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MAXIMUM LIKELIHOOD EQUILIBRIA OF RANDOM GAMES

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In this paper we introduce the concept of random game in order to incorporate the possible random structure of a game in an explicit way. Two definitions of maximum likelihood Nash equilibrium (MLNE) are given depending on the random structure of the game. Existence theorems of MLNE are proved in both setups.

KEY WORDS Game theory, Nash equilibrium, random game.

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1. INTRODUCTION

In many practical situations, a group of agents have to take strategical decisions in an environment of risk. The traditional approach of game theory to this problem is to embody such a risk (and the attitude of the players in front of it) in the utility functions of the players and, then, to solve the problem as a deterministic one (obtaining, for instance, the Nash equilibria–see 1 for the details about the Nash equilibrium concept–). Although this is the right approach in a number of occasions, sometimes it is also interesting to address the situation in a way which pays more attention to its random structure and to explicitily incorporate such a structure in the proposed solution (in Section 3 we develope a significant example that motivates the introduction of the new concept).

The approach we are adopting here is the following: it is conceivable that, in some conflictive situations, players are willing to play a combination of strategies if it is an equilibrium with a (fixed) considerably high probability (note that, in such a case, they only have an incentive to deviate with a negligible probability). Hence, it is interesting to study these *random games* from a probabilistic point of view. This study is only initiated in the present work. In fact, we mostly concentrate on a remarkable issue for random games: the existence of *maximum likelihood Nash*

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equilibria (combinations of strategies which are the most likely ones to be Nash equilibria looking at the random structure of the situation).

There are several reasons to consider this as an interesting problem. First, because the existence of an MLNE in a certain random game implies that players can measure the probability of the least probable piece of the sample space that they have to drop for the MLNE to be a standard Nash equilibrium (a Nash equilibrium of the expected payoff game). This is a useful information for the players because, in view of it, they may jointly decide to take the risk and agree on the MLNE. In other cases the maximum risk that players jointly accept to take can be fixed *a priori*. Even in this case the measure mentioned above is still interesting because it reveals if the MLNE is acceptable or not for the players (in this case, perhaps other combinations of strategies are acceptable, if the risk associated to them is small enough; however an MLNE seems to be an optimal suggestion for the players).

Another reason why studying the existence of MLNE in random games is an interesting issue is the following. An important problem in non-cooperative game theory is that of defining selfenforcing combinations of strategies (combinations of strategies from which the players do not have incentives to unilateraly deviate). One of the first approaches to this problem is 1. In this paper, the Nash equilibrium concept is proposed to select selfenforcing combinations of strategies in normal form games. However, in 3, Selten showed that not all Nash equilibria are really selfenforcing. For instance, in the bimatrix game

	B	b
A	(1,1)	(0,0)
a	(0,0)	(0,0)

(a, b) is a Nash equilibrium which is not selfenforcing because player 1 knows that choosing A instead of a is never bad for him and could be good if player 2 also deviates and thus, he will deviate; an analogous reasonement can be done by player 2, who will play B instead of b (in fact the only selfenforcing Nash equilibrium of this game is (A, B)). Hence, the Nash condition seems to be not sufficient in order to select rest-points in a non-cooperative game in a sensible way. Many possible refinements of the Nash equilibrium concept have been proposed in the last years (4 contains a good survey on the topic) because this is an important (and still not exhausted) problem in game theory. Now, our concept of MLNE can be useful to refine the Nash equilibrium concept. Namely, in the example above, if we consider a sequence of random games (each of them being slightly payoff-perturbed versions of the original one) which converges to the initial bimatrix game, the only combination of strategies which is a limit point of MLNE of games in the sequence is (A, B). This suggest that, in a non-cooperative game, we can select (as more selfenforcing) those Nash equilibria which are limit of a sequence of MLNE of a corresponding sequence of random games converging to the original game. These ideas are first stated and analyzed in 5 using very simple classes of random games. However, the development of the study of MLNE in more general classes of random games (as we do in this paper) would be very helpful for further applications of this probabilistic approach to the theory of refinements.

2. RANDOM GAMES AND MAXIMUM LIKELIHOOD EQUILIBRIA

In this section we present and study a solution concept for random games. We begin introducing our model.

