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Perturbation Theory for Games in Normal Form and Stochastic Games

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Abstract. In this paper, the effect on values and optimal strategies of perturbations of game parameters (payoff function, transition probability function, and discount factor) is studied for the class of zero-sum games in normal form and for the class of stationary, discounted, two-person, zero-sum stochastic games.

A main result is that, under certain conditions, the value depends on these parameters in a pointwise Lipschitz continuous way and that the sets of ϵ-optimal strategies for both players are upper semicontinuous multifunctions of the game parameters.

Extensions to general-sum games and nonstationary stochastic games are also indicated.

Key Words. Game theory, games in normal form, stochastic games, nonstationary stochastic games, perturbation of payoffs, perturbation of transition probability function, perturbation of discount factor.

1. Introduction and Summary

In this paper, the main question in various settings is the following: What is the influence of perturbation of the *game parameters* on the *solutions* of the game? This question is not only of theoretical importance, but also of practical utility, because *favorable* answers to this question will give greater confidence in the use of game models in applications. Roughly speaking, *favorable* means here that small changes in the game parameters induce only small changes in the values, and *good* strategies in the original game are not *bad* in a slightly perturbed game.

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For papers in the same spirit, but in a different context, we refer to Krabs (Ref. 1), Schweitzer (Ref. 2), Tijs (Ref. 3), and Whitt (Ref. 4). Here, we focus our attention upon two classes of games, namely, games in normal form and stochastic games.

The paper is organized as follows. In Section 2, subclasses of two-person, zero-sum games in normal form with fixed strategy spaces are considered. The value appears to depend in a Lipschitz continuous manner on the payoff function (Theorem 2.1), and ϵ-optimal strategies of the original game are $(\epsilon+2\delta)$-optimal in a δ-perturbed game (Theorem 2.2). Under additional topological conditions on strategy spaces and the payoff function, the space of ϵ-optimal strategies appears to depend in an upper semicontinuous manner, in the multifunction sense, on the payoff function (Theorem 2.3). Furthermore, special attention is paid to a subclass of games with unique optimal strategies for both players (Theorems 2.4 and 2.5). Extensions to general-sum games in normal form are given (Theorems 2.6 and 2.7).

In Section 3, for a family of discounted two-person, zero-sum stochastic games with fixed countable state space and compact metric action spaces, the effect of perturbations of the reward function and the transition probability function on the value and the set of stationary ϵ-optimal strategies is studied. *Favorable* answers are also obtained here in Theorems 3.1, 3.2, 3.3.

2. Perturbations of Two-Person Games in Normal Form

A *two-person game in normal form* is an ordered quadruplet $\langle \Pi_1, \Pi_2, p_1, p_2 \rangle$, in which Π_1 and Π_2 are nonempty sets and

$$p_1 : \Pi_1 \times \Pi_2 \to \mathbb{R} \qquad \text{and} \qquad p_2 : \Pi_1 \times \Pi_2 \to \mathbb{R}$$

are real-valued functions. Π_i is called the *strategy space* of player i, and p_i the *payoff function* of player i, $i \in \{1, 2\}$. Such a game is played as follows. Player 1 and player 2 choose, independently of one another, a strategy $\pi_1 \in \Pi_1$ and a strategy $\pi_2 \in \Pi_2$, respectively; then player 1 receives a payoff $p_1(\pi_1, \pi_2)$, and player 2 a payoff $p_2(\pi_1, \pi_2)$. If

$$p_1 + p_2 = 0$$

for a two-person game $\langle \Pi_1, \Pi_2, p_1, p_2 \rangle$, then we call the game a *two-person zero-sum game* (*in normal form*), and we denote it by the ordered triplet $\langle \Pi_1, \Pi_2, p \rangle$, where

$$p := p_1 = -p_2.$$

For such a game $\langle \Pi_1, \Pi_2, p\rangle$, the *lower value*

$$\sup_{\pi_1 \in \Pi_1} \inf_{\pi_2 \in \Pi_2} p(\pi_1, \pi_2)$$

is denoted by $\underline{V}(p)$ and the *upper value*

$$\inf_{\pi_2 \in \Pi_2} \sup_{\pi_1 \in \Pi_1} p(\pi_1, \pi_2)$$

by $\bar{V}(p)$. If

$$\underline{V}(p) = \bar{V}(p),$$

then we say that the *game is strictly determined*, and then

$$\mathrm{val}(p) := \underline{V}(p)$$

is called the *value of the game* $\langle \Pi_1, \Pi_2, p\rangle$. Denote by $B(\Pi_1, \Pi_2)$ the metric space of bounded real-valued functions on $\Pi_1 \times \Pi_2$, provided with the metric d defined by

$$d(p, q) := \|p - q\|, \qquad \text{for all } p, q \in B(\Pi_1, \Pi_2).$$

Throughout this paper, for a bounded function f on a set S, the number $\sup_{x \in S}|f(x)|$ is denoted by $\|f\|$.

Let $BV(\Pi_1, \Pi_2)$ be the set of those elements $p \in B(\Pi_1, \Pi_2)$ for which the zero-sum game $\langle \Pi_1, \Pi_2, p\rangle$ is strictly determined. It is easy to show that $BV(\Pi_1, \Pi_2)$ is a closed subset of $B(\Pi_1, \Pi_2)$.

