On the Negishi-Approach to Dynamic Economic Systems

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1. Introduction

This paper deals with the existence of general equilibria in economies endowed with natural exhaustible resources. In such economies dynamic considerations almost necessarily enter into the analysis: typically agents must decide to exploit the resources in the present or in the future. The appropriate general equilibrium concept is then intertemporal or temporary, according to the presupposed existence of the relevant markets in the future. Whatever equilibrium concept is chosen, one must also specify the time structure of the economy which is to be described: should one adopt a discrete-time setting or is a continuous-time analysis more appropriate? This choice is not only important from an economic point of view but also with regard to the mathematics involved. Here we will not address the former point, but we will focus on the latter observation. The issue is that with a discrete-time model and a finite horizon the commodity space will in general be of finite dimension. This implies that a proof of the existence of a general equilibrium for the finite horizon requires the same techniques as the standard Arrow/Debreu model and that its properties (such as e.g. the boundedness of equilibrium allocations) can be fruitfully used to establish the existence of an infinite horizon equilibrium (see VAN GELDROP et al. (1989)). For the continuous-time case the situation is far more complicated because even the existence of an equilibrium with a finite horizon poses great difficulties; the commodity space is of infinite dimension in spite of the finite horizon.

Admittedly there exists a growing literature on infinite dimensional commodity spaces, based on the seminal work of BEWLEY (1972), but it can be shown that for a large class of economies with exhaustible resources the results do not simply carry over. The purpose of the present paper is to provide an existence proof using an alternative technique, i.c. optimal control theory. Our attention will be restricted to the finite horizon case. This has some merits on its own if the economy has a limited life time or when at some given future instant of time the raw materials from the exhaustible resources become valueless because of the discovery of alternative resources which can be exploited at considerably lower expenses (solar energy, nuclear fusion). This is of course only a modest defense, since in a proper model such events should be incorporated from the outset. The main justification for looking at a finite horizon is that it provides an adequate starting point for an infinite horizon analysis. Space limitations prevent us to go into such an analysis here.

The method followed is rather traditional. It is based on NEGISHI'S (1960) idea to look for the set of Pareto efficient allocations and then to show that there exists an allocation which, together with the corresponding shadow prices, constitutes a general equilibrium. The point is that we employ this method in an intertemporal setting which severely complicates the analysis: a formal approach of the existence of Pareto allocations is in order as well as a rigorous derivation of their properties, in particular continuity and boundedness features.

The model is to some extent neo-classical in nature. The agents' instantaneous preferences can be represented by utility functions and the economy's technology is described by means of production functions. These functions however may be quite general and do not restrict too much the underlying preferences or production sets. The model is also on an aggregate level as far as commodities are concerned. Although the resources can be distinguished according to the extraction costs involved, the raw material once extracted is homogeneous. There is also one composite consumer commodity, which is perfectly malleable with capital. It cannot be denied that these assumptions are restrictive but they keep the model at a manageable level, without loosing the basic features of models involving exhaustible natural resources, known in the literature. It can even be argued that our model is more general than the existing general equilibrium models. See VAN GELDROP and WITHAGEN (1988) for a survey.

The model can be briefly outlined as follows. There is a given number of consumers, who maximize their welfare, a given number of firms that exploit the exhaustible resources and a given number of firms that transform the exploited raw material, together with capital, into the composite commodity. Associated with the exploitation there are costs originating from the use of capital. Finally there is a perfect capital market on which the agents can borrow and lend at will, provided their discounted expenditures do not exceed their discounted income. We assume the existence of a complete set of markets and perfect foresight.

A straightforward interpretation of the model is that it describes the trade between countries facing perfect world markets for raw materials, capital and consumer commodities. The importance of a general equilibrium approach lies in the observation that in such a model a crucial variable in resource economics, namely the rate of interest, is endogenous as well as the demand for the raw material, contrary to most exhaustible resource models.

In Section 2 a precise description of the model is given and a general equilibrium is defined. Section 3 formulates the problem of finding a Pareto-efficient allocation, where arbitrary weights are given to the consumers. We also go into the existence and the characterization of such allocations. Section 4 establishes the existence of an allocation which is compatible with the endowments of each individual agent and which constitutes a general equilibrium. Section 5 concludes.

2. THE MODEL

The model describes an economy with k resources and k resource exploiting firms, l firms that produce the non-resource commodity and m consumers. Time is denoted by t. The period over which the economy is considered is finite and given by $[0,T]\,(T>0)$. $p:[0,T]\to\mathbb{R}_+$ is a mapping which associates with every $t\in[0,T]$ the market price $p(t)$ of the raw material at that date. At the initial point of time, $t=0$, p is known to all agents. This also holds for $r:[0,T]\to\mathbb{R}_+$, the interest rate. The consumer commodity (the non-resource commodity) serves as the numéraire. The subsequent analysis will demonstrate that this choice of the numéraire is warranted, in the sense that that commodity is always desirable.

