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Classical vs wavelet-based filters

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Abstract

In this article we compare the performance of Hodrick-Prescott and Baxter-King …lters with a method of …ltering based on the multi-resolution properties of wavelets. We show that overall the three methods remain comparable if the theoretical cyclical component is de…ned in the usual waveband, ranging between six and thirty two quarters . However the approach based on wavelets provides information about the business cycle, for example, its stability over time which the other two …lters do not provide. Based on Monte Carlo Simulation experiments, our method applied to the American GDP using growth rate data shows that the estimate of the business cycle component is richer in information than that deduced from the level of GDP and includes additional information about the post 1980 period of great moderation.

Keywords: Filters HP, BK, Wavelets, Monte Carlo Simulation , Break, Business Cycles.

JEL classi…cation: C15, C22, C65, E32

Résumé

Dans cet article nous comparons les performances des …ltres de Hodrick et Prescott et de Baxter et King avec une méthode de …ltrage basée sur les propriétés multi résolution des ondelettes. Nous montrons que globalement les trois méthodes restent comparables si la composante cyclique théorique est dé…nie dans la bande de fréquences usuelle comprise entre six et trente et deux trimestres. En revanche l'approche basée sur les ondelettes fournit des informations sur le cycle des a¤aires, par exemple sa stabilité dans le temps, que les deux autres …ltres ne permettent pas. Nos résultats s'appuient sur des expériences de Monte Carlo. Notre méthode appliquée au PIB américain montre aussi que l'estimation de la composante du cycle économique basée sur les données du taux de croissance est meilleur et plus riche d'informations que celle déduite du PIB en niveau et apporte des éléments d'information sur la période de "grande modération" de l'après 1980

Mots clés: Filtres HP, BK, Ondelettes, Simulation MonteCarlo, Rupture, Cycle économique.

JEL: C15, C22, C65, E32

1 Introduction

The use of …lters is very widespread in research on business cycles (or activity cycles) based on the analysis of macroeconomic series. The challenge of such …lters is to isolate the various components which characterize the data: the tendency and the cyclical component. Among the most used …lters one …nds those proposed by Hodrick and Prescott (noted afterwards HP) (1997) and Baxter and King (afterwards BK) (1995). These …lters are systematically integrated in current software packages. Several papers have evaluated the performance of these …lters in extracting the cyclical component correctly. Some authors think that the Hodrick and Prescott …lter can generate false cycles (Harvey and Jaeger, 1993 and Cogley and Nason, 1995). But these conclusions strongly depend on how the concept of the business cycle is formally de…ned. Based on the de…nition suggested by Burns and Mitchell (1946), Guay and St-Amant (2005) compared the e¢ciency of the two …lters. They concluded that the performance of the …lters depends on the characteristics of the spectral density of the theoretical cyclical component. However, the theoretical cyclical component is usually not known. If this spectral density is too concentrated on low frequencies close to zero, then the two …lters give a wrong view of reality. The …lters, in this case, badly isolate the component of the business cycle. But if the density is concentrated on the waveband of the business cycle as de…ned by Burns and Mitchell (1946) then the performance of the …lters is correct. Many economic macro series are characterized by spectral density around frequency zero (Granger 1966). The results of Guay and St-Amant (2005) highlighted the weaknesses of the HP and BK …lters on macroeconomic data.

In this paper we compare the two …lters HP and BK but this time with another …lter based on the wavelets theory. We exploit the multi-resolution properties of the wavelets to isolate the cyclical component (Yogo, 2008). We also show how the multi-resolution property can be a tool of great e¤ec-tiveness when we study the business cycle over time, for example, the importance of economic shocks. The wavelets also o¤er information that other approaches do not give. This study analyses performance using Monte Carlo simulations and considers the business cycle based on the American GDP.

This paper is organized in the following way. The second section brie‡y presents the various …lters. The third section presents simulations and comments on results. Then in the fourth section we discuss the advantages of the wavelets method compared to the other methods. A concrete application to the American GDP is proposed in Section 5 before concluding in the last section.

2 Three filters

2.1 The HP filter

The HP filter is designed to break down a time series Y_t into two components in an additive way: a cyclical component Y_t^C and a trend component Y_t^G, $Y_t = Y_t^C + Y_t^G$. The principle of the HP filter is to compromise between the regularity of the trend component and the minimization of the variance of the cyclical component. More precisely, the component Y_t^G is obtained by minimizing the variance of Y_t^C under the constraint of a penalty of the derived second of Y_t^G:

$$\{Y_t^G\}_{t=1}^T = \arg\min \sum_{t=1} \left[(Y_t^C)^2 + \lambda \left((Y_{t+1}^G - Y_t^G) - (Y_t^G - Y_{t-1}^G) \right)^2 \right] \qquad (1)$$

The parameter λ is a factor of penalty allowing to control the smoothing of Y_t^G. A high value of λ will give a linear trend and a fluctuating cyclical component and conversely. For quarterly observations Hodrick and Prescott recommend the value $\lambda = 1600$. King and Rebelo (1993) showed that the HP filter can make the integrated evolutionary processes stationary up to the order four. Singleton (1988) showed that the HP filter is a good approximation of a high-pass filter when it is applied to a stationary series. For better understanding, let us recall that any stationary time series is a linear combination of cyclical components of periods included in the interval $[-\pi, \pi]$. The conclusions of Singleton ensure that the HP filter , when it is applied to a stationary process, makes it possible to obtain the component of the business cycle by removing the low frequencies which is included in the studied series (the low frequencies characterize the trend Y_t^G).

