $b_1^{k+1} = 0$*,* $k = 1, \cdots, K-2$ *and* $b_1^K > 0$*.*

The proof of Lemma 4 (in the appendix) has the following implication.

Corollary 5. *If $\zeta_1(s) < 0$ for some s, then 1) it is strictly optimal to use only the longest term debt, 2) after shock $G_1(s)$, the nominal interest rate is greater than zero in periods $t = 2, \cdots, K-1$.*

The corollary immediately implies that if the measurability constraint binds and $\zeta_1(s) < 0$, then the Friedman rule is not optimal. Moreover, the spending shock has persistent effect on nominal interest rates that is tied to the term of the outstanding debt in period 1.

Separable utility To obtain sharper results we further specialize preferences. Suppose that the household's preferences are given by $(1-\gamma)\log c_{1t} + \gamma \log c_{2t} + v(l_t)$. This renders the government's primary surplus values independent of the consumption allocation and allows us to focus on the effect of this allocation upon the government's liability values. We first show that in this case, $\zeta_1(s) > 0 > \zeta_1(s')$, for $s, s' \in S$. It then follows from the proof of the previous lemma that only long term debt is used.

Lemma 6. *Assume the preferences $(1-\gamma)\log c_1 + \gamma \log c_2 + v(l)$. Then, $\zeta_1(s) > 0 > \zeta_1(s')$.*

We now show that when $\zeta_1(s) < 0$, nominal interest rates rise over the term of the outstanding debt. Intuitively, the government exploits the flexibility afforded by long term debt to postpone costly nominal interest rate rises.

Lemma 7. *Assume the preferences $(1-\gamma)\log c_1 + \gamma \log c_2 + v(l)$. In the state s such that $\zeta_1(s) < 0$, $Q^1_{t+1}(s) < Q^1_t(s)$, $k = 1, \cdots, K-1$. For $t > K-1$, $Q^1_t(s) = 1$. In the state s such that $\zeta_1(s) > 0$, $Q^1_t(s) = 1$ for all t.*

Lemma 7 has immediate implications for the yield curve. If the measurability constraint does not bind and $\zeta_1(s) > 0$, then the yield curve remains at zero. On the other hand, if this constraint binds and $\zeta_1(s) < 0$, then the yield curve rises and tilts upwards. Additionally, Lemma 7 implies that the proportional variation in the period 1 debt price $\frac{Q_1^{k-1}(s)}{E_0[Q_1^{k-1}]}$ across states is increasing in its maturity k. Consequently, the holding return on this debt expected in period 0 is also increasing in the maturity k. The government sells debt that, under the optimal interest rate policy, is relatively expensive in order to hedge itself against shocks.[17] The next section derives a recursive formulation of the Ramsey problem that enables us to solve it in less stylized settings.

VIII. A general recursive formulation

We now look for a recursive formulation that will allow us to solve problems with infinite horizons and richer shock processes. We then put this formulation to work and compute solutions to Ramsey problems with nominal debt of various maturities. Recall that associated with any competitive allocation e^∞ are sequences $\{\xi_t\}_{t=0}^\infty$ and $\{D_{t+1}^k\}_{t=0}^\infty$, $k = 1, \cdots, K$. In addition, let $\phi_{t+1}(s^t) = E_{s^t}\left[\xi_{t+1}(s^{t+1})\right]$. Our approach treats tuples of the form $\{\phi_{t+1}, \{D_{t+1}^k\}_{k=1}^{K-1}, s_t\}$ as state variables that summarize relevant aspects of the past history of an allocation. Specifically, we may view ϕ_{t+1} and each D_{t+1}^k as representing implicit promises that the government has made at t concerning the value of its overall liability portfolio and the value of specific bonds within that portfolio. Future allocation choices must implement these promises.

We begin by defining a state space for our recursive formulation. This consists of the set of tuples $\{\phi, \{D^k\}_{k=1}^{K-1}, s\}$ consistent with a continuation competitive allocation. More formally, let $\mathcal{E}(\phi, \{D^k\}_{k=1}^{K-1}, s)$ denote all $e^\infty(s) = \{c_{1t}(s,\cdot), c_{2t}(s,\cdot), L_{f,t}(s,\cdot), L_{s,t}(s,\cdot)\}_{t=1}^\infty$ that satisfy:

$$\phi = E_s\left[\xi_1(s^1)\right], \tag{42}$$

$$D^k = \begin{cases} E_s\left[\prod_{j=0}^{k-2}\left\{\frac{N_{j+1}U_{2j+1}}{E_{s^j}[N_{j+1}U_{1j+1}]}\right\}\right] & k = 2, \cdots, K-1 \\ 1 & k = 1. \end{cases} \tag{43}$$

We then define the state space as $\mathbf{X} \equiv \{(\phi, \{D^k\}_{k=1}^{K-1}, s) : \mathcal{E}(\phi, \{D^k\}_{k=1}^{K-1}, s) \neq \varnothing\}$. For subsequent numerical analysis, it is useful to have bounds for the set $\mathbf{X}$. The absence of government lending implies that $\phi \geq 0$. We assume that the variables Λ_t are uniformly bounded above. This is the case if, for example, $U(c_1, c_2, l) = (1-\gamma)\log c_1 + \gamma \log c_2 + v(l)$, with v decreasing. Such a bound, coupled with (42), implies an upper bound on the ϕ variables which we denote $\phi_{\max}$. Next, note (43) coupled with the no arbitrage constraint, implies that each $D^k \in [0,1]$. Also, for $k = 1$, D^k is normalized to 1. It follows that $\mathbf{X} \subset Z \equiv [0, \phi_{\max}] \times \{1\} \times\ [0,1]^{K-2} \times S$.

Define the correspondence Γ pointwise as follows.

Definition 8. *Let* $\Gamma(\phi, \{D^k\}_{k=1}^{K-1}, s)$ *equal all tuples* $\{a, \{b^k\}_{k=2}^{K}, c_1, c_2, L_f, L_s, \phi', \{D^{k\prime}\}_{k=1}^{K-1}\}$ *such that* $c_i \geq 0$, $i = 1, 2$, $L_i \in [0, 1]$, $i = f, s$, *and*

1. *for all* s',
$$U_1(s') \geq U_2(s'); \tag{44}$$

2. *for all* s',
$$G(s') + c_1(s') + c_2(s') = \left[(1-\rho)L_f^{\frac{\alpha}{\mu}}(s') + \rho L_s^{\frac{\alpha}{\mu}}(s')\right]; \tag{45}$$

3. $$\sum_{s' \in S} \pi(s'|s)U_l(s')\Phi(s') = 0; \tag{46}$$

4. $$\phi = E_s\left[\Lambda(s') + \beta\phi'(s')\right]; \tag{47}$$

5. *for all* s',
$$\left\{U_1(s')a + U_2(s')\sum_{k=1}^{K-1} D^{k\prime}(s')b^{k+1}\right\} N(s') = \Lambda(s') + \beta\phi'(s'); \tag{48}$$

6. *for each* $k = 2, \cdots, K-1$,
$$D^k = \frac{E_s\left[D^{k-1\prime}(s')N(s')U_2(s')\right]}{E_s\left[N(s')U_1(s')\right]}; \tag{49}$$

7. *for each* s'
$$(\phi'(s'), \{D^{k\prime}(s')\}_{k=1}^{K-1}, s') \in \mathbf{X}. \tag{50}$$

We say that $\{c_{1t}(s, \cdot),\ c_{2t}(s, \cdot),\ L_{f,t}(s, \cdot),\ L_{s,t}(s, \cdot)\}_{t=1}^{\infty}$ is *generated* by Γ from $(\phi, \{D^k\}_{k=1}^{K-1}, s)$ if there exists sequences $\{x_{t+1}\}_{t=0}^{\infty} = \{\phi_{t+1}, \{D_{t+1}^k\}_{k=1}^{K-1}, s_t\}_{t=0}^{\infty}$ with $x_1(s) = (\phi, \{D^k\}_{k=1}^{K-1}, s)$, and

$\{a_t,\ \{b_t^k\}_{k=2}^K\}_{t=1}^\infty$ such that for all (t, s^{t-1}),

$$\begin{aligned}\Gamma(x_t(s^{t-1})) = \{&a_t(s^{t-1}), \{b_t^k(s^{t-1})\}_{k=2}^K,\\ &c_{1t}(s^{t-1},\cdot), c_{2t}(s^{t-1},\cdot), L_{f,t}(s^{t-1},\cdot), L_{s,t}(s^{t-1},\cdot), x_{t+1}(s^{t-1},\cdot)\},\end{aligned} \tag{51}$$

Our final definition gives a counterpart of Γ for the initial period of the government's problem.

Definition 9. *Let $\{a_0, \{b_0^{k+1}\}_{k=1}^{K-1}\} \in \mathbb{R}_+^K$. Define $\Gamma_0(a_0, \{b_0^{k+1}\}_{k=1}^{K-1})$ to be all those tuples $\{c_{10}$, c_{20}, L_{f0}, L_{s0}, ϕ_1, $\{D_1^k\}_{k=1}^{K-1}\}$ that satisfy $c_{i0} \geq 0$, $L_{i0} \in [0, T]$, (44), (45), (50) and*

1. for each s',

$$\left\{U_{10}(s')a_0 + U_{20}(s') \sum_{k=1}^{K-1} D_1^k(s') b_0^{k+1}\right\} N_0(s') = \Lambda_0(s') + \beta\phi_1(s'). \tag{52}$$

Lemma 10 establishes the recursivity of competitive allocations.

Lemma 10. *A sequence $e^\infty = \{c_{1t}, c_{2t}, L_{f,t}, L_{s,t}\}_{t=0}^\infty$ is a competitive allocation at $\{P_{s0}, M_0, \{B_0^k\}_{k=1}^K\}$ if there exists a tuple $\{\phi_1, \{D_1^k\}_{k=1}^{K-1}\}$ such that*

1. $\{c_{10}, c_{20}, L_{f0}, L_{s0}, \phi_1, \{D_1^k\}_{k=1}^{K-1}\} \in \Gamma_0(\frac{M_0+B_0^1}{P_{s0}}, \{\frac{B_0^{k+1}}{P_{s0}}\}_{k=1}^{K-1})$;

2. each continuation allocation $e^\infty(s)$ is generated by Γ from $\{\phi_1(s),\ \{D_1^k(s)\}_{k=1}^{K-1}, s\}$.

If $e^\infty = \{c_{1t}, c_{2t}, L_{f,t}, L_{s,t}\}_{t=0}^\infty$ is a competitive allocation at $\left\{P_{s0}, M_0, \{B_0^k\}_{k=1}^K\right\}$ with $c_{jt} > 0$, $j = 1, 2$ and $(1-v)L_{f,t} + vL_{s,t} \in (0, T)$, then there exists tuple $\{\phi_1, \{D_1^k\}_{k=1}^{K-1}\}$ such that e^∞ and $\{\phi_1, \{D_1^k\}_{k=1}^{K-1}\}$ satisfy conditions (1) and (2) above.

