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CHOICE OF TECHNIQUE IN A PUTTY-CLAY MODEL OF PRODUCTION

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I. Introduction

Under the putty-clay hypothesis,[1] it is of interest to identify and estimate the ex ante production possibilities available to the firm from observed ex post data. Fuss [1], and Fuss and McFadden [2] have shown that the ex ante production technology can be identified through the choice of technique (or design) of the firm.[2] In Fuss [1], it is assumed that there is no uncertainty in prices. In Fuss and McFadden [2], the choice of technique is considered under uncertainty in prices. However, their framework assumes that the expected profit function of each technique is *linear* in the technique-specific parameters.

In this paper, we re-examine the choice of technique for a risk-neutral firm under uncertainty in prices. Under the assumption of expected cost minimization, the optimal technique can be derived by certainty equivalence at the means of the prices. However, under the assumption of expected profit maximization and gamma distributed prices, the optimal technique is the same as that of certainty eqivalence at the means *if and only if* the ex ante production function is of a specific Cobb-Douglas form. It is also shown that the expected profit function of each technique is not *linear* in the technique-specific parameters.

II. The Model

We assume that, other than capital, there are n variable inputs denoted by $x_1, \dots, x_n$. Capital is heterogeneous. The production technology embodied in each type of capital equipment is characterized by a fixed maximum output capacity,[3] and fixed input coefficients which are the quantities of variable inputs required per unit output. A given vector of input coefficients, denoted by $\mathbf{a} = (a_1, \dots, a_n)$, is referred to as a technique. We assume constant

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[1]For a discussion of the putty-clay hypothesis, the reader is referred to Fuss [1], or Fuss and McFadden [2].

[2]Fuss [1] and Fuss and McFadden [2] in fact show that the ex ante production technology can be identified in a more general putty- semi-putty model.

[3]We assume that it is feasible to produce *any* quanity of output up to the maximum output capacity.

capital cost per unit output capacity, i.e., capital cost per unit output capacity does not vary with output capacity *nor* across techniques.[4]

Under our assumptions, the ex ante production function exhibits constant returns and may be written as:

$$y = F(x_1, \ldots, x_n),$$

where y is the quantity of output and $F(x_1, \ldots, x_n)$ is linear homogeneous.[5]

A given technique $\mathbf{a}$ is simply a point on the unit isoquant of the ex ante production function. It is more convenient to represent $\mathbf{a}$ in the 'price' space by a change of variables: let $C(\mathbf{p}, y) \equiv h(\mathbf{p})y$ be the ex ante cost function dual to the ex ante linear homogeneous production function $F(x_1, \ldots, x_n)$, where $\mathbf{p} = (p_1, \ldots, p_n)$ is the price vector. $h(\mathbf{p})$ is then the ex ante unit cost function. Assuming $h(\mathbf{p})$ is twice continuously differentiable *and* that its Hessian matrix has rank of $n-1$, then for a given technique $\mathbf{a}$, there exists a price vector $\mathbf{q}$, unique up to a scalar multiple,[6] such that

$$\partial h(\mathbf{q})/\partial p_i \equiv h_i(\mathbf{q}) = a_i, \quad i = 1, 2, \ldots, n. \tag{1}$$

The vector $\mathbf{q}$ may be referred to as the *'planning'* price vector of the technique $\mathbf{a}$. It is clear that any given technique $\mathbf{a}$ can be equivalently represented by the corresponding planning price vector $\mathbf{q}$.

III. Expected Cost Minimization

The input prices are assumed to be nonnegative random variables with known distributions. The firm's problem is to find the type of capital equipment with the minimum expected cost of producing a given quantity of output. For a given technique $\mathbf{a}$ and input price vector $\mathbf{p}$, the unit cost of production is:

$$\sum_{i=1}^{n} p_i a_i.^7$$

The expected cost of production may be obtained as

$$\sum_{i=1}^{n} \mu_i a_i$$

where $\mu_i = E(p_i)$, $i = 1, 2, \ldots n$. Hence the firm's problem becomes

$$\min_{a_i s} \sum_{i=1}^{n} \mu_i a_i \quad \text{s.t. } F(\mathbf{a}) = 1 \tag{2}$$

which is simply the familiar cost minimization problem with input prices equal to their means with certainty. It follows that the optimal technique can be derived by certainty equivalence at the means.

[4]This assumption implies that there is no substitution between capital and other inputs.

[5]In fact the ex ante production function may be represented as $G(x_1, \ldots, x_n, K) = \min(F(x_1, \ldots, x_n), K)$ where K is the quantity of capital measured in terms of capital cost per unit output capacity at constant prices. Given our assumptions, $G(x_1, \ldots, x_n, K)$ is linear homogeneous. Hence $F(x_1, \ldots, x_n)$ is linear homogeneous.