2.2 The BK filter

The BK filter is an approximation of a pass band filter, i.e. letting pass frequencies between high and low frequency. If one refers to Burns and Mitchell (1946) the components of the business cycle are located in a waveband ranging between 6 and 32 quarters. This definition suggests removing the highest and lowest frequencies of the studied series. Therefore it is necessary to extract the component from the business cycle on a well defined waveband. It is this approach which was adopted by Baxter and King (1995). Applied to quarterly data, the BK filter extracts the cyclical component Y_t^C in the following way.

$$Y_t^C = \sum_{k=-12}^{k=12} a_k Y_{t-k} = a(L)Y_t \tag{2}$$

where L is the operator delay. The cyclical component then takes the form of a moving average over 24 quarters. The coefficients $\{a_k\}$ result from the problem of minimization according to:

$$\min_{a_j} Q = \int_{-\pi}^{\pi} |\beta(\omega) - \alpha(\omega)|^2 \text{ and } \alpha(0) = 0 \tag{3}$$

where $\alpha(\omega)$ is the Fourier transform of the $\{a_k\}$, i.e. $\alpha(\omega) = \sum_{k=-12}^{k=12} a_k e^{-i\omega k}$ and $|\beta(\omega)|$ the gain of the the ideal filter $|\beta(\omega)| = I(\omega \epsilon [\omega_1, \omega_2])$ with $I(.)$ characteristic function. The waveband $[\omega_1, \omega_2]$ delimits the components, whose periodicity lies between 6 and 32 quarters, ($\omega_1 = \pi/16$ and $\omega_2 = \pi/3$). The constraint $\alpha(0) = 0$ is used to isolate any tendency from Y_t^C.

2.3 The wavelets approach

The Fourier theory makes it possible to break up a function in a trigonometric base. In a similar way, the wavelet theory makes it possible to break up a time series $\{Y_t, t = 1, ..., n = 2^P\}$ in the form:

$$Y_t = \sum_{k=0}^{2^{j_0}-1} s_{j_0 k} \phi_{j_0 k}(n_t) + \sum_{j=1}^{j_0} \sum_{k=0}^{2^{P-j}-1} d_{jk} \psi_{jk}(n_t), \tag{4}$$

or in an equivalent way in the form

$$Y_t = S_{j_0} + \sum_{j=1}^{j_0} D_j \tag{5}$$

$n_t = t/n$, $S_{j_0} = \sum_{k=0}^{2^{j_0}-1} s_{j_0 k} \phi_{j_0 k}(n_t)$, $D_j = \sum_{k=0}^{2^{P-j}-1} d_{jk} \psi_{jk}(n_t)$ and $\beta = \left\{\phi_{j_0 k}(t), \psi_{jk}(t)\right\}$ is a wavelet basis. The elements $\phi_{j_0 k}(t)$ and $\psi_{jk}(t)$ of the base are built by initially choosing two functions ϕ and ψ then by using the following transformations: $\psi_{jk}(t) = 2^{j/2}\psi(2^j t - k)$ and $\phi_{j_0 k}(t) = 2^{j/2}\phi(2^j t - k)$. The function ϕ is called a scaling function and the function ψ is called a mother wave function. The parameter k is used to relocate the wavelets in the temporal scale. The parameter j is used as the parameter of dilation of the waves' functions. The parameter j adjusts the support of $\psi_{jk}(t)$ in order to locally capture the characteristics of high or low frequencies. In frequencies, D_j is an approximation of an ideal …lter of the series Y_t, in the waveband $[1/2^{j+1}, 1/2^j]$. Therefore the component D_j captures roughly the components of Y_t of periodicities ranging between 2^j and 2^{j+1}. The component S_{j_0} locates the components of periodicity higher than 2^{j_0+1}.It is an approximation of a low-pass …lter (frequencies lower than $1/2^{j_0+1}$)[1]. The representation (5) allows an analysis of the studied series Y_t known as multi-resolution. Indeed each component D_j highlights details (various resolutions) of Y_t localized in the waveband $[1/2^{j+1}, 1/2^j]$.

In this paper we use the de…nition of Burns and Mitchell (1946) by considering the business cycle in the waveband range between 6 quarters and 32 quarters. Therefore for quarterly observations we will estimate the components of the business cycle (Y_t^C) and the tendency (Y_t^G) by

$$\begin{aligned} Y_t^C &= D_4 + D_3 + D_2 \qquad (6) \\ Y_t^G &= S_{j_0} + D_{j_0} + D_{j_0 - 1} + ... + D_5 \end{aligned}$$

For quarterly observations, Y_t^C captures the components of periodicity ranging between 4 and 32 quarters. The band captured by Y_t^C is a little broader than the one corresponding to the business cycle described by Burns and Mitchell (1946), i.e. between 6 and 32 quarters[2]. The equations (6) show that $j_0 > 5$ is good enough if the sample size allows it. The choice of a high value for j_0 will make it possible to visualize more details on the trend component Y_t^G. It does not deteriorate the characteristics of the cyclical component Y_t^C. In short one can break down Y_t in the form

$$Y_t = Y_t^G + Y_t^C + \upsilon_t \qquad (7)$$

where $\upsilon_t = D_1$ is regarded as noise.

3 Simulation

In this section, we are initially interested in the e¤ectiveness of each method discribed above in extracting the business cycle. The approach adopted is based on simulations representing various scenarios. We

[1] See Crowley 2005 for more intuitive details.

[2] The results obtained by simulations when the component D2 is removed are de…nitely less powerful than if it is taken into account.

compare the performance of the three filters when the cyclical component dominates the tendency and when the cycle is concentrated in different frequencies.