The first k firms exploit the resources. Exploitation patterns are given by $E:=(E_1,E_2,\cdots,E_k):[0,T]\to\mathbb{R}^k_+$, where $E_j(t)$ is the rate of exploitation of the j-th resource at time t by firm $j\,(j=1,2,\cdots,k)$. Each firm j is characterized by an extraction technology $G_j:\mathbb{R}_+\to\mathbb{R}_+$ which associates to each rate of exploitation the minimally required amount of capital to do so, denoted by K^e_j. We define $G:\mathbb{R}^k_+\to\mathbb{R}^k_+$ by $G(E)=(G_1(E_1),G_2(E_2),\cdots,G_k(E_k))$. About $G_j\,(j=1,2,\cdots,k)$ we assume

G^1. $\quad G_j(E_j)=a_jE_j$ with a_j a positive constant.

G^2. $\quad 0<a_1<a_j$ for all $j>1$.

The aim of firm j is to carry out a production plan by choosing $(E_j,K^e_j,\tilde{S}_{0j})\in L^+_\infty[0,T]\times L^+_\infty[0,T]\times\mathbb{R}_+$ so as to maximize total discounted profits

$$\int_0^T e^{-\int_0^s r(\tau)d\tau}(p(s)E_j(s)-r(s)K^e_j(s))ds-p_{0j}\tilde{S}_{0j} \tag{2.1}$$

subject to

$$K^e_j(t)\geqslant a_jE_j(t),\ t\in[0,T] \tag{2.2}$$

$$\int_0^T E_j(s)ds\leqslant\tilde{S}_{0j} \tag{2.3}$$

$$E_j(t)\geqslant 0,\ t\in[0,T]. \tag{2.4}$$

Here $\tilde{S}_{0j}$ is the stock of resource j which firm j buys at the outset of the planning period at a price p_{0j}. Strictly speaking one should allow for the possibility that a firm buys stocks along the entire period. But in the absence of uncertainty and with a perfect capital market, one can evidently do as if the markets for stocks are open at $t=0$ only. Conditions 2.3 and 2.4 say that a firm cannot extract more than its resource initially contains.

Firms $k+1, k+2, \cdots, k+l$ are endowed with a technology to convert capital and the raw material into a commodity that can be used for consumption and investments. The technology of non-resource producing firm $k+i$ is represented by a production function $F_i:\mathbb{R}^2\to\mathbb{R}_+$, which assigns to every amount of capital K and every amount of raw material inputs R a produced amount $F_i(K,R)$ of the composite commodity. Let $K^y:=(K_1^y, \cdots, K_l^y):[0,T]\to\mathbb{R}_+^l$ be inputs of capital in the non-resource producing firms for each moment of time and, similarly, let $R:=(R_1, \cdots, R_l):[0,T]\to\mathbb{R}_+^l$ be the raw material inputs of those firms at each moment of time. Then we define the function $F:\mathbb{R}_+^{2l}\to\mathbb{R}_+^l$ by

$$F(K^y, R):=(F_1(K_1^y, R_1), F_2(K_2^y,R_2), \cdots, F_l(K_l^y, R_l)).$$

About F_i $(i=1,2, \cdots, l)$ we assume

$F.^1$ F_i is continuous on $\mathbb{R}_+^2$ and continuously differentiable on $\mathbb{R}_{++}^2$

$F.^2$ F_i is strictly increasing on $\mathbb{R}_{++}^2$, concave and homogeneous of order 1

$F.^3$ $F_i(K,0)=F_i(0,R)=0$ for all $K\in\mathbb{R}_+$ and all $R\in\mathbb{R}_+$

$F.^4$ $\lim_{K\to\infty} \dfrac{F_i(K,R)}{K}=0$; for all R; $\lim_{K\to 0} \dfrac{F_i(K,R)}{K}=\infty$ for all $R>0$.

The objective of firm i is to choose $(K_i^y,R_i):[0,T]\to\mathbb{R}_+^2$ so as to maximize the present value of total profits:

$$\int_0^T e^{-\int_0^s r(\tau)d\tau} (F_i(K_i^y(s), R_i(s))-r(s)K_i^y(s)-p(s)R_i(s))ds. \tag{2.5}$$

Consumer h $(h=1,2, \cdots, m)$ is at $t=0$ endowed with a stock of capital $K^h(>0)$, resource stocks $S^h=(S_1^h,S_2^h, \cdots, S_k^h)$ and shares $\theta^h:=(\theta_1^h, \theta_2^h, \cdots, \theta_{k+l}^h)$ in the firms. The consumer chooses consumption so as to maximize his welfare. The instantaneous utility function is given by $u_h:\mathbb{R}_+\to\mathbb{R}$ which satisfies for all h

$U.^1$ u_h is continuous on $\mathbb{R}_+$ and continuously differentiable on $\mathbb{R}_{++}$

$U.^2$ u_h is strictly increasing and strictly concave

$U.^3$ $u'_h(0)=\infty$, $0<\eta_h \leqslant -Cu''_h(C)/u_h'(C)<\infty$ for all $C>0$ and some constant η_h.