[6]Since $h(\mathbf{p})$ is linear homogeneous, its gradient is homogeneous of degree zero and its Hessian matrix is of rank at most $(n-1)$. Hence the vector $\mathbf{q}$ in (1) is unique to a scalar multiple. The existence with rank equal to $(n-1)$ of the Hessian matrix of $h(\mathbf{p})$ implies and is implied by twice differentiability and strict convexity of the unit isoquant.

[7]Capital cost is not included since it is fixed and by assumption does not vary across techniques.

IV. Expected Profit Maximization

Prices are assumed to be independent,[8] nonnegative random variables with known distributions. The output price has an exponential distribution and input prices have gamma distributions.[9] The firm has a fixed capital budget and its problem is to find the type of capital equipment (subject to the capital budget) with the maximum expected profit. We ask whether the optimal technique can also be derived by certainty equivalence at the means.

For a given output price p_0, input price vector $\mathbf{p}$, and technique $\mathbf{a}$, profit is given by:

$$\prod(p_0, \mathbf{p}; \mathbf{a}) = \max\left\{0, p_0 - \sum_{i=1}^{n} p_i a_i\right\}. \tag{3}$$

The form of (3) reflects the fact that a profit-maximizing firm will simply not operate when profit is negative. The expected profit may be obtained by the following lemma:

Lemma. Let the distribution of p_0 be exponential with density function $\alpha_0 \exp(-\alpha_0 p_0)$, $\alpha_0 > 0$, and the distribution of p_i $(i = 1, 2, \ldots, n)$ be gamma with density function

$$\alpha_i^{n_i} p_i^{n_i - 1} \exp(-\alpha_i p_i)/\Gamma(n_i),$$

where $\Gamma(\cdot)$ is the gamma function, $\alpha_i > 0$, and $n_i > 0$; then the expected profit is given by

$$E\left(\prod(p_0, \mathbf{p}; \mathbf{a})\right) = \mu_0 / \prod_{i=1}^{n} (1 + \mu_i a_i / \mu_0 n_i)^{n_i}$$

where $\mu_i \equiv n_i/\alpha_i = E(p_i)$, $i = 1, 2, \ldots n$, and $\mu_0 \equiv E(p_0) = 1/\alpha_0$.[10]

In the terminology of Fuss and McFadden [2], the $a_i s$ $(i = 1, 2, \ldots, n)$ in the expected profit function $E(\prod(p_0, \mathbf{p}; \mathbf{a}))$ are the technique (design)-specific parameters. It is apparent that $E(\prod(p_0, \mathbf{p}; \mathbf{a}))$ is not linear in the $a_i s$.

The firm's problems is to find the technique with the maximum expected profit:

$$\max_{a_i s} \mu_0 / \prod_{i=1}^{n} (1 + \mu_i a_i / \mu_0 n_i)^{n_i} \tag{4}$$

$$\text{s.t. } F(\mathbf{a}) = 1$$

Using the fact that $\mathbf{a}$ can be represented by the corresponding planning price vector q, (4) may be rewritten as:

$$\max_{q_i s} \mu_0 / \prod_{i=1}^{n} (1 + \mu_i h_i(\mathbf{q}) / \mu_0 n_i)^{n_i} \tag{5}$$

over the region $\mathbf{q} > 0$. If for any given $\mu > 0$, $\mathbf{q} = \mu$ $(\mu = (\mu_1, \ldots, \mu_n))$ solves (5), the optimal technique can be derived by certainty equivalence at the means. We now state the principal result of our paper:

Theorem. Suppose the Hessian matrix of the unit cost function,

$$[\partial^2 h(\cdot)/\partial p_i \partial p_j \equiv h_{ij}(\cdot)], \quad i, j = 1, 2, \ldots, n,$$

[8]The assumption that prices are statistically independent of one another does not exclude the possibility that the means of the prices may be related to one another.

[9]The exponential distribution is a special case of gamma distribution. Hence all prices are gamma distributed.

[10]For a proof of the Lemma, see Ma [4].

has rank $n-1$, then for *any* given $\mu > 0$, $\mathbf{q} = \mu$ solves (5) if and only if

$$h(\mathbf{q}) = A \prod_{i=1}^{n} q_i^{n_i/\prod_{j=1}^{n} n_j} \tag{6}$$

where A is an arbitrary positive constant.

Proof: It is sufficient to show that for *any* given $\mu > 0$, the minimum of

$$\prod_{i=1}^{n} (1 + \mu_i h_i(\mathbf{q})/n_i\mu_0)^{n_i} \tag{7}$$

is attained at $\mathbf{q} = \mu$ if and only if $h(\mathbf{q})$ is of the form in (6).