For each $p \in BV(\Pi_1, \Pi_2)$ and $\epsilon \geq 0$, let

$$O_1^\epsilon(p) := \{\pi_1 \in \Pi_1 \mid \inf_{\pi_2 \in \Pi_2} p(\pi_1, \pi_2) \geq \mathrm{val}(p) - \epsilon\},$$

$$O_2^\epsilon(p) := \{\pi_2 \in \Pi_2 \mid \sup_{\pi_1 \in \Pi_1} p(\pi_1, \pi_2) \leq \mathrm{val}(p) + \epsilon\}.$$

For each $\epsilon > 0$, the set

$$O_i^\epsilon(p) \neq \varnothing;$$

the elements of $O_i^\epsilon(p)$ are called *ϵ-optimal strategies* for player i, $i \in \{1, 2\}$. The elements of the possibly empty set

$$O_i(p) := O_i^0(p)$$

are called *optimal strategies* for player i. Note that

$$O_1(p) = \bigcap_{\epsilon > 0} O_1^\epsilon(p), \qquad O_2(p) = \bigcap_{\epsilon > 0} O_2^\epsilon(p). \tag{1}$$

The first two theorems are easy to show, and we leave it to the reader to give a proof (see Ref. 5, pp. 65–67). The first theorem states that the value function is Lipschitz continuous with constant 1; and the second theorem states that ϵ-optimal strategies of a game are $(\epsilon+2\delta)$-optimal in a δ-perturbed game.

Theorem 2.1. For each $p, q \in BV(\Pi_1, \Pi_2)$, we have

$$|\mathrm{val}(p) - \mathrm{val}(q)| \leq d(p, q).$$

Theorem 2.2. Let $\epsilon \geq 0$, $\delta > 0$. Let $p, q \in BV(\Pi_1, \Pi_2)$ such that $d(p, q) \leq \delta$. Then,

$$O_i^\epsilon(p) \subset O_i^{\epsilon+2\delta}(q), \qquad \text{for each } i \in \{1, 2\}.$$

Now we look at two-person zero-sum games for which the strategy spaces Π_1 and Π_2 are topological Hausdorff spaces. We shall call a function $p: \Pi_1 \times \Pi_2 \to \mathbb{R}$ semicontinuous if, for each $\pi_2 \in \Pi_2$, the function

$$\pi_1 \mapsto p(\pi_1, \pi_2)$$

is an upper semicontinuous function on Π_1, and if, for each $\pi_1 \in \Pi_1$, the function

$$\pi_2 \mapsto p(\pi_1, \pi_2)$$

is lower semicontinuous on Π_2. Let

$$SBV(\Pi_1, \Pi_2) := \{p \in BV(\Pi_1, \Pi_2) \mid p \text{ is semicontinuous}\}.$$

Then, for each $\epsilon > 0$, the set $O_1^\epsilon(p)$ is a closed subset of Π_1 if $p \in SBV(\Pi_1, \Pi_2)$, because

$$O_1^\epsilon(p) = f^{-1}([\mathrm{val}(p) - \epsilon, \infty)),$$

where f is the upper semicontinuous function on Π_1, defined by

$$f(\pi_1) := \inf_{\pi_2 \in \Pi_2} p(\pi_1, \pi_2).$$

If, moreover, Π_1 is compact, then it follows from (1) that $O_1(p) \neq \emptyset$. Analogously, $O_2^\epsilon(p)$ is closed if $p \in SBV(\Pi_1, \Pi_2)$ and $O_2(p) \neq \emptyset$ if Π_2 is compact.

Now, we devote some attention to multifunctions. Let X and Y be Hausdorff spaces, and let Y be compact. Let $f: X \to Y$ be a multifunction assigning, to each $x \in X$, a nonempty compact subset $f(x)$ of Y. Following Berge (Ref. 6, pp. 114, 115), we call such a multifunction *upper semicontinuous* if, for each open set $U \subset Y$, the set

$$\{x \in X \mid f(x) \subset U\}$$

is an open subset of X. It is well known that $f: X \to Y$ is upper semicontinuous iff the graph

$$\{(x, y) \in X \times Y \mid y \in f(x)\}$$

of f is a closed set in $X \times Y$ (see Ref. 6, p. 118). This can be reformulated as follows (see Ref. 6, p. 117 and p. 65). The multifunction $f: X \to Y$ is upper semicontinuous iff the following holds: for each $(x, y) \in X \times Y$ and each net $\langle x^\alpha, y^\alpha \rangle_{\alpha \in I}$ in $X \times Y$, where $(I, >)$ is a directed set, which converges to (x, y) and for which $y^\alpha \in f(x^\alpha)$ for each $\alpha \in I$, we have $y \in f(x)$. Especially this last characterization of upper semicontinuity of multifunctions indicates the usefulness of this concept for the situations in which we are involved, namely situations in which we have a multifunction $f: X \to Y$, where the topological space X represents a family of games, where Y is a compact set of strategies (or strategy pairs), and where f assigns to each $x \in X$ the compact set of ϵ-optimal strategies, or ϵ-equilibrium points, of the game x (see Theorems 2.3, 2.7, and 3.3).

We have proved the following theorem in such a way that the proof can easily be extended to general-sum games (dealt with in Theorem 2.7).

Theorem 2.3. Let Π_1 and Π_2 be compact Hausdorff spaces, and let $\epsilon \geq 0$. Then,

$$O_i^\epsilon: SBV(\Pi_1, \Pi_2) \to \Pi_i$$

is an upper semicontinuous multifunction for $i \in \{1, 2\}$.

On $SBV(\Pi_1, \Pi_2)$, we have the topology induced by the metric d.

Proof. Let

$$(p, \pi_1^*) \in SBV(\Pi_1, \Pi_2) \times \Pi_1,$$

and let $\langle p^\alpha, \pi_1^\alpha \rangle_{\alpha \in I}$ be a net in $SBV(\Pi_1, \Pi_2) \times \Pi_1$ converging to (p, π_1^*); and suppose that

$$\pi_1^\alpha \in O_1^\epsilon(p^\alpha), \qquad \text{for each } \alpha \in I.$$