Definition 1: *A random game is a three-tuple*

$$\langle(\Omega, \mathscr{A}, P), X, \rangle$$

where:

1. $(\Omega, \mathscr{A}, P)$ *is a probability space,*
2. X *is the set of combination of strategies of the n players. It has the form*

$$X = \prod_{i=1}^{n} X_i$$

where each X_i*, the set of strategies of player i, is a separable topological space (i.e., each* X_i *is a topoligical space containing a countable subset* S_i *which verifies that* $\bar{S}_i = X_i$*) and*

3. H *is the payoff function given by:*

$$H: X \times \Omega \to R^n$$

$$(x, \omega) \to H(x, \omega) := (H_1(x, \omega), H_2(x, \omega), \ldots H_n(x, \omega))$$

where, for every $i \in \{1, \ldots, n\}$, $x \in X$ *and* $\omega \in \Omega$, $H_i(x, \omega)$ *is the payoff for the i-th player if x is played and the state of nature is* ω. *We suppose that H is measurable as a function of* ω*, for all x, and lower semicontinuous as a function of x, for all* ω*, (i.e., the sets* $\{x / H_i(x, \omega) > r\}$ *are open in* R^n*, for every* $r \in R$ *and all* ω*).*

Observe that the model described above is quite general. The condition of separability for the X_i is only a technical one. The properties of measurability and semicontinuity of H are not very restrictive but necessary if we want the model to be reasonably handy. On the other hand, observe that a random game is a very peculiar incomplete information game where no player has private information. Note that, in this situation, the bayesian analysis is not really useful for the players.

For every $\omega \in \Omega$, we denote by H_ω the function which assigns $H(x, \omega)$ to every $x \in X$. Obviously, $\langle X, H_\omega \rangle$ is a normal form game for every ω. Remind that x is a Nash equilibrium of $\langle X, H_\omega \rangle$ if $H_i(x, \omega) \geqslant H_i((x_{-i}, x'_i), \omega)$ for all $x'_i \in X_i$ and all $i \in \{1, 2, \ldots, n\}$. Here (x_{-i}, x'_i) denotes the combination of strategies $(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)$. Bearing all this in mind, we give the following definition.

Definition 2: *For every* $x \in X$, *the Nash equilibrium indicator of x is the function* $N_x: \Omega \to \{0, 1\}$ *given by:*

$$N_x(\omega) = \begin{cases} 1 & \text{if and only if } x \text{ is a Nash equilibrium of } \langle X, H_\omega \rangle \\ 0 & \text{otherwise} \end{cases}$$

This Nash equilibrium indicator function describes the possibilities of a particular combination of strategies x to be a Nash equilibrium. In fact, it would be desirable that $N_x \equiv 1$. If such an x exists, it would be a good solution for the random game.

Nevertheless, that will rarely be the case. Then, as we have argued in the introduction, it is interesting to introduce and study a concept which picks up, for a random game, the combinations of strategies having the largest possibilities of being Nash equilibria. This is what we do next but first let us prove that, in a random game as in Definition 1, the set $\{\omega \in \Omega / N_x(\omega) = 1\}$ is in $\mathscr{A}$ (for all $x \in X$). Namely, fix $x \in X$. Now, taking into account the lower semicontinuity of every H_ω, we can write:

$$\{\omega \in \Omega / N_x(\omega) = 1\}$$
$$= \{\omega \in \Omega / \forall i \in \{1, \ldots, n\}, \forall x'_i \in X_i, H_i(x, \omega) \geqslant H_i((x_{-i}, x'_i), \omega)\}$$
$$= \{\omega \in \Omega / \forall i \in \{1, \ldots, n\}, \forall x'_i \in S_i, H_i(x, \omega)$$
$$\geqslant H_i((x_{-i}, x_i'), \omega)\} = \bigcap_{i=1}^{n} \bigcap_{x_i' \in S_i} \{\omega \in \Omega / H_i(x, \omega) \geqslant H_i((x_{-i}, x_i'), \omega)\}.$$

which clearly belongs to $\mathscr{A}$ because H is measurable as a functional of ω and every S_i is a countable set.

Now we can define our new concept.

Definition 3: *A combination of strategies $x \in X$ is said to be a maximum likelihood Nash equilibrium (MLNE) if and only if*

$$N(x) \geqslant N(y) \ \forall y \in X,$$

where the function $N: X \to [0, 1]$ is given by:

$$N(x) := P\{\omega \in \Omega / N_x(\omega) = 1\}.$$

Clearly, the only MLNE in the example proposed in the introduction of this paper is, as desired, (A, B). Next we present a more elaborated example concerning an infinite game.

