SIAM J. DISC. MATH.
Vol. 5, No. 3, pp. 305–320, August 1992

ON THE POSITION VALUE FOR COMMUNICATION SITUATIONS*

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Abstract. A new solution concept for communication situations is considered: the position value. This concept is based on an evaluation of the importance of the various communication links between the players. An axiomatic characterization of the position value is provided for the class of communication situations where the communication graphs contain no cycles. Furthermore, relations with the Myerson value are discussed, and, for special classes of communication situations, elegant calculation methods for their position values are described.

Key words. game theory, graphs, communication, Myerson value, position value

AMS(MOS) subject classification. 90D

1. Introduction. A *communication situation* corresponds to a triple (N, v, A), where $N := \{1, \cdots, n\}$ is the set of agents (players), (N, v) is a *coalitional game* having player set N and characteristic function $v : 2^N \rightarrow \mathbb{R}$ with $v(\emptyset) = 0$, and where (N, A) is an undirected *communication graph*.

The game (N, v) describes the economic possibilities of each coalition (= subgroup of players) that decides to cooperate. However, cooperation is restricted because communication is. The possibilities for communication are described by the graph (N, A), where the arc set A consists of (unordered) pairs of players. If $\{i, j\} \in A$, then the interpretation is that players i and j can communicate directly. Indirect communication between i and j is possible if there is a path in (N, A) from i to j. Because of this interpretation and since we will implicitly assume that each player is able to communicate with himself, we may restrict our attention to communication graphs without parallel arcs and loops. For convenience, we assume throughout this paper that the underlying game (N, v) is zero-normalized; i.e., $v(\{i\}) = 0$ for all $i \in N$.

Communication situations were first studied in Myerson (1977), who introduced corresponding *graph-restricted games* or *communication games* and who characterized the Shapley value (Shapley (1953)) of these games in terms of efficiency and fairness. An alternative proof of this characterization is provided in Aumann and Myerson (1988), who also address the following question. Given a coalitional game, what communication links may be expected to form between the players? Here the Shapley value of the corresponding communication games is used as a criterion. In the present paper, Myerson's communication games will be called *point games*, and the Shapley value of these games will be called the *Myerson value* (cf. Aumann and Myerson (1988)) for the corresponding communication situation. Point games and the Myerson value were also investigated in Owen (1986), who concentrated on situations where the communication graph is a tree. Other types of graph-restricted games were introduced in Rosenthal (1988a), (1988b) by putting weights on the communication arcs representing costs of communication and measures of trust or friendship, respectively. Finally, we note that Myerson (1980) generalized the idea of direct communication between two players toward direct communication between the players of certain subgroups of N (*conferences*).

The present paper offers an alternative approach to evaluate a communication situation. While Myerson's point game focuses on the role of a node of the communication

* Received by the editors February 10, 1990; accepted for publication (in revised form) May 2, 1991.
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graph (a player) in establishing communication within the various possible coalitions, this paper proposes a dual point of view and concentrates on the role of an arc (communication link). The communicative strength of an arc is measured by means of the Shapley value of a kind of "dual" game on the arcs of the communication graph: a so-called arc game. Then, assuming each player has veto power of the use of any arc that he is an endpoint of, it seems reasonable to divide the worth of an arc equally between the two players who are at its endpoints. The total amount that a player obtains in this way is called the *position value* for the player in the corresponding communication situation. This value was first introduced in Meessen (1988). Formal definitions are provided in § 2.

It may be noted that, in general, the Myerson value and the position value differ. In § 3 it is shown that for the class of those communication situations in which the communication graph contains no cycles, the position value can be characterized by four properties: additivity, component efficiency, the superfluous arc property, and the degree property. Furthermore, for the same class of communication situations, a new axiomatic characterization of the Myerson value is provided in terms of the first three properties mentioned above and the so-called communication ability property.

In § 4 we derive a relation between the *dividends* (cf. Harsanyi (1959)) of an arc game and the dividends of the coalitional game underlying the communication situation. For the position value, this leads to computational results in the manner of Owen (1986) for special subclasses of communication situations in which the underlying coalitional game is a pure overhead game or a quadratic measure game. These results are described in § 5. The paper concludes with some remarks for the case when the communication graph does contain cycles.

Preliminaries. Let $N := \{1, \cdots, n\}$ and $2^N := \{S | S \subset N\}$. By G^N we denote the class of all coalitional games (N, v) and by G_0^N, the subclass of all zero-normalized games. A game (N, v) often will be identified with its characteristic function v.

Let $v \in G^N$. Then the *Shapley value* $\Phi(v)$ of v (cf. Shapley (1953)) is defined by

$$\Phi_i(v) = \frac{1}{n!} \sum_{\sigma \in P(N)} (v(PR_\sigma(i) \cup \{i\}) - v(PR_\sigma(i)))$$

for all $i \in N$, where $P(N)$ is the set of all permutations of N and

$$PR_\sigma(i) := \{j \in N | \sigma(j) < \sigma(i)\}$$

denotes the set of *predecessors* of player i according to σ. Furthermore, having the *unanimity game* $u_S \in G^N$ on S defined by

$$u_S(T) = \begin{cases} 1 & \text{if } T \supset S, \\ 0 & \text{else,} \end{cases}$$

for all $S \in 2^N \setminus \{\emptyset\}$, we find that $v = \sum_{S \in 2^N \setminus \{\emptyset\}} \Delta_v(S) u_S$, where the *dividends* $\Delta_v(S)$ (cf. Harsanyi (1959)) are given by

$$\Delta_v(S) = \sum_{T \subset S} (-1)^{|S| - |T|} v(T).$$

For the Shapley value, it readily follows that

$$\Phi_i(u_S) = \begin{cases} \dfrac{1}{|S|} & \text{if } i \in S, \\ 0 & \text{else} \end{cases}$$

for all $S \in 2^N \setminus \{\emptyset\}$ and, consequently,

$$\Phi_i(v) = \sum_{S \subset N : i \in S} \frac{\Delta_v(S)}{|S|}$$

for all $i \in N$. Finally, we define the empty sum to be zero.

2. Myerson value and position value. Let CS^N denote the class of all communication situations (N, v, A) with fixed player set N as described in § 1. So, especially, (N, v) is zero-normalized, and (N, A) is an undirected graph without parallel arcs and loops.

Let $(N, v, A) \in CS^N$. It is clear that the communication possibilities within N, given by the graph (N, A), determine a partition N/A of N into (communication) *components*. So a coalition T is a component within N if and only if all players in T can communicate and if there is no coalition $\bar{T}$ with $T \subset \bar{T}$ and $T \neq \bar{T}$ in which all players can communicate. Similarly, we can define components within each given coalition S by only allowing the communication possibilities given by the subgraph $(S, A(S))$, where

$$(1) \qquad A(S) := \{\{i,j\} \in A \mid i \in S, j \in S\}.$$

Then a partition of S results, which will be denoted by S/A.

The following notation will be used frequently. Let the player set N and the game $v \in G_0^N$ be fixed. Then, for each $S \subset N$ and each $L \subset \{\{i,j\} \mid i \in N, j \in N\}$,

$$(2) \qquad r^v(S,L) := \sum_{T \in S/L} v(T)$$

will denote the reward for the coalition S having the communication arcs in $L(S) \subset L$ available (cf. (1)). Note that $r^v(\emptyset, L) = r^v(S, \emptyset) = 0$ for all S and L.

Now we can formulate the following definition.

DEFINITION (cf. Myerson (1977)). Let $(N, v, A) \in CS^N$. Then the *point game* (N, r_A^v) corresponding to (N, v, A) is given by

$$(3) \qquad r_A^v(S) := r^v(S,A) \quad \text{for all } S \in 2^N.$$

Furthermore, the *Myerson value* $\mu(N, v, A) \in \mathbb{R}^N$ corresponds to the Shapley value of (N, r_A^v), so

$$(4) \qquad \mu(N,v,A) := \Phi(N, r_A^v).$$

Another type of game corresponding to a communication situation is introduced in the definition below.