We have to show that

$$\pi_1^* \in O_1^\epsilon(p).$$

Let

$$\delta(\alpha) := d(p, p^\alpha), \qquad \text{for each } \alpha \in I.$$

Then, by Theorem 2.2, we have

$$\pi_1^\alpha \in O_1^{\epsilon + 2\delta(\alpha)}(p).$$

Because the net $\langle\delta(\alpha)\rangle_{\alpha\in I}$ in $\mathbb{R}$ converges to 0, for each $k\in\mathbb{N}$, there is an $\alpha(k)\in I$ such that

$$\delta(\alpha)\leq 1/k, \qquad \text{for each } \alpha>\alpha(k).$$

This implies that

$$\pi_1^\alpha\in O_1^{\epsilon+(2/k)}(p), \qquad \text{for each } \alpha\in I, \qquad \text{with } \alpha>\alpha(k),$$

because

$$O_1^r(p)\subset O_1^s(p), \qquad \textit{if } 0\leq r\leq s.$$

But then

$$\pi_1^*\in O_1^{\epsilon+(2/k)}(p), \qquad \text{for each } k\in\mathbb{N},$$

since $O_1^{\epsilon+(2/k)}(p)$ is closed, and π_1^* is the limit of the net $\langle\pi_1^\alpha\rangle_{\alpha\in I}$. So,

$$\pi_1^*\in O_1^\epsilon(p)=\bigcap_{k\in\mathbb{N}} O_1^{\epsilon+(2/k)}(p). \qquad \square$$

From Theorem 2.3, we infer the following corollary.

Corollary 2.1. Let Π_1 and Π_2 be Hausdorff spaces, and suppose that Π_1 is compact. Let $p_1, p_2, p_3, \ldots$ be a sequence in $SBV(\Pi_1, \Pi_2)$, and let $\pi_1, \pi_2, \pi_3, \ldots$ be a sequence in Π_1 such that $\pi_n\in O_1(p_n)$ for each $n\in\mathbb{N}$. Let $p\in SBV(\Pi_1,\Pi_2)$ such that $O_1(p)$ consists of exactly one element, say π, and suppose that

$$\lim_{n\to\infty} d(p, p_n)=0.$$

Then, $\lim_{n\to\infty}\pi_n$ exists and is equal to π.

Let

$$\pi_1^*\in\Pi_1, \qquad \pi_2^*\in\Pi_2.$$

It is well known that

$$(\pi_1^*, \pi_2^*)\in O_1(p)\times O_2(p)$$

iff

$$p(\pi_1, \pi_2^*)\leq p(\pi_1^*, \pi_2^*)\leq p(\pi_1^*, \pi_2), \qquad \text{for all } \pi_1\in\Pi_1,\ \pi_2\in\Pi_2. \qquad (2)$$

In view of (2), the elements of $O_1(p)\times O_2(p)$ are called *saddle points* of the game $\langle\Pi_1,\Pi_2,p\rangle$. Now, we want to study the subset $US(\Pi_1,\Pi_2)$ of $SBV(\Pi_1,\Pi_2)$ consisting of those functions p for which $\langle\Pi_1,\Pi_2,p\rangle$ has a unique saddle point. Note that, for each

$$(\pi_1^*, \pi_2^*)\in\Pi_1\times\Pi_2,$$

the bounded semicontinuous function

$$s_{(\pi_1^*,\pi_2^*)}:\Pi_1\times\Pi_2\to\mathbb{R},$$

defined by

$$s_{(\pi_1^*,\pi_2^*)}(\pi_1,\pi_2):=\begin{cases}-1, & \text{if } \pi_2=\pi_2^* \text{ and } \pi_1\neq\pi_1^*,\\ 1, & \text{if } \pi_1=\pi_1^* \text{ and } \pi_2\neq\pi_2^*,\\ 0, & \text{elsewhere,}\end{cases} \tag{3}$$

is an element of $US(\Pi_1,\Pi_2)$ with unique saddle point (π_1^*,π_2^*). For the proof of the next theorem, we need a lemma. The proof of this lemma is a simple exercise and is left to the reader.

Lemma 2.1. Let $p\in SBV(\Pi_1,\Pi_2)$, for which $(\pi_1^*,\pi_2^*)\in O_1(p)\times O_2(p)$. Then,

$$p+\epsilon s_{(\pi_1^*,\pi_2^*)}\in US(\Pi_1,\Pi_2),\qquad \text{for each } \epsilon>0,$$

and

$$O_i(p+\epsilon s_{(\pi_2^*,\pi_2^*)})=\{\pi_i^*\},\qquad \text{for } i\in\{1,2\}.$$

Theorem 2.4. Let Π_1, Π_2 be compact Hausdorff spaces. Then, (a) the restriction of $O_i: SBV(\Pi_1,\Pi_2)\to\Pi_i$ to the subset $US(\Pi_1,\Pi_2)$ is a continuous map; and (b) $US(\Pi_1,\Pi_2)$ is a dense subset of $SBV(\Pi_1,\Pi_2)$.

Proof. (a) This follows from the fact that a single-valued map, which is upper semicontinuous in the multi-valued sense, is continuous.

(b) Let $p\in SBV(\Pi_1,\Pi_2)$ and $\epsilon>0$. We have to prove that there is a $q\in US(\Pi_1,\Pi_2)$ such that

$$d(p,q)<\epsilon.$$

Take

$$(\pi_1^*,\pi_2^*)\in O_1(p)\times O_2(p).$$

Note that

$$O_1(p)\times O_2(p)\neq\varnothing,$$

because Π_1 and Π_2 are compact. Now, let

$$q:=p+\tfrac{1}{2}\epsilon s_{(\pi_1^*,\pi_2^*)},$$

where $s_{(\pi_2^*,\pi_2^*)}$ is the function defined in (3). Then,

$$q\in US(\Pi_1,\Pi_2),$$

by Lemma 2.1, and

$$d(p, q) \leq \tfrac{1}{2}\epsilon < \epsilon. \qquad \square$$

Theorem 2.5. Let Π_1, Π_2 be compact metric spaces with metrics d_1, d_2. Then, $US(\Pi_1, \Pi_2)$ is connected iff both strategy space Π_1 and Π_2 are connected.