Example 1: Take the two-person game $G^{\alpha,\beta} = (X, Y, H_1{}^{\alpha,\beta}, H_2{}^{\alpha,\beta})$, given by:

a) $X = Y = [0, 1]$

b) $H_i(x, y) = (x - \alpha)(y - \beta), \forall (x, y) \in X \times Y, \forall i \in \{1, 2\}$, where α and β are unknown random parameters independently distributed according to a $U(-a, a)$ distribution and to a $U(-b, b)$ distribution respectively ($a < 1$ and $b < 1$).

If we take the expected payoff function we can construct a standard two-person normal form game $G = (X, Y, H_1, H_2)$, where $H_1(x, y) = H_2(x, y) = xy \, \forall (x, y) \in X \times Y$. G has two Nash equilibria: $(0, 0)$ and $(1, 1)$.

If we analyze $G^{\alpha,\beta}$ as a random game, we observe that $N(0,0) < 1$ and that $N(1, 1) = 1$. It is also easy to check that $(1, 1)$ is the only MLNE.

An interesting point is when a certain random game has at least one MLNE. This is what we deal with next. First we prove the following lemma.

Lemma 1: *Let $< (\Omega, \mathscr{A}, P), X, H >$ be a random game as in Definition 1 and $\{x_n\}_{n \geqslant 1}$ a sequence in X such that $\lim_{n \to \infty} x_n = x_0 \in X$. Then, if the sequence $\{N(x_n)\}$ converges,*

$$N(x_0) \geqslant \lim_{n \to \infty} N(x_n)$$

Proof: For all $n \in \{0, 1, 2, \ldots\}$ define $A_n := \{\omega \in \Omega / N_{x_n}(\omega) = 0\}$, and take $\omega_0 \in A_0$. Clearly, x_0 is not an equilibrium of the game $G_{\omega_0} = \langle X, H_{\omega_0} \rangle$. Besides, the lower semicontinuity of H_{ω_0} implies that the set of Nash equilibria of G_{ω_0} is closed in X and, hence, there exists a neighbourhood E_{ω_0} of x_0 such that

$$N_x(\omega_0) = 0 \quad \forall x \in E_{\omega_0}.$$

On the other hand, the convergence of $\{x_n\}$ to x_0 implies that

$$\exists k \in N / n \geq k \Rightarrow x_n \in E_{\omega_0}.$$

Now, the last two conditions lead to

$$\exists k \in N / n \geq k \Rightarrow N_{x_n}(\omega_0) = 0.$$

In summary:

$$\forall \omega_0 \in A_0 \; \exists k(\omega_0) \in N / n \geq k(\omega_0) \Rightarrow \omega_0 \in A_n.$$

This means that

$$A_0 \subset \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n = \liminf_{n \to \infty} A_n,$$

which implies that

$$1 - N(x_0) = P(A_0) \leq P(\liminf_{n \to \infty} A_n) = \liminf_{n \to \infty} P(A_n) = 1 - \lim_{n \to \infty} N(x_n).$$

From this fact the result can be immediately derived. □

Now we are able to prove the following theorem.

Theorem 1: *Every random game $\langle (\Omega, \mathscr{A}, P), X, H \rangle$ satisfying that X is compact has at least one MLNE.*

Proof: Since the image of N is bounded, it has a supremum M. Hence, we can construct a sequence $\{x_n\}$ in X such that $\lim_{n \to \infty} N(x_n) = M$. The compactness of X ensures the existence of a subsequence $\{x_{n_k}\}$ and a point $x_0 \in X$ such that $\lim_{k \to \infty} x_{n_k} = x_0$. Then, applying Lemma 1 we conclude:

$$N(x_0) \geq \lim_{k \to \infty} N(x_{n_k}) = M \geq N(x) \text{ for every } x \in X.$$

In other terms, x_0 is an MLNE. □

Summarizing, we have introduced the MLNE and proved its existence for a special class of random games. In this process, we have defined a function N which measures the possibilities of every combination of strategies x to be a Nash equilibrium. Such a function allows, for instance, to give not only an MLNE but also its *level of likelihood.*

However there are situations where the function N is identically equal to zero. In that case, every $x \in X$ is trivally an MLNE. In the following theorem we present two conditions sufficient to assure that this is not the case.