3.1 Experimental design

We experiment with the three filters by considering the following data-generating process (noted DGP):

$$Y_t = Y^C + Y_t^G,\ t = 1, ..., T \tag{8}$$

where Y_t^C indicates the cyclical component and Y_t^G the tendency. To generate observations resulting from Y_t^C, we will use the properties of the processes $AR(2)$:

$$Y_t^C = \phi_1 Y_{t-1}^C + \phi_2 Y_{t-2}^C + \varepsilon_t \text{ where } \varepsilon_t \sim BB(0, \sigma_\varepsilon^2) \tag{9}$$

To make sure of the stationarity of Y_t^C we will suppose that $\phi_1 + \phi_2 < 1$ and $\phi_2 < 1$. The theoretical spectral density of such a process is given by

$$f(\omega) = \sigma_\varepsilon^2 \left[1 + \phi_1^2 + \phi_2^2 - 2\phi_1 (1 - \phi_2) \cos(\omega) - 2\phi_2 \cos(2\omega) \right]^{-1} \tag{10}$$

Peaks of $f(\omega)$ are localized at frequencies given by the equation

$$\omega = Arc \cos\left(-\phi_1 (1 - \phi_2) / 4\phi_2 \right) \tag{11}$$

Therefore one can choose the parameters ϕ_1 and ϕ_2 so that peaks of $f(\omega)$ are localized in the desired frequencies. In that case the periodicity of Y_t^C is of $2\pi/\omega$ units of time.

For the trend component Y_t^G we will take a stochastic tendency in accordance with many macroeconomic observations.

$$Y_t^G = Y_{t-1}^G + \eta_t \tag{12}$$

where $\eta_t \sim BB(0, \sigma_\eta^2)$. We choose ε_t and η_t independent. The conditional variance of Y_t, knowing the historical trajectories of Y_t, Y_t^C and Y_t^G, is $var_t(Y_t) = var_t(Y_t^C) + var_t(Y_t^G) = \sigma_\varepsilon^2 + \sigma_\eta^2$. In the composition of Y_t one can impose the predominance of the cycle Y_t^C on the tendency Y_t^G while choosing $var(\varepsilon_t) = \sigma_\varepsilon^2 > var(\eta_t) = \sigma_\eta^2$ (i.e. by imposing values $\sigma_\eta^2/\sigma_\varepsilon^2 < 1$). Conversely, more tendencies of the cycle will be obtained while choosing $\sigma_\eta^2/\sigma_\varepsilon^2 > 1$. They are the same DGP as those chosen by Guay and

St-Amant (2005), but we extend these comparisons to the wavelets. Moreover, their comparisons are somewhat skewed since they are based on theoretical calculations of autocorrelations of Y_t^C having some inconsistencies.

The experiment proceeds in the following way:

1. Initially we fix the parameters $\Omega = \{\phi_1, \phi_2, \sigma_\varepsilon^2, \sigma_\eta^2, T\}$ according to the characteristics of the cycle Y_t^C which we want to study. Example: long, medium or short period for the cycle Y^C, trend dominating or dominated, etc. Then we calculate theoretical autocorrelations of Y^C, noted $\rho(i)$ according to ϕ_1, ϕ_2 and σ_ε^2.

2. We generate the DGP (8), (9) and (12)[3]. Then we filter Y by using the HP, BK and wavelet filters. Therefore, one obtains for each method of filtering an estimate of the cyclical component, $\widehat{Y^C}$. For each of the three estimates $\widehat{Y^C}$, we then estimative the $\hat{\rho}(i)$ and the linear correlation with the true cyclical component $\left\langle Y^C, \widehat{Y^C} \right\rangle = corr(Y_t^C, \widehat{Y^C})$.

3. We repeat stage 2 a number of times equal to M, which enables us to obtain, for different i and with each of the three methods of filtering, the following quantities which will be used as indices of measurement of the filters' qualities:

$$\rho^*(i) = M^{-1} \sum_{j=1}^{M} \hat{\rho}_j(i) \tag{13}$$

$$d(\rho, \hat{\rho})^* = M^{-1} \sum_{j=1}^{M} \left[\sum_{i=1}^{12} \left(\rho(i) - \hat{\rho}_j(i) \right)^2 \right] \tag{14}$$

$$\left\langle Y^C, \widehat{Y^C} \right\rangle^* = M^{-1} \sum_{j=1}^{M} corr(Y_t^C, \widehat{Y^C})_j \tag{15}$$

where $\hat{\rho}_j(i)$ and $\sum_{j=1}^{M} corr(Y_t^C, \widehat{Y^C})_j$ represent the same quantities as in step 2, calculated at the j repetition, $j = 1, ..., M$.

The $\rho^*(i)$ (equation 13) allows us to make a specific comparison with the theoretical correlations of Y_t^C. The index $d(\rho, \hat{\rho})^*$ provides a total comparison between the dynamics of Y_t^C and of $\widehat{Y^C}$ starting from the first twelve correlations. We will appreciate the good quality of the studied filter by the proximity between $\rho^*(i)$ and $\rho(i)$ measured by the proximity of $d(\rho, \hat{\rho})^*$ to zero. In addition, two processes can

[3] Naturally the DGP (8) is reproduced after the DGP (9) and (12).

have the comparable dynamic while being independent. This is why we also use the index of correlation between theoretical and empirical cycles (equation 15) to con…rm the results obtained by (13) and (14).

3.2 Analysis

Tables 1,2 and 3, respectively show the results from simulations for the HP, BK and wavelets …lters.

Table 1 Simulation results for HP …lter

$\sigma_\eta/\sigma_\varepsilon$	ϕ_1	ϕ_2	ρ_1^*	ρ_2^*	ρ_3^*	ρ_4^*	ρ_5^*	$\left\langle Y^C, \widehat{Y^C} \right\rangle^*$	$d(\rho,\widehat{\rho})^*$	Period
										¼ years
5	1.2	-0.4800	0.71	0.45	0.24	0.08	-0.04	0.236	0.030	3
			0.81	049	0.21	0.01	-0.09			
5	1.2	-0.3726	0.71	0.46	0.25	0.09	-0.04	0.215	0.068	8
			0.87	0.68	0.49	0.33	0.22			
5	1.2	-0.3662	-0.53	0.48	0.17	0.13	-0.05	0.33	0.246	12
			-0.88	0.69	-0.48	0.35	-024			
1	1.2	-0.4800	0.73	0.38	0.08	-0.12	-0.22	0.737	0.025	3
			0.81	049	0.21	0.01	-0.09			
1	1.2	-0.3726	0.75	0.54	0.19	-0.01	-0.14	0.681	0.087	8
			0.87	0.68	0.49	0.33	0.22			
1	1.2	-0.3662	-0.48	0.62	-0.31	0.28	-0.18	0.87	0.028	12
			-0.88	0.69	-0.48	0.35	-024			
0.2	1.2	-0.4800	0.74	0.34	-0.02	-0.25	-0.34	0.916	0.036	3
			0.81	049	0.21	0.01	-0.09			
0.2	1.2	-0.3726	0.78	0.45	0.16	-0.06	-0.20	0.85	0.104	8
			0.87	0.68	0.49	0.33	0.22			
0.2	1.2	-0.3662	-0.86	0.67	-0.48	0.33	-0.22	0.99	0.011	12
			-0.88	0.69	-0.48	0.35	-024			