DEFINITION. Let $(N, v, A) \in CS^N$. Then the *arc game* (A, r_N^v) corresponding to (N, v, A) is given by

$$(5) \qquad r_N^v(L) = r^v(N,L) \quad \text{for all } L \in 2^A.$$

Furthermore, the *position value* $\pi(N, v, A) \in \mathbb{R}^N$ is given by

$$(6) \qquad \pi_i(N,v,A) := \sum_{a \in A_i} \frac{1}{2} \Phi_a(A, r_N^v) \quad \text{for all } i \in N,$$

where

$$(7) \qquad A_i := \{\{i,j\} \in A \mid j \in N\}$$

denotes the set of all arcs of which player i is an endpoint.

If there can be no misunderstanding, the upper index v will be omitted from the notation above. The following example illustrates the various concepts introduced above.

Example 2.1. For the communication situation (N, v, A), let $N = \{1, 2, 3\}$, $A = \{\{1, 3\}, \{2, 3\}\}$, and let v equal the unanimity game $u_{\{1,2\}}$. This situation is schematically represented in Fig. 1. Then (N, r_A) is given by

$$r_A(S) = \begin{cases} 1 & \text{if } S = N, \\ 0 & \text{else.} \end{cases}$$

So $\mu(N, v, A) = \Phi(N, r_A) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Furthermore, (A, r_N) is given by

$$r_N(L) = \begin{cases} 1 & \text{if } L = A, \\ 0 & \text{else.} \end{cases}$$

So, with $a := \{1, 3\}$ and $b := \{2, 3\}$,

$$\Phi_a(A, r_N) = \Phi_b(A, r_N) = \tfrac{1}{2}$$

and

$$\pi_1(N, v, A) = \tfrac{1}{2}\Phi_a(A, r_N) = \tfrac{1}{4}, \qquad \pi_2(N, v, A) = \tfrac{1}{2}\Phi_b(A, r_N) = \tfrac{1}{4},$$
$$\pi_3(N, v, A) = \tfrac{1}{2}\Phi_a(A, r_N) + \tfrac{1}{2}\Phi_b(A, r_N) = \tfrac{1}{2}.$$

Note that, for the game in Example 2.1, the position value for each player $i \in N$ is the same multiple of the *degree* $d_i(N, A) := |A_i|$ of the corresponding point in the graph (N, A), which in some sense is a "natural" measure for the importance of a point (player) in the (communication) graph. This relation between the position value and degree will play an important role in the axiomatic characterization of the position value given in § 3.

This section is concluded with an example showing that even if there are no restrictions on communication at all, i.e., if $A = \{\{i, j\} \in 2^N \mid i \neq j\}$, the Myerson value (which then just is the Shapley value of the underlying game (N, v)) may differ from the position value.

Example 2.2. Let (N, v, A) be given by

$$N = \{1, 2, 3\}, v = u_{\{1,2\}} \quad \text{and} \quad A = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\},$$

as represented in Fig. 2. Then $r_A = v$ and, with $a := \{1, 2\}$, $b := \{1, 3\}$, and $c := \{2, 3\}$,

$$r_N(L) = \begin{cases} 1 & \text{if } a \in L \text{ or } \{b, c\} \subset L, \\ 0 & \text{else.} \end{cases}$$

Hence $\mu(N, v, a) = \Phi(N, u_{\{1,2\}}) = (\frac{1}{2}, \frac{1}{2}, 0)$, $\Phi_a(r_N) = \frac{2}{3}$, $\Phi_b(r_N) = \Phi_c(r_N) = \frac{1}{6}$, and $\pi(N, v, A) = (\frac{5}{12}, \frac{5}{12}, \frac{2}{12})$.

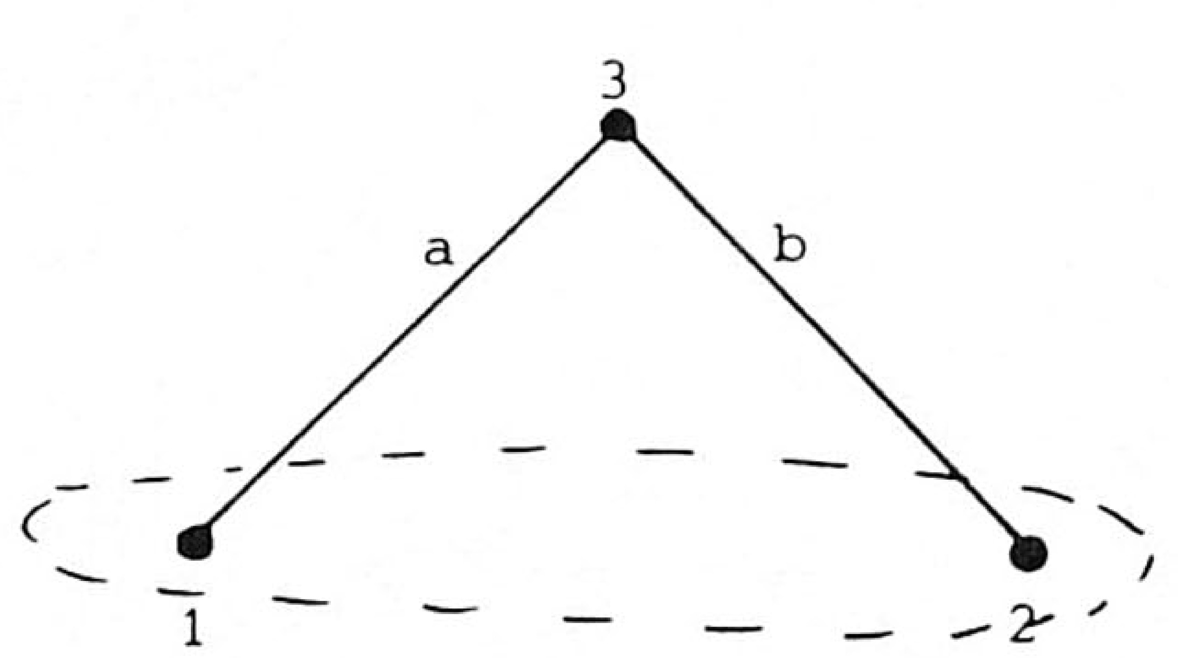

FIG. 1

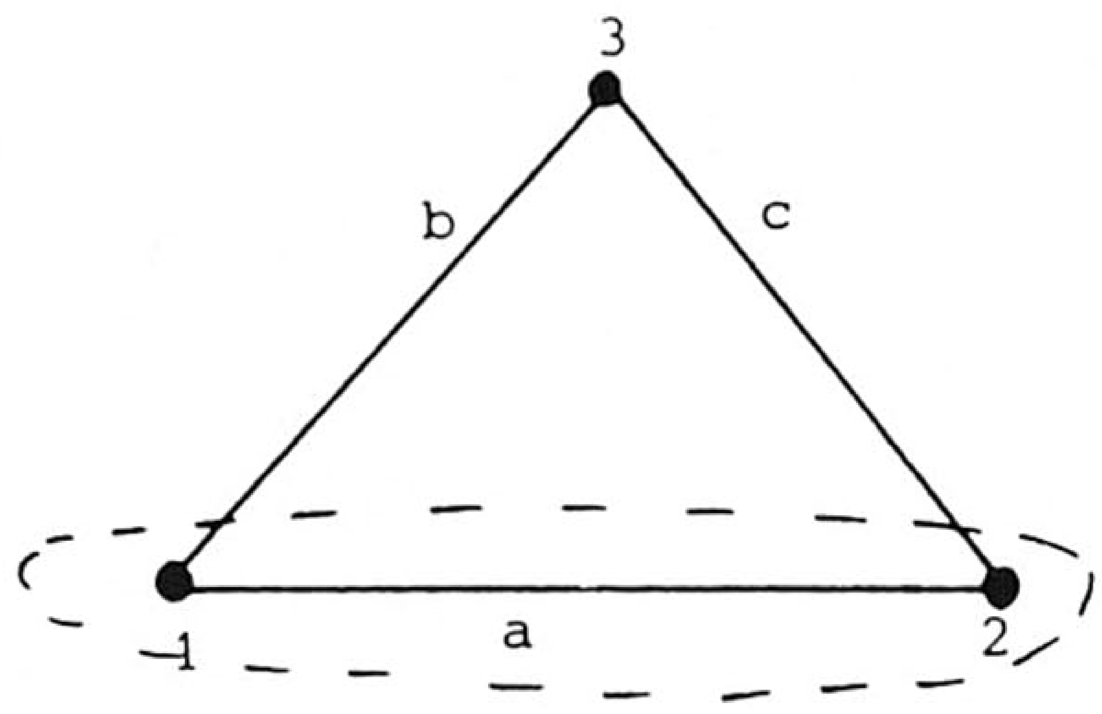

FIG. 2

3. Axiomatic characterizations. In this section we consider properties of allocation rules $\gamma : CS^N \to \mathbb{R}^N$ and, in particular, of the rules μ and π introduced in the previous section. Myerson (1977) showed the following result.