The second part of U^3 says that the elasticity of marginal utility is bounded. The (constant) rate of time preference of consumer h is given by $\rho_h(>0)$. Consumer h's objective is to choose $C_h:[0,T]\to\mathbb{R}_+$ so as to maximize

$$\int_0^T e^{-\rho_h s}u_h(C_h(s))ds \tag{2.6}$$

subject to

$$\int_0^T e^{-\int_0^s r(\tau)d\tau} C_h(s)ds \leqslant K^h + \sum_{j=1}^{k} p_{oj}S_j^h + \sum_{f=1}^{k+l} \theta_f^h \Pi_f. \tag{2.7}$$

Here Π_f stands for the present value of the profits of firm $f(f=1,2,\cdots,k+l)$. The left-hand side of the budget condition is the present value of total expenditures and the right-hand side is the present value of total income, where it is to be understood that the markets for resource stocks are open at $t=0$ only, so that it is in the interest of the consumer to supply his entire stock (even if parts of it would have a zero price) since resource stocks do not enter into the welfare functional.

Along a general competitive equilibrium each producer maximizes its profits and each consumer his welfare, subject to the relevant constraints. Furthermore demand is met by supply for each commodity at each instant of time t. In order to formulate this latter condition in a concise way, we define

$$C:=(C_1, C_2, \cdots, C_m):[0,T]\to\mathbb{R}_+^m$$

$$K_0:=\sum_h K^h,\ S_0:=(\sum_h S_1^h, \cdots, \sum_h S_k^h)$$

and

$$\tilde{S}_0=(\tilde{S}_{01}, \tilde{S}_{02}, \cdots, \tilde{S}_{0k}).$$

Furthermore, for $\nu\in\mathbb{N}$ and $x=(x_1,x_2,\cdots,x_\nu)\in\mathbb{R}^\nu$ we denote $\sum_{i=1}^{\nu} x_i$ by σx; so $\sigma=(1,1,\cdots,1)\in\mathbb{R}^\nu$ without specification of the size of σ. Then we must have

$$\sigma E(t)-\sigma R(t)\geqslant 0, \qquad t\in[0,T] \tag{2.8}$$

$$p(t)(\sigma E(t)-\sigma R(t))=0, \qquad t\in[0,T] \tag{2.9}$$

$$S_0-\tilde{S}_0\geqslant 0 \tag{2.10}$$

$$p_0(S_0-\tilde{S}_0)=0 \tag{2.11}$$

and, if we define $K(t):=\sigma K^e(t)+\sigma K^y(t)$,

$$\dot{K}(t)=\sigma F(K^y(t), R(t))-\sigma C(t),\ t\in[0,T],\ K(0)=K_0. \tag{2.12}$$

The central question we address in this paper is: does there exist a general equilibrium in the economy described above? We shall tackle this question with the aid of optimal control theory. One might wonder why we do not try to apply existence theorems from the vast literature on economies with infinite-dimensional commodity spaces, developed since BEWLEY (1972). The main obstacle is, even for a finite horizon, the absence of any reasonable boundedness conditions on the production side, more specifically the level of consumption-output C. The best thing one can do is to impose some artificial upper bounds and afterwards show that bounds can be chosen such that they are not binding. In order to do so, one tries to get more insight in the nature of equilibria - the existence of which is forthcoming from the literature - by means of control theory. But provided that all other assumptions in these theorems are satisfied price systems are to be found then in the dual space of $L_\infty[0,T]$. In the best case we find an L_1 function. However, control theory requires at least some continuity properties. This is the main reason why we have chosen for a different method. The Negishi-approach yields, as a result of control theory and in a natural way, absolutely continuous shadow-prices. But even then the main concern is to show that it is possible to define non-binding upper bounds on C.

3. Pareto efficiency

In this section we consider the problem of finding Pareto-efficient allocations in the economy described in the previous section. That is: do there exist allocations such that the welfare of a consumer can only be increased at the expense of the welfare of other consumers? Mathematically, such allocations can be found by maximizing a weighted sum of the welfare functionals subject to the technological constraints. To that end we attach weights $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ to the consumers. For the time being we shall take $\alpha_h > 0$ for all h and $\sum_{h=1}^{m} \alpha_h = 1$.

Moreover we introduce the aggregate production function $Q(K,E)$ defined on the set $A := \{(K,E) | E \geqslant 0, aE \leqslant K\}$ by

$$Q(K,E) := \max \sigma F(K^y, R)$$

subject to

$$\sigma K^y + aE \leqslant K \tag{3.1}$$

$$\sigma R - \sigma E \leqslant 0. \tag{3.2}$$

It is clear that there exists some continuous function $\hat{F}: \mathbb{R}^2_+ \to \mathbb{R}_+$ satisfying conditions F^2 and F^3 such that $Q(K,E) = \hat{F}(K - aE, \sigma E)$. Though we are aware of the fact that Q may not be a C^2 function, we assume that $\hat{F}$ is C^2 on the open orthant $\mathbb{R}^2_{++}$ and that $\hat{F}$ satisfies F^4.

$Q(K,E)$ gives the maximal production of the composite commodity if the economy's capital stock is K and the available raw material is E. In the definition of Q we have assumed that $K^e_j = a_j E_j$ which is of course innocuous. We are now looking for a solution of the following problem. Choose $(E,C):[0,T] \to \mathbb{R}^{k+m}_+$ such that (E,C) maximizes

$$\int_0^T \sum_{h=1}^{m} \alpha_h e^{-\rho_h s} u_h(C_h(s)) ds \tag{3.3}$$

subject to

$$\dot{S} = -E, \ S(0) = S_0, \ S(T) \geqslant 0 \tag{3.4}$$

$$\dot{K} = Q(K,E) - \sigma C, \ K(0) = K_0, \ K(T) \geqslant 0, \ (K,E) \in A. \tag{3.5}$$

$S(t)$ can be considered as the vector of stocks of the exhaustible resources at t. The differential equations (3.4-3.5) should hold almost everywhere (a.e.). This problem will be referred to as the Pareto-problem.