Table 2 Simulation results for BK …lter

$\sigma_\eta/\sigma_\varepsilon$	ϕ_1	ϕ_2	ρ_1^*	ρ_2^*	ρ_3^*	ρ_4^*	ρ_5^*	$\left\langle Y^C, HP^C \right\rangle^*$	$d(\rho,\widehat{\rho})^*$	Period
										¼ years
5	1.2	-0.4800	0.84	0.50	0.21	0.04	-0.06	0.24	0.03	3
			0.81	049	0.21	0.01	-0.09			
5	1.2	-0.3726	0.84	0.51	0.21	0.05	-0.06	0.22	0.08	8
			0.87	0.68	0.49	0.33	0.22			
5	1.2	-0.3662	0.84	0.51	0.22	0.07	-0.04	0.01	0.35	12
			-0.88	0.69	-0.48	0.35	-024			
1	1.2	-0.4800	0.82	0.41	0.04	-0.17	-0.26	0.74	0.03	3
			0.81	049	0.21	0.01	-0.09			
1	1.2	-0.3726	0.84	0.49	0.16	-0.05	-0.17	0.68	0.10	8
			0.87	0.68	0.49	0.33	0.22			
1	1.2	-0.3662	0.81	0.45	0.17	0.06	-0.02	0.06	0.3	12
			-0.88	0.69	-0.48	0.35	-024			
0.2	1.2	-0.4800	0.80	0.36	-0.05	-0.28	-0.37	0.90	0.04	3
			0.81	049	0.21	0.01	-0.09			
0.2	1.2	-0.3726	0.84	0.48	0.12	-0.10	-0.24	0.84	0.12	8
			0.87	0.68	0.49	0.33	0.22			
0.2	1.2	-0.3662	0.59	-0.06	-0.23	0.02	0.11	0.16	0.27	12
			-0.88	0.69	-0.48	0.35	-024			

Table 3 Simulation results for levels D2, D4, D4

$\sigma_\eta/\sigma_\varepsilon$	ϕ_1	ϕ_2	ρ_1^*	ρ_2^*	ρ_3^*	ρ_4^*	ρ_5^*	$\left\langle Y^C, HP^C \right\rangle^*$	$d(\rho,\widehat{\rho})^*$	Period
										¼years
5	1.2	-0.4800	0.86	0.58	035	0.17	0.03	0.18	0.05	3
			0.81	049	0.21	0.01	-0.09			
5	1.2	-0.3726	0.86	0.58	0.35	0.18	0.03	0.16	0.06	8
			0.87	0.68	0.49	0.33	0.22			
5	1.2	-0.3662	0.86	0.58	0.36	0.19	0.04	0.01	0.37	12
			-0.88	0.69	-0.48	0.35	-024			
1	1.2	-0.4800	0.83	0.49	0.18	-0.03	-0.16	0.62	0.03	3
			0.81	049	0.21	0.01	-0.09			
1	1.2	-0.3726	0.85	0.55	0.28	0.07	-0.08	0.56	0.08	8
			0.87	0.68	0.49	0.33	0.22			
1	1.2	-0.3662	0.83	0.53	0.32	0.18	0.05	0.04	0.36	12
			-0.88	0.69	-0.48	0.35	-024			
0.2	1.2	-0.4800	0.80	0.40	0.03	-0.23	-0.34	0.85	0.04	3
			0.81	049	0.21	0.01	-0.09			
0.2	1.2	-0.3726	0.84	0.52	0.20	-0.04	-0.20	0.75	0.11	8
			0.87	0.68	0.49	0.33	0.22			
0.2	1.2	-0.3662	0.57	0.01	-0.04	0.11	0.06	0.13	0.26	12
			-0.88	0.69	-0.48	0.35	-024			

Let us raise the following points:

1. For each fixed value taken by (ϕ_1, ϕ_2), one can note that the values taken by $\left\langle Y^C, \widehat{Y}^C \right\rangle^*$ increase as $\sigma_\eta/\sigma_\varepsilon$ drop in each table. For example for $(\phi_1, \phi_2) = (1.2, -0.48)$ in table 1, we have $\left\langle Y^C, \widehat{Y}^C \right\rangle^* =$ 0.236, 0.737, 0.916 respectively for $\sigma_\eta/\sigma_\varepsilon = 5$, 1, 0.2. Concretely that means that the correlation between the estimated cycle and the theoretical cycle is stronger when the true cyclical component has more influence compared to the tendency.

2. Whatever the value taken by $\sigma_\eta/\sigma_\varepsilon$, one can note that the values of $d(\rho, \widehat{\rho})^*$ are relatively close to zero (especially for the 12 quarters period) in the three tables. For the BK and wavelet filters it is when the cycle is 12 quarters that correlation $\left\langle Y^C, \widehat{Y}^C \right\rangle^*$ is stronger. This result means that all the filters behave well when the period of the theoretical cyclical component is included within the periodic interval of the business cycle (i.e., between 6 quarters and 32 quarters).

