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A Note on The Musgravian Transformation

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INTRODUCTION

An issue of current interest in the area of commodity taxation is the sales or ad valorem tax, i.e., a tax expressed as a constant fraction of the price. Despite the fact that this topic has been well-documented in the literature [Bishop, 1968], the relationship between the ad valorem tax on revenues (demand) and costs (supply) deserves further exploration. While a selective excise tax rate, v, is typically levied upon the net price of the goods, the factor tax rate, u, is usually calculated based on the gross price. Since Professor Musgrave's treatment of the subject is normally cited as a primary source [1959], we will refer to this relationship, $v = u/(1 - u)$ or $u = v/(1 + v)$, as the Musgravian transformation. We shall analyze, in particular, the concavity of the corresponding tax revenue curves under such a transformation. To the best of our knowledge, the central result of the analysis is not known. This transformation can have several interesting applications in the theory of taxation, especially in the spatial equilibrium models as illustrated by Irwin and Yang [1981, 1983] and Yang and Labys [1981, 1985] as well as in the well known Averch-Johnson model of a regulated monopolist [1962]. For purpose of illustration, we employ a set of linear demand and supply functions with the results being applied to the more general case.

THE PROPERTIES OF THE MUSGRAVIAN TRANSFORMATION

Given a set of linear demand and supply schedules of $p = a - bx$ and $p = c + dx$ with $a > c$, $a > 0$, $b > 0$, and $d > 0$, an ad valorem tax u on the revenue is equivalent to pivoting down the demand schedule. Algebraically, the imposition of the tax gives the following equilibrium condition [Musgrave, 1959, p. 293]:

$$(1 - u)(a - bx) = c + dx \tag{1}$$

Similarly, an ad valorem tax v on cost payments may be viewed as raising the supply schedule. Algebraically, the imposition of such a tax gives the following equilibrium condition [Musgrave, 1959, p. 307]:

$$(1 + v)(c + dx) = a - bx \tag{2}$$

The relationship between the demand ad valorem tax rate u and the supply ad valorem tax rate v in a competitive industry (hereafter the Musgravian transformation) has been shown by Musgrave [1959, pp. 306–307] to be:

$$u = v/(1 + v) \tag{3}$$

or

$$v = u/(1 - u) \tag{4}$$

Mathematically, the transformation is a one to one mapping from R_+ to R_+, i.e., f: $R_+ \rightarrow R_+$ or vice versa[1] where R_+ is a set of non-negative real numbers. These functional relations may be derived based upon a

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common gross price that consumers have to pay or p = a − bx and hence a common equilibrium quantity under both demand and supply ad valorem tax systems. As a result, equations (3) and (4) hold under a common set of prices and quantities. Note that u is a real number bounded by 1 and zero, and v is some nonnegative number whose corresponding u is for all practical purposes, less than one. For instance, a 10% supply ad valorem tax is equivalent to a 9.09% demand ad valorem tax. Clearly, the Musgravian transformation as shown in Figure 1 is not linear but rather monotonic. This may be seen using the following differentiation:

(5) $$du/dv = 1/(1+v)^2 > 0,\ d^2u/dv^2 = -2/(1+v) < 0$$

(6) $$dv/du = 1/(1-u)^2 > 0,\ d^2v/du^2 = 2/(1-u) > 0$$

It was shown by Musgrave [1959, p. 306] that under competition, an ad valorem tax on cost payments must result in the same price and output as an ad valorem tax on revenue of equal yield. It is indeed true that the price and output remain invariant under the Musgravian transformation. However, it has not been established whether the concavity and the maxima of the tax revenue curve (or Laffer curve) remain invariant under such a transformation. In order to investigate these properties, we may solve for price and quantity using equilibrium condition (1):

(7) $$q^* = (a - c - au)/(b + d - bu)$$

(8) $$\begin{aligned} p^* &= (1-u)(bc + ad)/(b + d - bu) \\ &= \text{price net of tax that producers receive} \\ r &= p^*/(1-u) = \text{gross price consumers pay} \end{aligned}$$