The existence of a solution is by no means straightforward. In general, existence theorems require boundedness of the state variables and the control variables, in casu K and S, and E and C. We proceed by constructing such bounds and by showing in the sequel that they are not binding.

1. It follows from (3.4) that S is non-increasing. So it is easy to define an upper bound on S. Furthermore $S \geqslant 0$, because of (3.4) and the fact that $E \geqslant 0$. Let $\epsilon_s > 0$ be given. Define

$$\overline{S} := (\max_j \sum_h S_j^h + \epsilon_s, \max_j \sum_h S_j^h + \epsilon_s, \cdots, \max_j \sum_h S_j^h + \epsilon_s)$$

$$\underline{S} := (-\epsilon_s, -\epsilon_s, \cdots, -\epsilon_s).$$

2. It follows from (3.1) that along an optimum $a_j E_j \leqslant K$ for all j and $K_i^r \leqslant K$ for all i. We furthermore have

$$Q(K,E) \leqslant \hat{F}(K, K/a_1) = \gamma K$$

where $\gamma = \hat{F}(1, 1/a_1)$. Next, consider the differential equation $\dot{K} = \gamma K$, $K(0) = K_0$ and define

$$\overline{K} = 2K_0 e^{\gamma T}; \underline{K} = -1.$$

It is evidently true that, if the problem given above has a solution, the optimal K is at all t smaller than $\overline{K}$. And it is strictly larger than $\underline{K}$ because of the requirement that $K(T) \geqslant 0$.

3. E is bounded:

$$0 \leqslant E(t) \leqslant (\overline{K}/a_1, \cdots, \overline{K}/a_1) := \overline{E}\sigma.$$

4. C is bounded from below by zero. For the time being we impose an opperbound $\overline{C}\sigma$, where $\overline{C}$ is assumed to be non-sustainable on $[0,T]$. This means that $\dot{K} = Q(K,E) - \overline{C}$ generates negative K before T.

So we add the restrictions

$$\underline{S} \leqslant S(t) \leqslant \overline{S}, \underline{K} \leqslant K(t) \leqslant \overline{K}, 0 \leqslant E(t) \leqslant \overline{E}\sigma, 0 \leqslant C(t) \leqslant \overline{C}\sigma, \text{ for } t \in [0,T]. \quad (3.6)$$

Problem (3.3)-(3.6) allows for a direct application of Filippov's existence theorem (see e.g. CESARI (1983) p.199). This is so because the set of the feasible state values and the set of feasible controls are compact, the set of feasible endpoints is closed and all the functions involved are concave and continuous. Therefore we have

THEOREM 3.1. *There exist absolutely continuous functions K and S and measurable E and C which maximize (3.3) subject to (3.4)-(3.6).*

It will be shown in the sequel that $\overline{C}$ can be chosen such that it is never binding. This will be done by using the necessary conditions for an optimum. To formulate these conditions we need the differentiability of the functions involved. This will be our first concern.

LEMMA 3.1.

1) $K(T)=0$

2) $K(t)>0$ *for all* $0 \leq t < T$.

PROOF. $K(T)=0$ *is trivial. The proof of the second statement is not given here. Basically it exploits the fact that the utility functions are concave and that the elasticity of the marginal utility is bounded.* □

LEMMA 3.2. *Let* $t_0 \in [0,T]$ *be given. There exists* $\delta > 0$ *such that* $K-aE > \delta$ *and* $\sigma E > \delta$ *on* $[0,t_0]$ *a.e.* □

The lemma states that $\hat{F} > 0$ a.e. on $[0,T]$. That this holds true is intuitively clear. A formal proof is not given here.

We are now ready to formulate the necessary conditions. Define $T_n := \frac{n}{n+1}T$. Consider the Pareto problem with horizon T_n and end points $K(T_n)=\hat{K}(T_n), S(T_n)=\hat{S}(T_n)$, where $\hat{K}(T_n)$ is the value of K at T_n along the optimal trajectory of the problem with horizon T and $\hat{S}(T_n)$ is defined analogously. So we modify the problem so as to have a smaller horizon and fix the final state at the optimal state of the original problem. Later we shall take limits for $n \to \infty$. It follows from Lemma 3.1 that along the optimal trajectory of the modified problem $K>0$. Furthermore, the function Q is defined (a.e) in view of Lemma 3.2. Then we can pretend that it is defined thoughout since it is always possible to reduce E without violating any of the feasibility constraints.