Proof. (a) First, suppose that (say) Π_1 is not connected. Let Π_{11} and Π_{12} be two disjunct nonempty open subsets of Π_1, with

$$\Pi_1 = \Pi_{11} \cup \Pi_{12}.$$

Let

$$US_i := \{p \in US \mid O_1(p) \subset \Pi_{1i}\}, \qquad \text{for each } i \in \{1, 2\}.$$

It is obvious that

$$US = US_1 \cup US_2, \qquad US_1 \cap US_2 = \varnothing.$$

Further, US_1 and US_2 are open in US, because O_1 is an upper semicontinuous multifunction. If we can show that

$$US_1 \neq \varnothing, \qquad US_2 \neq \varnothing,$$

then we have proved that $US(\Pi_1, \Pi_2)$ is not connected if Π_1 is not connected. Now,

$$\Pi_{11} \neq \varnothing, \qquad \Pi_{12} \neq \varnothing.$$

Take

$$\pi' \in \Pi_{11}, \qquad \pi'' \in \Pi_{12}, \qquad \pi \in \Pi_2.$$

Then, it is obvious that

$$s_{(\pi', \pi)} \in US_1, \qquad s_{(\pi'', \pi)} \in US_2,$$

where $s_{(\pi', \pi)}$ and $s_{(\pi'', \pi)}$ are defined in an analogous manner to $s_{(\pi_1^*, \pi_2^*)}$ in (3). So, $US_i \neq \varnothing$ for $i \in \{1, 2\}$; and we have proved the implication to the right in the theorem.

(b) Now, we suppose that Π_1 and Π_2 are connected sets. Let U_1 and U_2 be disjunct open subsets of the metric space $US(\Pi_1, \Pi_2)$, such that

$$US(\Pi_1, \Pi_2) = U_1 \cup U_2.$$

If we can show that

$$U_1 = \varnothing \quad \text{or} \quad U_2 = \varnothing,$$

then $US(\Pi_1, \Pi_2)$ is connected.

(b1) For each $(\pi_1^*, \pi_2^*) \in \Pi_1 \times \Pi_2$, let

$$z_{(\pi_1^*, \pi_2^*)} : \Pi_1 \times \Pi_2 \to \mathbb{R}$$

be the function defined by

$$z_{(\pi_1^*, \pi_2^*)}(\pi_1, \pi_2) = d_2(\pi_2, \pi_2^*) - d_1(\pi_1, \pi_1^*), \qquad \text{for each } (\pi_1, \pi_2) \in \Pi_1 \times \Pi_2.$$

Then,

$$z_{(\pi_1^*, \pi_2^*)} \in US(\Pi_1, \Pi_2) \quad \text{and} \quad O_i(z_{(\pi_1^*, \pi_2^*)}) = \{\pi_i^*\}, \qquad \text{for } i \in \{1, 2\}.$$

Let

$$F: \Pi_1 \times \Pi_2 \to US(\Pi_1, \Pi_2)$$

be the map defined by

$$F(\pi_1^*, \pi_2^*) = z_{(\pi_1^*, \pi_2^*)}.$$

Then, it is straightforward to show that

$$\|F(\pi_1^*, \pi_2^*) - F(\pi_1^{**}, \pi_2^{**})\| \le d_1(\pi_1^*, \pi_1^{**}) + d_2(\pi_2^*, \pi_2^{**}),$$

for all

$$(\pi_1^*, \pi_2^*), (\pi_1^{**}, \pi_2^{**}) \in \Pi_1 \times \Pi_2.$$

Hence, F is a continuous map from the connected set $\Pi_1 \times \Pi_2$ into $US(\Pi_1, \Pi_2)$. This implies that

$$\text{either } F(\Pi_1 \times \Pi_2) \subset U_1 \quad \text{or } F(\Pi_1 \times \Pi_2) \subset U_2.$$

Without loss of generality, we suppose that

$$F(\Pi_1 \times \Pi_2) \subset U_1,$$

i.e.,

$$z_{(\pi_1^*, \pi_2^*)} \in U_1, \qquad \text{for each } (\pi_1^*, \pi_2^*) \in \Pi_1 \times \Pi_2. \tag{4}$$

(b2) Now, take an arbitrary

$$p \in US(\Pi_1, \Pi_2).$$

Let

$$O_1(p) \times O_2(p) = \{(\pi_1, \pi_2)\}.$$

For each $t \in [0, 1]$, let

$$G(t) := tp + (1-t) z_{(\pi_1, \pi_2)}.$$

Then, it is easy to show that

$$G(t) \in US(\Pi_1, \Pi_2), \qquad \text{for each } t \in [0, 1].$$

Furthermore,

$$\|G(s)-G(t)\|\leq|s-t|(\|p\|+\|z_{(\pi_1,\pi_2)}\|), \qquad \text{for each } s,t\in[0,1].$$

Hence,

$$G:[0,1]\to US(\Pi_1,\Pi_2)$$

is continuous. Since $[0, 1]$ is connected and

$$G(0)=z_{(\pi_1,\pi_2)}\in U_1,$$

by (4), we may conclude that

$$G(1)=p\in U_1,$$

as well. So, we have proved that

$$US(\Pi_1,\Pi_2)\subset U_1.$$

Thus,

$$U_2=\varnothing;$$

this completes the proof of the theorem. □

Note that the metric property of Π_1 and Π_2 in Theorem 2.5 is used only in the proof of the implication to the left of that theorem.

Remark 2.1. (a) It can be shown by examples that $US(\Pi_1, \Pi_2)$ is not necessarily an open subset of $SBV(\Pi_1, \Pi_2)$.

(b) In Ref. 7, Bohnenblust, Karlin, and Shapley proved that the set U_{mn} of those $m\times n$ matrix games ($m, n\in\mathbb{N}$), for which the mixed extension has a unique saddle point, is an open and dense subset of the set of all $m\times n$ matrix games (provided with the usual topology). With some labor, one can prove that U_{mn} is not connected for all

$$(m,n)\neq(1,1).$$

We will not do this here, but remark that, in case

$$(m,n)=(1,2),$$

we have

$$U_{12}=\{[a,b]\,|\,a\neq b\}=\{[a,b]\,|\,a>b\}\cup\{[a,b]\,|\,a<b\}.$$

Hence, U_{12} is the union of two disjunct open subsets. Thus, U_{12} is not connected.