Proof: Suppose e^∞ satisfies the conditions in the lemma. The definitions of Γ_0 and Γ imply that e^∞ satisfies the no arbitrage conditions (19), resource conditions (20), sticky price conditions (21) and no government lending conditions (23) at each date. Since for each $s \in S$, $e^\infty(s)$, is generated by Γ from $\{\phi_1(s),\ \{D_1^k(s)\}_{k=1}^{K-1}, s\}$, there exists associated sequences $\{\phi_{t+1}(s,\cdot)$,

$\{D^k_{t+1}(s,\cdot)\}_{k=1}^{K-1}\}_{t=0}^{\infty}$ and $\{a_t(s,\cdot), \{b^k_t(s,\cdot)\}_{k=2}^{K}\}_{t=1}^{\infty}$ such that these sequences and $e^{\infty}(s)$ satisfy (47), (48) and (49) after each s^t. Hence, iterating forward from t on (47) gives for all R,

$$\phi_{t+1}(s^t) = E_{s^t} \sum_{j=0}^{R-1} \beta^j [\Lambda_{t+1+j}] + \beta^R E_{s^t} [\phi_{t+R+1}] .$$

Using (50) and the boundedness of $\mathbf{X}$ this implies:

$$\phi_{t+1}(s^t) = E_{s^t} \sum_{j=0}^{\infty} \beta^j [\Lambda_{t+1+j}] . \tag{53}$$

Also, iterating on (49) gives for all k, t, and s^t,

$$D^k_{t+1}(s^t) = \begin{cases} E_{s^t} \left[\prod_{j=0}^{k-2} \left\{ \frac{N_{t+j+1} U_{2t+j+1}}{E_{s^{t+j}}[N_{t+j+1} U_{1t+j+1}]} \right\} \right] & k = 2, \cdots, K-1 \\ 1 & k = 1. \end{cases} \tag{54}$$

Combining (48) for $t \geq 1$ or (52) for $t = 0$ (with $a_0 = (M_0 + B^1_0)/P_{s0}$ and $b^k_0 = B^k_0/P_{s0}$) with (53) and (54) implies that e^{∞} satisfies the implementability and the measurability constraints (22). The result then follows from Proposition 2.

For the converse, suppose that e^{∞} is a competitive allocation. Then, it satisfies the no arbitrage, implementability/ measurability and no lending constraints. An associated state variable sequence $\{\phi_{t+1}, \{D^k_{t+1}\}_{k=1}^{K}, s_t\}_{t=0}^{\infty}$ can be obtained using the earlier definitions of ϕ_{t+1} and D^k_{t+1} variables. It is straightforward to verify that after each $t \geq 1$, s^{t-1}, $\{c_{1t}(s^{t-1},\cdot)$, $c_{2t}(s^{t-1},\cdot)$, $L_{f,t}(s^{t-1},\cdot)$, $L_{f,t}(s^{t-1},\cdot)$, $\phi_{t+1}(s^{t-1},\cdot)$, $\{D^k_{t+1}(s^{t-1},\cdot)\}_{k=1}^{K-1}\} \in \Gamma(\phi_t(s^{t-1})$, $\{D^k_t(s^{t-1})\}_{k=1}^{K-1}$, $s_{t-1})$ and that at $t = 0$, $\{c_{10}$, c_{20}, L_{f0}, L_{f0}, ϕ_1, $\{D^k_1\}_{k=1}^{K-1}\} \in \Gamma_0(a_0$, $\{b^{k+1}_0\}_{k=1}^{K-1})$. ■

In light of Lemma 10, we can divide the government's recursive Ramsey problem into an initial period problem and a family of continuation problems. In the initial period problem the government inherits a sticky price and a liability portfolio $\{P_{s0}, M_0, \{B^k_0\}_{k=1}^{K}\}$. It then chooses a period 0 allocation and a tuple of continuation state variables $\{\phi', \{D^{k\prime}\}_{k=1}^{K-1}\}$ from $\Gamma_0(\frac{M_0+B^1_0}{P_{s0}}, \{\frac{B^{k+1}_0}{P_{s0}}\}_{k=1}^{K-1})$

to solve:

$$V_0(\frac{M_0+B_0^1}{P_{s0}}, \{\frac{B_{-1}^{k+1}}{P_{s0}}\}_{k=1}^{K-1}) = \sup_{\Gamma_0(\frac{M_0+B_0^1}{P_{s0}}, \{\frac{B_0^{k+1}}{P_{s0}}\}_{k=1}^{K-1})} E[U(c_1, c_2, (1-\rho)L_f+\rho L_s)+\beta V(\phi', \{D^{k\prime}\}_{k=1}^{K-1}, s')]. \tag{55}$$

The government arrives in subsequent periods with a vector of state variables $\{\phi, \{D^k\}_{k=1}^K\}$. It then chooses a current period allocation and a tuple of continuation state variables from $\Gamma(\phi, \{D^k\}_{k=1}^K, s)$ to solve:

$$V(\phi, \{D^k\}_{k=1}^{K-1}, s) = \sup_{\Gamma(\phi, \{D^k\}_{k=1}^{K-1}, s)} E_s[U(c_1, c_2, (1-\rho)L_f + \rho L_s) + \beta V(\phi', \{D^{k\prime}\}_{k=1}^{K-1}, s')]. \tag{56}$$

The dynamic programming problems in (55) and (56) can be solved numerically.

IX. A Calibrated Example

We use the above recursive formulation to solve for the Ramsey equilibrium in a fully calibrated version of our economy.

A. Numerical method and parameter values

Numerical method Our approach is to solve the dynamic programming problems (55) and (56) and then back out the implied optimal policies. The state space $\mathbf{X}$ for these problems is endogenous and of dimension $K+1$. In our calculations we restrict the state space to be a $K+1$-dimensional rectangular subset $\tilde{X}$ of the bounding set Z. We check that we can numerically solve the dynamic programming problems at each point in $\tilde{X}$ and that enlarging $\tilde{X}$ so that it remains in Z does not significantly alter the numerical results we report. The dynamic programming problem is then solved by a value iteration. We approximate the value functions with cubic splines.

Calibration To permit comparability of our results to those in Siu (2004) and Chari, Christiano and Kehoe (1991), we first compute a baseline case with parameter values that are close to theirs.

In this baseline case, we assume preferences of the form:

$$U(c_1, c_2, l) = \log\left\{\left[(1-\gamma)c_1^{\phi} + \gamma c_2^{\phi}\right]^{\frac{1}{\phi}}\right\} + \psi \log(T - l). \tag{57}$$

We set the preference parameters γ, ϕ and β to 0.58, 0.79 and 0.96. We choose ψ so that approximately 30% of an agent's time is spent working. The values of γ and ϕ are similar to those used by Siu (2004) and Chari, Christiano and Kehoe (1991). These authors take logs in the expression $U_{1t}/U_{2t} = 1/Q_t^1$, identify cash good consumption with total money balances and then run a regression to obtain estimates of γ and ϕ. The value of ϕ used implies an elasticity of substitution between cash and credit goods of approximately 4.8 and thus a high degree of substitutability between these goods. We also compute a version of the model with preferences that are log in both cash and credit goods. This version thus has a lower unit elasticity of substitution. We follow Siu (2004) and set the production parameters α, μ and ρ to 1.0, 1.05 and 0.08 respectively. Government spending takes on two values $\underline{G}$ and $\overline{G}$. The government spending process has a mean of around 20% of GDP in a complete markets model with a debt to GDP ratio of 60%. We set the standard deviation of this process to be 6.7%. The government spending shocks follow a symmetric Markov process and it has an autocorrelation coefficient of 0.95. These values for shocks and the shock process are close to those estimated from the data and conform with the values used in Siu. We also consider a version of the model with a larger standard deviation for shocks of 14%.

We allow the maximal maturity limit K to vary between 1 and 4. The number of state variables in our dynamic programming problems equals $K + 1$. As a practical computational matter, we keep the maximal maturity relatively short. Nonetheless, as we shall see moving from a maximal maturity of one to a maximal maturity of four periods has an impact on the amount of hedging that the government can do.

B. Results

All numerical calculations confirm our earlier analytical results that the government uses only the longest maturity debt available. At the first round of trading in each period it buys up all of its outstanding nominal liabilities and sells long term debt. In the remainder of this section, we focus on the implications of optimal policy for nominal interest rates, inflation and debt holding returns. We illustrate these implications with short run impulse responses to shocks and with sample moments from long simulations.

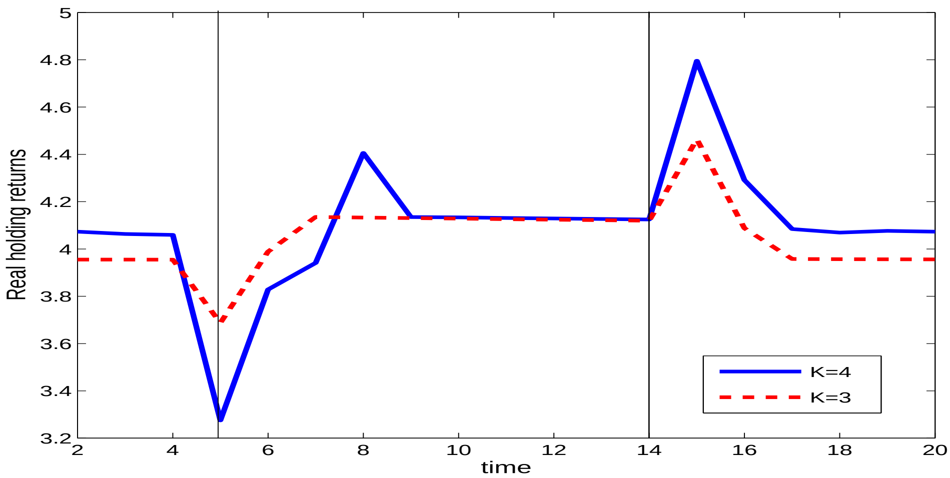
B.1. Impulse responses

In each of the impulse responses presented in this section, the government is assumed to have an initial debt value to output ratio of about 40%. The government then draws low spending shocks until period 4, high spending shocks from periods 5 to 14 and low spending shocks thereafter. In the figures below the first and last periods in which spending shocks are high are marked with vertical lines.

Holding returns Figure 1 shows the evolution of the real holding return on the government's portfolio for economies with baseline preferences. The solid line is drawn for the case $K = 4$, the dashed line for $K = 3$. In both cases, the holding return falls in period 5 contemporaneously with the high spending shock. However, the reduction is significantly greater in the $K = 4$ case. Over the next few periods, the holding returns rise. In period 15, government spending falls back to the lower level and both holding returns rise sharply. This increase is about twice as large in the $K = 4$ case. The quantitative difference between the two cases indicates that the government is better able to hedge fiscal shocks by devaluing its debt when the maturity of that debt is larger.

Nominal capital gains and inflation Since the government uses only the longest term debt, the real holding return on its portfolio at t can be decomposed as: $\mathrm{HR}_t = q_t^K - \pi_t$, where $q_t^K = \log Q_t^{K-1} - \log Q_{t-1}^K$ gives the rate of nominal capital gains on the K-th maturity bond and π_t is the inflation rate. Figure 2 below illustrates the impulse responses of q_t^K and π_t for the cases $K = 3$ and $K = 4$. Qualitatively, they are similar. In both cases, the K-maturity nominal bond price decreases coincidentally with the onset of high government spending shocks in period 5. In this way, the government reduces the real holding return on its portfolio by delivering a nominal

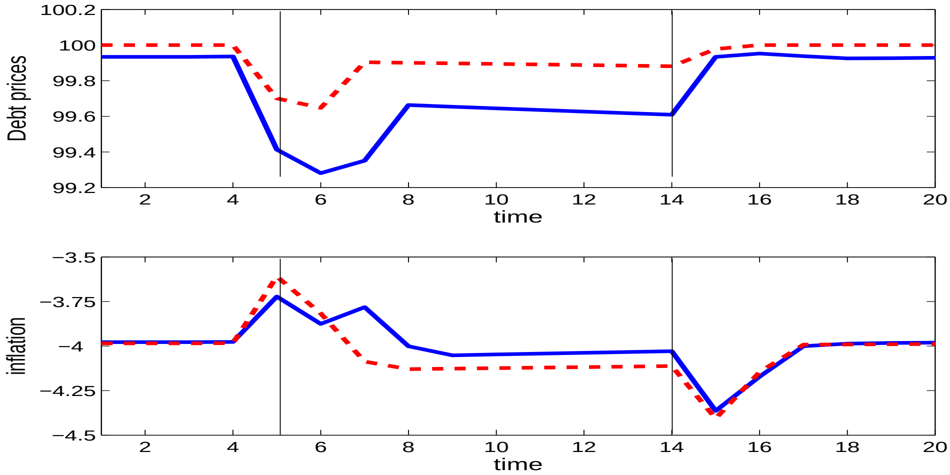
Figure 1. Debt holding returns



capital loss to bond holders. This bond price subsequently rises as the debt outstanding at the time of the initial high spending shock matures. Conversely, when government spending falls back in period 15, there is an increase in the nominal bond price and a nominal capital gain for investors. These nominal capital gains and losses are reinforced by contemporaneous changes in inflation. This rises sharply at the beginning of the high government spending spell, and decreases sharply at the end of this spell.