Theorem 2: *If $\langle (\Omega, \mathscr{A}, P), X, H \rangle$ is a random game and one of following two conditions is verified*

1. X is countable and the event $A := \{\omega \in \Omega / \langle X, H_\omega \rangle$ *has at least one Nash equilibrium*$\}$ *satisfies that* $P(A) > 0$,

2. Ω *is finite and, for an* $\omega \in \Omega$ *with* $P(\{\omega\}) > 0, \langle X, H_\omega \rangle$ *has at least one Nash equilibrium,*

then the function N, as in Definition 3, is not identically equal to zero.

Proof: If Condition 1 is fulfilled then

$$0 < P(A) = \int_A dP \leqslant \int_A \sum_{x \in X} N_x(\omega) dP(\omega) = \sum_{x \in X} \int_A N_x(\omega) dP(\omega)$$

$$= \sum_{x \in X} P\{\omega \in A / N_x(\omega) = 1\} \leqslant \sum_{x \in X} N(x)$$

and hence we can conclude that there exists $x \in X$ such that $N(x) > 0$. If Condition 2 is fulfilled then, if x is an equilibrium of $\langle X, H_\omega \rangle$ and $P(\{\omega\}) > 0$, we can obviously assure that $N(x) > 0$. □

So, in Theorem 2 above, we have proved that, in many practical situations, the concept of MLNE is not a trivial one (in fact, we will often deal with random games of type "*one game is going to be played out of a finite list of games (with Nash equilibria) each of them having a positive and known probability of being played*" which clearly falls in Condition 2. However, it is convenient to modifiy the Definition 3 if, in the corresponding random game, N is identically equal to zero. A nontrivial case when this can happen is when some of the random variables H_x (we denote by H_x the function which assigns $H(x, \omega)$ to every $\omega \in \Omega$) are absolutely continuous. Observe that, although in these situations $N(x) = 0$ for all $x \in X$, some x can be such that their corresponding events $\{\omega \in \Omega / N_x(\omega) = 1\}$ bear more density of probability than others' corresponding events do and, hence, still makes sense to select a maximum likelihood Nash equilibrium. In the next section we present the redefinition of the MLNE when $N \equiv 0$.

3. AN ALTERNATIVE MAXIMUM LIKELIHOOD EQUILIBRIUM CONCEPT

Let us consider a random game $\langle (\Omega, \mathscr{A}, P), X, H \rangle$ such that its corresponding N, as in Definition 3, is identically equal to zero. Now, let us make the following suppositions:

S1 Each X_i is a metric space (with distance d_i) which verifies that, if we denote the open and closed balls with center x_i and ratio δ by $B(x_i, \delta)$ and $B[x_i, \delta]$ respectively, $\overline{B(x_i, \delta)} = B[x_i, \delta]$, for all $x_i \in X_i$ and $\delta > 0$. We denote by $C(x, \delta)$ and $C[x, \delta]$ the sets $\Pi_{i=1}^n B(x_i, \delta)$ and $\Pi_{i=1}^n B[x_i, \delta]$ respectively.

S2 There is a measure $\mu : X \to [0, \infty]$ satisfying:

1. $\mu(C[x, \delta]) > 0$ for every $x \in X$ and every $\delta > 0$, and
2. For every $\varepsilon > 0$, there exist $\rho > 0$ and $r > 0$ such that, for every $\delta \in (0, \rho]$ and every $x, y \in X$ verifying that

$$\max_{1 \leqslant i \leqslant n} d_i(x_i, y_i) \leqslant r,$$

it results that

$$\left|\frac{\mu(C[x,\delta])}{\mu(C[y,\delta])}-1\right|\leqslant\varepsilon.$$

The existence of a measure defined on X is a necessary supposition to define a kind of probability density fuction containing the information about the Nash equilibria. Apart from that, S1 and S2 are only technical conditions and not very restrictive; for instance, if the sets X_i are euclidean spaces and μ is the Lebesgue measure, S1 and S2 are fulfilled.

Now we can redefine the MLNE for this particular case.

Definition 4: *Let us consider a random game $R=\langle(\Omega,\mathscr{A},P),X,H\rangle$ satisfying S1 and S2 and such that its corresponding N, as in Definition 3, is identically equal to zero. Assume that X is compact. Then, $x\in X$ is an MLNE of R if*

$$f(x)\geqslant f(y)\quad\forall y\in X,$$

where

$$f(x):=\limsup_{\delta\to 0^+}\frac{P(C[x,\delta]^*)}{\mu(C[x,\delta])}$$

(being $Y^*:=\{\omega\in\Omega/\exists y\in Y, N_y(\omega)=1\}$ for any $Y\subset X$).