3. For the BK and wavelet filters, whatever the value of $\sigma_\eta/\sigma_\varepsilon$, the worst values of $d(\rho, \widehat{\rho})^*$ and $\left\langle Y^C, \widehat{Y}^C \right\rangle^*$ are noted when the period of the theoretical cycle is equal to 48 quarters (values of $d(\rho, \widehat{\rho})$ are relatively high, whereas values $\left\langle Y^C, \widehat{Y}^C \right\rangle^*$ are relatively weak). This result shows that the two filters, BK and wavelet, isolate the periodicity component at 32 quarters less than one could expect. This observation is not always true for the HP filter. Indeed when $\sigma_\eta/\sigma_\varepsilon \leq 1$, values of $d(\rho, \widehat{\rho})^*$ are relatively weak and

those of $\left\langle Y^C,\widehat{Y}^C\right\rangle$ are strong when the theoretical cycle is 48 quarters. In other words, the HP filter which is regarded as a high pass filter can let pass the components from 48 quarters especially when such components are influential. This result goes in the direction of that obtained by Singleton (1988) showing that the HP acts on stationary data (the case when there is no tendency, i.e $\sigma_\eta/\sigma_\varepsilon \leq 1$) as a good high pass filter.

4. "Typical spectral shape". In an article, Granger (1966) affirms that many economic time series measured by level were characterized by a strong concentration of their spectral density near zero and are also accompanied by a continuous decrease towards the horizontal axis . One can obtain the phenomenon "Typical spectral shape" when the trend component dominates the series using a cyclical component of low frequency. For example, spectral density concentrations of the DGP obtained while choosing $\sigma_\eta/\sigma_\varepsilon$ = 5 and a cyclical component of very low frequency, i.e. long periods such as 48 quarters, well illustrate this phenomenon described by Granger. The results of the three tables show that the extraction of the cyclical component in the case of "Typical spectral shape" is overall poor. The best indices are obtained with the HP filter with $d(\rho,\mathbf{b})$ = 0.246 and $\left\langle Y^C,\widehat{Y}^C\right\rangle$ = 0.33. For the BK and wavelet filter this result is not surprising since these filters must theoretically authorize components of periods maximum of 32 quarters.

5. When the theoretical cyclical component is defined around the low frequencies (for example, a 48-quarter period), we can note that the results of the three filters are better if there is less of a tendency ($\sigma_\eta/\sigma_\varepsilon \leq 1$) than more of it. As an example, here are the results for the BK filter $\left\langle Y^C,\widehat{Y}^C\right\rangle$ = 0.01 if $\sigma_\eta/\sigma_\varepsilon$ = 5 (strong tendency) and $\left\langle Y^C,\widehat{Y}^C\right\rangle$ = 0.16 if $\sigma_\eta/\sigma_\varepsilon$ = 0.2 (no tendency). The observation remains valid for the other two filters. This is why, in our study of the stability of the business cycle (Section 5) we will take the growth rate of the GDP (without tendency) instead of the GDP level (strong tendency).

4 Advantages of the multi-resolution approach

Although the performance of the approach based on the wavelets remains overall comparable with that of the two filters HP and BK, (especially if the cycle is well defined inside the usual waveband), filtering by wavelets allows us to perform the temporal and frequency analyses of the cyclical component at the same time. More precisely we can supply brief replies to the following questions:

(a) Between two periods $t = t_{k-1}, ..., t_k$ and $t = t_{j-1}, ..., t_j$ which are the frequencies which remain or become influential in the cyclical component? The assumption of only one dominant frequency in the cyclical component of a macroeconomic data series observed over a very long period remains improbable. For example the literature on "The great moderation" suggests significant modification of the cyclic characteristics after the 1980's. An examination of the variance of Y_t^C in each sub-period using the formula $var(Y_t^C) \simeq var(D_2) + var(D_3) + var(D_4)$[4] can allows us to identify the dominant components among $D_i, i = 2, 3, 4$.

(b) The traditional tests for regime changes in the cyclical component Y_t^C do not make it possible to appreciate the extent of the shock which triggered the change. Is it a shock modifying the long run, medium term or short term characteristics of the series? Instead of applying the regime change tests directly to the cyclical component $Y_t^C = D_2 + D_3 + D_4$, one can apply them to each component D_2, D_3 and D_4. It is then obvious that a change occurring in D_4, for example, has consequences on the low frequencies of Y_t^C and therefore affects the long run. We use the method suggested by Bai and Perron (2003) with D_i^2. Let us use the following regression with multiple breaks:

$$D_{it}^2 = \sigma_{ik} + v_t \tag{16}$$

where $\sigma_{ik} = E(D_{it}^2) = var(D_i)$ and $t = t_{ik-1}, ..., t_{ik}$. These $\{t_{ik}, k = 1, ..., m\}$ give the breaking dates. The model (16) is called a pure structural breaks model by the authors analyzing only breaks on the mean level of the endogenous variable D_{it}^2. Bai and Perron give an effective algorithm of the minimization of the sums of residual squares which includes a determination of the number of breaks and offers tests under general noise v_t conditions.

[4] The breakdown comes from the orthogonality of the wavelet basis. There is perfect equality if the sample is to a power of two.

5 Application to the American GDP

We now apply the three methods to the American GDP, with quarterly data, between 1958-Q1 and 2008-Q4 . We have 204 data. In order to use the discrete wavelet transform based on a size of sample equal to a power of 2, here 256 points, we used the simplest techniques to supplement the sample, as recommended by Percival and Walden (2000). In the case of the GDP growth we extend the series downstream and upstream with similar data. For the level of GDP this is done using the growth rate of the four extreme years and generating the data using a deterministic series with the same growth rate. Of course, once the decomposition in wavelets has been carried out we truncate the results to return to the initial sample's size and dates.