Despite these qualitative similarities, the response of bond prices and inflation differs quantitatively across the two cases. The reduction in the debt price coincident with the high government spending shock in period 5 is more than twice as great when $K = 4$ relative to $K = 3$ and, since the initially outstanding debt takes longer to mature in the $K = 4$ case, debt prices remain at lower levels for longer. In contrast, the contemporaneous rise in inflation in period 5 is slightly greater when the maximal debt maturity is shorter and $K = 3$. (Although, inflation remains higher for longer in the $K = 4$ case). This indicates that the government relies more heavily on nominal capital gains and losses and movements in the yield curve to hedge fiscal shocks as the maximal debt maturity increases.

Figure 2. Nominal capital gains and inflation



Short run nominal interest rates As noted previously, absent uncertainty, the formula for the equilibrium price of the K period nominal bond at t is simply the product of the reciprocal of gross one period nominal bond returns between t and $t+K-1$. When uncertainty is introduced, the relationship between current bond prices and future short term nominal interest rates is more complicated. Inspection of the formula for the k-th period debt price given Section III B, reveals that:

$$Q_t^k(s^t) = \tilde{E}_{s^t}\left[\prod_{j=0}^{k-1} \frac{U_{2t+j}}{U_{1t+j}}\right] \tag{58}$$

where the expectation $\tilde{E}_t$ is constructed using the adjusted probability distribution

$$\tilde{\pi}^k(s^{t+k}|s^t) = \pi(s^{t+k}|s^t) \prod_{j=0}^{k-2} \frac{N_{t+j+1}(s^{t+j+1})U_{1t+j+1}(s^{t+j+1})}{E_{s^{t+j}}[N_{t+j+1}(s^{t+j+1})U_{1t+j+1}(s^{t+j+1})]}. \tag{59}$$

This adjusted probability distribution weights states in which the realized value of N_{t+j} and the cash good marginal utility U_{1t+j} are high more heavily than does the true probability distribution $\pi^t(\cdot|s^t)$. Since the reciprocal of the gross one period nominal interest rate equals U_{2t}/U_{1t}, it follows from these equations that a low debt price Q_t^k is associated with a high conditional expectation of future short term nominal interest rates (with respect to the adjusted probability distribution) over the horizon t to $t+k-1$.

Figure 3. One period nominal interest rates

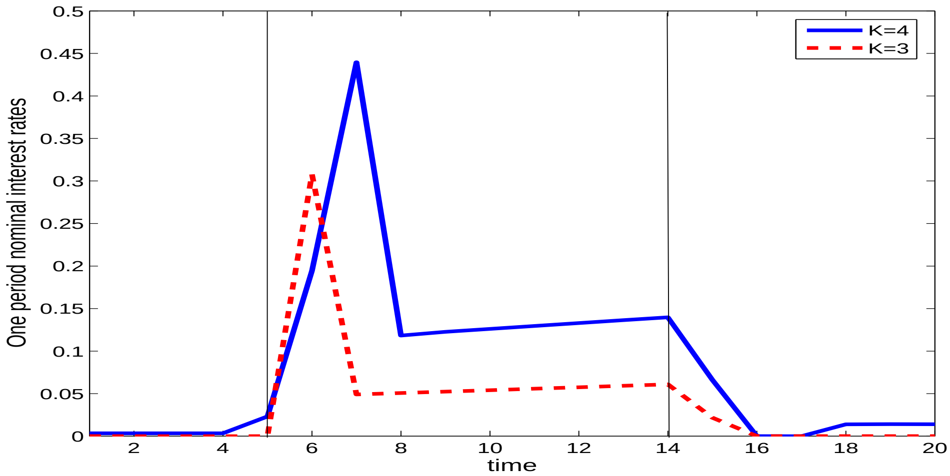


Figure 3 shows the impulse response of one period nominal interest rates for the $K = 4$ and $K = 3$ cases. In each case, this interest rate gradually rises to a peak after period 5 and the advent of the high government spending shocks. It then falls back (at date $5 + K - 2$) as the debt outstanding at the time of the first high spending shock matures. The gradual increase is consistent with efforts to delay the distortion from positive nominal interest rates identified in the example from Section VII. Quantitatively, the initial rise in nominal interest rates is larger and more protracted when $K = 4$, relative to $K = 3$. In the former case, rates peak in period 7 at about 0.45%, in the latter case at 0.30% in period 6. We conjecture that as K is increased further, short term nominal interest rates would peak at a higher value and at a progressively later date. We take these results as further indication that the government is better able to hedge using longer term nominal debt, by delaying the associated costly nominal interest rate distortions.

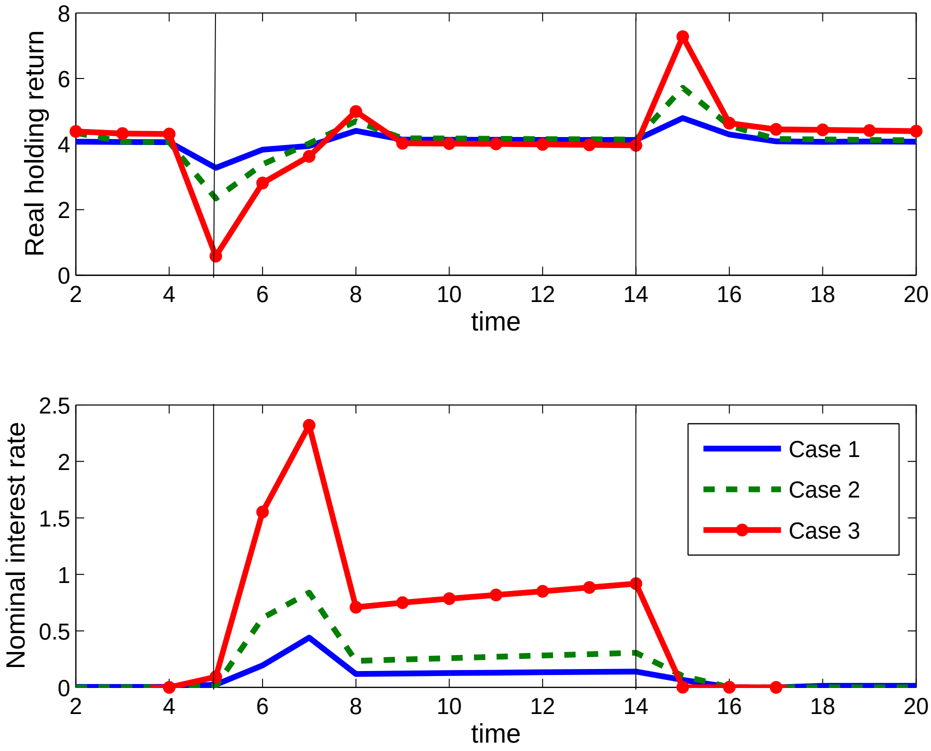
Sensitivity analysis: Variations in preferences and shock volatility Figure 4 shows the evolution of one period holding returns on the government's portfolio and nominal interest rates for the baseline economy (Case 1, solid line), an economy with a large shock volatility (a standard deviation of 14%, dashed line) and an economy with a large shock volatility and preferences that are log-log in cash and credit goods (Case 3, solid line with dots). After the high government spending shock occurs, the holding return falls by 0.8% in Case 1, 1.8% in Case 2 and 4% in Case 3. Similarly, the peak in nominal short term nominal interest rates climbs from 0.45% in Case 1, to 0.8% in Case 2, to 2.3% in Case 3. As the volatility of shocks rises, the government hedges to a greater degree, in part by raising nominal interest rates further and distorting the cash-credit good consumption more. When preferences are log-log in cash and credit goods, the elasticity of substitution between these goods is reduced relative to the baseline case. Distortions to the cash-credit good consumption margin are less costly and the government is prepared to distort this margin even further. Thus, the holding return falls most in Case 3, and nominal interest rates rise most in this case, after a high spending shock.

B.2. Long simulations

In this section we report results from long simulations of various economies. Each simulation is of length $T_{\text{sample}} = 20,000$. Figure 5 shows simulated values for the debt to output ratio and nominal interest rates for the first 1,000 periods of the baseline economy. The figure shows that the volatility of nominal interest rates is increasing in the debt level. This is largely because a given interest rate volatility induces a greater variation in the government's total liability value at high debt levels. Hence, it is more effective hedging against shocks.

Tables 1 and 2 below summarize the remainder of our results. Table 1 gives a rough measure of the impact of a positive fiscal shock on the value of the government's debt. In particular, the first row of this table gives the average variation in the real value of government debt across high and

Figure 4. Debt holding returns and nominal interest rates

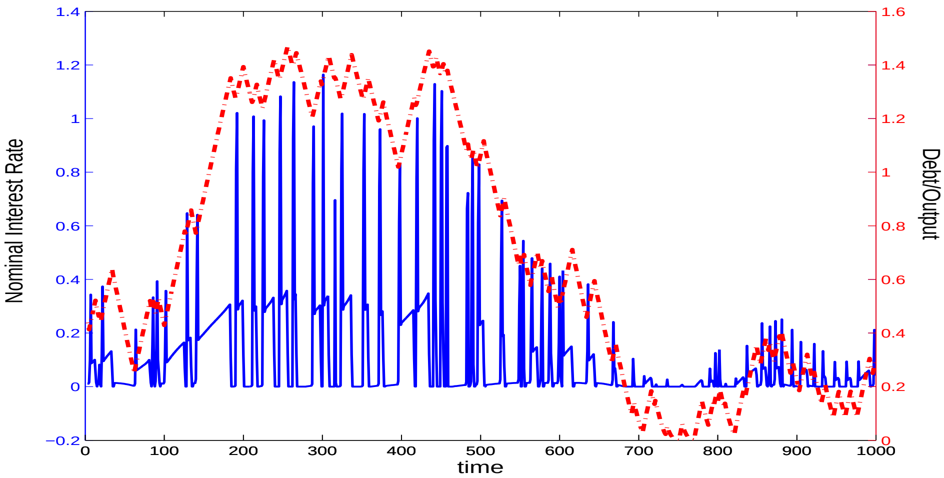


low spending states:

$$\Delta B = \frac{1}{T_{\text{sample}}} \sum_{t=0}^{T_{\text{sample}}} \left[\left[\frac{Q_t^{K-1}(\overline{G})}{P_t(\overline{G})} - \frac{Q_t^{K-1}(\underline{G})}{P_t(\underline{G})} \right] B_t^K \right],$$

The next two rows break this adjustment down into a component that comes purely from nominal capital losses and a component that comes from a contemporaneous price inflation. The remainder of the table gives these values normalized in the variation in government spending $\Delta G = \overline{G} - \underline{G}$. Table 1 reports results for the baseline economy with $K = 1$, 3 and 4. As K rises both the degree of hedging (as measured by ΔB) and the extent to which this hedging is obtained from movements in debt prices rather than contemporaneous inflations increases. When $K = 4$, ΔB equals about 30%

Figure 5. Debt levels and nominal interest rates



of the variation in government spending ΔG. Over 60% of this variation comes from a movement in the nominal debt price and less than 40% from a contemporaneous inflation. The table also reports results for the economy with more volatile shocks and with log-log preferences. There is more hedging of shocks in both cases, with considerably increased reliance on adjustments in debt price in the latter case.

