Eastern Economic Journal, Volume XIV, No. 2, April–June 1988

A Note About The Interest Rate and The Revenue Function

Neri Salvadori* and Ian Steedman**

The concern of this note is to reexamine the generally accepted notion that

> "... the revenue function provides a very general tool for modelling production so long as there are no distortions. (Dixit and Norman, 1980, p 160). In particular, and contrary to popular belief, there is no need to restrict the analysis to the case where final goods are produced directly by primary factors. There may be any pattern of goods being used as inputs to the production of other goods."

Whether or not there is, indeed, a 'popular belief' of the kind described by Dixit and Norman (who do not tell us where it is to be found), it is our contention such a belief is ill-founded. Yet it is no less true that the presence of produced inputs can cause certain problems for the use of the revenue function, as that is normally presented, when a positive rate of interest is paid on the value of those inputs. (And economic systems with a uniform and constant interest rate have, of course, been widely studied; c.f., for example, Malinvaud (1953), Mirrlees (1969), Starrett (1970), Gale and Rockwell (1975).)

The usual derivation of the revenue function is based on the claim that "Production decisions will maximize total profit ... the problem will be to ... maximize the value of [net] output" (ibid p. 31). But, if entrepreneurs have to pay a positive rate of interest on the capital advanced for the purchase of produced inputs, the maximization of profit will not maximize the value of net output; rather it will maximize that value minus the value of total interest payments. What follows from this for the properties of the 'revenue' function?

The Linear Programming Case

Consider first a linear programming representation of a competitive economy, in which the technical possibilities are shown by an output matrix, $B \geq 0$, a produced input matrix, $A \geq 0$, and a primary input matrix, $E \geq 0$. The vectors $p > 0$ and $e > 0$ represent a given commodity price vector and a given primary input endowment vector, respectively; $r \geq 0$ is a given interest rate. An activity vector, x, and a competitive primary input price vector, w, are to be chosen to solve:

$$V(p, r, e) \equiv \text{Max } p[B-(1+r)A]x \quad \text{s.t.} \begin{cases} Ex \leq e \\ x \geq 0 \end{cases} \qquad U(p, r, e) \equiv \text{Min } we \quad \text{s.t.} \begin{cases} wE \geq p[B-(1+r)A] \\ w \geq 0 \end{cases}$$

*Facoltà di Scienze Politiche, via Reclusorio del Lume 44, Catania, Italy.
**Department of Economics, Dover Street, Manchester, M13 9PL, UK.
We thank A.K. Dixit and H. Gram for their comments.

Since there are always feasible solutions, there are always optimal solutions and $V = U$ are uniquely determined.

It is readily seen that $V(p, r, e)$ is continuous; non-decreasing, linearly homogeneous and convex in p; decreasing in r; non-decreasing, linearly homogeneous and concave in e. Moreover

$$\Delta V \geq \Delta p_j[(B^j - A^j) - rA^j]x \qquad \Delta p_j \gtrless 0$$
$$\Delta V \geq -\Delta rp\,Ax \qquad \Delta r \gtrless 0$$
$$\Delta V \leq \Delta e_i\, w_i \qquad \Delta e_i \gtrless 0$$

where Δs is the increment of s and M^j is the jth row of matrix M. Therefore, where $V(p, r, e)$ is differentiable (i.e., where x and e are uniquely determined),

$$\frac{\partial V}{\partial p_j} = y_j - rk_j \qquad (1)$$

where y_j is the net output of j and k_j is the capital stock of j;

$$\frac{\partial V}{\partial r} = -pk \qquad (2)$$

where k is the capital stock *vector;*

$$\frac{\partial V}{\partial e_i} = w_i \qquad (3)$$

It will be noted at once, from (1), that $(\partial V/\partial p_j) = y_j$ if *and only if* either the interest rate is zero or commodity j is not used as a produced input.

It can, of course, also be seen from the original statement of the LP problem that while

$$\Delta w \Delta e \leq 0, \qquad (4)$$

from the Min problem, the Max problem yields only

$$\Delta p(\Delta y - r\Delta k) \geq 0$$

or

$$\Delta p \Delta y \geq r \Delta p \Delta k \qquad (5)$$

If the interest rate is zero, or if there are no produced inputs, we can be sure, from (5), that $\Delta p \Delta y \geq 0$; but in general this is not known. Hence the presence of produced inputs *does matter* for the use of the revenue function *when combined with the presence of a positive interest rate.*