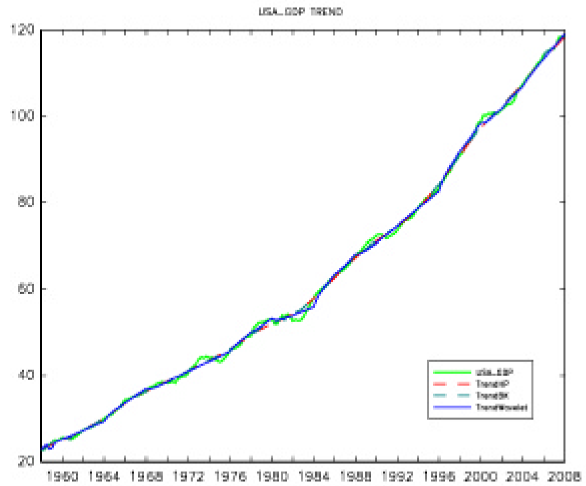

Figure 1 GDP Trends

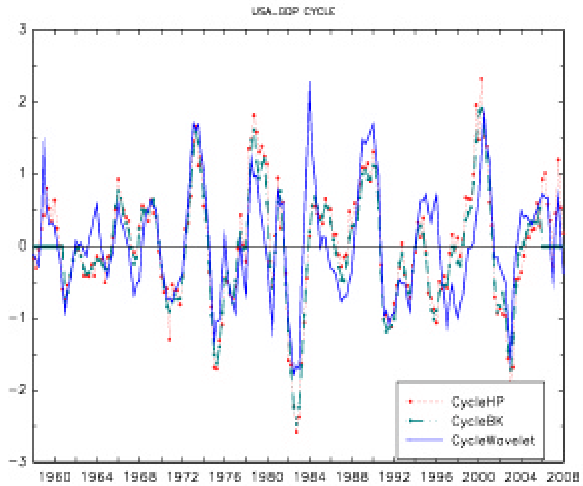

Figure 2 GDP Cycles

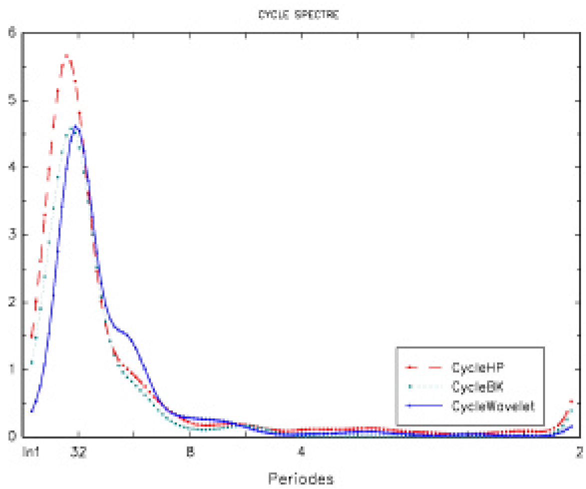

Figure 3 GDP: Three spectrums

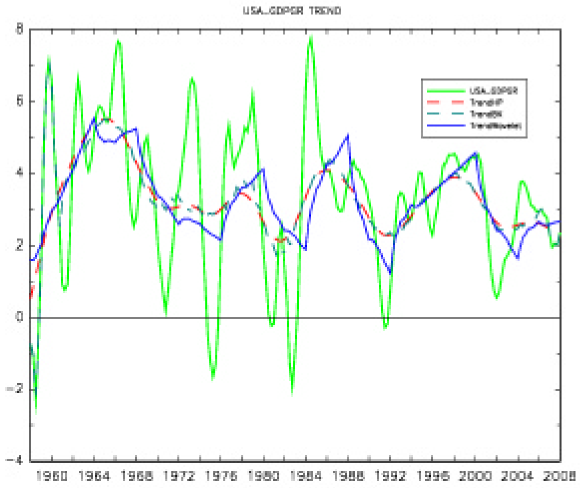

Figure 4 GDP growth Trends

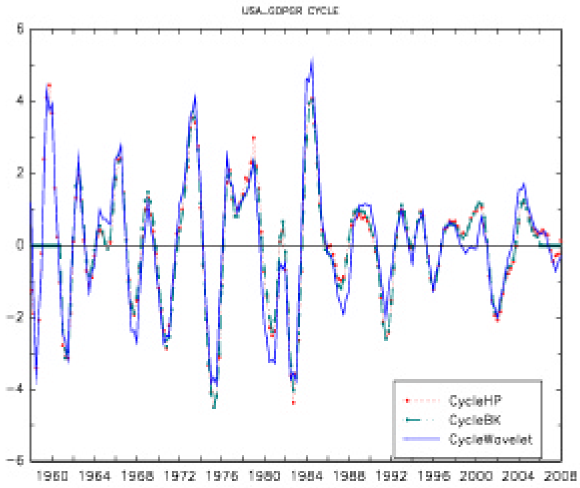

Figure 5 GDP growth Cycles

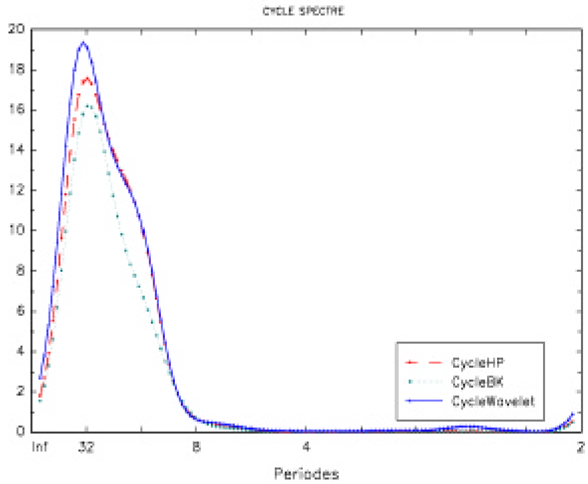

Figure 6 GDP growth: Three spectrums

The GDP (Figure 1) is characterized by the predominance of the tendency, whereas for the growth rate (figure 4) the tendency is less marked compared to the cyclical component. Therefore these two types of data enable us to take into account the various types of Data Generating Process used in our simulations. In the case of the GDP, estimates of the tendencies are presented in Figure 1, the cycles in Figure 2 and the spectral concentrations of the cyclical components in Figure 3. For the GDP growth rate, the corresponding results are represented in Figures 4, 5 and 6. In both cases, the spectral concentrations of frequencies correspond to periods slightly higher than 32 quarters. The cyclical components resulting from the three approaches are close, for the growth rate and for the GDP. This result conforms to the analyses made Section 3.2.5.