Table 1: Financing Government Spending

	$K=1$	$K=3$	$K=4$		
	Base	Base	Base	High volatility	High vol.; log preferences
$\triangle$ *in real value of debt*	**-0.0065**	**-0.0125**	**-0.0157**	**-0.0347**	**-0.0766**
change in inflation	**-0.0065**	**-0.0066**	**-0.0057**	**-0.0137**	**-0.0136**
change in price of debt	**0.0000**	**-0.0058**	**-0.010**	**-0.0212**	**-0.0634**
$\triangle$ *in real value of debt (norm.)*	**-0.1293**	**-0.2491**	**-0.3139**	**-0.3472**	**-0.7665**
change in inflation	**-0.1293**	**-0.1328**	**-0.1147**	**-0.1370**	**-0.1357**
change in price of debt	**0.0000**	**-0.1189**	**-0.2000**	**-0.2120**	**-0.6341**

Table 2 reports statistics from long simulations of different versions of the model. For reasons of space, we focus below on the contrast between the baseline economies with K equal to one and

K equal to four. First, the correlation between government spending shocks and the one period nominal interest rates is increasing in the maximal debt maturity K. Relatedly, the correlation between government spending shocks and the nominal price of the government's debt portfolio is decreasing (towards -1) in K. This captures the fact that the government raises nominal interest rates further from 0 for longer in the aftermath of an adjustment from low to high spending as K rises. The correlation between interest rates and spending shocks is negative for $K = 1$, but positive for $K > 1$. This stems from the fact that when $K = 1$, the measurability constraints are of the form: $\xi_t = N_t U_{1t} a_{t-1}$. When a high spending shock occurs, ξ_t falls. The government partly accommodates this by decreasing U_{1t} (relative to U_{2t}) and, hence, reducing the current nominal interest rate. For $K > 1$, the measurability constraint at the optimal debt portfolio takes the form: $\xi_t = N_t U_{2t} D_{t+1} b^K_{t-1}$. In this case, to accommodate the fall in ξ_t, the government depresses U_{2t} (relative to U_{1t}) and D_t. Hence, it raises current and future nominal interest rates. Consistently, the mean, standard deviation and autocorrelation coefficient for nominal interest rates increases with K. Conversely, the standard deviation of tax rates and their correlation with government spending shocks decreases slightly as K increases from 1 to 4, indicating greater tax smoothing at higher K values.

Table 2: Statistics from Long Simulations

Statistics	$K=1$	$K=3$	$K=4$		
	Base	Base	Base	High volatility	High vol.; log preferences
inflation					
mean	**-4.016**	**-3.998**	**-3.947**	**-3.790**	**-3.288**
st. deviation	**0.227**	**0.290**	**0.268**	**0.634**	**0.997**
autocorrelation	**0.315**	**0.430**	**0.595**	**0.584**	**0.762**
correlation with G-shock	**0.167**	**0.369**	**0.462**	**0.478**	**0.654**
taxes					
mean	**23.68**	**23.66**	**23.60**	**23.50**	**23.80**
st.deviation	**1.934**	**1.922**	**1.888**	**2.380**	**2.120**
autocorrelation	**0.995**	**0.995**	**0.996**	**0.985**	**0.982**
correlation with G-Shock	**0.229**	**0.217**	**0.217**	**0.372**	**0.268**
price of debt					
mean	**0.999**	**0.998**	**0.996**	**0.991**	**0.977**
st. deviation	**0.002**	**0.003**	**0.009**	**0.013**	**0.027**
autocorrelation	**0.082**	**0.600**	**0.433**	**0.656**	**0.727**
correlation with G-shock	**0.398**	**-0.192**	**-0.293**	**-0.527**	**-0.744**
1-period nom. interest rate					
mean	**0.067**	**0.084**	**0.135**	**0.292**	**0.800**
st. deviation	**0.168**	**0.181**	**0.223**	**0.495**	**1.160**
autocorrelation	**0.081**	**0.152**	**0.548**	**0.565**	**0.648**
correlation with G-shock	**-0.396**	**0.420**	**0.499**	**0.504**	**0.667**

X. Conclusion

We have explored optimal debt management and taxation when the government is restricted to using non-contingent nominal debt of various maturities and is limited in its ability to lend. Our results imply that long term nominal debt can allow the government to hedge fiscal shocks through less distortionary nominal interest rate policies. Consequently, our model prescribes the use of long term nominal debt. Others have argued against the use of such debt on the grounds that it is excessively risky. In our model, the holding return on long term nominal debt is more volatile than that on short term debt, but this volatility is deliberate and the government uses it to hedge fiscal shocks.

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Appendices

A. Proofs

Proof of Proposition 1

Part 1: Necessity Suppose $\{c_{1t}, c_{2t}, L_{f,t}, L_{s,t}\}_{t=0}^{\infty}$ is an interior competitive allocation with no government lending at $\left\{P_{s0}, A_0, \{B_0^{k+1}\}_{k=1}^{K-1}\right\}$. We show that it satisfies the conditions in the proposition.

There is no loss of generality in replacing the no lending constraints on the household with an alternative sequence of constraints $\forall t, s^t, k,\ B_t^k(s^{t-1}) \geq -B,\ \widetilde{B}_t^k(s^t) \geq -B,\ B > 0$ and in assuming that at the equilibrium bond prices, households have no desire to borrow. Thus, these relaxed no lending constraints are non-binding for the household. The interiority of the competitive allocation implies that the constraints $c_{jt} \geq 0$, $j = 1, 2$ and $L_t \in [0, T]$ are non-binding. For expositional purposes, we will assume the existence of optimal Lagrange multipliers on the households' budget and cash-in-advance constraints and state the household's first order conditions using these.[18] Specifically, let $\mu_t(s^t)$ denote the multiplier on the household's period t cash-in-advance constraint. It represents the shadow price of liquidity. Let $\widetilde{\lambda}_{t+1}(s^t)$ denote the multiplier on the household's first round budget constraint (3) at $t+1$ after history s^t. Similarly, let $\lambda_{t+1}(s^{t+1})$ denote the multiplier on the second round budget constraint (4) after history s^{t+1}. $\widetilde{\lambda}_{t+1}$ and λ_{t+1} are the shadow prices of nominal wealth at each trading round. Applying arguments of Kamihigashi (2003), we obtain the necessity of the transversality condition:[19]

$$\lim_{t\to\infty} \beta^t E_{s^0}\left[\lambda_t(s^t)\left\{\sum_{k=0}^{T} Q_t^k(s^t)\widetilde{B}_t^{k+1}(s^t) + \widetilde{M}_t(s^t)\right\}\right] = 0.$$

Using the optimal multipliers, the first order conditions for consumption and labor supply may be stared as: are then:

$$c_{1t}: \quad \{\beta\widetilde{\lambda}_{t+1}(s^t) + \mu_t(s^t)\}P_t(s^t) = U_{1t}(s^t) \tag{A1}$$

$$c_{2t}: \quad \beta\widetilde{\lambda}_{t+1}(s^t)P_t(s^t) = U_{2t}(s^t) \tag{A2}$$

$$l_t: \quad \beta\widetilde{\lambda}_{t+1}(s^t)(1-\tau_t(s^t))\frac{\partial I_t}{\partial l_t}(s^t) = -U_{lt}(s^t). \tag{A3}$$

The first order conditions for each of the asset levels are:

$$\widetilde{M}_t: \quad \lambda_t(s^t) = \mu_t(s^t) + \beta\widetilde{\lambda}_{t+1}(s^t) \tag{A4}$$

$$M_{t+1}: \quad \widetilde{\lambda}_{t+1}(s^t) = E_{s^t}[\lambda_{t+1}] \tag{A5}$$

$$\widetilde{B}_t^k: \quad Q_t^k(s^t)\lambda_t(s^t) = \beta\widetilde{Q}_{t+1}^k(s^t)\widetilde{\lambda}_{t+1}(s^t) \tag{A6}$$

$$B_{t+1}^k: \quad \widetilde{Q}_{t+1}^k(s^t)\widetilde{\lambda}_{t+1}(s^t) = E_{s^t}[Q_{t+1}^{k-1}\lambda_{t+1}] \tag{A7}$$

Combining (A1) and (A2), we obtain: $\frac{U_{1t}}{U_{2t}} = \frac{\beta\widetilde{\lambda}_{t+1}+\mu_t}{\beta\widetilde{\lambda}_{t+1}} \geq 1$. This establishes (19). Adding the household's and the government's first round budget constraints and the profit conditions of firms in any period gives (20), the aggregate resource constraint.

The first order condition from the final goods firm implies $\frac{P_{it}}{P_t} = \left(\frac{Y_t}{Y_{it}}\right)^{\frac{\mu-1}{\mu}}, \quad i = f, s$. Thus, we have

$$P_{s,t}(s^{t-1}) = \left(\frac{Y_t(s^t)}{Y_{s,t}(s^t)}\right)^{\frac{\mu-1}{\mu}} P_t(s^t) = \left(\frac{Y_{f,t}(s^t)}{Y_{s,t}(s^t)}\right)^{\frac{\mu-1}{\mu}} P_{f,t}(s^t). \tag{A8}$$

The flexible price intermediate goods firm's first order conditions gives $P_{f,t}(s^t) = \frac{\mu}{\alpha}W_t(s^t)L_{f,t}(s^t)^{1-\alpha}$. Combining this with (A8) we obtain:

$$P_{s,t}(s^{t-1}) = \left(\frac{Y_{f,t}(s^t)}{Y_{s,t}(s^t)}\right)^{\frac{\mu-1}{\mu}} \frac{\mu}{\alpha}W_t(s^t)L_{f,t}(s^t)^{1-\alpha}. \tag{A9}$$

The first order condition from the sticky price firm's problem is, $t > 0$:

$$\sum_{s^t|s^{t-1}} \pi(s^t|s^{t-1}) \left\{(1-\tau_t(s^t))\frac{U_{2t}(s^t)}{P_t(s^t)}\right\} \left[P_{s,t}(s^{t-1}) - \frac{\mu}{\alpha}W_t(s^t)L_{s,t}(s^t)^{1-\alpha}\right] P_t(s^t)^{\frac{\mu}{\mu-1}}Y_t(s^t) = 0. \quad \text{(A10)}$$

From the household's first order condition for labor, we have $(1-\tau_t)\frac{U_{2t}}{P_t} = \beta\widetilde{\lambda}_{t+1}(1-\tau_t) = -\frac{U_{lt}}{W_t}$. Applying this condition to the curly bracket term in (A10) and substituting (A9) into (A10) gives

$$\sum_{s^t|s^{t-1}} \pi(s^t|s^{t-1})U_{lt}(s^t) \left[\left(\frac{Y_{f,t}}{Y_{s,t}}\right)^{\frac{\mu-1}{\mu}} L_{f,t}(s^t)^{1-\alpha} - L_{s,t}(s^t)^{1-\alpha}\right] P_{s,t}(s^{t-1})^{\frac{\mu}{\mu-1}}Y_{s,t}(s^t) = 0.$$

Substituting for $Y_{s,t}$, $Y_{f,t}$ and canceling the $P_{s,t}$ term gives (21).