THEOREM 3.1. *The Myerson value $\mu : CS^N \to \mathbb{R}^N$ is the unique allocation rule that satisfies component efficiency and fairness.*

Here, a rule $\gamma : CS^N \to \mathbb{R}^N$ is called *component efficient* if

$$\sum_{i \in C} \gamma_i(N, v, A) = v(C) \tag{8}$$

for all $(N, v, A) \in CS^N$ and all components $C \in N/A$. Note that, for all $C \in N/A$, we have that $v(C) = \sum_{T \in C/A} v(T) = r(C, A)$. Furthermore, $\gamma : CS^N \to \mathbb{R}^N$ is called *fair* (cf. Myerson (1977)) if

$$\gamma_i(N, v, A) - \gamma_i(N, v, A \setminus \{\{i, j\}\}) = \gamma_j(N, v, A) - \gamma_j(N, v, A \setminus \{\{i, j\}\}) \tag{9}$$

for all $(N, v, A) \in CS^N$ and $\{i, j\} \in A$. So, if we use a fair allocation criterion in the manner of Myerson and an arc is removed from the communication graph, then the two players connected by this arc lose (or gain) the same amount.

The following example shows that the position value $\pi : CS^N \to \mathbb{R}^N$ does not satisfy this fairness criterion.

Example 3.1. Consider the three-person communication situation (N, v, A) of Example 2.1. Then

$$\pi_1(N, v, A) - \pi_1(N, v, A \setminus \{\{1, 3\}\}) = \tfrac{1}{4} - 0 = \tfrac{1}{4},$$

while

$$\pi_3(N, v, A) - \pi_3(N, v, A \setminus \{\{1, 3\}\}) = \tfrac{1}{2} - 0 = \tfrac{1}{2}.$$

A rule $\gamma : CS^N \to \mathbb{R}^N$ is called *additive* if

$$\gamma(N, v + w, A) = \gamma(N, v, A) + \gamma(N, w, A) \tag{10}$$

for all $v, w \in G_0^N$ and communication graphs (N, A).

An arc $a \in A$ is called *superfluous* for the communication situation (N, v, A) if

$$r(N, L) = r(N, L \cup \{a\}) \quad \text{for all } L \subset A. \tag{11}$$

This means that in each communication subsystem the presence of a superfluous arc does not affect the gains of the grand coalition. A rule $\gamma : CS^N \to \mathbb{R}^N$ is said to have the *superfluous arc property* if

$$\gamma(N, v, A) = \gamma(N, v, A \setminus \{a\}) \tag{12}$$

for all $(N, v, A) \in CS^N$ and superfluous arcs $a \in A$.

Now we can formulate the following lemma.

LEMMA 3.1. *Both the Myerson value* $\mu : CS^N \to \mathbb{R}^N$ *and the position value* $\pi : CS^N \to \mathbb{R}^N$ *satisfy component efficiency, additivity, and the superfluous arc property.*

Proof. For both μ and π, additivity follows trivially from the additivity of the Shapley value, since $r_A^{v+w} = r_A^v + r_A^w$ and $r_N^{v+w} = r_N^v + r_N^w$ for all $v, w \in G_0^N$ and all communication graphs (N, A).

For μ, component efficiency follows from Theorem 3.1. To prove the component efficiency for π, let $(N, v, A) \in CS^N$ and $C \in N/A$. Then

$$\sum_{i\in C} \pi_i(N,v,A) = \sum_{i\in C}\sum_{a\in A_i} \frac{1}{2}\Phi_a(A,r_N) = \sum_{a\in A(C)} \Phi_a(A,r_N)$$

$$\overset{(*)}{=} \sum_{a\in A(C)} \Phi_a(A(C),r_N) = r_N(A(C)) = v(C),$$

where equality $(*)$ follows from the definition of the Shapley value and the fact that

$$r(N,L\cup\{a\}) - r(N,L) = r(N,(L\cap A(C))\cup\{a\}) - r(N,L\cap A(C))$$

for all $L \subset A$ and $a \in A(C)$.

Now let $(N, v, A) \in CS^N$ have a superfluous arc $a \in A$. To prove the superfluous arc property for μ, it suffices to show that $r_A = r_{A\setminus\{a\}}$.

Let $S \subset N$. Then

$$r_A(S) = \sum_{T\in S/A} v(T) = \sum_{T\in S/A(S)} v(T) = \sum_{T\in N/A(S)} v(T) = r(N,A(S)),$$

where the third equality follows from the fact that v is zero-normalized and, similarly,

$$r_{A\setminus\{a\}}(S) = r(N,A(S)\setminus\{a\}).$$

So (11) immediately implies that $r_A(S) = r_{A\setminus\{a\}}(S)$ for all S.

Proving the superfluous arc property for π, expression (11) directly implies that the arc a is a zero-player in the game (A, r_N) and $\Phi_a(A, r_N) = 0$. Furthermore, it follows that $\Phi_b(A, r_N) = \Phi_b(A\setminus\{a\}, r_N)$ for all $b \in A\setminus\{a\}$. Hence, $\pi(N, v, A) = \pi(N, v, A\setminus\{a\})$. □

A communication situation $(N, v, A) \in CS^N$ is called *arc anonymous* if there exists a function $f: \{0, 1, \cdots, |A|\} \to \mathbb{R}$ such that

$$(13) \qquad r(N,L) = f(|L|) \quad \text{for all } L\subset A.$$

Since in an arc anonymous communication situation all arcs are equally important (cf. Lemma 3.2), the communicative strength of a node (player) can be measured by its degree, and therefore it seems reasonable to allocate the gains of the grand coalition proportional to this degree. Formally, an allocation rule $\gamma : CS^N \to \mathbb{R}^N$ is said to have the *degree property* if for all arc anonymous communication situations (N, v, A), we have that $\gamma(N, v, A)$ is a multiple of the degree vector $d(N, A) := (d_i(N, A))_{i\in N}$, i.e.,

$$(14) \qquad \gamma(N,v,A) = \alpha d(N,A) \quad \text{for some } \alpha\in\mathbb{R}.$$

An equivalent formulation of (13) is given in the following lemma. The proof is straightforward and is therefore omitted.

LEMMA 3.2. *Let* $(N, v, A) \in CS^N$. *Then* (N, v, A) *is arc anonymous if and only if* $r(N, L\setminus\{a\}) = r(N, L\setminus\{b\})$ *for all* $L \subset A$ *and* $a, b \in L$.

The Myerson value μ does not have the degree property as is seen in the next example.

Example 3.2. The communication situation (N, v, A) of Example 2.1 is arc anonymous since (13) holds with $f: \{0, 1, 2\} \to \mathbb{R}$ given by $f(0) = f(1) = 0, f(2) = 1$. Furthermore,

$$d(N,A)=(1,1,2), \qquad \pi(N,v,A)=(\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{2})=\tfrac{1}{4}d(N,A),$$

and $\mu(N, v, A) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, which is not a multiple of $d(N, A)$.

However, we have the following lemma.