Hence, the tax revenue (TR1) is:

(9) $$TR1 = up^*q^*/(1-u) = urq^* = u(bc + ad)(a - c - au)/(b + d - bu)^2$$

The revenue-maximizing tax rate can be derived from the interior maximization for a positive u:

(10) $$dTR1/du = (ad + bc)(ab + ad - bc - cd - abu - bcu - 2adu)/(b + d - bu)^3 = 0$$

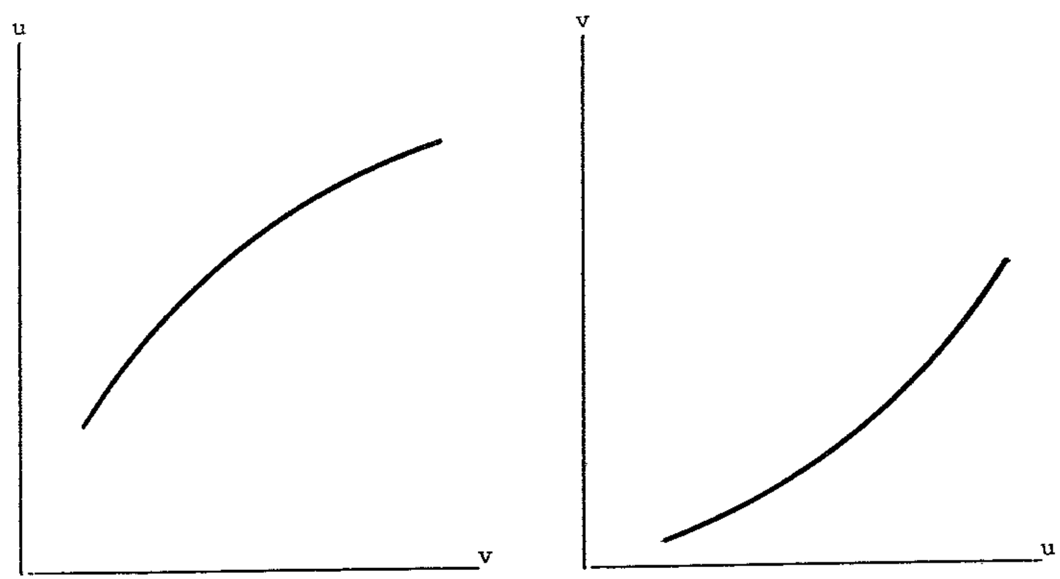

Figure 1. The Musgravian Transformation

and

(11) $$u^* = (a - c)(b + d)/(2ad + bc + ab)$$

The concavity of the tax revenue curve may be determined from the second derivative:

(12) $$\frac{d^2TR1}{du^2} = \frac{(ad + bc)(2ab^2 - 4b^2c - 4bcd - 2ad^2 - 2ab^2u - 2b^2cu - 4abdu)}{(b + d - bu)^4}$$

The sign of (12) cannot be determined, i.e., it is more likely to be concave for larger c's, d's and u's and smaller a's and b's. Similarly, quantity, price, tax revenue, revenue-maximizing tax rate and derivatives under the supply ad valorem tax (or based on cost payments) may be obtained as follows:

(13) $$q^{**} = (1 - c - cv)/(b + d + dv)$$

(14) $$\begin{aligned} p^{**} &= (1+v)(ad + bc)/(b + d + dv) \\ &= \text{gross price consumers have to pay} \end{aligned}$$

(15) $$TR2 = vp^{**}q^{**}/(1+v) = (ad + bc)(av - cv - cv^2)/(b + d + dv)^2$$

(16) $$dTR2/dv = (ad + bc)(ab + ad - bc - cd - adv - cdv - 2bcv)/(b + d + dv)^3 = 0$$

(17) $$v^* = (a - c)(b + d)/(ad + cd + 2bc)$$

(18) $$\frac{d^2TR2}{dv^2} = \frac{(ad + bc)(-4abd - 2b^2c - 4ad^2 + 2cd^2 + 2ad^2v + 4bcdv + 2cd^2v)}{(b + d + dv)^4}$$

