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On Pure Equilibria for Bimatrix Games

PETER JURG and MATHŸS JANSEN

Department of Mathematics, University of Nÿmegen, Toernooiveld, 6525 ED Nÿmegen, The Netherlands

STEF TIJS

Faculty of Economics, University of Brabant, P.O. Box 90 15 3, 5000 LE Tilburg, The Netherlands

Abstract: In Shapley (1964) several conditions are given for the existence of pure saddlepoints for a matrix game. In this paper we show that only a few of these conditions, when translated to the situation of a bimatrix game guarantee the existence of pure equilibria. Further, we associate with a bimatrix game a directed graph as well as a so-called 'binary game'. If this graph has no cycles, then the bimatrix game in question has a pure equilibrium. It is shown that the binary game for a bimatrix game without a pure equilibrium possesses a 'fundamental' subgame, which can be characterized by means of 'minimal' cycles.

1 Introduction

In Shapley (1964) various (sufficient) conditions are given for the existence of pure saddlepoints for matrix games. Each of these conditions requires that every submatrix of some fixed size has a pure saddlepoint. These conditions can easily be extended to bimatrix games, but what can be said about the existence of pure equilibria? In section 3 we give an example of a bimatrix game which satisfies all the various conditions of Shapley, translated to the situation for bimatrix games. This game however, does not have a pure equilibrium. Since this example concerns a square game, it does not fully account for the general situation. Indeed, in section 4, we show that some of Shapley's conditions are still useful to obtain pure equilibrium results for non-square bimatrix games. Furthermore we introduce for a bimatrix game the binary game. This is a bimatrix game which assigns for a player a payoff 1 to a pair of pure strategies if (in the original game) the strategy of this player is a pure best reply to the strategy of his opponent. To other pairs of pure strategies a payoff 0 is assigned. Next a directed graph is associated with the binary game. We give another condition for the existence of a pure equilibrium for a bimatrix game in terms of cycles in this graph. Note that also Smadici (1979) used graph theoretical methods to generalize a result of Shapley (1964) concerning generalized saddlepoints of matrix games to the case of a bimatrix game. In section 5 we study games without a pure equilibrium. For such games the binary game possesses a specific subgame, which is called a fundamental subgame. These fundamental subgames are characterized by means of 'minimal' cycles and their form is described in section 6.

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2 Preliminaries

Let $A = [a_{ij}]_{i=1}^{m}{}_{j=1}^{n}$ and $B = [b_{ij}]_{i=1}^{m}{}_{j=1}^{n}$ be two $m \times n$-matrices. The $m \times n$-*bimatrix game* (A, B) is defined as the two-person game in strategic form where player 1 chooses a *row*, i.e. an $i \in M := \{1, \dots, m\}$ and independently player 2 chooses a *column*, i.e. a $j \in N := \{1, \dots, n\}$. Accordingly player 1 obtains a payoff a_{ij} and player 2 obtains a payoff b_{ij}. These payoffs can be found in the ij^{th} entry (a_{ij}, b_{ij}) of the bimatrix (A, B).

A pair (i_0, j_0) of a row and a column is called an *equilibrium* for (A, B) if unilateral deviation does not yield a higher payoff, i.e.

$$\begin{aligned} a_{i_0 j_0} &= \max_i a_{i j_0} \\ b_{i_0 j_0} &= \max_j b_{i_0 j} . \end{aligned} \tag{2.1}$$

Since we don't consider the mixed extension of a game in this paper, we don't use the word pure. For non-empty subsets I of M and J of N we denote by $(A, B)_{I,J}$ the *subgame* of (A, B) where player 1 and 2 restrict their choices to a row in I and to a column in J respectively. For $1 \leq r \leq m$ and $1 \leq s \leq n$ an $r \times s$-subgame of (A, B) is a subgame $(A, B)_{I,J}$ with $|I| = r$ and $|J| = s$. A bimatrix game (A, B) with $B = -A$ is called a *matrix game* and denoted by A. An equilibrium for a matrix game is called a *saddlepoint*.

3 Some Results of Shapley on Saddlepoints

In Shapley (1964) the following results concerning the existence of saddlepoints for matrix games are presented.