LEMMA 3.3. *The position value* $\pi : CS^N \to \mathbb{R}^N$ *satisfies the degree property.*

Proof. Let $(N, v, A) \in CS^N$ be arc anonymous. If $A = \varnothing$, then $\pi(N, v, A) = 0 = d(N, A)$. So assume that $A \neq \varnothing$. Let f be as in (13) and let $a \in A$. Since (A, r_N) is a symmetric game (i.e., all arcs are substitutes) we have that

$$\Phi_a(A,r_N)=\frac{1}{|A|}r_N(A)=\frac{1}{|A|}f(|A|).$$

Hence

$$\pi_i(N,v,A)=\sum_{a\in A_i}\frac{1}{2}\Phi_a(A,r_N)=\sum_{a\in A_i}\frac{1}{2}\cdot\frac{f(|A|)}{|A|}=\frac{1}{2}\frac{f(|A|)}{|A|}\cdot|A_i|=\frac{1}{2}\frac{f(|A|)}{|A|}\cdot d_i(N,A)$$

for all $i \in N$. □

Now we prove that the above-mentioned properties of the position value characterize π completely on the subclass CS_*^N of *cycle-free* communication situations where the communication graphs contain no cycles.

THEOREM 3.2. *The (restriction of the) position value* π *is the unique allocation rule on* CS_*^N *that satisfies component efficiency, additivity, the superfluous arc property, and the degree property.*

Proof. Let $\gamma : CS_*^N \to \mathbb{R}$ satisfy the four properties stated in the theorem.

Because of Lemmas 3.1 and 3.3, it suffices to show that $\gamma(N, v, A) = \pi(N, v, A)$ for all $(N, v, A) \in CS_*^N$. Let (N, A) be a communication graph without cycles. By additivity and by the fact that $\{u_S \mid |S| \geqq 2\}$ is a basis of the class G_0^N of zero-normalized games, it remains to prove that

$$\gamma(N,\beta u_S,A)=\pi(N,\beta u_S,A) \tag{15}$$

for all $\beta \in \mathbb{R}$ and $S \in 2^N$ with $|S| \geqq 2$. Let $\beta \in \mathbb{R}$ and $S \in 2^N$, $|S| \geqq 2$, be fixed throughout the proof. Furthermore, for notational convenience we define $w := \beta u_S$. To prove (15) we will distinguish between two cases.

The first case is that there is no component $C \in N/A$ with $S \subset C$. Then $r(N, L) = 0$ for all $L \subset A$. So $\Phi_a(A, r_N^w) = 0$ for all $a \in A$ and, consequently, $\pi_i(N, v, A) = 0$ for all $i \in N$. Furthermore, since in this case each arc is superfluous, the superfluous arc property implies that $\gamma(N, w, A) = \gamma(N, w, \varnothing)$. Then, since $(N, w, \varnothing)$ trivially is arc anonymous, the degree property implies that there is an $\alpha \in \mathbb{R}$ such that

$$\gamma_i(N,w,\varnothing)=\alpha d_i(N,\varnothing)=0$$

for all $i \in N$. Hence $\pi = \gamma$.

Second, let $C \in N/A$ be such that $S \subset C$. Then $(C, A(C))$ is a tree, and there exists a (unique) set $H(S) \subset C$ defined by

$$H(S):=\cap\{T \mid S\subset T\subset C, (T,A(T)) \text{ is a connected (sub)graph}\}, \tag{16}$$

which is called the *connected hull* of S (cf. Owen (1986, Thm. 5)).

It is easy to verify that

$$(17)\qquad r^{w}(N,L)=\begin{cases}\beta & \text{if } A(H(S))\subset L,\\ 0 & \text{else.}\end{cases}$$

Hence

$$\Phi_a(A,r_N^{w})=\begin{cases}|A(H(S))|^{-1}\beta & \text{if } a\in A(H(S)),\\ 0 & \text{else,}\end{cases}$$

which implies that

$$(18)\qquad \begin{aligned}\pi_i(N,w,A)&=\sum_{a\in A_i\cap A(H(S))}\frac{1}{2}|A(H(S))|^{-1}\beta\\ &=\frac{d_i(N,A(H(S)))}{2|A(H(S))|}\beta=\frac{d_i(N,A(H(S)))}{\sum_{j\in N}d_j(N,A(H(S)))}\beta\end{aligned}$$

for all $i\in N$. Furthermore, (17) implies that each arc $a\notin A(H(S))$ is superfluous, and so the superfluous arc property implies that

$$(19)\qquad \gamma(N,w,A)=\gamma(N,w,A(H(S))).$$

The communication situation $(N, w, A(H(S)))$ is arc anonymous because of Lemma 3.2 and the fact that

$$r^{w}(N,L\setminus\{a\})=r^{w}(N,L\setminus\{b\})=0$$

for all $L\subset A(H(S)))$ and $a, b\in L$. Therefore the degree property implies that there is an $\alpha\in\mathbb{R}$ such that

$$(20)\qquad \gamma_i(N,w,A(H(S)))=\alpha d_i(N,A(H(S)))\quad \text{for all } i\in N.$$

So, especially, $\gamma_i(N, w, A(H(S))) = 0$ for all $i\in N\setminus H(S)$.

Using component efficiency, we find that

$$\sum_{i\in C}\gamma_i(N,w,A(H(S)))=\sum_{i\in H(S)}\gamma_i(N,w,A(H(S)))=w(C)=\beta.$$

So from (20) we may conclude that

$$(21)\qquad \alpha=\left(\sum_{i\in H(S)}d_i(N,A(H(S)))\right)^{-1}\beta.$$

Combining these results gives $\gamma=\pi$. □

By introducing one other property for allocation rules, we will be able to provide a new axiomatic characterization of the restriction of the Myerson value to communication situations in CS_*^N.

A communication situation $(N, v, A)\in CS^N$ is called *point anonymous* if there exists a function $f\colon\{0, 1, \cdots, |D|\}\to\mathbb{R}$ such that

$$(22)\qquad r(S,A)=f(|S\cap D|)$$

for all $S\in 2^N$, where

$$(23)\qquad D:=\{i\in N\,|\,d_i(N,A)>0\}.$$

An allocation rule $\gamma: CS^N\to\mathbb{R}^N$ is said to have the *communication ability property* if, for all point anonymous communication situations (N, v, A), there exists an $\alpha\in R$

such that

$$(24) \qquad \gamma_i(N,v,A)=\begin{cases}\alpha & \text{for all } i\in D,\\ 0 & \text{else.}\end{cases}$$

Then we have the following lemma.

LEMMA 3.4. *The Myerson value* $\mu : CS^N \to \mathbb{R}^N$ *satisfies the communication ability property.*

Proof. Let (N, v, A) be point anonymous and let f and D be as in (22) and (23), respectively. Because each player $i \in N\backslash D$ is a dummy player in (N, r_A) and all players in D are substitutes, it is found that

$$\mu_i(N,v,A)=\Phi_i(N,r_A)=\begin{cases}\frac{r_A(N)}{|D|} & \text{if } i\in D,\\ 0 & \text{else}\end{cases}=\begin{cases}\frac{f(|D|)}{|D|} & \text{if } i\in D,\\ 0 & \text{else.}\end{cases} \qquad \square$$

THEOREM 3.3. *The (restriction of the) Myerson value* μ *is the unique allocation rule on* CS^N_* *that satisfies component efficiency, additivity, the superfluous arc property, and the communication ability property.*

Proof. Let $\gamma : CS^N_* \to \mathbb{R}^N$ satisfy the four properties above. Let the cycle-free communication graph (N, A), the real $\beta \in \mathbb{R}$, and the coalition $S \in 2^N$, $|S| \geqq 2$, be fixed, and define $w := \beta u_S$. Then Lemmas 3.1 and 3.4, and additivity, imply that it suffices to show that $\gamma(N, w, A) = \mu(N, w, A)$.