Next take the household's first round budget constraint (3) at $t+1$, multiply it by $\widetilde{\lambda}_{t+1}(s^t)$ and add $\mu_t(s^t)\widetilde{M}_t(s^t)$ to obtain:

$$\begin{aligned}
&\beta\sum_{k=1}^{K}\widetilde{Q}^k_{t+1}(s^t)\widetilde{\lambda}_{t+1}(s^t)\widetilde{B}^k_t(s^t) + \{\beta\widetilde{\lambda}_{t+1}(s^t) + \mu_t(s^t)\}\widetilde{M}_t(s^t) \\
&= \mu_t(s^t)\widetilde{M}_t(s^t) + \beta\widetilde{\lambda}_{t+1}\{P_t(s^t)c_{1t}(s^t) + P_t(s^t)c_{2t}(s^t) - (1-\tau_t(s^t))I(s^t)\} \\
&\quad + \beta\widetilde{\lambda}_{t+1}(s^t)\left\{\sum_{k=1}^{K}\widetilde{Q}^k_{t+1}(s^t)B^k_{t+1}(s^t) + M_{t+1}(s^t)\right\}
\end{aligned}$$

The household's first order conditions and the second round budget constraint (4) at $t+1$ then implies:

$$\begin{aligned}
&\beta\widetilde{\lambda}_{t+1}(s^t)\sum_{k=1}^{K}\widetilde{Q}^k_{t+1}(s^t)\widetilde{B}^k_t(s^t) + \{\beta\widetilde{\lambda}_{t+1}(s^t) + \mu_t(s^t)\}\widetilde{M}_t(s^t) \\
&= U_{1t}(s^t)c_{1t}(s^t) + U_{2t}(s^t)c_{2t}(s^t) + U_{lt}(s^t)I_t(s^t)/W_t(s^t) \\
&\quad + \beta E_{s^t}\left[\lambda_{t+1}(s^{t+1})\left\{\sum_{k=0}^{K}Q^k_{t+1}(s^{t+1})\widetilde{B}^{k+1}_{t+1}(s^{t+1}) + \widetilde{M}_{t+1}(s^{t+1})\right\}\right] \\
&= U_{1t}(s^t)c_{1t}(s^t) + U_{2t}(s^t)c_{2t}(s^t) + U_{lt}(s^t)I_t(s^t)/W_t(s^t) \\
&\quad + \beta E_{s^t}\left[\beta\widetilde{\lambda}_{t+2}(s^{t+1})\sum_{k=1}^{K}\widetilde{Q}^k_{t+2}(s^{t+1})\widetilde{B}^k_{t+1}(s^{t+1}) + \{\beta\widetilde{\lambda}_{t+2}(s^{t+1}) + \mu_{t+1}(s^{t+1})\}\widetilde{M}_{t+1}(s^{t+1})\right]. \quad \text{(A11)}
\end{aligned}$$

Using the expressions for profits from the intermediate goods firms problems, $I_t(s^t)/W_t(s^t) = \Upsilon_t(s^t)$. Iterating on (A11) and using the household's first order condition, transversality and no lending conditions gives:

$$U_{1t}(s^t)\left[\sum_{k=1}^{K}Q^k_t(s^t)\frac{\widetilde{B}^k_t(s^t)}{P_t(s^t)} + \frac{\widetilde{M}_t(s^t)}{P_t(s^t)}\right] = \xi_t(s^t), \quad \text{(A12)}$$

where

$$\xi_t(s^t) \equiv E_{s^t}\left[\sum_{j=0}^{\infty}\beta^{t+j}\left\{U_{1t+j}c_{1t+j}(s^{t+j})+U_{2t+j}c_{2t+j}(s^{t+j})+U_{lt+j}\Upsilon_{t+j}(s^{t+j})\right\}\right].$$

The household's second round budget constraint at t and (A12) implies:

$$\begin{aligned}\frac{A_t(s^{t-1})}{P_t(s^t)}+\sum_{k=1}^{K-1}Q_t^k(s^t)\frac{B_t^{k+1}(s^{t-1})}{P_t(s^t)} &= \left[\sum_{k=1}^{K}Q_{t+1}^k(s^t)\frac{\widetilde{B}_t^k(s^t)}{P_t(s^t)}+\frac{\widetilde{M}_t(s^t)}{P_t(s^t)}\right]\\ &= \frac{\xi_t(s^t)}{U_{1t}(s^t)}.\end{aligned} \tag{A13}$$

Define $a_t(s^{t-1})$ and $b_t^k(s^{t-1})$, $t \geq 0$, according to (13). From (A8), we have

$$\frac{P_t(s^t)}{P_{t+1}(s^{t+1})}=\frac{P_t(s^t)}{P_{s,t+1}(s^t)}N_{t+1}(s^{t+1}). \tag{A14}$$

The household's first order conditions imply $\frac{U_{2t}}{U_{1t}}-\frac{\beta}{U_{1t}}E_t\left(\frac{P_t}{P_{t+1}}U_{1t+1}\right)=0$. So, $\frac{U_{2t}}{U_{1t}}-\frac{\beta}{U_{1t}}\frac{P_t}{P_{s,t+1}}\times E_t[N_{t+1}U_{1t+1}]$ $=0$. Combining this equality with (A14), we have:

$$\frac{P_t}{P_{t+1}}=\frac{1}{\beta}\frac{N_{t+1}U_{2t}}{E_t[N_{t+1}U_{1t+1}]}. \tag{A15}$$

We can also use the first order conditions and (A15) to obtain:

$$\begin{aligned}Q_t^k &= \beta\frac{E_t[Q_{t+1}^{k-1}\lambda_{t+1}]}{\lambda_t}=\beta^2\frac{E_t\left[Q_{t+2}^{k-2}\lambda_{t+2}\right]}{\lambda_t}=\dots\\ &= \beta^{k-1}E_t\left[\frac{U_{2t+k-1}}{U_{1t}}\frac{P_t}{P_{t+k-1}}\right]\\ &= E_t\left[\frac{U_{2t+k-1}}{U_{1t}}\prod_{j=0}^{k-2}\left\{\frac{N_{t+j+1}U_{2t+j}}{E_{t+j}[N_{t+j+1}U_{1t+j+1}]}\right\}\right]=\frac{U_{2t}}{U_{1t}}D_{t+1}^k.\end{aligned} \tag{A16}$$

Combining (A13), (A14), (A16) and the definitions of a_t and b_t^k we have the implementability/ measurability constraints (22):

$$\left[\frac{A_t(s^{t-1})}{P_{s,t}(s^{t-1})}+\frac{U_{2t}(s^t)}{U_{1t}(s^t)}\sum_{k=1}^{K-1}D_{t+1}^k(s^t)\frac{B_t^{k+1}(s^{t-1})}{P_{s,t}(s^{t-1})}\right]N_t(s^t)=\frac{\xi_t(s^t)}{U_{1t}(s^t)}, \tag{A17}$$

Finally, the definitions of b_t^k and a_t and the fact that $B_t^k \geq 0$ and $A_t \geq 0$ gives the no lending constraints (23).

Part 2: Sufficiency We construct a candidate competitive equilibrium from an allocation and a portfolio weight sequence satisfying the conditions in the proposition. First we set prices. Specifically, for

$t > 0$, set the relative sticky price to:

$$\frac{P_{s,t}}{P_{t-1}} = \frac{\beta}{U_{2t-1}} E_{t-1} \left(\left(\frac{Y_t}{Y_{s,t}} \right)^{\frac{\mu-1}{\mu}} U_{1t} \right) \tag{A17}$$

and set the gross (final goods) rate of inflation to:

$$\frac{P_t(s^t)}{P_{t-1}(s^{t-1})} = \frac{P_{s,t}(s^{t-1})}{P_{t-1}(s^{t-1})} \left(\frac{Y_{s,t}(s^t)}{Y_t(s^t)} \right)^{\frac{\mu-1}{\mu}}. \tag{A18}$$

At date 0, P_{s0} is a parameter, while P_0 is set equal to $P_0(s^0) = P_{s0} \left(\frac{Y_{s0}(s^0)}{Y_0(s^0)} \right)^{\frac{\mu-1}{\mu}}$. These conditions and the equations (A17) and (A18) allow us to recursively recover all goods prices. For $k > 0$ and $t \geq 0$, set the asset prices Q_t^k from the period t second round budget constraint to:

$$Q_t^k(s^t) = \frac{U_{2t}(s^t)}{U_{1t}(s^t)} D_t^k(s^t). \tag{A19}$$

Also, for $k > 0$ and $t \geq 0$, set the asset prices from the period $t+1$ first round budget constraint to be $\widetilde{Q}_{t+1}^k(s^t) = D_t^k(s^t)$. For $t > 0$, we set the portfolios purchased by households in the first round of asset trading as follows. The level of debt of $k > 1$ maturity is fixed at $B_t^k(s^{t-1}) = b_t^k(s^{t-1}) P_{s,t}(s^{t-1})$. $M_t(s^{t-1}) \geq 0$ and $B_t^1(s^{t-1}) \geq 0$ are chosen so that $M_t(s^{t-1}) + B_t^1(s^{t-1}) = a_t(s^{t-1}) P_{s,t}(s^{t-1})$. Next we turn to the portfolios purchased in the second round of trading. For $t \geq 0$, the money supply is set to $\widetilde{M}_t(s^t) = P_t(s^t) c_{1t}(s^t)$, We then choose each $\{\widetilde{B}_t^k(s^t)\}_{k=1}^K \in \mathbb{R}_+^K$ so that the second round budget constraints hold at each date t:

$$A_t(s^{t-1}) + \sum_{k=1}^{K-1} Q_t^k(s^t) B_t^{k+1}(s^{t-1}) \geq \widetilde{M}_t(s^t) + \sum_{k=1}^{K} Q_t^k(s^t) \widetilde{B}_t^k(s^t).$$

Given (23), these asset holdings are all non-negative. Moreover, by construction, they ensure that the second round budget and cash-in-advance constraints are satisfied. The government's debt holdings are set equal to the household's holdings of bonds.

Now, set the real wage to $\frac{W_t}{P_t} = \frac{\alpha}{\mu} L_{f,t}{}^{\alpha-1} \left(\frac{Y_t}{Y_{f,t}} \right)^{\frac{\mu-1}{\mu}}$, and the income tax rate to $(1-\tau_t) = -\frac{U_{lt}}{U_{2t}} \frac{P_t}{W_t}$. The Lagrange multipliers can be recovered from $\lambda_t P_t = U_{1t} \geq 0$, $\beta \widetilde{\lambda}_{t+1} P_t = U_{2t}$ and $\mu_t P_t = U_{1t} - U_{2t} \geq 0$. To check that we have defined a competitive allocation, we need to ensure that the first order conditions of the households and firms and their constraints are satisfied at the prices and tax rates constructed above. It is immediate from the definitions of multipliers that $\lambda_t = \mu_t + \beta \widetilde{\lambda}_{t+1}$, $\{\beta \widetilde{\lambda}_{t+1} + \mu_t\} P_t = U_{1t}$ and $\beta \widetilde{\lambda}_{t+1} P_t = U_{2t}$. The definition of taxes and these multiplier definitions gives $\beta \widetilde{\lambda}_{t+1} (1-\tau_t) \partial I_t / \partial l_t = -U_{lt}$. (A17) and (A18) imply $U_{2t} = \beta E_t \left(\frac{P_t}{P_{t+1}} U_{1t+1} \right)$. From our definitions of the multipliers, we then have

$\widetilde{\lambda}_{t+1} = \beta E_{s^t}[\lambda_{t+1}]$. Finally, the definitions of Q_t^k, $\widetilde{Q}_{t+1}^k$ and the multipliers gives $Q_t^k \lambda_t = \beta \widetilde{Q}_{t+1}^k \widetilde{\lambda}_{t+1}$ and $\widetilde{Q}_{t+1}^k \widetilde{\lambda}_{t+1} = E_{s^t}[Q_{t+1}^{k-1}\lambda_{t+1}]$. Hence, all of the household's first order conditions are satisfied.