Theorem 3.1: Let A be an $m \times n$-matrix game with $m, n \geq 2$. If every 2×2-subgame of A has a saddlepoint, then A has a saddlepoint.

Theorem 3.2: Let A be an $m \times n$-matrix game with $m, n \geq 2$ and with the property that no two collinear entries are equal. Let $2 \leq r \leq m$ and $2 \leq s \leq n$. Then A has a saddlepoint if every $r \times s$-subgame of A has a saddlepoint.

Here we give an example which shows that the statements in the theorems above cannot be extended to bimatrix games and equilibria. We consider the

3×3-bimatrix game given by

$$(A, B) = \begin{bmatrix} (2, \frac{1}{2}) & (\frac{1}{5}, 5) & (\frac{1}{10}, \frac{1}{11}) \\ (\frac{1}{8}, \frac{1}{9}) & (3, \frac{1}{3}) & (\frac{1}{6}, 6) \\ (\frac{1}{7}, 7) & (\frac{1}{12}, \frac{1}{13}) & (4, \frac{1}{4}) \end{bmatrix}.$$

With respect to theorem 3.1 first note that every 2×2-subgame of (A, B) has an equilibrium, where (A, B) itself has not. With respect to theorem 3.2 we note that all entries of A and B are different and that also every 2×3- and 3×2-subgame has an equilibrium. This example also shows that another result of Shapley concerning so-called detached rows and columns cannot be generalized to the case of bimatrix games.

4 On Sufficient Conditions for the Existence of an Equilibrium

In this section we state two theorems which give sufficient conditions for a bimatrix game to possess an equilibrium. The first theorem uses conditions on subgames. The second theorem concerns the existence of cycles in a directed graph associated with the game.

It may seem that the counterexample of the previous section exhausts all possibilities of finding, for bimatrix games, a result of the type of theorem 3.1 or 3.2. However, the example concerns a square game ($m = n = 3$), so that there remain some cases of theorem 3.2 still to be examined: if $m \le n$, those are the cases where $r = m$ and $m \le s \le n$. In theorem 4.1 we show that in those cases theorem 3.2 can be extended to bimatrix games, and even a version where the entries in the same row or column can be the same.

Theorem 4.1: Let (A, B) be an $m \times n$-bimatrix game with $m \le n$. Let $m \le s \le n$. Then (A, B) has an equilibrium if every $m \times s$-subgame of (A, B) has an equilibrium.

Proof: We show that if (A, B) has no equilibrium, then for all s with $m \le s \le n$ there is an $m \times s$-subgame of (A, B) that has also no equilibrium. So suppose (A, B) has no equilibrium and let $m \le s \le n$. We can choose for each $i \in M$ a $j \in N$ such that $b_{ij} = \max_{l \in N} b_{il}$. Let J' be the subset of N consisting of these columns. Then $|J'| \le m$. We add to J', if necessary, arbitrary columns in order to have s columns. By J we denote the subset of N obtained in this way. Then

$(A, B)_{M,J}$ is an $m \times s$-subgame of (A, B). Suppose that (i, j) is an equilibrium for $(A, B)_{M,J}$. Then $a_{ij} = \max_{k \in M} a_{kj}$ and $b_{ij} = \max_{l \in J} b_{il} = \max_{l \in N} b_{il}$ since $J' \subset J$. Hence (i, j) is also an equilibrium for (A, B). This contradicts the assumption. ◁

In order to find a new condition which quarantees existence of an equilibrium for a bimatrix game, we first look at a simplified version of a bimatrix game, which we call, for obvious reasons, the binary game.

For an $m \times n$-bimatrix game (A, B) we define the *binary game* (A^*, B^*) by $A^* = [a^*_{ij}]_{i=1}^{m}{}_{j=1}^{n}$ and $B^* = [b^*_{ij}]_{i=1}^{m}{}_{j=1}^{n}$ with

$$a^*_{ij} := \begin{cases} 1 & \text{if } a_{ij} = \max_k a_{kj} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad b^*_{ij} := \begin{cases} 1 & \text{if } b_{ij} = \max_l b_{il} \\ 0 & \text{otherwise .} \end{cases} \tag{4.1}$$