Let us now pay some attention to general-sum, two-person games in normal form. For such a game $\langle\Pi_1,\Pi_2,p_1,p_2\rangle$, a point

$$(\pi_1^*,\pi_2^*)\in\Pi_1\times\Pi_2$$

is called an *equilibrium point* if

$$p_1(\pi_1^*, \pi_2^*) = \max_{\pi_1 \in \Pi_1} p_1(\pi_1, \pi_2^*), \qquad p_2(\pi_1^*, \pi_2^*) = \max_{\pi_2 \in \Pi_2} p_2(\pi_1^*, \pi_2);$$

and is called an *ϵ-equilibrium point*, $\epsilon > 0$, if

$$p_1(\pi_1^*, \pi_2^*) \geq \sup_{\pi_1 \in \Pi_1} p_1(\pi_1, \pi_2^*) - \epsilon, \qquad p_2(\pi_1^*, \pi_2^*) \geq \sup_{\pi_2 \in \Pi_2} p_2(\pi_1^*, \pi_2) - \epsilon.$$

The set of equilibrium points of $\langle \Pi_1, \Pi_2, p_1, p_2 \rangle$ is denoted by $E(p_1, p_2)$, and the set of ϵ-equilibrium points by $E^\epsilon(p_1, p_2)$.

For fixed Π_1, Π_2, let $B^2(\Pi_1, \Pi_2)$ be the metric space of pairs (p_1, p_2) of bounded real-valued functions on $\Pi_1 \times \Pi_2$, provided with the metric d defined by

$$d((p_1, p_2), (p_1', p_2')) := \max\{\|p_1 - p_1'\|, \|p_2 - p_2'\|\},$$

$$\text{for all } (p_1, p_2) \in B^2(\Pi_1, \Pi_2), \qquad (p_1', p_2') \in B^2(\Pi_1, \Pi_2).$$

Let $BE(\Pi_1, \Pi_2)$ be the subset of $B^2(\Pi_1, \Pi_2)$, consisting of those pairs (p_1, p_2) for which

$$E^\epsilon(p_1, p_2) \neq \emptyset, \qquad \text{for all } \epsilon > 0.$$

It is well-known that

$$p \in BV(\Pi_1, \Pi_2)$$

iff

$$(p, -p) \in BE(\Pi_1, \Pi_2)$$

(see Ref. 8, Proposition 1.3) and that

$$O_1(p) \times O_2(p) = E(p, -p).$$

This implies that the following two theorems can be seen as extensions of previously mentioned results for zero-sum games. Proofs of these theorems can be obtained by slightly modifying foregoing proofs, and they will be omitted.

Theorem 2.6. (a) $BE(\Pi_1, \Pi_2)$ is a closed subset of $B^2(\Pi_1, \Pi_2)$; and (b) let

$$\epsilon \geq 0, \qquad \delta \geq 0, \qquad (p_1, p_2), \qquad (p_1', p_2') \in B^2(\Pi_1, \Pi_2),$$

$$d((p_1, p_2), (p_1', p_2')) \leq \delta.$$

Then,

$$E^\epsilon(p_1, p_2) \subset E^{\epsilon + 2\delta}(p_1', p_2').$$

See Theorems 3.7 and 3.6 in Tijs, Ref. 5, pp. 99–100.

Now, let Π_1 and Π_2 be topological spaces. Put

$$CBE(\Pi_1, \Pi_2) := \{(p_1, p_2) \in BE(\Pi_1, \Pi_2) \mid p_1 \text{ and } p_2 \text{ are continuous functions}\}.$$

Theorem 2.7. Let Π_1 and Π_2 be compact spaces and let $\epsilon > 0$. Then,
(a) $E(p_1, p_2) \neq \emptyset$ for each $(p_1, p_2) \in CBE(\Pi_1, \Pi_2)$;
(b) $(p_1, p_2) \to E(p_1, p_2)$ and $(p_1, p_2) \mapsto E^{\epsilon}(p_1, p_2)$ are upper semicontinuous multifunctions from $CBE(\Pi_1, \Pi_2)$ into $\Pi_1 \times \Pi_2$.

Remark 2.2. The results in Theorems 2.6 and 2.7 can be extended (only notational problem arise) to N-person games in normal form with $N \geq 2$, and even to countable-person games, which were studied in Vrieze (Ref. 9).

3. Perturbations in Stochastic Games

In this section, we extend some results of Section 2 to stochastic games. Stochastic games (or Markov games) were introduced in 1953 by Shapley, Ref. 10. For a recent survey of the theory of stochastic games, we refer to Parthasarathy and Stern (Ref. 11). In this section, we restrict our attention to *discounted two-person zero-sum stochastic games*, characterized by an ordered sixtuple $\langle S, A_1, A_2, r, q, \beta\rangle$, where:

(S1) S is a nonempty countable set, called the *state space*;

(S2) A_1 and A_2 are nonempty compact metric spaces, called the *action spaces* of player 1 and player 2, respectively;

(S3) $r: S \times A_1 \times A_2 \to \mathbb{R}$ is a bounded function, called the *reward function*, for which, for each $s \in S$, the map

$$(a_1, a_2) \mapsto r(s, a_1, a_2)$$

is a measurable function on $A_1 \times A_2$ (the measurability is taken with respect to the product σ-algebra of $\mathscr{A}_1$ and $\mathscr{A}_2$, where $\mathscr{A}_i$ is the σ-algebra generated by the Borel sets of A_i, $i = 1, 2$);

(S4) $q: S \times A_1 \times A_2 \to P$ is a function from $S \times A_1 \times A_2$ into the family P of probability measures on S, such that, for all $s, s' \in S$, the map

$$(a_1, a_2) \mapsto q(s' \mid s, a_1, a_2) := q(s, a_1, a_2)\{s'\}$$

is a measurable function on $A_1 \times A_2$; q is called the *transition probability function*;

(S5) β is a real number in $[0, 1)$, called the *discount factor*.