We now check that the household's first round budget constraints are satisfied. Combining (22), (A18) and (A19) we obtain:

$$\xi_t(s^t) = \frac{U_{1t}(s^t)}{P_t(s^t)}\left[A_t(s^{t-1}) + \sum_{k=1}^{K-1} Q_t^k(s^t) B_t^{k+1}(s^{t-1})\right]. \tag{A20}$$

Hence, using the second round budget constraint, dividing by $U_{2t}(s^t)$ and using the definitions of $Q_t^k(s^t)$ and $\widetilde{Q}_{t+1}^k(s^t)$, we have

$$\frac{\xi_t(s^t)}{U_{2t}(s^t)} = \frac{U_{1t}(s^t)}{U_{2t}(s^t)}\frac{\widetilde{M}_t(s^t)}{P_t(s^t)} + \sum_{k=1}^{K} \widetilde{Q}_{t+1}^k(s^t)\frac{\widetilde{B}_t^k(s^t)}{P_t(s^t)}.$$

Subtracting $\frac{U_{2t}(s^t)-U_{1t}(s^t)}{U_{2t}(s^t)}\frac{\widetilde{M}_t(s^t)}{P_t(s^t)}$ from each side, using the definition of $\xi_t(s^t)$ and $\tau_t(s^t)$ and $\frac{\widetilde{M}_t(s^t)}{P_t(s^t)} = c_{1t}(s^t)$ yields:

$$\frac{\widetilde{M}_t(s^t)}{P_t(s^t)} + \sum_{k=1}^{K} \widetilde{Q}_{t+1}^k(s^t)\frac{\widetilde{B}_t^k(s^t)}{P_t(s^t)} = c_{1t}(s^t) + c_{2t}(s^t) + (1-\tau_t(s^t))I_t(s^t) + \frac{\beta}{U_{2t}(s^t)}E_{s^t}[\xi_{t+1}(s^{t+1})]$$

Then, using the measurability conditions at $t+1$, the definitions of Q_{t+1}^k, $\widetilde{Q}_{t+1}^k$and P_{t+1} and the condition $U_{2t} - \beta E_{t+1}\left(\frac{P_t}{P_{t+1}}U_{1t+1}\right) = 0$, we obtain:

$$\frac{\widetilde{M}_t(s^t)}{P_t(s^t)} + \sum_{k=1}^{K} \widetilde{Q}_{t+1}^k(s^t)\frac{\widetilde{B}_t^k(s^t)}{P_t(s^t)} = c_{1t}(s^t) + c_{2t}(s^t) + (1-\tau_t(s^t))I_t(s^t) + \frac{A_{t+1}(s^t)}{P_t(s^t)} + \sum_{k=2}^{T-1} \widetilde{Q}_{t+1}^k(s^t)\frac{B_{t+1}^k(s^t)}{P_t(s^t)} \tag{A21}$$

The first round budget constraint at $t+1$ then follows from (A21) and the definition of $\widetilde{A}_t(s^t)$

Since the sequence $\{\xi_t\}_{t=0}^{\infty}$ satisfies the recursion

$$\xi_t(s^t) \equiv U_{1t}(s^t)c_{1t}(s^t) + U_{2t}(s^t)c_{2t}(s^t) + U_{lt}(s^t)\Upsilon_t(s^t) + \beta E_{s^t}[\xi_{t+1}(s^{t+1})].$$

Hence,

$$\xi_0 = \lim_{t\to\infty} E\left[\sum_{j=0}^{t} \beta^j \left\{U_{1j}c_{1j} + U_{2j}c_{2j} + U_{lj}\Upsilon_j\right\}\right] + \lim_{t\to\infty}\beta^{t+1}E[\xi_{t+1}]. \tag{A22}$$

But then from the definition of ξ_0, the measurability constraint, and (A20):

$$\lim_{t\to\infty}\beta^{t+1}E[\xi_{t+1}] = \lim_{t\to\infty}\beta^{t+1}E\left[U_{1t+1}\left(\sum_{k=1}^{K} Q_{t+1}^k\frac{B_{t+1}^k}{P_{t+1}} + \frac{M_{t+1}}{P_{t+1}}\right)\right] = 0,$$

which confirms the transversality condition. The no lending conditions (23) imply for all s^{t+1} that the solvency constraints hold. Hence, the allocation is feasible and optimal for the household's at the derived prices and tax rates. The household's budget constraints and the resource constraint guarantee that the government's budget constraints are satisfied. It is easy to verify that the derived choices of firms satisfy their first order conditions and are optimal. ■

B. Complete Markets Problems

Here we derive some properties of the complete markets Ramsey problem. We begin with the Ramsey problem from date 0 and then discuss a continuation problem from date 1.

A. Complete Markets Ramsey Problem

Following the discussion in the main text, the complete markets Ramsey problem is:

$$\sup_{\{c_{1t},c_{2t},L_{f,t},L_{s,t}\}_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty}\beta^t U(c_{1t},c_{2t},(1-\rho)L_{f,t}+\rho L_{s,t})\right] \tag{B1}$$

subject to:

$$\beta^t\eta_t(s^t)\pi^t(s^t) : U_{1t}(s^t)-U_{2t}(s^t)\geq 0; \tag{B2}$$

$$\beta^t\chi_t(s^t)\pi^t(s^t) : G(s_t)+c_{1t}(s^t)+c_{2t}(s^t)=\left[(1-\rho)L_{f,t}^{\frac{\alpha}{\mu}}(s^t)+\rho L_{s,t}^{\frac{\alpha}{\mu}}(s^t)\right]^{\mu}; \tag{B3}$$

$$\varsigma_t(s^{t-1})\pi^{t-1}(s^{t-1}) : E_{s^{t-1}}U_{lt}\left[L_{f,t}^{1-\frac{\alpha}{\mu}}L_{s,t}^{\frac{\alpha}{\mu}}-L_{s,t}\right]=0; \tag{B4}$$

$$\zeta_0 : f_0(s^0)N_{10}(s^0)U_{10}(s^0)=\xi_0(s^0). \tag{B5}$$

For $i=1,2$, let $H_{it}=\zeta_0[U_{1it}c_{1t}+U_{1t}+U_{2it}c_{2t}+U_{lit}\Upsilon_t]+\varsigma_t U_{lit}\Phi_t$. The first order condition for c_{it}, $t>0$, is:

$$U_{it}+\eta_t[U_{1it}-U_{2it}]-\chi_t-H_{it}=0, \tag{B6}$$

If $U_{1t}>U_{2t}$, then $\eta_t=0$. (B6) then implies that $U_{1t}-U_{2t}-H_{1t}+H_{2t}=0$. Under our assumed preferences, $\frac{H_{1t}}{U_{1t}}=\frac{H_{2t}}{U_{2t}}$, so that

$$(U_{1t}-U_{2t})\left[1-\frac{H_{it}}{U_{it}}\right]=0.$$

From (B6) for $i=1,2$ coupled with $\chi_t>0$, we have that $\frac{H_{it}}{U_{it}}\neq 1$. But, then $U_{1t}=U_{2t}$, a contradiction. We deduce that $U_{1t}=U_{2t}$, $t>0$.

Next we argue that from period 1 onwards, $L_{f,t}(s^t)=L_{s,t}(s^t)$. To see this, let $r_t(s^t)=\frac{L_{s,t}(s^t)}{L_{f,t}(s^t)}$ and let $l_t(s^t)=(1-\rho)L_{f,t}(s^t)+\rho L_{s,t}(s^t)$. The government's problem can be reformulated in terms of r_t and l_t. Specifically, the primary surplus value ξ_0 can be written as:

$$\xi_0(s^0)=E_{s^0}\left[\sum_{t=0}^{\infty}\left\{U_{1t}c_{1t}+U_{2t}c_{2t}+\frac{\mu}{\alpha}U_{lt}l_t\left(\frac{\rho r_t^{\alpha/\mu}+(1-\rho)}{(1-\rho)+\rho r_t}\right)\right\}\right] \tag{B7}$$

But, the sticky price constraint (B4) implies:

$$E_{s^{t-1}}\left[U_{lt}l_t\left(\frac{\rho r_t^{\alpha/\mu}+(1-\rho)}{(1-\rho)+\rho r_t}\right)\right]=\rho E_{s^{t-1}}\left[U_{lt}l_t\left(\frac{r_t^{\alpha/\mu}-r_t}{(1-\rho)+\rho r_t}\right)\right]+E_{s^{t-1}}\left[U_{lt}l_t\right]=E_{s^{t-1}}\left[U_{lt}l_t\right].$$

Combining this with (B5), we then have that any feasible allocation must satisfy:

$$f_0(s^0)N_{10}(s^0)=\Lambda_0(s^0)+\beta E_{s^0}\sum_{t=0}^{\infty}\beta^t\left[U_{1t+1}c_{1t+1}+U_{2t+1}c_{2t+1}+\frac{\mu}{\alpha}U_{lt+1}l_{t+1}\right].$$

Conversely, if an allocation satisfies the previous equation, the no arbitrage conditions (B2), the resource constraints

$$G(s_t)+c_{1t}(s^t)+c_{2t}(s^t)=\frac{\left[(1-\rho)+\rho r_t^{\frac{\alpha}{\mu}}(s^t)\right]^{\mu}}{(1-\rho)+\rho r_t(s^t)}l_t^{\alpha}(s^t), \tag{B8}$$

and the sticky price constraints:

$$E_{s^{t-1}}\left[U_{lt}l_t\left(\frac{r_t^{\alpha/\mu}-r_t}{(1-\rho)+\rho r_t}\right)\right]=0, \tag{B9}$$

then it is feasible. But setting $r_t=1$ satisfies the sticky price constraints (B9) and relaxes the resource constraints (B8). Thus, it is optimal to set r_t to this value. Consequently, after date 0, $L_{f,t}=L_{s,t}=l_t$ and the sticky price constraint can be dropped.

The optimal choice at date t after history (s^{t-1},s) solves:

$$\sup_{c_1,c_2,l}U(c_1,c_2,l)-\zeta_0\left[U_1c_1+U_2c_2+\frac{\mu}{\alpha}U_l l\right], \tag{B10}$$

subject to $U_1\geq U_2$ and $l^{\alpha}\geq c_1+c_2+G(s)$. Hence, the optimal allocation at t depends on s and ζ_0. Denote the optimal labor choice by $l^*(s)$ and the optimal cash good choice by $c_1^*(s_t)$ where the dependence on ζ_0 is suppressed. Under the more specific preferences $(1-\gamma)\log c_1+\gamma\log c_2+v(l)$, $U_1c_1+U_2c_2+(\mu/\alpha)U_l l^{\alpha}=1+(\mu/\alpha)v'(l)l^{\alpha}$. Let $F^*(s)=1+(\mu/\alpha)v'(l^*(s))l^*(s)^{\alpha}$. With $\zeta_0\leq 0$, the first order conditions for l, c_1 and c_2 immediately yield that if $G(s)>G(s')$, then $l^*(s)>l^*(s')$ and $F^*(s')>F^*(s)$. Additionally, since π is monotone, $\xi_t(s^{t-1},s')>\xi_t(s^{t-1},s)$, when ξ_t is evaluated at the optimal allocation. Finally, it also follows from the first order conditions that $U_{1t}(s^{t-1},s)\geq U_{1t}(s^{t-1},s')$. Since, by the no lending constraints, $\xi_t\geq 0$, $\frac{\xi_t(s^{t-1},s')}{U_{1t}(s^{t-1},s')}>\frac{\xi_t(s^{t-1},s)}{U_{1t}(s^{t-1},s)}$.