Clearly (A^*, B^*) is a bimatrix game. Let (i_0, j_0) be an equilibrium for (A^*, B^*). By (4.1), for every row i there is a column l such that $b^*_{il} = 1$ and for every column j there is a row k such that $a^*_{kj} = 1$. Then, by (2.1), the $i_0 j_0^{\text{th}}$ entry of (A^*, B^*) is (1, 1). By (4.1) again this implies that (i_0, j_0) is an equilibrium for (A, B). Conversely, if (i_0, j_0) is an equilibrium for (A, B), then by (2.1) and (4.1) we have that the $i_0 j_0^{\text{th}}$ entry of (A^*, B^*) is (1, 1), which implies that (i_0, j_0) is an equilibrium for (A^*, B^*). Hence we have a proof of

Lemma 4.1: Let (A, B) be a bimatrix game. Then a pair of a row and a column is an equilibrium for (A, B) if and only if this pair is an equilibrium for (A^*, B^*).

Next we will associate with (the binary game of) an $m \times n$-bimatrix game (A, B) a directed graph $\mathcal{G}(A^*, B^*)$. The *points* of this graph are the elements of $M \times N$. An *arc* in the graph is an ordered pair of points $((i, j), (i', j'))$ with the property

$$a^*_{ij} = 0, \, a^*_{i'j'} = 1 \quad \text{and} \quad j = j'$$

or

$$b^*_{ij} = 0, \, b^*_{i'j'} = 1 \quad \text{and} \quad i = i' \; .$$

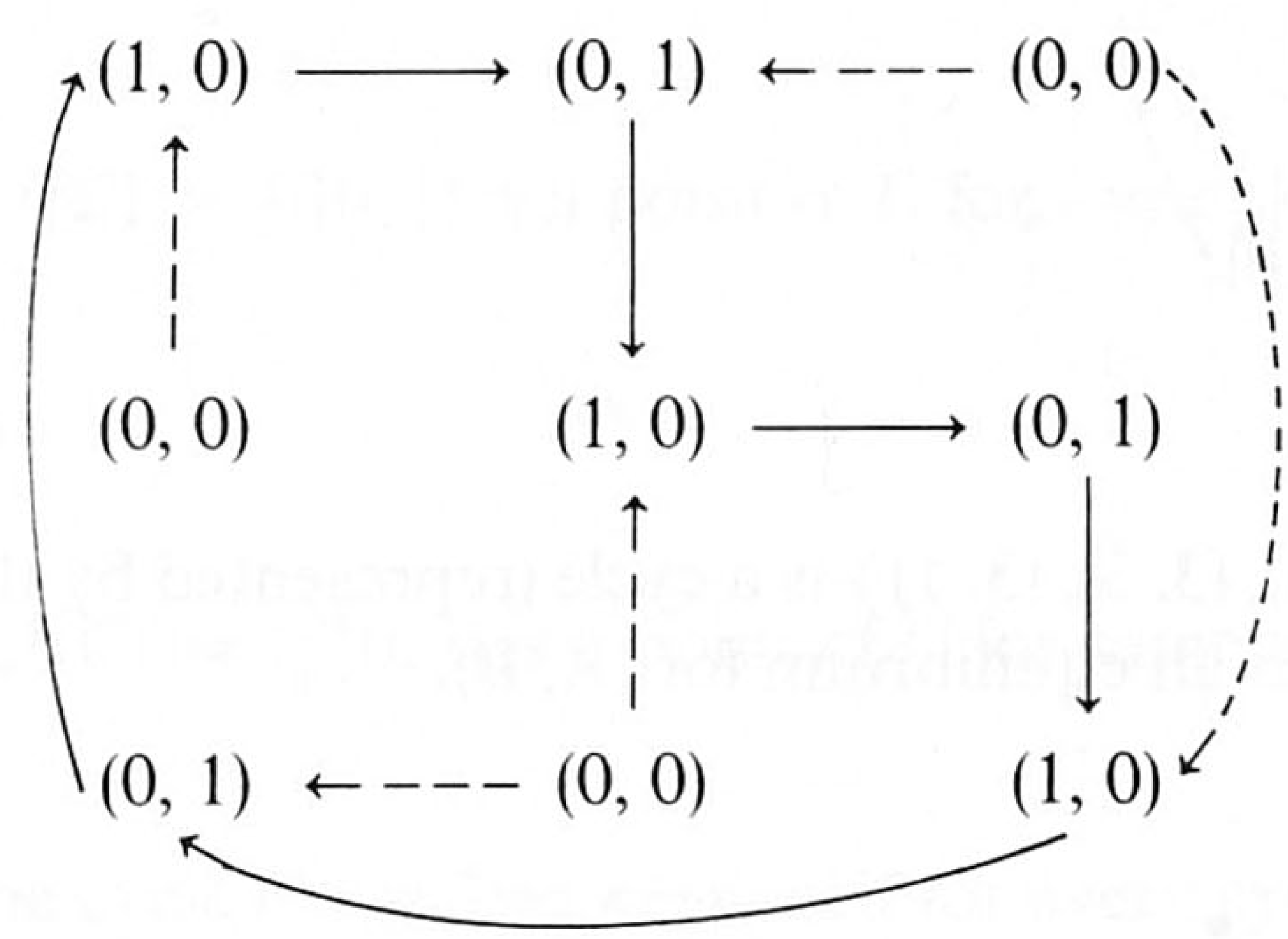
We say that the arc $((i, j), (i', j'))$ *leaves* (i, j) and *enters* (i', j'). Clearly an arc either reflects a unilateral deviation of player 1 in order to increase his payoff (from 0 to 1 in the binary game) or a unilateral deviation of the second player with the same purpose.