If there is no component $C \in N/A$ with $S \subset C$, then $r^w(T, A) = 0$ for all $T \in 2^N$. Consequently, $\mu_i(N, w, A) = \Phi_i(N, r^w_A) = 0$ for all $i \in N$. Furthermore, we trivially have that (N, w, A) is point anonymous, so there exists an $\alpha \in \mathbb{R}$ such that $\gamma_i(N, w, A) = \alpha$ for all $i \in N$ with $d_i(N, A) > 0$ and $\gamma_i(N, w, A) = 0$ otherwise. Using component efficiency, we may conclude that $\alpha = 0$ and $\gamma(N, w, A) = \mu(N, w, A)$.

Let $C \in N/A$ be such that $S \subset C$ and let $H(S)$ be as in the proof of Theorem 3.2. Then it is easily checked that

$$r^w(T,A)=\begin{cases}\beta & \text{if } H(S)\subset T,\\ 0 & \text{else}\end{cases}$$

and, consequently,

$$\mu_i(N,w,A)=\begin{cases}(1/|H(S)|)\beta & \text{if } i\in H(S),\\ 0 & \text{else.}\end{cases}$$

Furthermore, as we have seen in the proof of Theorem 3.2, the superfluous arc property implies that

$$\gamma(N,w,A)=\gamma(N,w,A(H(S))).$$

Then, since $(N, w, A(H(S)))$ is point anonymous with $D = H(S)$,

$$f(0)=\cdots=f(|D|-1)=0 \quad \text{and} \quad f(|D|)=\beta,$$

the communication ability property and component efficiency imply that $\gamma(N, w, A) = \mu(N, w, A)$. $\square$

4. A relation between dividends. In this section we derive a result that can be used to compute the position value for various subclasses of communication situations in which the communication graph contains no cycles. This result is based upon the fact that, for each communication situation (N, v, A) in CS^N_*, the coefficients (dividends) of the corresponding arc game r_N^v, with respect to the basis of (arc) unanimity games, can be expressed in terms of the dividends of the underlying game v. It may be noted that Owen (1986) has derived similar results for computing the Myerson value.

For a communication situation $(N, v, A) \in CS_*^N$ and the corresponding arc game (A, r_N^v), we can write

$$r_N^v = \sum_{L \in 2^A \setminus \{\varnothing\}} \Gamma_v(L) u_L,$$

where, for all $L \in 2^A \setminus \{\varnothing\}$, (A, u_L) is the (arc) unanimity game on L and

$$\Gamma_v(L) := \Delta_{r_N^v}(L) = \sum_{K \subset L} (-1)^{|L|-|K|} r_N^v(K)$$

denotes the dividend for L in the game r_N^v.

Then, extending definition (16) of the connected hull of a coalition S in a cycle-free communication graph (N, A) by setting $H(S) = \varnothing$ if there is no component $C \in N/A$ that contains S, the dividends of the arc game r_N^v and the underlying game v are related in the following way.

THEOREM 4.1. *Let $(N, v, A) \in CS_*^N$. Then*

$$\Gamma_v(L) = \sum_{S \in \Sigma(L)} \Delta_v(S) \tag{25}$$

for all $L \in 2^A \setminus \{\varnothing\}$, where

$$\Sigma(L) := \{S \in 2^N \setminus \{\varnothing\} \mid H(S) \neq \varnothing, L = A(H(S))\} \tag{26}$$

denotes the set of those coalitions that exactly must use all communication arcs in L to communicate.

Proof. Let $L \in 2^A \setminus \{\varnothing\}$. Then

$$\begin{aligned}
\Gamma_v(L) &= \sum_{K \subset L} (-1)^{|L|-|K|} \sum_{C \in N/K} v(C) = \sum_{K \subset L} (-1)^{|L|-|K|} \sum_{C \in N/K} \sum_{S \in 2^N \setminus \{\varnothing\}} \Delta_v(S) u_S(C) \\
&= \sum_{K \subset L} (-1)^{|L|-|K|} \sum_{C \in N/K} \sum_{S \in 2^C \setminus \{\varnothing\}} \Delta_v(S) \\
&= \sum_{S \in 2^N \setminus \{\varnothing\}} \Delta_v(S) \sum_{K \subset L} \sum_{C \in N/K : S \subset C} (-1)^{|L|-|K|} \\
&= \sum_{S \in 2^N \setminus \{\varnothing\}} \Delta_v(S) \sum_{K \subset L} |\{C \in N/K \mid S \subset C\}| (-1)^{|L|-|K|} \\
&= \sum_{S \in 2^N \setminus \{\varnothing\} : H(S) \neq \varnothing} \Delta_v(S) \sum_{K : A(H(S)) \subset K \subset L} (-1)^{|L|-|K|} \\
&= \sum_{S \in \Sigma(L)} \Delta_v(S),
\end{aligned}$$

where the last equality follows from the fact that, for each $S \in 2^N \setminus \{\varnothing\}$ such that $H(S) \neq \varnothing$, and with $l := |L|$ and $a := |A(H(S))|$,

$$\begin{aligned}
\sum_{K : A(H(S)) \subset K \subset L} (-1)^{|L|-|K|} &= \sum_{k=a}^{l} (-1)^{l-k} \binom{l-a}{k-a} \\
&= (-1)^{l-a} \sum_{k=0}^{l-a} (-1)^k \binom{l-a}{k} \\
&= \begin{cases} 1 & \text{if } l = a, \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

□

In Lemma 4.1, below, the set $\Sigma(L)$ of expression (26) is characterized in an easier way. Here we use the following definitions. Let (N, A) be a communication graph. Then, for $L \subset A$, $N(L) \subset N$ is defined to be the set of all players that are endpoints of an arc in L, and, if (N, A) is a *tree* (i.e., a connected graph that contains no cycles), then

$$\text{Ext}\,(N,A):=\{i\in N \mid d_i(N,A)=1\} \tag{27}$$

denotes the nonempty set of *extreme points* of (N, A).

LEMMA 4.1. *Let (N, A) be a (communication) graph that contains no cycles. Let $L \in 2^A \setminus \{\varnothing\}$. Then the following two assertions hold.*

(i) *If $(N(L), L)$ is not a tree, then $\Sigma(L) = \varnothing$.*

(ii) *If $(N(L), L)$ is a tree, then*

$$\Sigma(L)=\{S\subset N(L) \mid \text{Ext}\,(N(L),L)\subset S\}. \tag{28}$$

Proof. Condition (i) is trivial, and the proof of (ii) is a straightforward but tiresome exercise from the definitions. Its most important ingredients are that, for a tree $(N(L), L)$,

$$H(\text{Ext}\,(N(L),L))=H(N(L))=N(L) \quad \text{and} \quad A(N(L))=L. \qquad \square$$

The above results are illustrated in the following example.

Example 4.1. Consider the four-person communication situation (N, v, A) with $v = u_{\{1,2\}}$ and $A = \{a, b, c\}$, where $a = \{1, 4\}$, $b = \{2, 3\}$, and $c = \{3, 4\}$, as represented in Fig. 3. With, e.g., $L = \{b, c\}$, we find that $N(L) = \{2, 3, 4\}$ and $\text{Ext}\,(N(L), L) = \{2, 4\}$. Lemma 4.1 then implies that $\Sigma(\{b, c\}) = \{\{2, 4\}, \{2, 3, 4\}\}$. In a similar way, we obtain

$$\Sigma(\{a,b\})=\varnothing, \quad \Sigma(\{a\})=\{1,4\}, \quad \Sigma(\{b\})=\{2,3\}, \quad \Sigma(\{c\})=\{3,4\},$$

$$\Sigma(\{a,c\})=\{\{1,3\},\{1,3,4\}\}, \quad \text{and}$$

$$\Sigma(\{a,b,c\})=\{\{1,2\},\{1,2,3\},\{1,2,4\},\{1,2,3,4\}\}.$$

Therefore, since

$$\Delta_v(S)=\begin{cases}1 & \text{if } S=\{1,2\},\\ 0 & \text{else,}\end{cases}$$

we have

$$\Gamma_v(L)=\sum_{S\in\Sigma(L)}\Delta_v(S)=\begin{cases}1 & \text{if } L=\{a,b,c\},\\ 0 & \text{else.}\end{cases}$$

5. Examples. (i) *Unanimity games*. Using Theorem 4.1, we will re-prove expression (18) (for $\beta = 1$), which describes the position value for each cycle-free communication situation in which the underlying game is a unanimity game.