B. Complete Markets Continuation Problem

This problem is of the form:

$$\sup_{\{c_{1t+1},c_{2t+1},L_{ft+1},L_{st+1}\}_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty}\beta^t U(c_{1t+1},c_{2t+1},(1-\rho)L_{ft+1}+\rho L_{st+1})\right] \tag{B11}$$

subject to, for $t > 0$,

$$\beta^t\eta_t(s^t)\pi^t(s^t) : U_{1t}(s^t) - U_{2t}(s^t) \geq 0; \tag{B12}$$

$$\beta^t\chi_t(s^t)\pi^t(s^t) : G(s_t) + c_{1t}(s^t) + c_{2t}(s^t) = \left[(1-\rho)L_{f,t}^{\frac{\alpha}{\mu}}(s^t) + \rho L_{s,t}^{\frac{\alpha}{\mu}}(s^t)\right]^{\mu}; \tag{B13}$$

$$\varsigma_t(s^{t-1})\pi^{t-1}(s^{t-1}) : E_{s^{t-1}}\left[U_{lt}(s^t)\left(L_{f,t}(s^t)^{1-\frac{\alpha}{\mu}}L_{s,t}(s^t)^{\frac{\alpha}{\mu}} - L_{s,t}(s^t)\right)\right] = 0; \tag{B14}$$

$$\zeta_1 : \phi_1(s^0) = E_{s^0}[\xi_1(s^1)]. \tag{B15}$$

In this problem the government selects a continuation allocation subject to the constraints above, and, in particular, subject to the implementability constraint (B15). Arguments similar to those above can be used to establish that 1) the Friedman rule holds at all dates, 2) $N_t = 1$ at all dates and that the sticky price constraint (B14) is non-binding. This proves Proposition 3 in the text.

C. The role of the no lending constraint

Example Suppose the household's preferences are given by: $(1-\gamma)\log c_1 + \gamma \log c_2 + v(l)$, where v is a decreasing, smooth, concave function. These preferences have the convenient feature that the value of the government's primary surplus at each date, ξ_t, is given by: $\xi_t(s^t) = E_{s^t}\left[\sum_{j=0}^{\infty}\beta^j\{1 - v'(l_{t+j})\Upsilon_{t+j}\}\right]$ and is, therefore, independent of consumption. Let $\{c_{1t}^*, c_{2t}^*, L_{f,t}^*, L_{s,t}^*\}_{t=1}^{\infty}$ denote the allocation that solves the complete markets continuation problem and let $\{\xi_t^*(s^t)\}$ denote the corresponding sequence of primary surplus values. Now turn to the economy with only non-contingent nominal debt of two period maturity, but without lending restrictions. Fix an arbitrary s^{t-1}, let $\overline{s}(s^{t-1}) = \arg\max_s \xi_t^*(s^{t-1}, s)/[U_{1t}^*(s^{t-1}, s)]$. Set the government's holdings of one and two period nominal liabilities after history s^{t-1} to

$$a_t(s^{t-1}) = \left[\frac{1}{2}\frac{\xi_t^*(s^{t-1},\overline{s}(s^{t-1}))}{U_{1t}^*(s^{t-1},\overline{s}(s^{t-1}))} - X_t(s^{t-1})\right] \tag{60}$$

$$b_t^2(s^{t-1}) = \left[\frac{1}{2}\frac{\xi_t^*(s^{t-1},\overline{s}(s^{t-1}))}{U_{1t}^*(s^{t-1},\overline{s}(s^{t-1}))} + X_t(s^{t-1})\right], \tag{61}$$

where $X_t(s^{t-1})$ is a large positive number. For the optimal complete markets *labor* allocation $\{L_{f,t}^*, L_{s,t}^*\}_{t=1}^{\infty}$ to be supported in the economy with non-contingent nominal debt, it must be the case that this allocation, the government's debt portfolio and the household's consumption allocation jointly satisfy the measurability constraints:

$$\forall s^t, \quad \left[U_{2t}(s^t)b_t^2(s^{t-1}) + U_{1t}(s^t)a_t(s^{t-1})\right] = \xi_t^*(s^t). \tag{62}$$

Where in (62) we use the fact that $\{L_{f,t}^*, L_{s,t}^*\}_{t=1}^{\infty}$ satisfies $N_t(s^t) = 1$ at all s^t. Next consider, $\{c_{1t}^*, c_{2t}^*\}_{t=1}^{\infty}$, the optimal consumption allocation from the complete markets economy. The definitions of a_t and b_t^2 in (60) and (61), coupled with the Friedman rule, imply that $\{c_{1t}^*, c_{2t}^*, L_{f,t}^*, L_{s,t}^*\}_{t=1}^{\infty}$ and a_t and b_t^2 satisfy the measurability condition (62) after histories $(s^{t-1}, \overline{s}(s^{t-1}))$. However, for the other histories (s^{t-1}, s), $s \neq \overline{s}(s^{t-1})$, the measurability constraint may be violated since the Friedman rule, the definition of $\overline{s}(s^{t-1})$ and the definitions of a_t and b_t^2 imply:

$$\begin{aligned}\left[U_{2t}^*(s^{t-1}, s)b_t^2(s^{t-1}) + U_{1t}^*(s^{t-1}, s)a_t(s^{t-1})\right] &= U_{1t}^*(s^{t-1}, s)\left[b_t^2(s^{t-1}) + a_t(s^{t-1})\right] \\ &= U_{1t}^*(s^{t-1}, s)\frac{\xi_t^*(s^{t-1}, \overline{s})}{U_{1t}^*(s^{t-1}, \overline{s})} \geq \xi_t^*(s^{t-1}, s)\end{aligned}$$

Suppose that the above inequality is strict and consider slightly raising credit good consumption and slightly reducing cash good consumption in this state so that resource constraint continues to be satisfied. This will reduce $\left[U_{2t}(s^{t-1}, s)b_t^2(s^{t-1}) + U_{1t}(s^{t-1}, s)a_t(s^{t-1})\right]$. If $X_t(s^{t-1})$ is large, only a very small alteration in the pattern of consumption will be required to equate $[U_{2t}(s^{t-1}, s)b_t^2(s^{t-1}) + U_{1t}(s^{t-1}, s)a_t(s^{t-1})]$ to $\xi_t^*(s^{t-1}, s)$. Consequently, by choosing the $\{X_t\}_{t=1}^{\infty}$ sequence appropriately, a government in an economy with only non-contingent nominal debt markets can attain the labor allocation that is optimal and a consumption allocation arbitrarily close to the optimal one in an economy with complete markets. Clearly, this will require the government to lend an arbitrarily large amount of one period debt and borrow an arbitrarily large amount of two period debt. ■

D. An illustrative example

PROOF OF LEMMA 4: First, we show that there exists a solution in which $b_1^K > 0$, then that there is a solution in which $a_1 = 0$, $b_1^k = 0$, $k = 2, \cdots, K-1$ and $b_1^K > 0$. Under our assumptions, in particular our assumption that the household's utility function is continuously differentiable and that it's total labor supply lies within the set $[0, T]$, it may be shown that the set of feasible government choices is compact. The continuity of the household's objective guarantees the existence of a solution. Let $\{a_1^*, \{b_1^{k*}\}_{k=2}^K\}$ denote an optimal portfolio. Since $\phi_1 > 0$, either $a_1^* > 0$ or $b_1^{k*} > 0$ for some k. Let $\hat{k}$ denote the smallest k such that for all $k > \hat{k}$, $b_1^{k*} = 0$. Suppose $\hat{k} < K$. Then, for $t \geq \max\{2, \hat{k}\}$, the first order condition for c_{it} reduces to:

$$0 = U_{it} + \eta_t\left[U_{1it} - U_{2it}\right] - \chi_t - H_{it}, \tag{63}$$

Suppose that $U_{1t} > U_{2t}$. Since under our assumed preferences, $\frac{H_{1t}}{U_{1t}} = \frac{H_{2t}}{U_{2t}}$, (63) then implies that $(U_{1t} - U_{2t})\left(1 - \frac{H_{2t}}{U_{2t}}\right) = 0$. It may be verified that $\chi_t > 0$ and, from (63) with $\eta_t = 0$, that $1 - \frac{H_{2t}}{U_{2t}} > 0$. Thus, we deduce that, in fact, $U_{1t} = U_{2t}$. It then follows from the measurability constraint that the optimal allocation can be implemented with a portfolio in which either $b_1^K = b_1^{\hat{k}*}$ and $b_1^{\hat{k}} = 0$ or, if $\hat{k} = 1$, $b_1^K = a_1^*$ and $a_1 = 0$. All other portfolio weights remain the same.

Wlog, assume that $b_1^{K*} > 0$. Combining the first order conditions for c_{1t} and c_{2t}, $t \in \{2, \cdots, K-1\}$, we obtain:

$$\begin{aligned} 0 &= -\chi_t\left(\frac{1}{U_{1t}} - \frac{1}{U_{2t}}\right) + \eta_t\left[\left(\frac{U_{11t}}{U_{1t}} - \frac{U_{12t}}{U_{1t}}\right) + \left(\frac{U_{22t}}{U_{2t}} - \frac{U_{12t}}{U_{2t}}\right)\right] \\ & \quad -\beta^{-(t-1)}\zeta_1 U_{21} N_1 \sum_{k=t}^{T-1}\left[b_0^{k+1}\prod_{j=2}^{k}\frac{U_{2j}}{U_{1j}}\right]\left[\frac{1}{U_{1t}}\left(\frac{U_{11t}}{U_{1t}} - \frac{U_{21t}}{U_{2t}}\right) + \frac{1}{U_{2t}}\left(\frac{U_{22t}}{U_{2t}} - \frac{U_{21t}}{U_{1t}}\right)\right]. \end{aligned} \tag{64}$$

By the quasiconcavity of U,

$$\frac{1}{U_{1t}}\left(\frac{U_{11t}}{U_{1t}} - \frac{U_{21t}}{U_{2t}}\right) + \frac{1}{U_{2t}}\left(\frac{U_{22t}}{U_{2t}} - \frac{U_{21t}}{U_{1t}}\right) < 0. \tag{65}$$

By homotheticity,

$$\text{sign}\left[\left(\frac{U_{11t}}{U_{1t}} - \frac{U_{12t}}{U_{2t}}\right)\right] = \text{sign}\left[\left(\frac{U_{22t}}{U_{2t}} - \frac{U_{12t}}{U_{1t}}\right)\right].$$

This relationship and (65) yield $\left(\frac{U_{iit}}{U_{it}} - \frac{U_{ijt}}{U_{jt}}\right) < 0$, $i, j = 1, 2$, $i \neq j$. Combining this, (65) and (64), we obtain that $U_{1t} - U_{2t} > 0$ if and only if $\zeta_1 < 0$, and $U_{1t} - U_{2t} = 0$ if and only if $\zeta_1 \geq 0$. Since, $b_1^{K*} > 0$, it follows from the first order conditions (41) that either A) $\zeta_1(s) = 0$ for each s or B) $\zeta_1(s) > 0 > \zeta_1(s')$ for some pair s, s'. Now suppose $b_1^{k*} > 0$ for $k < K$. Then, $\sum_{s\in S}\zeta_1(s)N_1(s)U_{21}(s)\prod_{j=2}^{k}\frac{U_{2j}(s)}{U_{1j}(s)}\pi(s) = 0$. If Case B holds, it follows that:

$$\sum_{s\in S}\zeta_1(s)N_1(s)U_{21}(s)\prod_{j=1}^{k}\frac{U_{2j}(s)}{U_{1j}(s)}\prod_{j=k+1}^{K}\frac{U_{2j}(s)}{U_{1j}(s)}\pi(s) > 0 \tag{66}$$

