Local asymptotic stability of optimal steady states

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This note provides a sufficient condition for steady states arising from economic optimal control models to be locally asymptotically stable. In particular, we show that if some submatrix of the matrix of eigenvectors of the corresponding Jacobian is invertible, the stable manifold is locally y-parametrizable.

1. Introduction

The use of optimal control theory is quite common in economics. There is a large variety of fields where this technique is employed: growth economics, environmental economics and many others. It often occurs that there is a steady state solution (e.g. modified golden rule) and much effort is put into its analysis, justified by the fact that the steady state may be stable in some sense. The stability issue itself has also been subject to intensive research [see e.g. Brock and Malliaris (1989) for an overview]. The emphasis is on global asymptotic stability (GAS) rather than on local asymptotic stability (LAS); as early as 1976, Brock and Scheinkman (1976, p. 166) have put forward that the literature on LAS is 'fairly complete'. In this area the standard references are Kurz (1968), Levhari and Leviatan (1972) and Samuelson (1972). Roughly speaking one proceeds as follows. The necessary conditions for optimality can be cast into a system of differential equations, which is linearized around the steady state, if any, and which is subsequently analysed in terms of the eigenvalues of the Jacobian. The occurrence of as many eigenvalues with negative real parts as the number of state variables is then generally deemed sufficient for LAS, the basic idea being that under that condition there exists a stable manifold containing the steady state [see Burmeister (1968, 1980) for applications]. This reasoning implicitly [1] assumes that the stable manifold is locally y-parametrizable 'non-degenerate' in the sense that for all values of the state variables (y) in a neighbourhood of the steady state there exist values of the co-state variables so that they lie all on the stable manifold. However, Scheinkman (1976) provides an example where this assumption does not hold true (in a discrete-time growth model). There are several ways to deal with the degeneracy issue. One is to make particular assumptions guaranteeing that degeneracy does not occur, as is done by Scheinkman (1976) in the sequel of his article; in the case of one state variable it is often

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[1] Kurz (1968) makes this assumption explicitly.

easy to depict the manifold in a phase diagram; finally, one could try to construct the manifold for the particular model at hand and to study its properties. The latter approach has proven useful in Van Marrewijk et al. (1992), but the analysis turns out to be rather cumbersome.

The above observations make clear that LAS is not yet completely settled and raise the need for a sufficient condition for LAS which, preferably, is easy to check. It is the aim of the present note to provide such a condition.

2. A sufficient condition for LAS

In this section we shall study a system of differential equations of the following type:

$$\dot{x}(t) = Ax(t) + f(t, x(t)).\tag{2.1}$$

Here t denotes time. x is a $2n$-vector. A is a given real $2n * 2n$ matrix with n characteristic roots having negative real part and n characteristic roots having positive real part. f has the following properties.

F.1. f is continuous in (t, x) for small $|x|$ and $t \geq 0$ and continuously differentiable in x for t sufficiently large.

F.2. $f(t, 0) = 0$.

The relation of this differential equation with optimal control problems can be sketched as follows. Consider an optimal control problem (OCP) over an infinite horizon with n state variables (y) and m instruments. Assume that the OCP has a solution with continuous state variables and piece-wise continuous instruments. For the situation at hand Toman (1985) and Cesari (1983) provide existence theorems. It will also be assumed that the maximization of the Hamiltonian with respect to the instruments yields the instruments as continuous functions of the state variables and the co-state variables (q). See Gale and Nikaido (1965) on the issue of global univalence. Then the set of necessary conditions reduces to a system of $2n$ differential equations in $z := (y, q)$: $\dot{z} = g(t, z)$. Assume next that $g(t, z^*) = 0$ for some z^*. By defining $x := z - z^*$ one obtains $\dot{x} = k(t, x)$ for some k. Finally, k is linearized around 0 and (2.1) results. It is worth stressing that the steps described here are by no means trivial. But since we are interested here in LAS only, we depart from (2.1) with its properties mentioned, without going into the basic assumptions underlying these properties.

The following lemma plays a crucial role in the LAS literature. It is a paraphrase of Theorem 4.1 in Coddington and Levinson (1955).

Lemma. Under the assumptions made with respect to A and f and for any large t^* there exists in x-space a real n-dimensional manifold S containing the origin such that any solution ϕ of (2.1) with $\phi(t^*)$ on the manifold S satisfies $\phi(t) \to 0$ as $t \to \infty$.

Remark. For a more modern geometric treatment, see e.g. Abraham and Marsden (1978), Palis and De Melo (1982), Shub (1987).

Without loss of generality we may partition x such that the first n elements refer to the state variables (y) and the final n elements refer to the co-state variables (q). A is partitioned accordingly.

We assume that A can be written as

$$A\begin{pmatrix} S & P \\ -Q & \rho I - S^T \end{pmatrix},$$

with P positive definite, Q symmetric and ρ a nonnegative constant. It will moreover be assumed that A has n eigenvalues with negative real part and n eigenvalues with positive real part and that all eigenvalues are different. These assumptions are valid in a large variety of economic models, for example those which satisfy some concavity/convexity conditions and where there is a nonnegative rate of discount or rate of time preference ρ.