Such a stochastic game corresponds with a dynamic system with state space S, where the dynamic behavior as well as the rewards are influenced by the players at discrete points in time, say $t=0, 1, 2, \ldots$, in the following way. At each time $t \in \{0, 1, 2, \ldots\}$, the players observe the current state of the system. Then, they have to select, independently of one another, an action. If at time t the system is in state s, and if player 1 selects action $a_1 \in A_1$ and player 2 selects action $a_2 \in A_2$, then two things happen:

(a) player 1 obtains an immediate reward $r(s, a_1, a_2)$ from player 2;

(b) the system moves with probability $q(s'|s, a_1, a_2)$ to the state $s' \in S$, which is observed at time $t+1$.

Furthermore, one supposes that a reward r to player 1 (or 2) at time t has worth $\beta^t r$ at time 0 ($\beta^t r$ is called the *discounted reward*) and that player 1 (player 2) wants to maximize (minimize) the total discounted expected reward. In this context, we can restrict our attention to stationary strategies (see Blackwell, Ref. 12, Theorem 6, p. 232).

Definition 3.1. Let $\langle S, A_1, A_2, r, q, \beta\rangle$ be a stochastic game. Let P_i be the set of probability measures on $\langle A_1, \mathcal{A}_i\rangle$, $i=1, 2$. Then, each map $\pi_i: S \to P_i$ is called a *stationary strategy* for player i. The set of stationary strategies is denoted by Π_i.

Playing a stationary strategy $\pi_i \in \Pi_i$ means for player i that, each time $t \in \{0, 1, 2, \ldots\}$ that the system is in state $s \in S$, he chooses his action according to the probability measure $\pi_i(s)$. Suppose that the players 1 and 2 play

$$\pi_1 \in \Pi_1 \quad \text{and} \quad \pi_2 \in \Pi_2$$

and that the initial state, the state at $t=0$, of the system is $s \in S$. Then, the expected reward of player 1 at time $t \in \{0, 1, 2, \ldots\}$ exists and is denoted by $f^t_{srq}(\pi_1, \pi_2)$; the *total discounted expected reward*

$$\sum_{t=0}^{\infty} \beta^t f^t_{srq}(\pi_1, \pi_2)$$

is denoted by $f_{srq}(\pi_1, \pi_2)$. Note that

$$\|f_{srq}\| \leq \sum_{t=0}^{\infty} \beta^t \|r\| = (1-\beta)^{-1}\|r\|$$

and that the function

$$s \mapsto f_{srq}(\pi_1, \pi_2)$$

satisfies the relation

$$f_{srq}(\pi_1, \pi_2) = \tilde{r}(s, \pi_1(s), \pi_2(s)) + \beta \sum_{s' \in S} \tilde{q}(s'|s, \pi_1(s), \pi_2(s)) f_{s'rq}(\pi_1, \pi_2), \quad (5)$$

for all $s \in S$, where

$$\tilde{r}(s, \pi_1(s), \pi_2(s)) := \int_{A_2}\int_{A_1} r(s, a_1, a_2)\, d\pi_1(s)(a_1)\, d\pi_2(s)(a_2),$$

$$\tilde{q}(s'|s, \pi_1(s), \pi_2(s)) := \int_{A_2}\int_{A_1} q(s'|s, a_1, a_2)\, d\pi_1(s)(a_1)\, d\pi_2(s)(a_2).$$

Definition 3.2. Let $\langle S, A_1, A_2, r, q, \beta\rangle$ be a stochastic game and $\epsilon \geq 0$. A pair of stationary strategies

$$(\pi_1^\epsilon, \pi_2^\epsilon) \in \Pi_1 \times \Pi_2,$$

such that

$$-\epsilon + f_{srq}(\pi_1, \pi_2^\epsilon) \leq f_{srq}(\pi_1^\epsilon, \pi_2^\epsilon) \leq f_{srq}(\pi_1^\epsilon, \pi_2) + \epsilon$$

for all $s \in S$ and all $(\pi_1, \pi_2) \in \Pi_1 \times \Pi_2$ is called an *ϵ-saddle point* if $\epsilon > 0$, and is called a *saddle point* if $\epsilon = 0$. If, for each $\epsilon > 0$, there are ϵ-saddle points, then we say that the *stochastic game is strictly determined.* In that case, for each $s \in S$, the two-person game in normal form $\langle \Pi_1, \Pi_2, f_{srq}\rangle$ is strictly determined, and the function $V_{rq}: S \to \mathbb{R}$, where $V_{rq}(s)$ is the value of $\langle \Pi_1, \Pi_2, f_{srq}\rangle$, is called the *value* of the stochastic game. By an *ϵ-optimal (optimal) strategy $\pi_i \in \Pi_i$ for player i in the stochastic game,* we mean a strategy such that $\pi_i(s)$ is ϵ-optimal (optimal) in $\langle \Pi_1, \Pi_2, f_{srq}\rangle$ for all $s \in S$.

For the remainder of this section, S, A_1, A_2, β are fixed. Let DV be the family of pairs of functions (r, q) satisfying (S3) and (S4), such that, for each bounded function $Y: S \to \mathbb{R}$ and each $s \in S$, the *dummy game* in normal form

$$\langle P_1, P_2, \tilde{r}(s, \ldots) + \sum_{s' \in S} \tilde{q}(s'|s, \cdot, \cdot) Y(s')\rangle$$

has a value. The following proposition can be seen as a modification of Theorem 4.1 in Maitra and Parthasarathy (Ref. 13).