But this contradicts the first order condition for $b_1^{K*} > 0$. Thus, $b_1^{k*} = 0$ for $k < K$. By a similar argument, $a_1^* = 0$ as well. Suppose that Case A holds. It follows that $\frac{U_{1t}}{U_{2t}} = 1$, $t \geq 1$. The left hand side of the measurability constraint may then be written as: $[\sum_{k=1}^{K-1} b_1^{k+1} + a_1]U_{11}(s)N_1(s)$ and the maturity structure is irrelevant. It follows that there is a solution to the government's problem in which $a_1 = 0$, $b_1^k = 0$, $k = 2, \cdots, K-1$ and $b_0^K > 0$ as required. ■

PROOF OF LEMMA 6: Wlog, assume that $b_1^{K*} > 0$ and a_1^* and b_1^{k*}, $k = 2, \cdots, K-1$, equal 0. From the first order condition for b_1^{K*}, either $\zeta_1(s) = 0$ for each s or $\zeta_1(s) > 0 > \zeta_1(s')$. We must rule out the first case. So suppose it is true. Then from the proof of Lemma 4, $U_{ct}(s) := U_{1t}(s) = U_{2t}(s)$ each s and t. Additionally, it can be shown that in this case, $N_1(s) = N_1(s')$. Consequently, for each s and t, $L_{f,t}(s) = L_{s,t}(s) = L_t(s)$. We can then re-express the government's choice problem as one involving only $\{c_t, L_t\}_{t=1}^{\infty}$ and b_1^K, where $c_t = (1-\gamma)c_{1t} + \gamma c_{2t}$. With $\zeta_1(s) = 0$, the first order condition for $L_t(s)$ is:

$$U_{lt}(s) + \chi_t(s)\alpha L_t^{\alpha-1} - \zeta_0 \frac{\mu}{\alpha}(U_{llt}(s)L_t^{\alpha}(s) + U_{lt}(s)\alpha L_t^{\alpha-1}(s)) = 0 \tag{67}$$

It follows that the optimal labor supply $L_t(s)$ is increasing in the shadow price of resources $\chi_t(s)$. Similarly, inspection of the first order condition for consumption reveals that $c_t(s)$ is decreasing in $\chi_t(s)$. It then follows from the resource constraint $L_t^{\alpha}(s) = G_t(s) + c_t(s)$ that $\chi_t(s)$, $L_t(s)$ and $-c_t(s)$ are increasing in $G_t(s)$. Thus,

$$\frac{1}{U_{c1}(\overline{s})} \sum_{t=1}^{\infty} [U_{ct}(\overline{s})c_t(\overline{s}) + U_{lt}(\overline{s})L_t(\overline{s})] < \frac{1}{U_{c1}(\underline{s})} \sum_{t=1}^{\infty} [U_{ct}(\underline{s})c_t(\underline{s}) + U_{lt}(\underline{s})L_t(\underline{s})]. \tag{68}$$

But this violates the measurability constraints since when $U_{1t}(s) = U_{2t}(s) = U_{ct}(s)$ and $L_{f,t}(s) = L_{s,t}(s) = L_t(s)$, these take the form, for each s.

$$b_1^K = \frac{1}{U_{c1}(s)} \sum_{t=1}^{\infty} [U_{ct}(s)c_t(s) + U_{lt}(s)L_t(s)]. \qquad \blacksquare \tag{69}$$

PROOF OF LEMMA 7 It follows from the proof of Lemma 4, that if s is such that $\zeta_1(s) \geq 0$ or if $t \geq K$, then $U_{1t}(s) = U_{2t}(s)$ and $Q_t^1(s) = 1$. On the other hand, if s is such that $\zeta_1(s) < 0$ and if $t = 2, \cdots, K-1$, then $U_{1t}(s) - U_{2t}(s) > 0$ and $Q_t^1(s) < 1$. Under the assumed preferences, $H_{it} = 0$, $U_{ijt} = 0$, $j \neq i$. Additionally, $a_1 = 0$, $b_1^k = 0$, $k = 2, \cdots, K-1$ at the optimal allocation. Consequently, the first order conditions for c_{1t} and c_{2t}, $t \in \{2, \cdots, K-1\}$ imply:

$$U_{1t} - U_{2t} = \beta^{-(t-1)} \zeta_1 b_1^K \prod_{j=1}^{K-1} \frac{U_{2j}}{U_{1j}} \left[\frac{U_{11t}}{U_{1t}} + \frac{U_{22t}}{U_{2t}}\right] U_{11} N_1. \tag{70}$$

Also, $U_{iit}/U_{it} = -d_i U_{it}$, where $d_1 = \frac{1}{1-\gamma}$ and $d_2 = \frac{1}{\gamma}$. Hence, from (70),

$$\frac{\frac{\gamma}{1-\gamma}\frac{U_{1t}}{U_{2t}}}{\frac{\gamma}{1-\gamma}\frac{U_{1t}}{U_{2t}} + 1} - 1 + \beta^{-(t-1)} \zeta_1 b_1^K \prod_{j=1}^{K-1} \left[\frac{U_{2j}}{U_{1j}}\right] U_{11} N_1 = 0 \tag{71}$$

Since $\beta \in (0,1)$, and $\zeta_1 < 0$, we deduce that for $t = 2, \cdots, K-2$, $\frac{U_{1t+1}}{U_{2t+1}} > \frac{U_{1t}}{U_{2t}}$. Hence, $1 > Q_t^1 > Q_{t+1}^1$, for $t = 2, \cdots, K-2$. If $Q_1^1 = 1$, we are finished. Suppose this not the case. The household's first order conditions then imply:

$$\begin{aligned} U_{11} - U_{21} &= \zeta_1 b_1^K \prod_{j=1}^{K-1} \frac{U_{2j}}{U_{1j}} \left[\frac{U_{111}}{U_{11}} + \frac{U_{221}}{U_{21}} \right] U_{11} N_1 - \zeta_1 b_1^K \prod_{j=1}^{K-1} \frac{U_{2j}}{U_{1j}} U_{111} N_1 \\ &< \zeta_1 b_1^K \prod_{j=1}^{K-1} \frac{U_{2j}}{U_{1j}} \left[\frac{U_{111}}{U_{11}} + \frac{U_{221}}{U_{21}} \right] U_{11} N_1. \end{aligned}$$

Thus, for $t = 2, \cdots, K-1$.

$$\frac{\frac{\gamma}{1-\gamma}\frac{U_{11}}{U_{21}}}{\frac{\gamma}{1-\gamma}\frac{U_{11}}{U_{21}} + 1} - 1 < -\zeta_1 b_1^K \prod_{j=1}^{K-1} \left[\frac{U_{2j}}{U_{1j}} \right] U_{11} N_1 < \frac{\frac{\gamma}{1-\gamma}\frac{U_{1t}}{U_{2t}}}{\frac{\gamma}{1-\gamma}\frac{U_{1t}}{U_{2t}} + 1} - 1. \tag{72}$$

We deduce that $\frac{U_{1t}}{U_{2t}} > \frac{U_{11}}{U_{21}}$, for $t = 2, \cdots, K-1$. Thus, $Q_{t+1}^1 < Q_t^1$ for $t = 1, \cdots, K-2$. ■

Notes

[1]The two are connected in the model via a risk-augmented Fisher equation.

[2]See Bohn (1988), Schmitt-Grohé and Uribe (2004) and Siu (2004) amongst others.

[3]Chari and Kehoe (1993) make the same assumption. Much of our analysis would continue to hold if government lending was limited, rather than completely ruled out.

[4]We also restrict the government to using income taxation only. Correia et al (2002) have shown that if a government can implement state contingent taxes on consumption as well as income, it can hedge fiscal shocks and replicate a complete markets allocation.

[5]Additionally, a substantial literature has documented the practice of adjusting the short term interest rate gradually in response to shocks, see for example Sack (2000). Our optimal policy also incorporates elements of such gradualism.

[6]The absence of two trading rounds in period 0 simplifies the analysis without loss of generality. We will call the budget constraint in period 0 a second round budget constraint to be consistent with the labeling in subsequent periods.

[7]Models with only the first stage of asset trading are said to exhibit Svensson timing. This timing convention prevents households from adjusting their cash holdings in light of their current state-contingent cash needs. Models with only the second stage of asset trading are said to exhibit Lucas timing. In our incomplete market model, this timing convention restricts the government's ability to insure itself against shocks by forcing households and the government to hold part of their portfolio in cash.

[8]At this stage one period bonds and money are equivalent: they are both claims to a unit of cash after the realization of the state s_t. Hence, $\widetilde{Q}_t^1(s^t) = 1$.

[9]However, as described above, we assume that shoppers can buy goods from local stores on credit.

[10]Other weaker restrictions on household borrowing/government lending of the form $B_t^k(s^{t-1}) \geq -B$ would lead to qualitatively similar results.

[11]They have counterparts in the work of Siu (2004) and SU (2004).

[12]Reductions in $\frac{U_{2t}(s^t)}{U_{1t}(s^t)} D_t^k(s^t)$ are also associated with anticipated inflations over the horizon t to $t+k-1$. The household's first order conditions imply a risk-augmented Fisher equation: $\beta E_t[U_{1t+1}/U_{1t}] E_t[P_t/P_{t+1}] Q_t^1 +$

$\beta \text{Cov}_t[U_{1t+1}/U_{1t}, P_t/P_{t+1}] = 1$ that links anticipated inflations to reductions in one period debt prices and positive nominal interest rates.

[13]In this model, the government possesses enough policy instruments to separate debt management from monetary policy. As we discuss further below, optimal monetary policy will conform to the Friedman rule and $U_{1t} = U_{2t} = U_t$.

[14]To see this explicitly, note that the measurability constraints in this case are of the form: $\xi_t(s^t) = U_{1t}(s^t)N_t(s^t)$ $f_t(s^t)$, $t > 0$. The $f_t(s^t)$ terms appear in no other constraints and can be chosen to ensure that these constraints hold at any desired allocation.

[15]Note that the assumption that claims are of only one period maturity ensures that continuation allocations from period $t+1$ onwards do not affect the government's period t liability value. Thus, the period 1 continuation allocations appear in the implementability constraint (29) only insofar as they influence the primary surplus value ξ_0. ϕ_1 is then a sufficient state variable for Continuation Problem 2.

[16]See Appendix B for an example and further discussion.

[17]If we assume prices are completely sticky in period 1, then the period 0 expected holding return is given by:

$$\frac{1}{HR_0^k} = \beta E_0\left[\frac{U_{11}}{U_{10}}\right] + \beta Cov_0\left(\frac{U_{11}}{U_{10}}, \frac{Q_1^{k-1}}{E_0\left[Q_1^{k-1}\right]}\right).$$

When the high government spending shock occurs, $\frac{Q_1^{k-1}}{E_0\left[Q_1^{k-1}\right]}$ falls and $\frac{U_{11}}{U_{10}}$.

[18]The existence of optimal (summable) Lagrange multipliers is not essential for the argument. Weitzman (1973) gives sufficient conditions for such existence in a related deterministic context.

[19]Specifically, the general Theorem 2.2 of Kamihigashi (2003) can be specialized to this case.