A sequence $C := \langle (i_1, j_1), (i_2, j_2), \ldots, (i_t, j_t) \rangle$ of $t > 1$ points is called a *cycle* in $\mathscr{G}(A^*, B^*)$ if, for each $s \in \{1, \ldots, t-1\}$, $((i_s, j_s), (i_{s+1}, j_{s+1}))$ is an arc and if $((i_t, j_t), (i_1, j_1))$ is an arc. We say that these are arcs in C and for $s \in \{1, 2, \ldots, t\}$, we call (i_s, j_s) a point of C. The thus defined graph can be represented by a picture of the entries of the binary game, with an arrow pointed from one entry to another if there exists an arc that leaves the (point corresponding to the) first of these two entries and enters the (point corresponding to the) second one.

Example 1: The binary game of the game following theorem 3.2 is

$$\begin{bmatrix} (1,0) & (0,1) & (0,0) \\ (0,0) & (1,0) & (0,1) \\ (0,1) & (0,0) & (1,0) \end{bmatrix}.$$

A representation of the graph associated with this game is



The non-dotted arrows in this figure represent the only cycle in the graph. ◁

Now we can formulate our second result on the existence of equilibria.

Theorem 4.2: If for a bimatrix game (A, B) *there is no cycle in* $\mathscr{G}(A^*, B^*)$*, then* (A, B) *has an equilibrium.*