We define the Hamiltonian:

$$H(t, K, S, C, E, \lambda_0, \lambda_1, \cdots, \lambda_k, \phi) :=$$

$$\lambda_0 \sum_{h=1}^{m} \alpha_h e^{-\rho_h t} u_h(C_h) + \sum_{j=1}^{k} \lambda_j(-E_j) + \phi(\hat{F}(K-aE, \sigma E) - \sigma C).$$

Necessary conditions are (see CESARI (1983) p.197):

P_1. There exists an absolutely continuous vector function $\nu = (\lambda_0, \lambda_1, \cdots, \lambda_k, \phi) = (\lambda_0, \lambda, \phi)$ which is never zero in $[0, T_n]$ with λ_0 a constant ($\lambda_0 \geq 0$) such that

$$\dot{\lambda} = 0 \text{ a.e.} \tag{3.7}$$

$$\dot{\phi} = -\partial H / \partial K, \text{a.e., so } \dot{\phi} = -\phi \hat{F}_K(K-aE, \sigma E). \tag{3.8}$$

P_2. For every fixed t in $[0, T_n]$, H is maximized a.e. with respect to C and E

subject to $0 \leqslant C \leqslant \bar{C}\sigma, 0 \leqslant E \leqslant \bar{E}\sigma$, where Q is, if necessary, defined by $Q(K, E)=0$ outside A.

LEMMA 3.3. *$\lambda_0 > 0$, and $\phi(t) > 0$ for all $0 \leqslant t \leqslant T_n$.*

PROOF. Suppose $\lambda_0 = 0$. If there exists $t_1 \leqslant T_n$ such that $\phi(t_1)=0$ then $\phi(t)=0$ for all $0 \leqslant t \leqslant T_n$ because of (3.8). Then it is clear that $\lambda_j \geqslant 0 (j=1, \cdots, k)$, since otherwise $\hat{F}=0$ along an optimum. Let J be the subset of $\{1,2, \cdots, k\}$ such that, for $j \in J$, $\lambda_j > 0$. It follows from the maximization of the Hamiltonian that, for $j \in J$, $E_j = 0$ a.e. Next consider problem (3.3)-(3.6) with resources $j \in J$ omitted. This problem has virtually the same solution as the original problem. But that implies that in the modified problem $\lambda_j = 0$ for $j \in \{1,2, \cdots, k\}/J$ if $\lambda_0 = \phi(t) = 0$. This is not allowed. Therefore, if $\lambda_0 = 0$ then $\phi(t) > 0$ for all $t \leqslant T_n$. But then the maximization of the Hamiltonian with respect to C yields $C(t)=0$ a.e. which cannot be optimal. Hence $\lambda_0 > 0$. $\phi(t) > 0$ then follows trivially, due to the choice of $\bar{C}$. □

Henceforth we take $\lambda_0 = 1$. Now define $\tilde{C}_h(t)$ as the solution of

$$\alpha_h e^{-\rho_h t} u'_h(C_h(t)) = \phi(t)$$

and $\hat{C}_h(t) := \min\{\bar{C}, \tilde{C}_h(t)\}$. In view of the properties of u_h and since $\phi(t) > 0$, $\hat{C}(t) = (\hat{C}_1(t), \hat{C}_2(t), \cdots, \hat{C}_m(t))$ is well-defined. Define also $\hat{E}(t)$ as the solution of

$$\max_{E \in \mathbb{R}^k} \phi(t) Q(K(t), E(t)) - \lambda(t) E(t)$$

$$\text{subject to } 0 \leqslant E(t) \leqslant \bar{E} \text{ for all } t.$$

By the construction of $\bar{E}$, $\hat{E}(t) < \bar{E}$ a.e.

Clearly $\hat{C}(t) = C(t)$ a.e. and $\hat{E}(t) = E(t)$ a.e. So $(\hat{C}, \hat{E})$ constitutes an optimal trajectory. Furthermore $\hat{C}$ is continuous.

In order to get a better insight in the solution of the Pareto problem we derive some necessary conditions on $[0, T)$. However, we must keep in mind, that the function F, or Q, is not differentiable in the origin, while $K(T) = \sigma E(T) = 0$. In spite of this inconvenience, we have the following

LEMMA 3.4. *For every non-sustainable $\bar{C}$ there is an absolutely continuous function ϕ, defined on $[0, T)$, such that for the solution of (3.3)-(3.6)*

$$\dot{\phi} = -\phi \hat{F}_K(K - aE, \sigma E) \text{ a.e. on } [0, T) \tag{3.4.1}$$

$$C_h \text{ maximizes } \alpha_h e^{-\rho_h t} u_h(C_h) - \phi C_h \text{ on } 0 \leqslant C_h \leqslant \bar{C} \tag{3.4.2}$$

for all $t \in [0, T]$, and all $h = 1, \cdots, m$.

as a consequence of (3.4.2): $\phi(t) > 0$ *on* $[0, T]$. (3.4.3)

PROOF. For every horizon $T_n = \dfrac{n}{n+1} T$ we are entitled to make a similar statement for the function ϕ_n, associated with horizon T_n, keeping the values of the state variables at T_n fixed. See (P_1). But there is some $t_0 \in [0,T)$ where $C_h(t_0) < \bar{C}$ for all h. As a consequence $\alpha_h e^{-\rho_h t_0} u'_h(C_h(t_0)) = \phi_n(t_0)$ for all h, and n large enough. Then the function ϕ, defined by $\phi(t) := \phi_n(t) : (T_n > t)$ has the desired properties. □

The next lemma captures some of the until now neglected necessary conditions for the control variable E.