Proposition 3.1. Let $(r, q) \in DV$ and $\epsilon \geq 0$. Then, $\langle S, A_1, A_2, r, q, \beta\rangle$ is strictly determined. The value of the stochastic game is the unique solution of the following functional equation in $Y: S \to \mathbb{R}$:

$$Y(s) = \mathrm{val}(\tilde{r}(s, \cdot, \cdot) + \beta \sum_{s' \in S} \tilde{q}(s'|s, \cdot, \cdot) Y(s')), \qquad \text{for all } s \in S.$$

Furthermore if, for each $s \in S$, an ϵ-optimal strategy $\pi_i^\epsilon(s)$ is given for player i in the game in normal form

$$\langle P_1, P_2, \tilde{r}(s, \cdot, \cdot) + \beta \sum_{s' \in S} \tilde{q}(s'|s, \cdot, \cdot) V_{rq}(s')\rangle,$$

then the map

$$s \mapsto \pi_i^\epsilon(s)$$

is a $(1-\beta)^{-1}\epsilon$-optimal strategy for the stochastic game.

Now, we provide DV with the metric d defined by

$$d((r, q), (r', q')) := \max\{\|r - r'\|, \rho(q, q')\},$$

where

$$\rho(q, q') := \sup_{s', s, a_1, a_2} |q(s'|s, a_1, a_2) - q'(s'|s, a_1, a_2)|.$$

Theorem 3.1. The map

$$(r, q) \mapsto V_{rq}$$

from DV into B(S) is a continuous map (even pointwise Lipschitz continuous).

Proof. Let

$$(r, q), (r', q') \in DV.$$

First, note that, in view of Theorem 2.1, we have

$$|V_{rq}(s) - V_{r'q'}(s)| = |\text{val}(f_{srq}) - \text{val}(f_{sr'q'})| \leq \|f_{srq} - f_{sr'q'}\|. \quad (6)$$

Take

$$(\pi_1, \pi_2) \in \Pi_1 \times \Pi_2,$$

and put

$$x(s) := f_{srq}(\pi_1, \pi_2), \qquad x'(s) := f_{sr'q'}(\pi_1, \pi_2),$$

for each $s \in S$. Then, it follows from (5) that, for each $s \in S$,

$$|x(s) - x'(s)| \leq \|r - r'\| + \beta\|x - x'\| + \beta\rho(q, q')\|x\|,$$

and so

$$\|x - x'\| \leq \|r - r'\| + \beta\|x - x'\| + \beta\rho(q, q')\|x\|.$$

Recall that

$$\|x\| \leq (1-\beta)^{-1}\|r\|,$$

and put

$$C_r := (1-\beta)^{-1}(1 + \beta(1-\beta)^{-1}\|r\|). \quad (7)$$

Then,

$$\|f_{srq} - f_{sr'q'}\| \leq C_r\, d((r, q), (r', q')), \qquad \text{for each } s \in S. \tag{8}$$

Combining (6) and (8), we obtain

$$\| V_{rq} - V_{r'q'}\| \leq C_r\, d((r, q), (r', q')),$$

and this implies that V is pointwise Lipschitz continuous in (r, q). □

Let

$$\epsilon > 0 \quad \text{and} \quad (r, q) \in DV.$$

Denote the set of ϵ-optimal strategies for player i of the game

$$\left\langle P_1, P_2, \tilde{r}(s, \cdot, \cdot) + \beta \sum_{s' \in S} \tilde{q}(s' | s, \cdot, \cdot) V_{rq}(s') \right\rangle$$

by $O_i^\epsilon(s, r, q)$ and the set of optimal strategies by $O_i(s, r, q)$, $i = 1, 2$. Then, $X_{s \in S} O_i^\epsilon(s, r, q)$ can be seen as a subset of the set of $(1-\beta)^{-1}$ ϵ-optimal strategies of the stochastic game $\langle S, A_1, A_2, r, q, \beta\rangle$, and $X_{s \in S} O_i(s, r, q)$ can be identified with the set of optimal strategies (see Proposition 3.1). The influence of perturbations of (r, q) on this subset $X_{s\in S} O_i^\epsilon(s, r, q)$ of the set of $(1-\beta)^{-1}$ ϵ-optimal strategies can be studied by looking at $O_i^\epsilon(s, r, q)$ for each $s \in S$. The following theorem is a direct consequence of (8) and Theorem 2.2.

Theorem 3.2. Let

$$\epsilon \geq 0 \quad \text{and} \quad (r, q), (r', q') \in DV,$$

such that

$$d((r, q), (r', q')) \leq \delta.$$

Then, for each $s \in S$, we have

$$O_i^\epsilon(s, r, q) \subset O_i^{\epsilon + 2C_r\delta}(s, r', q'),$$

with C_r as defined in (7).

Let CDV be the subset of DV consisting of the elements (r, q), such that, for each $s, s' \in S$, the real-valued functions on $A_1 \times A_2$:

$$(a_1, a_2) \mapsto r(s, a_1, a_2) \quad \text{and} \quad (a_1, a_2) \mapsto q(s' | s, a_1, a_2)$$

are continuous. Now, endow P_i with the weak topology. Then, P_i is compact (see Parthasarathy, Ref. 14, Theorem 6.4, p. 45), and so

$$\Pi_i = P_i^S,$$

provided with the product topology, is also compact.

The following proposition is well known (see Refs. 15 and 13).

Proposition 3.2. Let $(r, q) \in CDV$. Then,

(a) for each $s \in S$, the function $f_{srq} : \Pi_1 \times \Pi_2 \to \mathbb{R}$ is continuous;

(b) $O_i(s, r, q) \neq \varnothing$ for each $s \in S$ and $i \in \{1, 2\}$;

(c) there is a one-to-one correspondence between the set of optimal stationary strategies for player i in the stochastic game and the set $X_{s\in S} O_i(s, r, q)$, $i \in \{1, 2\}$.

As a direct consequence of Theorem 2.3 we obtain the following theorem.