Again, the curvature of the tax revenue function is dependent on the value of a, b, c, d, and the tax rate v. Using (17) and (11) one may easily verify:

(19) $$v^* = u^*/(1 - u^*) \text{ and } u^* = v^*/(1 + v^*)$$

It is significant to note that equation (19) is a special case of equation (3) and (4) when u^* and v^* are two corresponding revenue-maximizing tax rates either for linear or nonlinear demand and supply functions. Since the Musgravian transformation is clearly a one to one monotonic mapping within a given domain, the revenue-maximizing tax rate of the demand side corresponds uniquely to the revenue-maximizing rate of the supply side or cost payment.[2] In addition, it must be equally true that the two revenue-maximizing tax revenues be identical. This may be seen by comparing (9) and (15) at $v = v^*$ and $u = u^*$, even for the case of nonlinear demand and supply schedules, i.e., since $v^* = u^*/1 - u^*)$ holds for a common set of gross price and quantities, it follows from (9) and (15):

(20) $$u^*rq^* = (v^*/1 + v^*)p^{**}q^{**}$$

because $r = p^{**}$ = common gross price, $q^* = q^{**}$ = common quantity and $u^* = v^*/(1 + v^*)$.

The preceding proves the equivalence of two maximum revenues under the transformation for general demand and supply functions. It will also be observed that under the Musgravian transformation, i.e., $u = v/(1 + v)$, the corresponding tax revenues must all be identical, that is, TR1 = TR2 for each $u = v/(1 + v)$. We will use this property in investigating the concavity property under the transformation.

Finally, if the tax revenue curve derived from the revenue side is strictly concave, question arises as to whether the concavity will remain invariant under the Musgravian transformation. Besides being of theoretical interest, the concavity of the tax revenue curve implies that a slight increase in the tax rate before u^* will lead to an increase in tax revenue, but at a decreasing rate. In contrast, a slight decrease in the tax rate will result in a decrease in the tax revenue at an increasing rate, resulting in a larger loss in tax revenue for a locality. In addition, the concavity of the curve usually suggests that the changes in tax revenues are relatively greater in the neighborhood of a zero or a 100 percent tax rate than that of a convex curve. Unfortunately, the property of concavity may not be preserved as one switches from a demand to supply ad valorem tax system or vice versa.[3] For instance, given a concave tax revenue curve of the demand ad valorem tax, the concavity of the corresponding tax revenue curve of the supply ad valorem tax may or may not be preserved depending on the parameter values of the demand and supply functions. The answer depends critically on the signs of (12) and (18). In order to prove the latter for general nonlinear cases, we use the chain rule in the monotonic Musgravian transformation (hence a unique du/dv is guaranteed to exist) to yield:

$$\frac{dTR2}{dv} = \frac{dTR1}{du}\frac{du}{dv} \text{ since TR1} = \text{TR2} \tag{21}$$

$$\frac{d^2TR2}{dv^2} = \frac{d^2TR1}{du^2}\left(\frac{du}{dv}\right)^2 + \frac{dTR1}{du}\frac{d^2u}{dv^2} \tag{22}$$

It is clear from (6) and (22) that the concavity of the tax revenue curve derived from a supply ad valorem tax (or d^2TR^2/dv^2) is not guaranteed even if the tax revenue curve derived from a demand ad valorem tax is strictly concave everywhere, i.e., $d^2TR1/du^2 < 0$ for $0 < u < 1$. However, it is sufficient that the concavity of the curve is preserved under the transformation for $dTR1/du > 0$. That is to say, if the tax revenue curve from the demand side is everywhere concave, then the tax revenue curve from the supply side is also concave, at least for the v's whose corresponding u's are before the Laffer hill, i.e., $u < u^*$ or $v < v^*$. For many applicable environments, this range is what is relevant. According to Blinder [1981] price elasticities are to be as high as five for us to encounter the down side of the Laffer hill. Note that this is only a sufficient condition since it is possible that the sign of (22) could also be negative for some $dTR1/du < 0$. This is indeed the case in our simulation shown in Figure 2. It is to be noted that the horizontal axis is measured in terms of both tax rates u and v. For instance, $v = 3$ (point C) corresponds to $u = 0.75$ (point D) for an identical values of price, quantity and tax revenue.