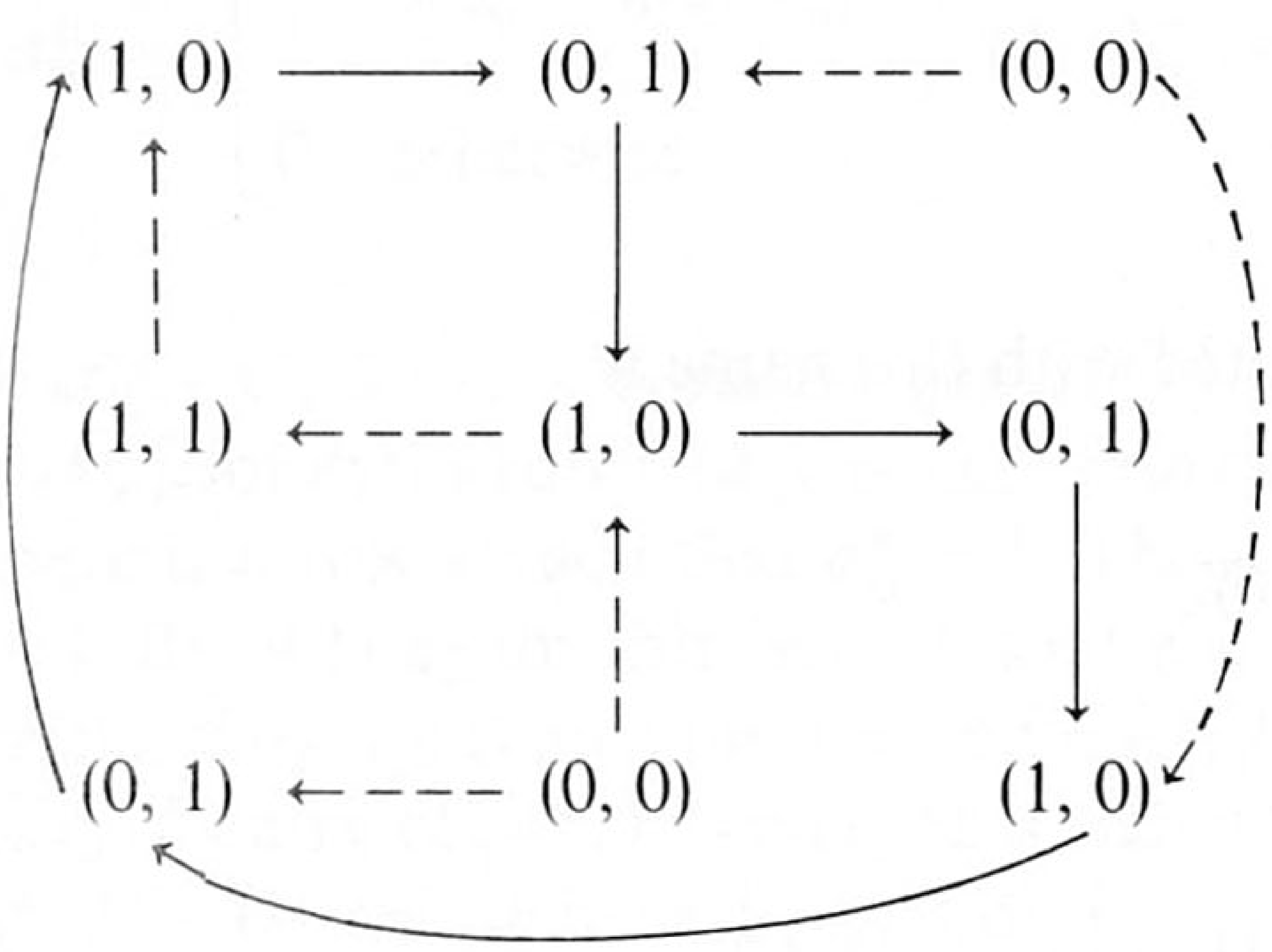
Proof: If there is no cycle in $\mathscr{G}(A^*, B^*)$, one can find a point (i, j) of this graph for which there is no arc leaving it. Then by lemma 4.1, $(a^*_{ij}, b^*_{ij}) \neq (0, 0)$. If $(a^*_{ij}, b^*_{ij}) = (1, 0)$, the fact that there is no arc leaving (i, j) implies that $b^*_{il} \neq 1$ for all l. This however is impossible in view of the definition of the binary game. In an analogous way one shows that the case $(a^*_{ij}, b^*_{ij}) = (0, 1)$ cannot occur. So $(a^*_{ij}, b^*_{ij}) = (1, 1)$ and by lemma 4.1 the proof is complete. ◁

In the next example we show that if a bimatrix game has an equilibrium, there can be a cycle in the graph associated with this game.

Example 2: For the 3×3-bimatrix game

$$(A, B) := \begin{bmatrix} (1,0) & (0,1) & (0,0) \\ (1,1) & (1,0) & (0,1) \\ (0,1) & (0,0) & (1,0) \end{bmatrix}$$

$(A^*, B^*) = (A, B)$ and the representation of $\mathscr{G}(A^*, B^*)$ is



Clearly $C := \langle(1, 1), (1, 2), (2, 2), (2, 3), (3, 3), (3, 1)\rangle$ is a cycle (represented by the non-dotted arrows). However, $(2, 1)$ is an equilibrium for (A, B). $\lhd$

5 Games Without an Equilibrium

The game in example 1 has a special property: although the game itself has no equilibrium, every subgame of it has one. We will consider games with this property – or more generally subgames with this property – more closely.