The first order conditions for a minimum of (7) are:

$$\sum_{i=1}^{n} ((\mu_i/\mu_0)/(1 + (\mu_i/\mu_0 n_i)h_i(\mathbf{q})))h_{ij}(\mathbf{q}) = 0, \quad j = 1, 2, \ldots, n \tag{8}$$

If $\mathbf{q} = \mu$ attains the minimum, $\mathbf{q} = \mu$ or any scalar multiple of μ solves (8). Let $\mathbf{q} = \mu/\mu_0$, then (8) becomes:

$$\sum_{i=1}^{n} (q_i/(1 + (q_i/n_i)h_i(\mathbf{q})))h_{ij}(\mathbf{q}) = 0, \quad j = 1, 2, \ldots, n. \tag{9}$$

Linear homogeneity of $h(\cdot)$ implies that:

$$\sum_{i=1}^{n} q_i h_{ij}(\mathbf{q}) = 0, \quad j = 1, 2, \ldots, n. \tag{10}$$

But the Hessian matrix is of rank $(n-1)$, hence (9) and (10) together imply that:

$$q_i/(1 + (q_i/n_i)h_i(\mathbf{q})) = \lambda q_i, \quad i = 1, 2, \ldots, n. \tag{11}$$

where λ is a scalar, which simplifies to

$$\lambda(n_i + q_i h_i(\mathbf{q})) = n_i, \quad i = 1, 2, \ldots, n. \tag{12}$$

Summing (12) over i, and using the fact that $h(\cdot)$ is linear homogeneous, we obtain:

$$\lambda = \left(\sum_{i=1}^{n} n_i\right) \Big/ \left(h(\mathbf{q}) + \sum_{i=1}^{n} n_i\right). \tag{13}$$

Substituting (13) into (12) and simplifying, we obtain:

$$q_i h_i(\mathbf{q}) = n_i h(\mathbf{q}) / \sum_{j=1}^{n} n_j, \quad i = 1, 2, \ldots, n \tag{14}$$

which may be integrated to yield (6).

To prove the converse, we show that if $h(\mathbf{q})$ is of the form given in (6), then $\mathbf{q} = \mu$ minimizes (7), i.e.,

$$\prod_{i=1}^{n} \left(1 + \left(\mu_i/q_i \sum_j n_j\right) A \prod_{j=1}^{n} q_j^{n_j/\sum_k n_k}\right)^{n_i} \geq \prod_{i=1}^{n} \left(1 + \left(1/\sum_j n_j\right) A \prod_{j=1}^{n} \mu_j^{n_j/\sum_k n_k}\right)^{n_i}$$

for any $\mathbf{q} > 0$. Taking the $(\sum_j n_j)$th root of both sides and simplifying, we obtain:

$$\prod_{i=1}^{n} \left(1 + \left(\mu_i/q_i \sum_j n_j\right) \prod_{j=1}^{n} q_j^{n_j/\sum_k n_k}\right)^{n_i/\sum_j n_j} \geq 1 + \left(1/\sum_j n_j\right) \prod_j \mu_j^{n_j/\sum_k n_k} \tag{15}$$

By an inequality theorem in Hardy, Littlewood, and Polya [3] (p.21, theorem 10), we have, if a_is, b_is and c_is $(i = 1, 2, \ldots, n)$ are nonnegative numbers and

$$\sum_{i=1}^{n} c_i = 1$$

then the following inequality holds:

$$\prod_{i=1}^{n} (a_i + b_i)^{c_i} \geq \prod_{i=1}^{n} a_i^{c_i} + \prod_{i=1}^{n} b_i^{c_i} \tag{16}$$

Let $a_i = 1$, $b_i = (\mu_i/q_i \sum_j n_j) \prod_j q_j n_j / \sum_k n_k$ and $c_i = n_i/\sum_j n_j$, $i = 1, 2, \ldots, n$, then (15) can be seen to be implied by (16). Q.E.D.

The theorem says that for independent gamma distributed prices there is only one ex ante unit cost function of the Cobb-Douglas form (up to an arbitrary multiplicative constant A) and hence by duality is ex ante production function of the Cobb-Douglas form for which the optimal technique can be derived by certainty equivalence at the means under the assumption of expected profit maximization.

V. Concluding Remarks

We have shown that the optimal technique under expected profit maximization unlike that under expected cost minimization, can not in general be obtained by a certainty equivalence argument. Our argument in this paper depends on the assumption of independent gamma distributed prices. However it is possible, under appropriate regularity conditions, to show that under expected profit maximization, there is essentially only one ex ante production function that can be consistent with certainty equivalence at the means corresponding to any given distributions of the prices. The general inconsistency between expected profit maximization and certainty equivalence may be intuitively understood as resulting from the possibility of shutdown if (variable) profit becomes negative.

Empirically, our result implies that the specification of the optimal choice of technique by a certainty equivalence at the means argument in an econometric model cannot, in general, be consistent with expected profit maximization. A second approach, proposed by Fuss and McFadden [2], appears to require the assumption that the expected profit function is linear in the technique-specific parameters, which cannot be expected to hold in general. Perhaps a third approach, based on maximizing an explicit closed form expression for the expected profit function, such as one derived in this paper, subject to a production function, may be useful in generating an estimable econometric specification.

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