To verify these properties, we simulate a linear model with $a = 150$, $b = 3.5$, $c = 15$, and $d = 3$ for u in the range 0% to 95%. The maximum revenue (points A and B) under both tax systems, $TR1^* = TR2^* = 700.96$, corresponds to $u^* = 0.5939$ and $v^* = 1.4625$, i.e., $u^* = v^*/(1 + v^*)$. In addition, the concavity of the tax revenue curve of the supply ad valorem tax is preserved at least before the revenue-maximizing tax rate. In actuality, the tax revenue curve of the supply ad valorem tax in this example is concave for approximately $v \leq 3$ (or EAC) given that the tax revenue curve (FBDG) of the demand ad valorem tax is everywhere concave.

POLICY IMPLICATIONS

Since price and quantity will be the same when revenue is maximized, the two taxation approaches will be exactly identical if the policy goal is set to maximize the tax revenue, at least in the competitive case.[4] The same is true for raising any given target tax revenue, i.e., 5% supply ad valorem tax always generates the identical revenue as does 4.76% demand ad valorem tax. However, the differences arise if the policy is to add some percentage points to the prevailing tax rates. For example, in the simulation one percentage point added to the existing 7% supply-based and demand-based tax will result in an increase in tax revenues of 13.755 units (118.403 − 104.648) and 15.875 units (127.806 − 111.931) respectively. There is a gain of 2.12 units in tax revenue if one percentage point is added to the existing 7% tax rate on the revenue side instead of on the cost side.

Similarly, if the policy is taken to involve a one percentage increase in an existing 9.09% demand-based tax or its corresponding 10% supply-based tax, one would witness an increase in tax revenues of 15.785 and 13.2 units respectively; a gain in the tax revenue of 2.585 units if one percentage increase is administered to the revenue side. Hence, Musgravian transformation shown in equation (3) or (4) indicates that one percentage point increase on the demand side before u^* always generates more tax revenues than that from the supply side.

After the revenue-maximizing tax rate u^*, the concavity of the tax revenue curve may not be preserved as one switches from a demand to a supply-based tax. For instance, for commodities with relatively large price elasticities, if the tax revenue curve on the revenue side is concave everywhere