LEMMA 3.5.

$$\hat{F}_R(K - aE, \sigma E) \geqslant a_1 \hat{F}_K(K - aE, \sigma E) \text{ a.e. on } [0,T).$$

PROOF. Since $K > aE > 0$ a.e. on $[0,T)$ we derive

$$\frac{\partial}{\partial E_j} \hat{F}(K - aE, \sigma E) = -a_j \hat{F}_K(K - aE, \sigma E) + \hat{F}_R(K - aE, \sigma E).$$

If $\hat{F}_R(K - aE, \sigma E) < a_1 \hat{F}_K(K - aE, \sigma E)$ then

$$\frac{\partial}{\partial E_j} \hat{F}(K - aE, \sigma E) < 0 \text{ for all } j.$$

As a consequence $\hat{F}$ could be increased by diminishing all E_j, for which $E_j > 0$. This contradicts optimality, if there is a set of positive measure where $F_R < a_1 F_K$. □

In the following lemma we explicitly make use of the homogeneity of F.

LEMMA 3.6. *There is a constant $\beta > 0$ such that*

$$\hat{F}_K(K - aE, \sigma E) \leqslant \beta, \text{ a.e. on } [0,T).$$

PROOF. Define $f : [0,\infty) \to \mathbb{R}_+$ by $f(x) := \hat{F}(x, 1)$. Then, for $K > 0$, $R > 0$, and introducing $x := \frac{K}{R}$, we have

$$F(K, R) = Rf(x),\ F_K(K, R) = f'(x),\ F_R(K,R) = f(x) - xf'(x).$$

Now consider the function $g(x) := f(x) - (a_1 + x)f'(x)$. Due to (F^4) we have: $\lim_{x \downarrow 0} g(x) = -\infty$ and $g'(x) = -(a_1 + x)f''(x) > 0$. So $g(x) \geqslant 0$ implies $x \geqslant \hat{x}$ for some $\hat{x} > 0$. Hence $f'(x) \leqslant f'(\hat{x})$. Take $\beta := f'(\hat{x})$. □

Now we are ready to compute an upperbound $\bar{C}$ for the consumption, which is not binding on $[0,T]$.

LEMMA 3.7. *There is an upperbound $\bar{C}$ such that $C_h(t) < \bar{C}$ for all h and all $t \in [0,T)$.*

PROOF. Let C^* be some non-sustainable level of consumption. Take $\overline{C}$ such that $e^{-(\rho_h+\beta)T}u'_h(\overline{C})<u'_h(C^*)$ for all h, where β is introduced in the foregoing lemma. Our claim is that $C_h(t)<\overline{C}$ for all h and all t, provided we take $C_h(t)=\min(\overline{C}, \hat{C}_h(t))$, where $\alpha_h e^{-\rho_h t}u'_h(\hat{C}_h(t))=\phi(t)$. The function ϕ is introduced in Lemma (3.4). Assume, on the contrary, that there are $t_0\in[0,T)$ and h such that $C_h(t_0)=\overline{C}$. Then $\alpha_h e^{-\rho_h t_0}u'_h(\overline{C})\geqslant\phi(t_0)$. From $\dot{\phi}=-\phi\hat{F}_K$ and $0\leqslant\hat{F}_K\leqslant\beta$ it follows, that $e^{\beta t}\phi(t)$ is increasing and that ϕ is decreasing. So we have:

$$\phi(t)\leqslant\phi(0)\leqslant e^{\beta t_0}\phi(t_0)\leqslant e^{\beta t_0}\cdot\alpha_h e^{-\rho_h t_0}\cdot u'_h(\overline{C})\leqslant$$
$$\leqslant\alpha_h e^{\beta t_0}\cdot e^{-\rho_h t_0}\cdot e^{-(\rho_h+\beta)T}u'_h(C^*)\leqslant\alpha_h e^{-\rho_h T}u'_h(C^*).$$

So $\phi(t)\leqslant\alpha_h e^{-\rho_h t}u'_h(C^*)$ for all $t\in[0,T]$.

But there are points t_1 where $C_h<C^*$ and consequently $C_h<\overline{C}$. For those values of t we have $\alpha_h e^{-\rho_h t}u'_h(C_h(t))=\phi(t)$, implying $u'_h(C_h(t))\leqslant u'_h(C^*)$ or $C_h(t)\geqslant C^*$. Contradiction. □

As a result of the previous lemma we state

THEOREM 3.2. *There exist continuously differentiable functions K and S and functions E and C which solve problem (3.3)-(3.5).* □

Now fix $h\in\{1,2,\cdots,m\}$ and consider

$$J^h:=-\int_0^T\phi C_h ds+\phi(0)K^h+\sum_j\lambda_j S_j^h.$$

In view of the homogeneity of $\hat{F}$, the (shadow) profits in the economy will be zero. Hence J^h can be interpreted as the present value of the excess demand by consumer h. In order to have a general equilibrium all excess demands should be non-positive. The question we focus on is: does there exist a vector of weights α such that this is the case? The answer is in the affirmative as will be shown below. The final step is then to derive the general equilibrium market prices.