Let T be the matrix having the eigenvectors of A as columns. Since all eigenvalues differ, T is invertible. T can be written as

$$T = \begin{pmatrix} T_1 & T_3 \\ T_2 & T_4 \end{pmatrix},$$

where $\overline{T} := (T_1^T, T_2^T)^T$ is the matrix of eigenvectors corresponding to the negative eigenvalues. According to Theorems 4.1 and 4.2 in Coddington and Levinson (1955) there exists an n-vector of functions $h := \mathrm{col}(h_1, h_2, \ldots, h_n)$ where $h_1, \ldots, h_n$ are continuously differentiable on an open set U in $\mathbb{R}^n$, containing the origin, such that the stable manifold is given by

$$x = T\begin{pmatrix} \eta \\ h(\eta) \end{pmatrix},$$

where $\eta \in U$ and $(\partial h / \partial \eta)(0) = 0$.

We call the stable manifold locally y-parametrizable if there exists a neighborhood $V \subset \mathbb{R}^n$ of the origin such that for every $y \in V$ there is a $q \in \mathbb{R}^n$ with the property that (y, q) is on the stable manifold. By the Implicit Function Theorem it follows that a sufficient condition for the stable manifold to be locally y-parametrizable is that T_1 is invertible. Our theorem can now be stated as follows.

Theorem. Under the assumptions stated above T_1 is invertible and hence the stable manifold is nondegenerate.

Proof. The proof follows the same lines as the proof of Theorem 7.2 in Francis (1988). Let Λ_- be the diagonal matrix with the stable eigenvalues of A as its diagonal elements. Decompose A according to

$$A = H + \rho B$$

where

$$H = \begin{pmatrix} S & P \\ -Q & -S^T \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}. \tag{2.2}$$

Introduce the symplectic matrix

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Then it is easy to check that JH is symmetric and $JB + BJ = J$. We have

$$A\bar{T} = \bar{T}\Lambda_-. \tag{2.3}$$

Premultiplying (2.3) by $\bar{T}^{\mathrm{T}}J$ we obtain

$$\bar{T}^{\mathrm{T}}JA\bar{T} = \bar{T}^{\mathrm{T}}J\bar{T}\Lambda_- \Leftrightarrow \bar{T}^{\mathrm{T}}JH\bar{T} = \bar{T}^{\mathrm{T}}J\bar{T}\Lambda_- - \rho\bar{T}^{\mathrm{T}}JB\bar{T}. \tag{2.4}$$

Now the left-hand side of (2.4) is symmetric, hence so is the right-hand side of (2.4). Using the skew-symmetry of J this yields

$$-\Lambda_-\bar{T}^{\mathrm{T}}J\bar{T} + \rho\bar{T}^{\mathrm{T}}BJ\bar{T} = \bar{T}^{\mathrm{T}}J\bar{T}\Lambda_{-1} - \rho\bar{T}^{\mathrm{T}}JB\bar{T} \Leftrightarrow -\Lambda_-\bar{T}^{\mathrm{T}}J\bar{T} + \rho\bar{T}^{\mathrm{T}}(BJ + JB)\bar{T}$$

$$= \bar{T}^{\mathrm{T}}J\bar{T}\Lambda_- \Leftrightarrow (\rho I - \lambda_-)\bar{T}^{\mathrm{T}}J\bar{T} = \bar{T}^{\mathrm{T}}J\bar{T}\Lambda_-. \tag{2.5}$$

Since $\rho \geq 0$, the set of eigenvalues of $\rho I - \Lambda_-$ is in $\mathbb{C}^+$. Moreover, the set of eigenvalues of Λ_- is in $\mathbb{C}^-$. Hence the sets of eigenvalues of $(\rho I - \Lambda_-)$ and Λ_- respectively are disjoint. Then (2.5) implies [cf. Gantmacher (1960)] that $\bar{T}^{\mathrm{T}}J\bar{T} = 0$, or

$$T_2^{\mathrm{T}}T_1 = T_1^{\mathrm{T}}T_2. \tag{2.6}$$

Now assume that T_1 is not ivertible, i.e., there is an $x \neq 0$ such that $T_1 x = 0$. Premultiply (2.3) by $(I0)$ to obtain

$$ST_1 + PT_2 = T_1\Lambda_-. \tag{2.7}$$

Premultiply (2.7) by $x^{\mathrm{T}}T_2^{\mathrm{T}}$ and postmultiply by x:

$$x^{\mathrm{T}}T_2^{\mathrm{T}}ST_1 x + x^{\mathrm{T}}T_2^{\mathrm{T}}PT_2 x = x^{\mathrm{T}}T_2^{\mathrm{T}}T_1\Lambda_- x \Leftrightarrow x^{\mathrm{T}}T_2^{\mathrm{T}}PT_2 x = x^{\mathrm{T}}T_1^{\mathrm{T}}T_2\lambda_- x = 0. \tag{2.8}$$

Since $P > 0$, (2.8) implies that

$$T_2 x = 0. \tag{2.9}$$

However, this implies that

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} x = 0,$$

which contradicts the fact that $\binom{T_1}{T_2}$ has full column rank. Hence T_1 is invertible.

3. Conclusion

This paper offers a sufficient condition for the local asymptotic stability of optimal steady states. The condition is easy to check in terms of the properties of the submatrices constituting the Jacobian of the linearized system of differential equations, but, indeed, many economic problems in e.g. growth theory produce a Jacobian with the desired properties, so that tedious calculations to establish that the stable manifold is locally *y*-parametrizable can be left undone.

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