The Continuous Case

Consider the problem

$$V(p, r, e) \equiv \underset{q,k}{\mathrm{Max}}\ ([pq - (1 + r)pk], \text{subject to } f(q, k, e) = 0,$$

where the notation is as above, except that q now appears as the vector of *gross* outputs. (The constraint $f(q, k, e) = 0$ has 'normal' properties.) First order conditions are naturally $p_j = L(\partial f/\partial q_j)$, $(1 + r)p_j = -L(\partial f/\partial k_j)$ and $f(q, k, e) = 0$, where L is a Lagrangean multiplier. If these can be solved for (q, k, L) we may write the maximum value of V as

$$V(p, r, e) \equiv [pg(p, r, e) - (1 + r)pk(p, r, e)] \qquad (6)$$

From (6),

$$(\partial V/\partial p_j) \equiv [q_j - (1 + r)k_j] + \sum_i p_i[(\partial q_i/\partial p_j) - (1 + r)(\partial k_i/\partial p_j)]$$
$$\equiv (y_j - rk_j) + L\sum_i [(\partial q_i/\partial p_j)(\partial f/\partial q_i) + (\partial k_i/\partial p_j)(\partial f/\partial k_i)] \qquad (1')$$
$$\equiv (y_i - rk_j)$$

In obtaining (1′) we have, of course, simply applied the envelope theorem; more important is the fact that (1′) in effect reproduces (1), above, but now for the continuous case. Even when $V(p, r, e)$ is differentiable everywhere, with respect to p, $(\partial V/\partial p_j)$ is *not* identical to the net output of j, unless either the interest rate is zero or j is not used as a produced input.

The same kind of argument as was used to derive (1′) also shows that

$$(\partial V/\partial r) \equiv -(pk) \qquad (2')$$

which reproduces (2), above, but now for the continuous case. Also, it is clear from the 'normal' properties of $f(q, k, e)$ that

$$(\partial V/\partial e_i) \equiv w_i \qquad (3')$$

It follows at once from (3′) that

$$(\partial w_i/\partial e_i) \equiv (\partial^2 V/\partial e_i^2) < 0 \qquad (4')$$

from the concavity of V with respect to e. But (1′) yields only

$$(\partial y_j/\partial p_j) - r(\partial k_j/\partial p_j) \equiv (\partial^2 V/\partial p_j^2) > 0$$

from the convexity of V with respect to p. Hence

$$(\partial y_j/\partial p_j) > r(\partial k_j/\partial p_j) \qquad (5')$$

But, $(\partial y_j/\partial p_j) > 0$ is not ensured when $r > 0$ and j is used as a capital good.

We note finally the effects of a positive interest rate on Samuelson's (1953) 'reciprocity conditions'. (Compare Gram, 1985.) From (1′) and (3′), the equality of $(\partial^2 V/\partial e_i \partial p_j)$ and $(\partial^2 V/\partial p_j \partial e_i)$ implies that

$$(\partial w_i/\partial p_j) = (\partial y_j/\partial e_i) - r(\partial k_j/\partial e_i);$$

it is *not* generally true that $(\partial w_i/\partial p_j) = (\partial y_j/\partial e_i)$. Similarly, from (1′), the symmetry of the matrix V_{pp} yields

$$(\partial y_j/\partial p_h) \equiv (\partial y_h/\partial p_j) + r[(\partial k_j/\partial p_h) - (\partial k_h/\partial p_j)];$$

therefore $(\partial y_j/\partial p_h) = (\partial y_h/\partial p_j)$ if and only if either $r = 0$ or the matrix $[(\partial k_j/\partial p_h)]$ is symmetric; note that the second condition holds if there are no produced inputs, or by a fluke. Only in a system having either a zero rate of interest, or no produced inputs, can we be sure that all the 'reciprocity conditions' will always hold.

Unless the presence of a positive interest rate is to be described as the presence of 'a distortion'—and why should it be?—it would seem that the above quote from Dixit and

Norman was insufficiently cautious. We have seen that some 'revenue function' results *are* affected by the presence of produced inputs *if* a positive rate of interest is paid on their value. And such a rate of interest is usually paid.

REFERENCES

Dixit, A.K. and Norman, V., 1980, *Theory of International Trade,* Welwyn, Nisbet and Co. and C.U.P.

Gale, D. and Rockwell, R., 1975, "On the interest rate theorems of Malinvaud and Starrett," *Econometrica,* Vol. 43, pp. 347–359.

Gram, H., 1985, Duality and positive profits, *Contributions to Political Economy,* Vol. 4, pp. 61–77.

Malinvaud, E., 1953, Capital accumulation and efficient allocation of resources, *Econometrica,* Vol. 21, pp. 233–268.

Mirrlees, J.A., 1969, The dynamic nonsubstitution theorem, *Review of Economic Studies,* Vol. 36, pp. 67–76.

Samuelson, P.A., 1953, "Prices of factors and goods in general equilibrium," *Review of Economic Studies,* Vol. 21, pp. 1–20.

Starrett, D., 1970, "The efficiency of competitive programs," *Econometrica,* Vol. 38, pp. 704–711.