LEMMA 3.8. $\Sigma J^h=0$.

PROOF. Since $\sigma C=\hat{F}-\dot{K}$, $K(0)=K_0=\Sigma K^h$ and $K(T)=0$, it follows that

$$\Sigma J^h=-\int_0^T(\phi\hat{F}+\dot{\phi}K)ds+\lambda S_0.$$

In view of the homogeneity of $\hat{F}$ and since $-\dot{\phi}=\phi\hat{F}_K$ we have

$$\Sigma J^h=-\int_0^T(\phi\hat{F}_R\sigma E-\phi\hat{F}_K aE)ds+\lambda S_0.$$

If, for some j, $\lambda_j=0$, then it follows from the maximization of the Hamiltonian with respect to E_j that $(F_R - a_j F_K)E_j = 0$. If, for some j, $\lambda_j > 0$, then there is a necessary condition saying that

$$\int_0^T E_j dt = \sum_h S_j^h.$$

In that case we also have that $E_j > 0$ implies $\phi(\hat{F}_R - a_j \hat{F}_K) = \lambda_j$. So $\Sigma J^h = 0$ whatever the value of λ. $\square$

Let $x(t;\alpha)$ be the value of the variable x at t when the weights in the welfare functional are given by α. Define $\Delta := \{\alpha \in \mathbb{R}_+^m \mid \sum_{k=1}^m \alpha_h = 1\}$. We then consider J^h as a mapping from Δ to $\mathbb{R}$.

LEMMA 3.9. *J^h is continuous on Δ, for all h.*

PROOF. Due to the strict concavity of u_h and $\hat{F}$ we can invoke Theorem 6 in SEIERSTAD (1985) to establish the continuity of $\phi(t;\alpha)$, $C(t;\alpha)$ and $\lambda(\alpha)$. Moreover

$$\phi(t;\alpha)C_h(t;\alpha) = \alpha_h u'_h(C_h(t;\alpha)) \leq u_h(\overline{C}).$$

It follows from Lebesgue's dominated convergence theorem that J^h is continuous. $\square$

Since $\alpha_h = 0$ implies $C_h = 0$, the next lemma holds by the definition of J^h.

LEMMA 3.10. *$J^h(\alpha) > 0$ if $\alpha_h = 0$ $(h = 1, 2, \cdots, m)$.*

LEMMA 3.11. *There exists $\hat{\alpha} \in \Delta$ such that $J^h(\hat{\alpha}) = 0$ for all h.*

PROOF. Consider the mapping $g: \Delta \to \Delta$ defined by

$$g_h = \frac{\alpha_h + \max(0, J^h(\alpha))}{1 + \sum_h \max(0, J^h(\alpha))}, \quad h = 1, 2, \cdots, m.$$

Then g is continuous. By Brouwer's fixed point theorem there exists $\hat{\alpha} \in \Delta$ such that $g(\hat{\alpha}) = \hat{\alpha}$. Hence

$$\max(0, J^h(\hat{\alpha})) = \hat{\alpha}_h \sum_{h=1}^m \max(0, J^h(\hat{\alpha})) \ (h = 1, 2, \cdots, m).$$

Suppose $J^h(\hat{\alpha}) > 0$ for some h. Then $\hat{\alpha}_h > 0$. Since $\sum_{h=1}^m J^h(\hat{\alpha}) = 0$, there exists j such that $J^j(\hat{\alpha}) < 0$. Then $\hat{\alpha}_j = 0$. But $\hat{\alpha}_j = 0$ implies $J^j(\hat{\alpha}) > 0$. Therefore $J^h(\hat{\alpha}) \leq 0$ for all h. $J^h(\hat{\alpha}) = 0$ for all h since $\sum_{h=1}^m J^h(\hat{\alpha}) = 0$. $\square$

4. General competitive equilibrium

In this section we establish the existence of a general competitive equilibrium. In view of the preceding analysis, this is relatively easy.

Theorem 4.1. *The economy described in Section 2 has a general equilibrium.*

Proof. According to Lemma 3.11 there exists $\hat{\alpha}$ such that $J^h(\hat{\alpha})=0$ for $h=1,2,\cdots,m$. Observe first that $\hat{\alpha}>>0$ in view of the fact that $K^h>0$.
Define

$$r(t):=\hat{F}_K(K(t;\hat{\alpha})-aE(t;\hat{\alpha}),\sigma E(t;\hat{\alpha})),p(t):=F_R(K(t;\alpha)-aE(t;\hat{\alpha}),\sigma E(t;\hat{\alpha}))$$

and

$$p_{0j}:=\lambda_j(\hat{\alpha})/\phi(0;\hat{\alpha})\ (j=1,2,\cdots,k).$$

The first claim is that $r(t),p(t),p_{0j}$, together with $C(t;\hat{\alpha}),K(t;\hat{\alpha})$ constitute a general equilibrium in the aggregated economy.

a) feasibility.

Clearly the proposed allocation is feasible.

b) profit maximization.