Let (A, B) be an $m \times n$-bimatrix game and let $I \subset M$ and $J \subset N$ be non-empty. We call the subgame $(A^*, B^*)_{I,J}$ of (A^*, B^*) *fundamental*, if firstly $(A^*, B^*)_{I,J}$ has no equilibrium and secondly if for every $I' \subset I$ and $J' \subset J$ with $I' \neq I$ or $J' \neq J$ the subgame $(A^*, B^*)_{I',J'}$ does have an equilibrium.

For a bimatrix game without an equilibrium, the binary game does not have an equilibrium either. If this binary game is not fundamental, it has a subgame without an equilibrium. If this subgame again is not fundamental, then there is a subgame of this subgame without an equilibrium. Since a bimatrix game has only finitely many subgames, this reasoning leads to

Theorem 5.1: Let (A, B) be a bimatrix game without an equilibrium. Then (A^*, B^*) has a fundamental subgame.

The converse of this theorem is not true. If we skip the first row and first column of the game

$$(A, B) = \begin{bmatrix} (1, 1) & (0, 0) & (0, 0) \\ (0, 0) & (1, 0) & (0, 1) \\ (0, 0) & (0, 1) & (1, 0) \end{bmatrix}$$

we obtain a fundamental subgame of $(A^*, B^*) = (A, B)$ and $(1, 1)$ is an equilibrium of (A, B).

By theorem 4.2, there exists at least one cycle in the graph associated with a bimatrix game without an equilibrium. In order to show that fundamental subgames are strongly related to special cycles, we define for a bimatrix game (A, B) and a cycle C in $\mathscr{G}(A^*, B^*)$

$$I(C) := \{i \mid (i, j) \text{ is a point of } C \text{ for some } j\}$$

and

$$J(C) := \{j \mid (i, j) \text{ is a point of } C \text{ for some } i\} .$$

The cycle C is called *minimal* if for every cycle C' with $I(C') \subset I(C)$ and $J(C') \subset J(C)$ we have $I(C') = I(C)$ and $J(C') = J(C)$.

In the following lemma we describe some useful properties of cycles.

Lemma 5.1: Let (A, B) be a bimatrix game and let C be a cycle in $\mathscr{G}(A^*, B^*)$. Then
1) *if (i, j) is a point of C, then $(a^*_{ij}, b^*_{ij}) \in \{(0, 1), (1, 0)\}$,*
2) *for every $i \in I(C)$ there is a $j \in J(C)$ such that $(a^*_{ij}, b^*_{ij}) = (0, 1)$,*
3) *for every $j \in J(C)$ there is an $i \in I(C)$ such that $(a^*_{ij}, b^*_{ij}) = (1, 0)$.*

Proof: We only prove (1) and (2).

1) For a point (i, j) of C, there is an arc entering (i, j) and an arc leaving (i, j). If $(a^*_{ij}, b^*_{ij}) = (1, 1)$, no arc leaves (i, j) and if $(a^*_{ij}, b^*_{ij}) = (0, 0)$, no arc enters (i, j).

2) Let $i \in I(C)$. Then there is a $j \in J(C)$ such that (i, j) is a point of C. According to 1), then either $(a^*_{ij}, b^*_{ij}) = (1, 0)$ or $(a^*_{ij}, b^*_{ij}) = (0, 1)$. In the latter case the proof is complete. So suppose that $(a^*_{ij}, b^*_{ij}) = (1, 0)$. Since (i, j) is a point of C, there is an arc in C that leaves (i, j). By the definition of an arc this implies that there is a j' such that $((i, j), (i, j'))$ is this arc and $b^*_{ij'} = 1$. Then, in view of 1) and the fact that (i, j') is a point of C, $a^*_{ij'} = 0$. ◅

The following lemma makes clear in which situation cycles appear.

Lemma 5.2: Let (A, B) be an $m \times n$-bimatrix game and let $I \subset M$ and $J \subset N$ be non-empty. If $(A^*, B^*)_{I,J}$ has no equilibrium, then there is a cycle C in $\mathcal{G}(A^*, B^*)$ such that $I(C) \subset I$ and $J(C) \subset J$.