Theorem 3.3. For each $s \in S$ and $i \in \{1, 2\}$, we have that

$$O_i(s, \cdot, \cdot) : CDV \to P_i$$

is an upper semicontinuous multifunction, and therefore also

$$X_{s\in S} O_i(s, \cdot, \cdot) : CDV \to \Pi_i.$$

Remark 3.1. Now, let A_1 and A_2 be finite sets consisting of m and n elements, respectively ($m, n \in \mathbb{N}$). Once again, let S be a countable set. Let $B(S, m, n)$ consist of the pairs (r, q) with r as in (S3) and q as in (S4). As Shapley (Ref. 10) proved, for each pair $(r, q) \in B(S, m, n)$, the stochastic game $\langle S, A_1, A_2, r, q, \beta\rangle$ has a value, and both players have stationary optimal strategies. Now, for each $s \in S$, the dummy game

$$\left\langle P_1, P_2, \tilde{r}(s, \cdot, \cdot) + \beta \sum_{s'\in S} \tilde{q}(s' | s, \cdot, \cdot) V_{rq}(s')\right\rangle,$$

with value $V_{rq}(s)$, can be seen as a mixed extension of an $m \times n$ matrix game. Let $U(S, m, n)$ be the subset of pairs $(r, q) \in B(S, m, n)$ for which the game $\langle S, A_1, A_2, r, q, \beta\rangle$ has a unique pair of optimal strategies. Now, if

$$(r, q) \in B(S, m, n),$$

then we can see from Remark 2.1(b) that, for each $\epsilon > 0$, there exists a pair

$$(r_u, q) \in B(S, m, n), \qquad \text{with } \|r - r_u\| < \epsilon,$$

and such that, for each $s \in S$, the game in normal form

$$\left\langle P_1, P_2, \tilde{r}_u(s, \cdot, \cdot) + \beta \sum_{s'\in S} \tilde{q}(s' | s, \cdot, \cdot) V_{rq}(s')\right\rangle$$

possesses a unique pair of optimal strategies; furthermore, we may even suppose that the game has value $V_{rq}(s)$. But this means (see Proposition 3.1) that the stochastic game $\langle S, A_1, A_2, r_u, q, \beta\rangle$ has value V_{rq} and possesses a unique pair of optimal strategies. So,

$$(r_u, q) \in U(S, m, n).$$

The following theorem is now immediate.

Theorem 3.4. The set $U(S, m, n)$ is an open and dense subset of $B(S, m, n)$.

Remark 3.2. Most results of this section can be extended to nonstationary, two-person zero-sum games, i.e., games in which the reward function, the transition probability function, and the discount factor are time dependent.

In the stationary case, we considered pairs of functions (r, q); in the nonstationary case, we have to consider sequences of triplets (r_t, q_t, β_t). If we take an appropriate subset of these sequences of triplets,

$$\|r_t\| \leq M \in \mathbb{R}, \qquad \beta_t \leq \beta^* < 1, \qquad \text{for all } t \in \{0, 1, \ldots\},$$

and if we let the set of Markov strategies take over the role of the stationary strategies, then nonstationary versions of Propositions 3.1 and 3.2 and of Theorems 3.1–3.3 can be formulated and proved in a similar way as above.

Remark 3.3. In a similar way to that in which we have extended the results obtained for zero-sum games in normal form to zero-sum discounted stochastic games, we could extend the results for general-sum games in normal form to general-sum discounted stochastic games.

References

1. KRABS, W., *Stetige Abänderung der Daten bei Nichtlinearer Optimierung und Ihrer Konsequenzen*, Operations Research Verfahren, Vol. 25, pp. 93–113, 1977.
2. SCHWEITZER, P. J., *Perturbation Theory and Finite Markov Chains*, Journal of Applied Probability, Vol. 5, pp. 401–413, 1968.
3. TIJS, S. H., *Semi-Infinite Programs and Semi-Infinite Matrix Games*, Catholic University, Nijmegen, Holland, Department of Mathematics, Report No. 7630, 1976.
4. WHITT, W., *Continuity of Markov Processes and Dynamic Programs*, Yale University, New Haven, Connecticut, 1975.
5. TIJS, S. H., *Semi-Infinite and Infinite Matrix Games and Bimatrix Games*, Catholic University, Nijmegen, Holland, PhD Thesis, 1975.
6. BERGE, C., *Espaces Topologiques*, Dunod, Paris, France, 1959.

7. BOHNENBLUST, H. F., KARLIN, S., and SHAPLEY, L. S., *Solutions of Discrete Two-Person Games*, Annals of Mathematics Studies, Vol. 24, pp. 51–72, 1950.
8. TIJS, S. H., *ϵ-Equilibrium Point Theorems for Two-Person Games*, Operations Research Verfahren, Vol. 26, pp. 755–766, 1977.
9. VRIEZE, O. J., *Noncooperative Countable-Person Games with Compact Action Spaces*, Mathematical Center, Amsterdam, Holland, Report No. BW 65/76, 1976.
10. SHAPLEY, L. S., *Stochastic Games*, Proceedings of the National Academy of Sciences of the USA, Vol. 39, pp. 1095–1100, 1953.
11. PARTHASARATHY, T., and STERN, M., *Markov Games: A Survey*, University of Illinois, Chicago, Illinois, 1976.
12. BLACKWELL, D., *Discounted Dynamic Programming*, Annals of Mathematical Statistics, Vol. 36, pp. 226–235, 1965.
13. MAITRA, A., and PARTHASARATHY, T., *On Stochastic Games*, Journal of Optimization Theory and Applications, Vol. 5, pp. 289–300, 1970.
14. PARTHASARATHY, K. R., *Probability Measures on Metric Spaces*, Academic Press, New York, New York, 1967.
15. VRIEZE, O. J., *The Stochastic Noncooperative Countable-Person Game with Countable State Space and Compact Action Spaces under the Discounted Payoff Criterion*, Mathematical Center, Amsterdam, Holland, Report No. BW 66/76, 1976.