Note that it is not evident that $C[x,\delta]^*\in\mathscr{A}$. Similarly to the arguments used shortly before Definition 3, the separability of every X_i and the semicontinuity of H can be used to prove it, provided the compactness of X.

Now we are able to prove an existence theorem for this version of the MLNE.

Theorem 3: *Every random game, R, in the conditions of Definition 4 and verifying that X is compact has at least one MLNE as introduced in that definition.*

Proof: Using the same arguments as in Theorem 1, it suffices to prove that if $\{x_n\}_{n\geqslant 1}$ a sequence in X such that $\lim_{n\to\infty}x_n=x_0\in X$ and verifying that the sequence $\{f(x_n)\}$ converges, then

$$f(x_0)\geqslant\lim_{n\to\infty}f(x_n).$$

To prove this fix $\delta>0$, $\varepsilon>0$ and $\omega\notin C[x,\delta+\varepsilon]^*$. It is straightforward to prove that $\exists k\in N/n\geqslant k\Rightarrow\omega\notin C[x_n,\delta]^*$, An alternative form for this result is that

$$\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}C[x_n,\delta]^*\subset C[x_0,\delta+\varepsilon]^*,$$

for every $\varepsilon>0$. Since $\bigcap_{\varepsilon>0}C[x_0,\delta+\varepsilon]^*=C[x_0,\delta]^*$, we conclude that

$$\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}C[x_n,\delta]^*\subset C[x_0,\delta]^*.$$

The monotonicity of the probability implies that

$$P(C[x_0,\delta]^*) \geqslant \limsup_{n\to\infty} P(C[x_n,\delta]^*) \geqslant \limsup_{\eta\to\infty}\ \limsup_{\eta\to 0^+} P(C[x_n,\eta]^*).$$

As a consequence,

$$\limsup_{\delta\to 0^+}\frac{P(C[x_0,\delta]^*)}{\mu(C[x_0,\delta])} \geqslant \limsup_{n\to\infty}\ \limsup_{\delta\to 0^+}\frac{P(C[x_n,\delta]^*)}{\mu(C[x_0,\delta])}.$$

Hence,

$$f(x_0) = \limsup_{\delta\to 0^+}\frac{P(C[x_0,\delta]^*)}{\mu(C[x_0,\delta])} \geqslant \left(\limsup_{n\to\infty}\ \limsup_{\delta\to 0^+}\frac{P(C[x_n,\delta]^*)}{\mu(C[x_n,\delta])}\right)$$
$$\times\left(\limsup_{n\to\infty}\ \limsup_{\delta\to 0^+}\frac{\mu(C[x_n,\delta])}{\mu(C[x_0,\delta])}\right) = \lim_{n\to\infty} f(x_n)$$

Now, using the compactness of X, we can immediately conclude the proof. □

Observe that the revised definition is only suitable for random games with $N \equiv 0$. Namely, if there exists $x \in X$ such that $N(x) > 0$, then

$$f(x) = \limsup_{\delta\to 0^+}\frac{P(C[x,\delta]^*)}{\mu(C[x,\delta])} \geqslant \limsup_{\delta\to 0^+}\frac{P(\{x\}^*)}{\mu(C[x,\delta])}$$
$$= \frac{N(x)}{\mu(\{x\})} = \infty \text{ if } \mu(\{x\}) = 0.$$

This means that, if μ is a non-atomic measure, for every $x \in X$ with $N(X) > 0$, $f(x) = \infty$. Hence, in this case, f is not a good criterium to select the MLNE.

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References

[1] Nash, J. F., Non-cooperative games. *Annals of Mathematics*, **54** (1951) 286–295
[2] Owen, G., Game Theory. 2nd Edition. Academic Press (1982)
[3] Selten, R., Reexamination of the perfectness concept for equilibrium points in extensive games. *International Journal of Game Theory*, 4 (1975) 25–55
[4] van Damme, E. E. C., Stability and perfection of Nash equilibria. 2nd edition. Springer-Verlag (1991)
[5] Borm P. E. M., R. Cao, I. García-Jurado and L. Mendez-Naya, Weakly strict equilibria in normal form games. To appear in OR Spektrum (1995)