Proof: Let $(A^*, B^*)_{I,J}$ have no equilibrium. Assume that there is no cycle C in $\mathcal{G}(A^*, B^*)$ such that $I(C) \subset I$ and $J(C) \subset J$. Then there exists a point (i, j) with $i \in I$ and $j \in J$ such that there is no arc which leaves (i, j) and enters some other point (i', j') with $i' \in I$ and $j' \in J$. Then, by the definition of an arc, $a^*_{ij} = 0$ implies $a^*_{kj} = 0$ for all $k \in I$ and $b^*_{ij} = 0$ implies $b^*_{il} = 0$ for all $l \in J$. Hence, if $a^*_{ij} = b^*_{ij} = 0$, then $a^*_{ij} = \max_{k \in I} a^*_{kj}$ and $b^*_{ij} = \max_{l \in J} b^*_{il}$. Clearly these equalities also hold if $a^*_{ij} = 1$ or $b^*_{ij} = 1$. So (i, j) is an equilibrium for $(A^*, B^*)_{I,J}$. This contradicts our assumption. Hence there is a cycle C in $\mathcal{G}(A^*, B^*)$ with $I(C) \subset I$ and $J(C) \subset J$. ◅

Theorem 5.2: Let (A, B) be an $m \times n$-bimatrix game without an equilibrium and let $I \subset M$ and $J \subset N$ be non-empty. Then $(A^*, B^*)_{I,J}$ is a fundamental subgame of (A^*, B^*) if and only if $I = I(C)$ and $J = J(C)$ for some minimal cycle C in $\mathcal{G}(A^*, B^*)$.

Proof: (a) First let $(A^*, B^*)_{I,J}$ be a fundamental subgame of (A^*, B^*). Then $(A^*, B^*)_{I,J}$ has no equilibrium. Hence, by lemma 5.2, there is a cycle C in $\mathcal{G}(A^*, B^*)$ such that $I(C) \subset I$ and $J(C) \subset J$. Let C' be a cycle in $\mathcal{G}(A^*, B^*)$ such that $I(C') \subset I(C)$ and $J(C') \subset J(C)$. Since $(A^*, B^*)_{I,J}$ has no equilibrium, it has no entry $(1, 1)$. If $(A^*, B^*)_{I(C'), J(C')}$ has an equilibrium, say (i, j), then by 2) and 3) of lemma 5.1, $a^*_{ij} = b^*_{ij} = 1$. Consequently, $(A^*, B^*)_{I(C'), J(C')}$ has no equilibrium. Since $(A^*, B^*)_{I,J}$ is fundamental, this implies $I(C') = I$ and $J(C') = J$. Hence $I(C) = I$ and $J(C) = J$ and C is minimal.

(b) Secondly, let C be a minimal cycle in $\mathcal{G}(A^*, B^*)$. As in part (a) and using the fact that (A^*, B^*) has no entry $(1, 1)$, one shows that $(A^*, B^*)_{I(C), J(C)}$ has no equilibrium. Let $I \subset I(C)$, $J \subset J(C)$ and $I \neq I(C)$ or $J \neq J(C)$. Suppose $(A^*, B^*)_{I,J}$ has no equilibrium. Then, by lemma 5.2, there is a cycle C' in $\mathcal{G}(A^*, B^*)$ such that $I(C') \subset I$ and $J(C') \subset J$. Since C is minimal, this is a contradiction. So $(A^*, B^*)_{I,J}$ has an equilibrium for every $I \subset I(C)$ and $J \subset J(C)$ with $I \neq I(C)$ or $J \neq J(C)$. This implies that $(A^*, B^*)_{I(C),J(C)}$ is a fundamental subgame. $\triangleleft$

In the second part of the proof of the foregoing theorem we explicitly used the assumption that (A, B) has no equilibrium. For the game in example 2, $C := \langle(1, 1), (1, 2), (2, 2), (2, 3), (3, 3), (3, 1)\rangle$ is a minimal cycle (represented by the non-dotted arrows) and $(A^*, B^*)_{I(C),J(C)} = (A^*, B^*)$. However, since (2, 1) is an equilibrium for (A^*, B^*), this game is not a fundamental subgame. So this example shows that the assumption of theorem 5.2 is essential.

6 The Fundamental Subgame

In this section we use the concept of a minimal cycle to describe the form of a fundamental subgame.