To show the wavelet contribution compared to the other two methods, we are interested in the stability of the business cycle during the period 1958-2008. We start with the growth rate. The choice of the GDP growth rate rather than the level of the GDP is explained in Section 3.2.5. In order to examine the stability of the cyclical component we apply the method of Bai and Perron (2003) to D2, D3 and D4, i.e. the components of the cycle estimated using the wavelets. The results are presented in Table 4 and are illustrated by Figure 7.

Table 4 Break dates and $\mathbf{b}_{ik}$ in sub periods

D_2		1960 : 4		1980 : 3		1983 : 1	
$\hat{\sigma}_{2k}$	1.17(0.11)		0.17(0.042)		1.32(0.12)		0.10(0.036)
D_3		1961 : 3		1973 : 1		1975 : 4	
$\hat{\sigma}_{3k}$	6.06(0.42)		1.33(0.24)		5.03(0.49)		0.44(0.14)
D_4		1971 : 4		1979 : 3		1985 : 1	
$\hat{\sigma}_{4k}$	0.63(0.18)		4.1(0.25)		2.33(0.29)		0.64(0.14)

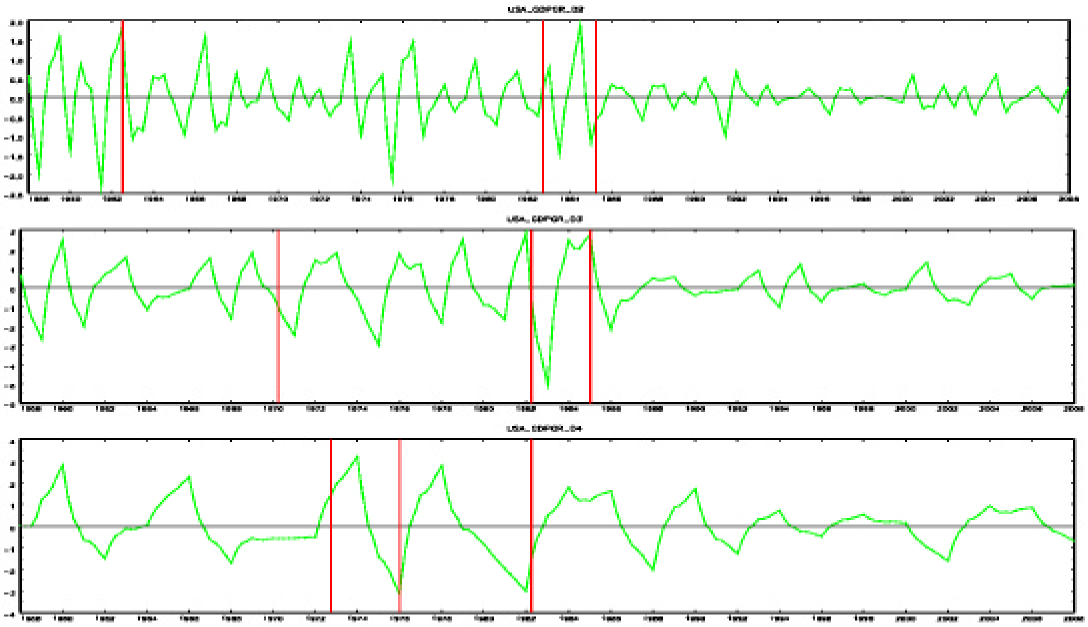

Figure 7 GDP growth: breaks in D2, D3, D4

We note the existence of several shocks of di¤erent scales.

The shocks which occurred at the beginning of the 1960s primarily a¤ect D2 (1960-Q4) and D3 (1961-Q3), which correspond to the components of periods between 4 quarters and 8 quarters (D2) and those ranging between 8 quarters and 16 quarters (D3). The beginning of the 1960s was an unstable period in the history of the US economy. Indeed, after the disappearance of Kennedy (elected in 1960 and assassineted in 1963), the administration of President Johnson mainly continued the Keynesian policies of its predecessor. During the …rst half of the 1960s, these policies were successful, with productivity growth and well controlled in‡ation (see, for example the analysis of Ahamada and Ben Aissa, 2005). It is necessary to wait until 1966, due to strong constraints on resources caused partly by the war in Vietnam, to notice the …rst alarm signals through the increase of interest rates and in‡ation. Table 4 shows that the economic policies applied by President Kennedy at the beginning of the 1960s and continued by President Johnson had wide e¤ects on nearly 16 quarters (shock on the D3 component in 1961-Q3).

The shocks of the 1970s have relatively greater widths than those of the 1960s, since these shocks a¤ect not only component D3 (1973-Q1 and 1975-Q4) but also components of long periods captured in D4 (1971-Q4 and 1979-Q4), i.e., cyclical components of periodicity ranging between 16 quarters and 32 quarters. The consequences of the shocks which occurred around 1971-Q4 and 1979-Q4 extended over relatively long periods. In August 1971 President Nixon announced very important measures intended to control prices and wages. Some authors such as Blinder (1979) and Brown (1985) would link these decisions to electoral strategy as Nixon for re-election in November 1972.

The shocks of the years 1973, 1975 and 1979 can be connected to the oil crises. The …rst oil crisis in 1973 was due to the consequences of the Yom Kippur war. In 1979 the Iranian revolution disturbed the oil transfers from the Arabic-Persian Gulf to the West (second oil crisis). From 1979, with the Volckler administration taking control of the Federal Bank, there started a policy of long term in‡ation reduction. In‡ation which was 12.8% in 1979, fell to 12.5% in 1980, 9.6% in 1981 and 4.5% in 1982.