Suppose there exists $t_1\in[0,T]$ and $(K,E)\geqslant 0$ such that

$$\hat{F}(K-aE,\sigma E)-r(t_1)(K-aE)-p(t_1)\sigma E>0.$$

Since $\hat{F}$ is homogeneous this is ruled out. Hence $r(t)$ and $p(t)$ are such that $K(t;\hat{\alpha})$ and $E(t;\hat{\alpha})$ are profit maximizing in the composite commodity sector. $r(t)$ and $p(t)$ ensure profit maximization in the resource sector as well. In view of the homogeneity all profits are zero.

c) utility maximization.

Since $\dot{\phi}/\phi=-r$, we may write $J^h(\hat{\alpha})$ as

$$J^h(\hat{\alpha})/\phi(0)=K^h+\sum_j p_{0j}S_j^h-\int_0^T e^{-\int_0^t r(\tau)d\tau}C_h(s)ds\geqslant 0 \tag{4.1}$$

where $\hat{\alpha}$ has been suppressed. Let $\tilde{C}_h(t)$ be an alternative consumption pattern for consumer h, satisfying (4.1). Then

$$\int_0^T e^{-\rho_h s}[u_h(C_h(s))-u_h(\tilde{C}_h(s))]ds\geqslant\int_0^T e^{-\rho_h s}u'_h(C_h(s))(C_h(s)-\tilde{C}_h(s))ds$$

$$=\int_0^T\frac{\phi}{\hat{\alpha}_h}(C_h(s)-\tilde{C}_h(s))ds=\int_0^T\frac{\phi(0)}{\hat{\alpha}_h}e^{-\int_0^t r(\tau)d\tau}(C_h(s)-\tilde{C}_h(s)ds\geqslant 0.$$

Therefore $C_h(t;\hat{\alpha})$ maximizes total utility for h, given the budget constraint. The second claim is that we have a general equilibrium for the disaggregated economy. Define $K_j^e(t)$ by $a_jR_j(t;\hat{\alpha})$ and $(K_i^y(t),R_i(t))$ as the solution of

$$\max_{K^y,R}\sum F_i(K_i^y,R_i)$$

$$\text{subject to } \sigma K^y \leqslant K(t;\hat{\alpha}) - aE(t;\hat{\alpha}), \sigma R \leqslant \sigma E(t;\hat{\alpha})$$

a) feasibility.
The proposed allocation is feasible by construction.
b) profit maximization.
Suppose there exist $t_1 \in [0,T]$, $i \in \{1,2, \cdots, l\}$ and $(K, R) \in \mathbb{R}^2_{++}$ such that

$$\Pi_i(K,R) := F_i(K,R) - r(t_1;\hat{\alpha})K - p(t_1;\hat{\alpha})R > 0.$$

Then there exist $0 < K \leqslant K(t_1;\hat{\alpha}) - aE(t_1;\hat{\alpha})$ and $0 < R \leqslant \sigma E(t_1;\hat{\alpha})$ such that $\Pi_i(K,R) > 0$, in view of the homogeneity of F_i. If

$$\Pi_i(K(t_1;\hat{\alpha}) - aE(t_1;\hat{\alpha}), \sigma E(t_1;\alpha)) > 0$$

then we have a contradiction because, as has been shown above, profits in the aggregated economy are zero. Therefore $K < K(t_1;\hat{\alpha}) - aE(t_1;\hat{\alpha})$ or $R < \sigma E(t_1;\hat{\alpha})$. But, in view of the homogeneity, it is not necessary that both inequalities are strict. If the first strict inequality holds then we have a contradiction because then $\partial \Pi_i / \partial K > 0$. If $R < \sigma E(t_1;\hat{\alpha})$ then we have a contradiction as well. Profit maximization on the part of the resource exploiting firms is evident.
c) utility maximization.
Given that profits are zero, this poses no problem. □

A remarkable feature of the general equilibrium is that the consumption rates are continuous over time, as well as the equilibrium prices. This is a result due to the special structure of the model. It would not have come about if a general existence theorem would have been applied.

5. Conclusion

This paper deals with the existence of a general competitive economy in a continuous-time finite horizon setting with natural exhaustible resources. It has been shown that under rather mild conditions with respect to technology and preferences an equilibrium exists. It has turned out to be impossible to apply existence theorems provided by the literature on infinite dimensional commodity spaces without restricting technology and preferences in such a way that cases usually dealt with in the literature on exhaustible resources would have been excluded. Our approach has been to follow Negishi in looking for a Pareto efficient allocation which can be sustained as an equilibrium. In doing so one has to be very careful in applying optimal control theory. Especially in the model at hand one has to overcome difficulties with respect to boundedness and differentiability before the maximum principle can be invoked.

No doubt the model can be generalized in several respects. The introduction of labour as a factor of production would present no problem. The extraction technology can be modified so as to depend on the remaining stock as well. It seems to be less straightforward to reduce the aggregation level by allowing for more than one consumer good. It is our conjecture however that with smooth enough preferences existence can still be shown.

A topic for further research is of course the infinite horizon case. For the discrete-time analogue existence has been established by VAN GELDROP et al. (1989), based on the fact that equilibrium outcomes of finite horizon economies are uniformly bounded. In a subsequent paper it will be shown that this approach is also useful in the continuous-time model. Finally we remark that in the present paper no attention is paid to a characterization of the equilibrium other than continuity properties. This has not been the aim here: it is also subject to further research.

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