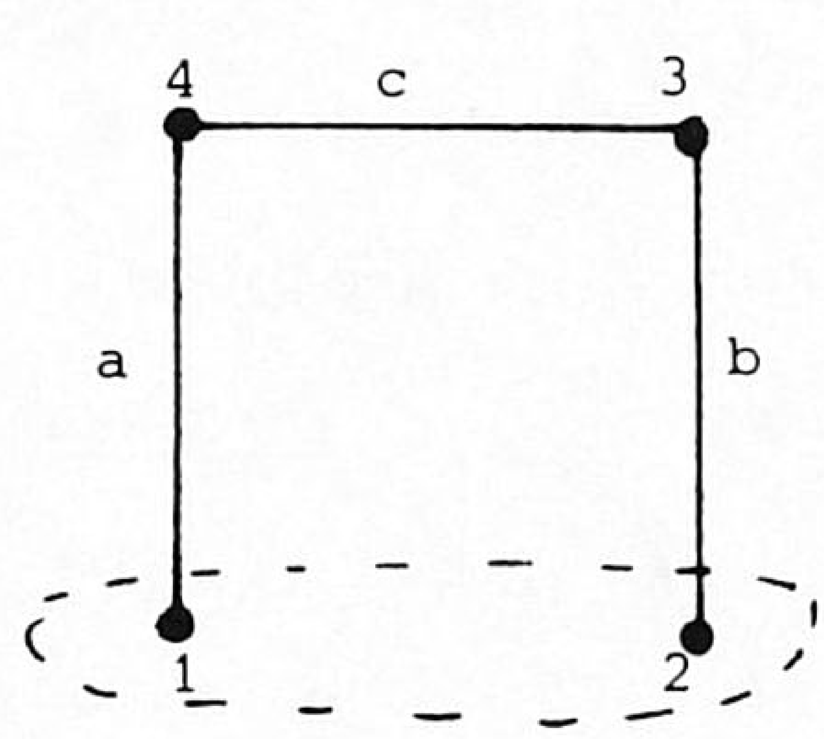

FIG. 3

Let $(N, v, A) \in CS_*^N$ be a fixed communication situation with $v = u_S$ for some $S \in 2^N$, $|S| \geqq 2$. It is clear that for the dividends $\Delta_v(T)$, we have

$$\Delta_v(T) = \begin{cases} 1 & \text{if } T = S, \\ 0 & \text{else.} \end{cases}$$

Furthermore, for all $L \in 2^A \setminus \{\varnothing\}$, the dividends $\Gamma_v(L)$ are given by

$$\Gamma_v(L) = \sum_{T \in \Sigma(L)} \Delta_v(T) = \begin{cases} 1 & \text{if } S \in \Sigma(L), \\ 0 & \text{else} \end{cases} = \begin{cases} 1 & \text{if } H(S) \neq \varnothing \text{ and } L = A(H(S)), \\ 0 & \text{else.} \end{cases}$$

So, if $H(S) = \varnothing$, then $\pi_i(N, v, A) = 0$ for all $i \in N$. Else, if $H(S) \neq \varnothing$, then

$$\pi_i(N, v, A) = \sum_{a \in A_i} \frac{1}{2} \Phi_a(A, r_N^v) = \sum_{a \in A_i} \frac{1}{2} \sum_{L: a \in L} \frac{\Gamma_v(L)}{|L|}$$

$$= \sum_{a \in A_i \cap A(H(S))} \frac{1}{2|A(H(S))|} = \frac{d_i(N, A(H(S)))}{\sum_{j \in N} d_j(N, A(H(S)))}$$

for all $i \in N$.

(ii) *Pure overhead games*. Let T be an arbitrary subset of N. Then the (zero-normalization of the) *pure overhead game* (N, p_T) on T (cf. Owen (1986)) is defined by

$$p_T(S) = \begin{cases} -1 + |S \cap T| & \text{if } S \cap T \neq \varnothing, \\ 0 & \text{else.} \end{cases}$$

It is easily verified that

$$\Delta_{p_T}(S) = \begin{cases} (-1)^{|S|} & \text{if } S \subset T \text{ and } |S| \geqq 2, \\ 0 & \text{else.} \end{cases}$$

Considering a communication situation $(N, p_T, A) \in CS_*^N$, Theorem 4.1 and Lemma 4.1 imply that, for $L \in 2^A \setminus \{\varnothing\}$,

$$\Gamma_{p_T}(L) = \begin{cases} \displaystyle\sum_{S: \text{Ext}(N(L), L) \subset S \subset (N(L) \cap T)} (-1)^{|S|} & \text{if } (N(L), L)) \text{ is a tree,} \\ 0 & \text{otherwise,} \end{cases}$$

(29)

$$= \begin{cases} (-1)^{|N(L) \cap T|} & \text{if Ext}(N(L), L) = N(L) \cap T, \\ 0 & \text{otherwise.} \end{cases}$$

A special, simple case occurs when $T = N$.

The condition Ext $(N(L), L) = N(L) \cap T$ then boils down to Ext $(N(L), L) = N(L)$, which is only satisfied if $|L| = 1$. Hence,

$$\Gamma_{p_N}(L) = \begin{cases} 1 & \text{if } |L| = 1, \\ 0 & \text{else.} \end{cases}$$

Consequently, we obtain the following simple expression for the position value:

(30) $$\pi_i(N, p_N, A) = \tfrac{1}{2} d_i(N, A)$$

for all $i \in N$. It follows (cf. Owen (1986)) that in this case ($T = N$), the Myerson value is equal to the position value. In the more general case, this need not be true. This is illustrated in the next example.

Example 5.1. Consider the five-person communication situation (N, p_T, A), where $T = \{1, 2, 3\}$ and the communication graph (N, A) is represented in Fig. 4. Then, using expression (29), it is found that

$$\Gamma_{p_T}(L) = \begin{cases} 1 & \text{if } L \in \{\{a,b\},\{a,c\},\{b,c\}\}, \\ -1 & \text{if } L = \{a,b,c\}, \\ 0 & \text{else.} \end{cases}$$

Consequently,

$$\pi_1(N, p_T, A) = \frac{1}{2} \sum_{L : a \in L} \frac{\Gamma_v(L)}{|L|} = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}.$$

Similarly, we find that

$$\pi_2(N, p_T, A) = \pi_3(N, p_T, A) = \tfrac{1}{3}, \quad \pi_4(N, p_T, A) = 1, \quad \text{and} \quad \pi_5(N, p_T, A) = 0.$$

Furthermore, with some work, we obtain

$$\mu(N, v, A) = (\tfrac{5}{12}, \tfrac{5}{12}, \tfrac{5}{12}, \tfrac{9}{12}, 0).$$

(iii) *Quadratic measure games*. Let $\omega = (\omega_1, \cdots, \omega_n)$ be a nonnegative (weight-) vector. Then the *quadratic measure game* $q_\omega \in G_0^N$ corresponding to ω (cf. Owen (1986)) is defined by

$$q_\omega(S) := \left(\sum_{i \in S} \omega_i\right)^2 - \sum_{i \in S} \omega_i^2 = \sum_{\{i,j\} \subset S : i \neq j} 2\omega_i \omega_j.$$

So in the quadratic measure game corresponding to ω, the worth of a coalition is completely determined by the worth of its various two-person subcoalitions, which, in turn, are completely determined by the product of the weights attached to each of its two players. It is easily seen that

$$\Delta_{q_\omega}(S) = \begin{cases} 2\omega_i \omega_j & \text{if } S = \{i, j\} \text{ with } i \neq j, \\ 0 & \text{else.} \end{cases}$$

Let us now consider a fixed communication situation $(N, q_\omega, A) \in CS_*^N$. Let $L \subset A$ be a subset of arcs. Then Theorem 4.1 implies that the dividend $\Gamma_{q_\omega}(L)$ is given by

$$\Gamma_{q_\omega}(L) = \begin{cases} 2\omega_s \omega_t & \text{if there are } s, t \in N \text{ such that } A(H(\{s,t\})) = L, \\ 0 & \text{else.} \end{cases}$$

In other words, the dividend $\Gamma_{q_\omega}(L) = 0$, unless L establishes a path in the graph (N, A). If this is a path from player s to player t, then the dividend $\Gamma_{q_\omega}(L)$ equals the worth of the coalition $\{s, t\}$ in the quadratic measure game.