The results of Table 4 show shocks at the beginning of the 1980s in the level D2 (1980-Q3 and 1983-Q1). These dates correspond to the heavy tax cuts proposed by President Reagan, elected in November 1980.

Amongst the most important results presented in Table 4, the period of 1985:1 is the most outstanding. The middle of the 1980s has been comprehensively dealt with the literature. It is commonly indicated as a period of great moderation when major macroeconomic variables of the G7 countries such as GDP, industrial production, unemployment rate etc. showed an large decrease in their volatilities. In the American case, it is undoubtedly Kim and Nelson (1999) and McConnell and Al (2000) who have identi…ed this phenomenon as the period around 1984. Stock and Waston (2003) showed that the growth rate variances in the G7 countries had fallen in considerable proportions from 50% to 80% starting from the middle of the 198's. The authors also concluded that the shocks to the GDP had become very persistent from this period.

The width of this phenomenon led them to the following question: was the business cycle modi…ed? The date of 1985:1 retained in Table 4 con…rms the results of these authors in a …ner way. Variances of D2, D3 and D4 fall respectively to 0.1,0.44 and 0.64 after 1985. This fall is accompanied by a modi…cation of the composition of the variance in the growth rate. For example, between 1980 and 1983, it is the D3 component which was in‡uential. Then after 1985:1, it is the long run component (D4) which becomes dominant. This result consolidates the idea of a modi…cation of the business cycle characteristics supported by Stock and Waston (2003). The predominance of D4, from the middle of the 1980s means that the cyclic periods became extended (between 16 and 32 quarters) and that the shocks became more severe.

6 Conclusion

In this paper we compared the traditional …lters of Hodrick - Prescott and Baxter-King with an approach to …ltering based on the multi-resolution properties of wavelets. We showed that the performance of the three …lters was comparable overall. Simulations showed that cycle estimations based on the growth rate re‡ected the business cycle better than cycle estimations based on level of the GDP. Through an example based on the American GDP, we showed that the …ltering based on wavelets is more powerful, allowing additional analyses: cycle's properties over time and a description of the changes in the business cycle over the last few years.

References

[1] Ahamada I. et Ben Aissa M.S.(2005). "Changements Structurels dans la Dynamique de l'In‡ation aux Etats-Unis, deux approches non paramétriques". Annales d'économie et Statistique, 77, 187-199.

[2] Bai,J. and Perron,P. (2003). "Computation and analysis of multiple structural change models". Journal of Applied Econometrics, 18, 1, 1-22.

[3] Baxter M. and King R.G. (1995). "Measuring Business-cycles: Approximate Band-Pass Filters for Economic Time Series". Working Paper No. 5022. National Bureau of Economic Research.

[4] Blinder A.S.(1979). "Economic Policy and the Great Stag‡ation ", Academic Press, London.

[5] Brown A.J. and Darby J. (1985). "World In‡ation Since 1950: An international Comparative Study ", Cambridge U.P. NIESR.

[6] Burns A.M. and Mitchell W.C. (1946). "Measuring Business-Cycles". New York: National Bureau of Economic Research.

[7] Cogley T. and Nason J. (1995). "E¤ects of the Hodrick-Prescott Filter on Trend and Di¤erence Stationary Time Series: Implications for Business-Cycle Research". Journal of Economic Dynamics and Control, 19: 253-78.

[8] Crowley P(2005). "An intuitive guide to wavelets for economists". Bank of Finland Research Discussion Papers.

[9] Guay A, St-Amant P.(2005). "Do the Hodrick-Prescott and Baxter-King Filters Provide a Good Approximation of Business Cycles". Annales d'Economie et de Statistique, n±77.

[10] Granger C.W.J. (1966)."The typical Spectral Shape of an Economic Variable". Econometrica, 37: 424-38.

[11] Hodrick R.J. and Prescotte E. (1997). "Post-war U.S. Business Cycles: An Empirical Investigation. Journal of Money". Credit, and Banking, 29(1): 1-16.

[12] Harvey A.C. and Jaeger A. (1993). "Detrending, Stylized Facts and the Business-Cycle". Journal of Applied Econometrics, 8, 231-47.

[13] Kim C-J and Nelson R.N. (1999). "Has the U.S. Economy Become More Stable? A Bayesian Approach Based on a Markov-Switching Model of the Business Cycle", The Review of Economics and Statistics 81, 608 – 616.

[14] King R.G. and Rebelo, S. (1993). "Low Frequency Filtering and Real Business-Cycles". Journal of Economic Dynamics and Control, 17: 207-31.

[15] McConnell, Margaret M, and Perez-Quiros G.(2000). “Output Fluctuations in the United States: What has Changed Since the Early 1980’s,” American Economic Review 90 (5), 1464 – 1476.

[16] Priestley M. (1981). Spectral Analysis and Time Series. Academic Press.

[17] Percival, D. Walden, A (2000). Wavelet Methods for Time Series Analysis. Cambridge University Press, UK.

[18] Ramsey, J. (2002). "Wavelets in Economics and Finance: Past and Future". Studies in Nonlinear Dynamics & Econometrics, 6, 3, 1-27

[19] Schleicher C(2002). "An introduction to Wavelets for Economists". Bank of Canada,Working Paper 2002-3.

[20] Singleton K. (1988). "Econometric Issues in the Analysis of Equilibrium Business-Cycle Models".Journal of Monetary Economics, 21: 361-86.

[21] Stock J.U. and Watson M.W. (2003). " Has the Business Cycle Changed? Evidence and Explanations ". Working paper, NBER.

[22] Yogo M.(2008)."Measuring Business Cycles: a wavelet Analysis of Economic Time Series". Economics Letters, 6, 2.