Let (A, B) be an $m \times n$-bimatrix game without an equilibrium and let $I \subset M$ and $J \subset N$ be such that $(A^*, B^*)_{I,J}$ is a fundamental subgame of (A^*, B^*). By theorem 5.2, there is a minimal cycle C in $\mathscr{G}(A^*, B^*)$ such that $I = I(C)$ and $J = J(C)$. After permuting, if necessary, the rows and columns of (A^*, B^*) we may suppose that $I = \{1, 2, \ldots, s\}$ and $J = \{1, 2, \ldots, t\}$ for some $s, t \in \mathbb{N}$ and that $(a^*_{11}, b^*_{11}) = (1, 0)$ (see lemma 5.1).

We will show that $s = t$ and that the rows and columns of (A^*, B^*) can be permuted in such a way that

$$(A^*, B^*)_{I,J} = \begin{bmatrix} (1,0) & (0,1) & \cdot & \cdots & \cdot & \cdot \\ \cdot & (1,0) & (0,1) & \cdots & \cdot & \cdot \\ \cdot & \cdot & (1,0) & \cdots & \cdot & \cdot \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \cdots & (0,1) & \cdot \\ \cdot & \cdot & \cdot & \cdots & (1,0) & (0,1) \\ (0,1) & \cdot & \cdot & \cdots & \cdot & (1,0) \end{bmatrix}, \tag{6.1}$$

where the entries that are not indicated are equal to (0, 0).

The proof consists of several steps.

1) By 2) of lemma 5.1 there is a $j \in J$ such that $(a^*_{1j}, b^*_{1j}) = (0, 1)$. We may suppose that $j = 2$. By 3) of lemma 5.1 there is an $i \in I$ such that $(a^*_{i2}, b^*_{i2}) = (1, 0)$. We may suppose that $i = 2$.

2) By 2) of lemma 5.1 there is a $j \in J$ such that $(a^*_{2j}, b^*_{2j}) = (0, 1)$. If $j = 1$, then $\tilde{C} := \langle(1, 1), (1, 2), (2, 2), (2, 1)\rangle$ is a cycle in $\mathcal{G}(A^*, B^*)$. Since $I(\tilde{C}) \subset I(C)$ and $J(\tilde{C}) \subset J(C)$, $I(C) = I(\tilde{C})$ and $J(C) = J(\tilde{C})$ in view of the minimality of C. So in this case $s = t = 2$ and $(A^*, B^*)_{I,J}$ has the form as described in (6.1). If $j > 2$, we may suppose that $j = 3$.
3) By 3) of lemma 5.1, there is an $i \in I$ such that $(a^*_{i3}, b^*_{i3}) = (1, 0)$. If $i = 1$, then $\tilde{C} := \langle(1, 2), (2, 2), (2, 3), (1, 3)\rangle$ is a cycle in $\mathcal{G}(A^*, B^*)$ with $I(\tilde{C}) \subset I(C)$ and $J(\tilde{C}) \subset J(C)$. The fact that $1 \in J(C)$ and $1 \notin J(\tilde{C})$ contradicts the minimality of C. So $i > 2$ and we may suppose that $i = 3$.
4) We can proceed in this way until we arrive at the situation that $(a^*_{tt}, b^*_{tt}) = (1, 0)$. Then by 2) of lemma 5.1 there is a $j \in J$ such that $(a^*_{tj}, b^*_{tj}) = (0, 1)$. However $j < t$. If $j \neq 1$, we can derive a contradiction like we did in part 3). So $j = 1$.
5) Also if one of the entries not indicated in (6.1) is not equal to (0, 0), we can derive a contradiction like in part 3).
6) So we have found that $s \geq t$ and that $\tilde{C} := \langle(1, 1), (1, 2), (2, 2), \ldots, (t - 1, t), (t, t), (t, 1)\rangle$ is a cycle in $\mathcal{G}(A^*, B^*)$. As in part 2) one can show that $I(C) = I(\tilde{C})$. So $s = t$.

Since one easily shows that a subgame of (A^*, B^*) with the form as described in (6.1) is fundamental, we have shown

Theorem 6.1: Let (A, B) be a bimatrix game without an equilibrium. Then a subgame of (A^*, B^*) is fundamental if and only if it is a square game with the form as in (6.1) *(if necessary after permuting the rows and columns of (A^*, B^*)).*

Note that in fact theorem 6.1 also gives a description of the form of a minimal cycle.

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