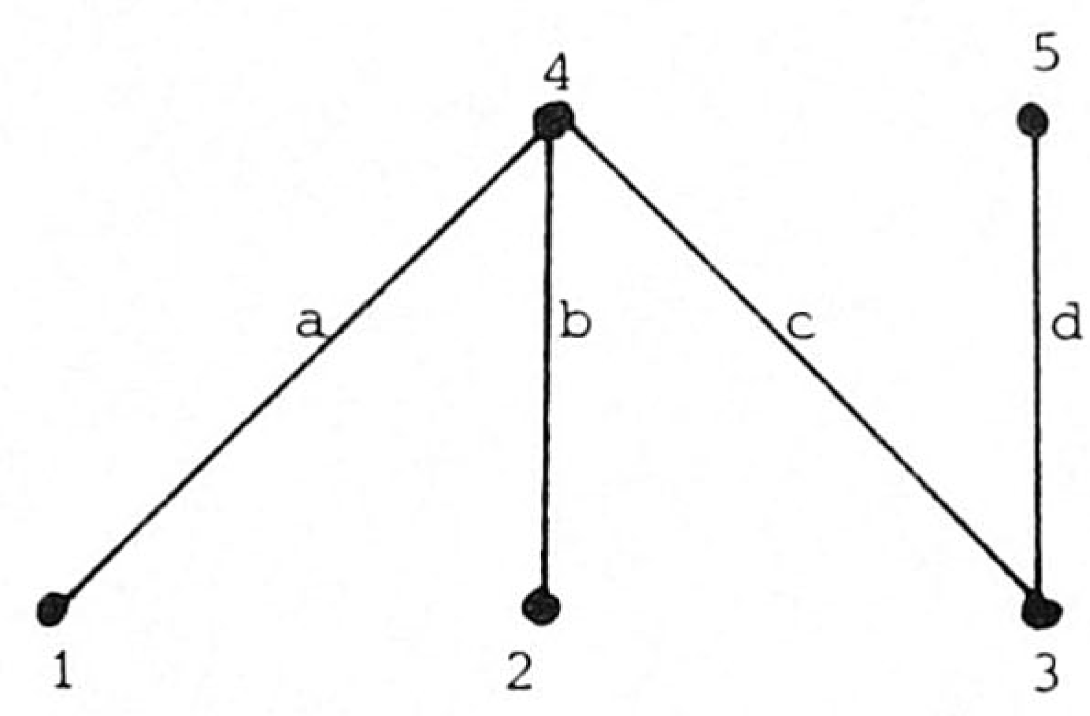

FIG. 4

We now introduce the following definition. Let $s, t \in N$. If $H(\{s, t\}) \neq \emptyset$, so if there exists a path from s to t in the (cycle-free) graph (N, A), then the *distance* $\delta(s, t)$ between s and t is defined by $\delta(s, t) := |A(H(\{s, t\}))|$, which represents the number of arcs supporting the path from s to t. Note that if $L \subset A$ establishes the path from player s to player t, we have that $|L| = \delta(s, t)$.

Computation of the position value will be based on the observation that

$$\Phi_a(A, r_N^{q_\omega}) = \sum_{\{s,t\} \subset N : a \in A(H(\{s,t\}))} \frac{2\omega_s \omega_t}{\delta(s,t)} \tag{31}$$

for all $a \in A$. Note that to apply (31), it is necessary to find all paths supported by a given arc. This can be done in a rather smooth way using (weighted) generating functions.

Let $a = \{i, j\} \in A$ be a fixed arc. If we were to cut this arc, then the component $C \in N/A$ with $i, j \in C$ would split up into two parts, say, C_i and C_j with $i \in C_i$ and $j \in C_j$. The *weighted generating function* θ_i^a describes the weighted number of points in C_i lying at a given distance from the point i. Formally,

$$\theta_i^a(x) = \sum_{k=0}^{d_i} \left(\sum_{s \in C_i : d(s,i) = k} \omega_s \right) x^k, \tag{32}$$

where $d_i := \max_{s \in C_i} d(s, i)$ is the maximum distance between a point in C_i and i. Similarly, we define d_j and θ_j^a.

The following theorem shows that it is possible to rewrite (31) in terms of generating functions.

THEOREM 5.1. *Let* $(N, q_\omega, A) \in CS_*^N$ *and* $a = \{i, j\} \in A$. *Let* θ_i^a *and* θ_j^a *be as in* (32). *Then*

$$\Phi_a(A, r_N^{q_\omega}) = 2 \int_0^1 \theta_i^a(x) \theta_j^a(x) dx. \tag{33}$$

Proof. Note that

$$\begin{aligned} 2\int_0^1 \theta_i^a(x)\theta_j^a(x)dx &= \sum_{k=0}^{d_i+d_j} \sum_{s \in C_i, t \in C_j : d(s,i) + d(t,j) = k} \int_0^1 2\omega_s \omega_t x^k dx \\ &= \sum_{k=0}^{d_i+d_j} \sum_{s \in C_i, t \in C_j : d(s,t) = k+1} \frac{2\omega_s\omega_t}{\delta(s,t)} \\ &\overset{[(\text{cf. } (31))]}{=} \Phi_a(A, r_N^{q_\omega}). \end{aligned}$$

□

The above concepts and results are illustrated in the following example.

Example 5.2. Consider the nine-person communication situation (N, q_ω, A) with $\omega_i = 1$ for all $i \in N$ and the communication graph (N, A) of Fig. 5.