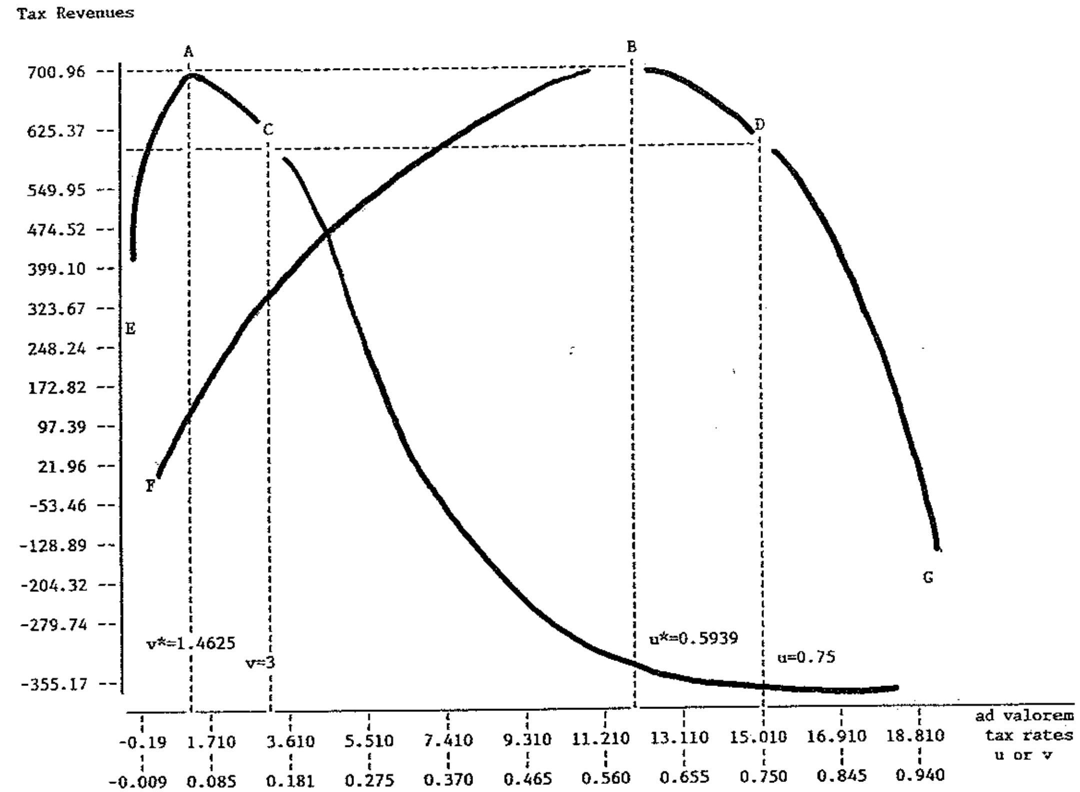

Figure 2. The Musgravian Transformation on the Laffer Curves

($0 < u < 1$), an increase of its tax rate after u^* will cause the tax revenue to decrease at increasing rates while it correspondingly increases in a supply-based tax rate may cause the tax revenue to decrease at a decreasing rate. This process can be illustrated by observing hypothetical tax rates from our simulation and associated results. One percentage point increase from $u = 0.83$ to $u = 0.84$ is equivalent to a 37 percentage point increase from $v = 4.88$ to $v = 5.25$ according to equation (4). The decrease in tax revenue of 39.1 units (338.849 − 299.749) can be spread out over 37 percentage points at *decreasing* rates on supply-based tax systems while the same amount of decrease in a demand-based tax revenue is made simply in 1 percentage point at *increasing* rates. The loss of concavity as one switches from a demand-based to a supply-based tax suggests that it may be politically more expedient to decrease a supply-based tax from $v = 0.59$ to $v = 0.54$ than from $u = 0.37$ to $u = 0.35$ of a demand-based tax to achieve a given amount of additional tax revenue.

CONCLUDING COMMENTARY

This note has demonstrated that revenue-maximizing tax rates generate identical maximum tax revenues under two ad valorem taxes via the Musgravian transformation. However, the property of concavity of the tax revenue curve is not preserved under such a transformation. Even if the tax revenue curve of the demand ad valorem tax is strictly concave everywhere, the tax revenue curve of the supply ad valorem tax is guaranteed to be concave only before its own revenue-maximizing tax rate. Hence, the property of concavity may be lost during the Musgravian transformation as one switches from a demand ad valorem to supply ad valorem tax. Other than this property, it appears that these two tax systems are mathematically equivalent, i.e., one is the re-numbering of the other in the case of a perfectly competitive market with zero transaction costs.

NOTES

1. The inverse relation f^{-1} is also a one to one function if f is a one to one function (Kasriel, 1971).
2. This may be seen from the following: Since these two tax systems yield the same price and quantity of equal yield [Musgrave, 1959], u* corresponds to v* for a given maximum tax revenue.
3. As pointed out by Professor Musgrave, by now most ad valorem taxes are imposed on price rather than cost payments. In addition, we do not consider the case of value added tax systems used in some European countries [Musgrave 1984], where the value added tax are of the supply side type.
4. In the case of monopoly, a tax on the gross receipts of the monopolist tends to be more efficient than a tax on total unit cost (Musgrave 1959, p. 309).

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