It is found that

$$\theta_1^a(x) = 1 + 2x + 2x^2, \qquad \theta_2^a(x) = 1 + 2x + x^2,$$

$$\theta_1^b(x) = 1 + 2x + 2x^2 + x^3, \qquad \theta_3^b(x) = 1 + 2x,$$

$$\theta_1^c(x) = 1 + 2x + 4x^2 + x^3, \qquad \theta_4^c(x) = 1,$$

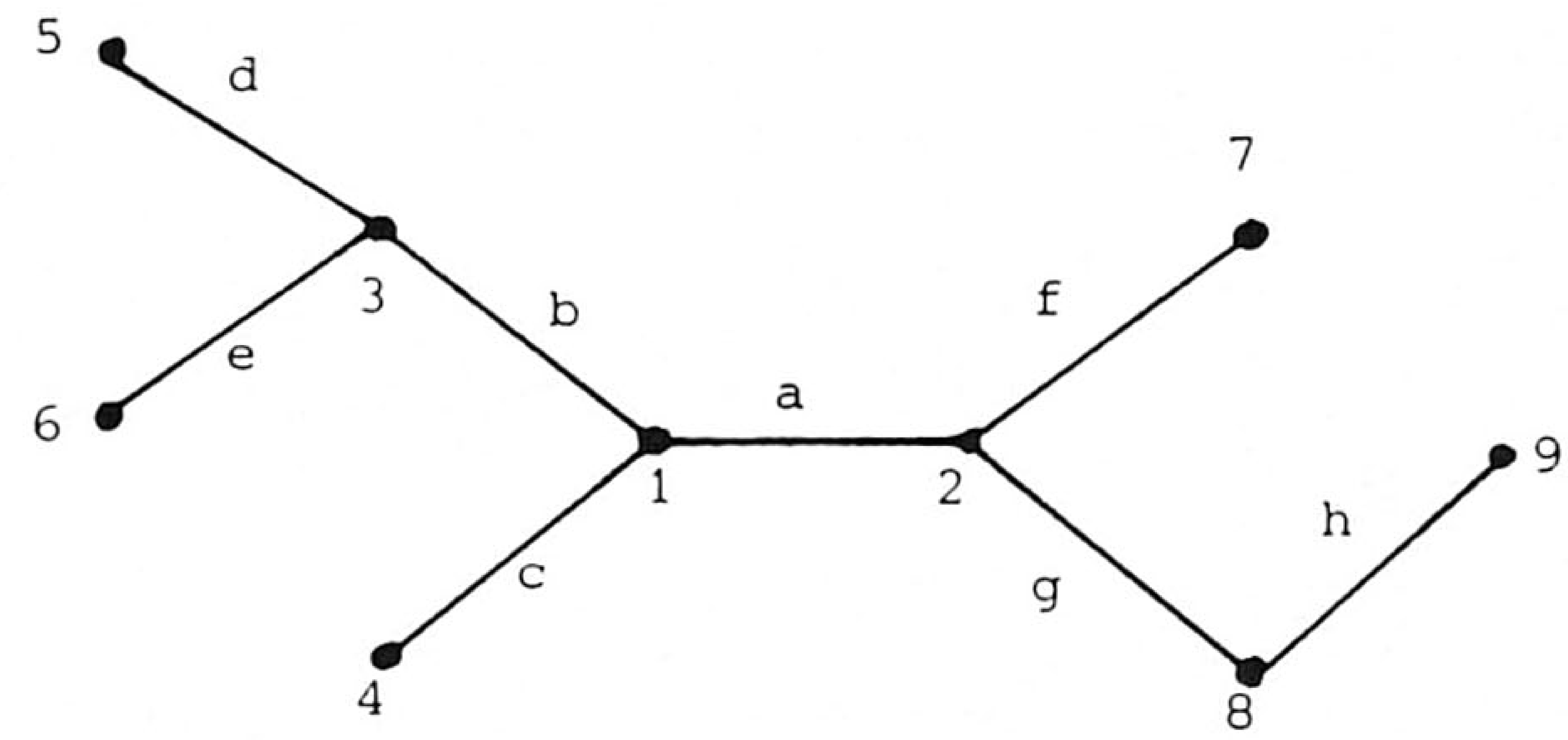

FIG. 5

and so

$$\pi_1(N, q_\omega, A) = \int_0^1 \theta_1^a(x)\theta_2^a(x)\,dx + \int_0^1 \theta_1^b(x)\theta_3^b(x)\,dx + \int_0^1 \theta_1^c(x)\theta_4^c(x)\,dx$$
$$= 7\tfrac{7}{30} + 6\tfrac{13}{20} + 3\tfrac{7}{12} = 17\tfrac{7}{15}.$$

In a similar way, it is possible to compute the position value for the other players.

6. Final remarks. (i) An open problem is how to characterize the position value axiomatically for the class CS^N of all communications situations. Furthermore, it would be interesting to find an axiomatic characterization of the position value for the class of all communication situations (N, v, A) with a full communication graph (N, A), i.e., $A = \{\{i, j\} \in 2^N | i \neq j\}$, because this class corresponds to the class of all (zero-normalized) games in coalitional form (cf. Example 2.2).

(ii) Having characterized the position value for the class of communication situations in which the communication graph contains no cycles, we might think of extending this concept to general communication situations by using *spanning trees*.

Let $(N, v, A) \in CS^N$. For each component $C \in N/A$, we consider the connected subgraph $(C, A(C))$ and the corresponding set $\mathcal{T}(C)$ of spanning trees, i.e.,

$$\mathcal{T}(C) := \{(C, L) \mid L \subset A(C), (C, L) \text{ is a tree}\}$$

and define an allocation rule $\rho : CS^N \to \mathbb{R}^N$ by

$$\rho_i(N, v, A) = \frac{1}{|\mathcal{T}(C_i)|} \sum_{L \subset A : (C_i, L) \in \mathcal{T}(C_i)} \pi_i(N, v, L)$$

for all $i \in N$, where $C_i \in N/A$ is the component to which player i belongs. It is clear that ρ equals π on the class CS_*^N of cycle-free communication situations. However, the following example shows that $\rho \neq \pi$.

Example 6.1. Consider the four-person communication situation (N, v, A) in which $v = u_{\{1,2,3\}}$ and (N, A) is as described in Fig. 6. There are three spanning trees corresponding

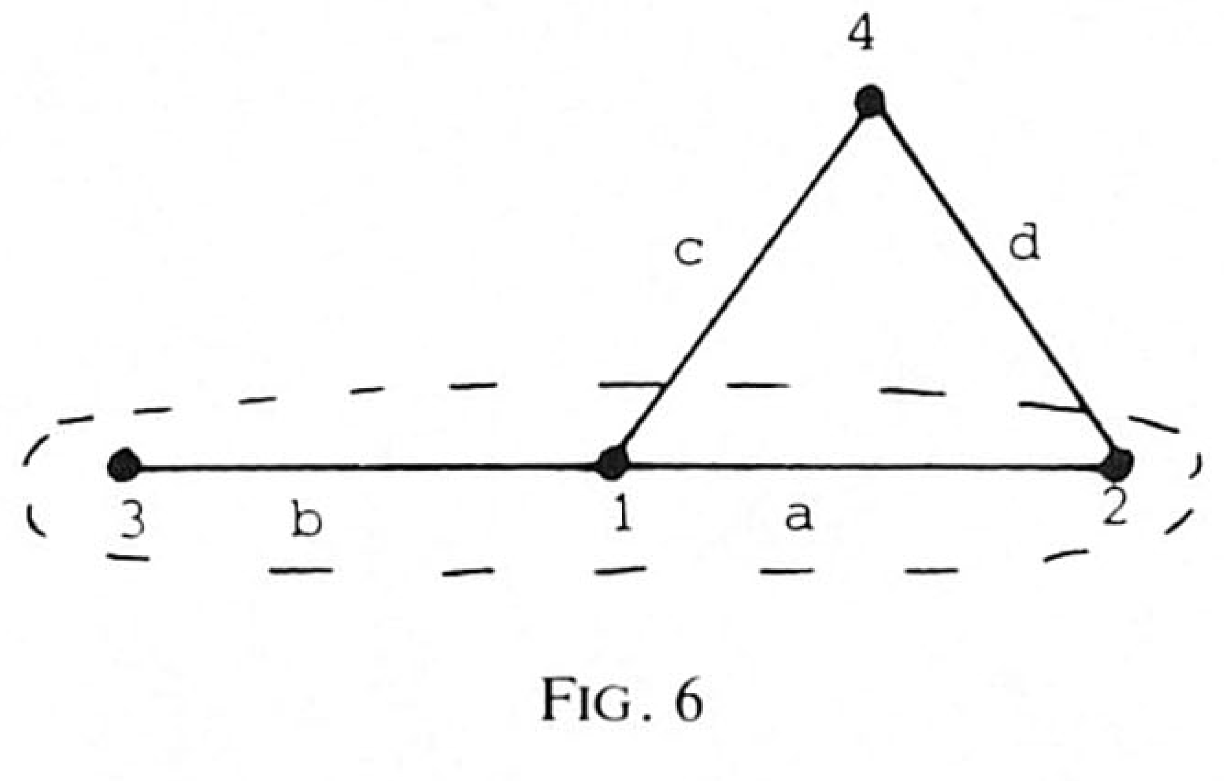

FIG. 6

to $L_1 = \{a, b, c\}$, $L_2 = \{a, b, d\}$, and $L_3 = \{b, c, d\}$, respectively. Then (e.g., using (18)) it is found that

$$\pi(N, v, L_1) = \pi(N, v, L_2) = (\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}, 0), \qquad \pi(N, v, L_3) = (\tfrac{1}{3}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{3}),$$

and so $\rho(N, v, A) = (\frac{4}{9}, \frac{2}{9}, \frac{2}{9}, \frac{1}{9})$, while some calculation shows that $\pi(N, v, a) = (\frac{11}{24}, \frac{4}{24}, \frac{7}{24}, \frac{2}{24})$.

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