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Economies with Land—A Game Theoretical Approach

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With an economy with land $\mathscr{E}$ (an economy of Debreu-type in which land is the unique commodity) we associate a cooperative game with transferable utility $v_{\mathscr{E}}$. The set of all TU-games of type $v_{\mathscr{E}}$ is investigated and the set of equilibrium payoffs (in the TU-sense) of the economy $\mathscr{E}$ is described as a subset of the core of $v_{\mathscr{E}}$. We prove that equilibrium payoffs can be extended to population monotonic allocation schemes in the sense of Sprumont. © 1994 Academic Press.

1. INTRODUCTION

As an economic commodity land differs in many aspects from other commodities normally considered in exchange economies. Berliant (1982) mentions in his Ph.D. thesis three main differences. Parcels of land can be subdivided and recombined into other parcels. Parcels of land are unique in the sense that if we want to make a particular parcel out of available parcels, this can be done in only one way. There is no room for substitution or duplication. Finally, the value of a piece of land is highly

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influenced by properties such as "shape," "connectivity," and "approximity." The first two characteristics can be met by *measure theoretic* considerations and that is what Berliant (1982) does. In this Ph.D. thesis Dunz (1984) considered more general utility functions defined on feasible parcels of land. In his setup there is no longer additivity and the only properties he assumes are monotonicity (a larger parcel has at least the value of a smaller one) and continuity with respect to the Lebesgue measure. However, even this more general idea of utility cannot cope with concepts like connectivity as the following example shows:

Let L be the square $\{(x, y) \mid |x| \leq 1, |y| \leq 1\}$ and let $A_\varepsilon \subset L$ be the set $\{(x, y) \mid |y| \geq \frac{1}{2} \text{ or } |x| < \varepsilon\}$ for some $\varepsilon \in [0, 1]$. Let B_ε be the complement of A_ε. Then the parcel A_ε consists of two pieces connected by a narrow path. If the path becomes too narrow the owner of the land cannot come from one side to the other side and his utility may dramatically decrease as soon as ε comes under a critical level ε_0. For the owner of B_ε the value of his parcel can increase (noncontinuously) when ε vanishes and his pieces grow together (the parcel can be used for more purposes).

For this reason we return in this paper to Berliant's model and assume that the utilities of agents in an economy with land are given by measures on a σ-algebra of subsets of L. An economy with land will consist of a set of agents N, and each agent $i \in N$ has an initial endowment $A_i \subset L$ and a measure μ_i defined on a σ-algebra $\mathcal{B}$ of subsets of $L := \bigcup_{i \in N} A_i$. The measure μ_i describes the appreciation of agent i for parcels $C \in \mathcal{B}$. Since the measures μ_i are absolutely continuous with respect to the measure $\nu := \Sigma_{i \in N} \mu_i$, we can write (by the famous Radon–Nikodym theorem) $\mu_i(C) = \int_C f_i d\nu$, where $\{f_i\}_{i \in N}$ are bounded ν-measurable functions on L. We call the function f_i the utility density of agent i. So an economy with land is a triple

$$\mathcal{E} := \{N, \{A_i, f_i\}_{i \in N}\},$$

where N is a finite set (of agents), $\{A_i\}_{i \in N}$ is a ν-measurable partition of L, and $\{f_i\}_{i \in N}$ is a set of bounded ν-measurable non-negative functions on L.

In this paper we investigate TU-games $(N, v_{\mathcal{E}})$ (*fair division games*) associated with economies with land $\mathcal{E}$. Since we consider the situation from a TU-point of view, the measures μ_i describe the monetary value for player i of parcels of land. A TU-approach always assumes the existence of an "ideal kind of money," equally appreciated by all agents and perfectly suited to transfer utility. Moreover, there are no restrictions in the transfer of money, every difference between selling and purchasing can be balanced by a transfer of money. So in a TU-setting utilities are expressed in money and can be compared therefore. Moreover, the agents have more action possibilities (than in an NTU-situation), namely the transfer of any amount of money.

Many papers in the fair division literature are closely related to the subject of this paper. In Kirman (1981) one can find a survey of the literature up to 1979. From the more recent papers we mention Weller (1985) who proves the existence of Pareto-optimal and envy-free divisions of a nonatomic measure space and Legut (1990), wherein the trade after a fair division mechanism is studied. The present paper can be understood as a continuation of the latter one.

The paper consists of the following parts. In Section 2 we introduce TU-games $(N, v_{\mathscr{E}})$ associated with economies $\mathscr{E}$: *fair division games*. The class of fair division games is a cone in the space of all cooperative games and we give a set of generators of this cone. In Section 3 we introduce and characterize the equilibrium payoffs of $\mathscr{E}$ as a subset of the core of the game $v_{\mathscr{E}}$. Finally, we prove that equilibrium payoffs can be extended to population monotonic allocation schemes (see Sprumont (1990)).

2. Economies with Land: The TU-Case

Let L be a set and $\mathscr{B}$ a σ-algebra of subsets of L. Let ν be a finite measure on $\mathscr{B}$. We call a collection of subsets $\{A_i\}_{i=1,\ldots,n}$ a *$\mathscr{B}$-partition of* L if $A_i \in \mathscr{B}$ for all $i \in N$, $\upsilon(A_i \cap A_j) = 0$ if $i \neq j$, and $\upsilon(L \backslash \cup_{i=1}^{n} A_i) = 0$.

An *economy with land* $\mathscr{E} = \{N, \{A_i, f_i\}_{i \in N}\}$ consists of a finite set of agents N, a $\mathscr{B}$-partition $\{A_i\}_{i \in N}$ of L and a collection of non-negative bounded υ-measurable functions $\{f_i\}_{i \in N}$ on L. The set A_i is the initial endowment of agent i and $\mu_i(C) := \int_C f_i \, d\nu$ gives the monetary appreciation of agent i for the measurable parcel C.

We study this kind of economy from a TU-point of view and define

$$v_{\mathscr{E}}(S) := \sup \left\{ \sum_{i \in S} \int_{Y_i} f_i \, d\nu \mid \{Y_i\}_{i \in S} \text{ is a } \mathscr{B}\text{-partition of } A(S) = \bigcup_{i \in S} A_i \right\}$$

for all $S \subset N$. We call the TU-game $(N, v_{\mathscr{E}})$ the *fair division game*[1] belonging to the economy $\mathscr{E}$.

In this section we investigate the properties of fair division games as

[1] This name does not seem to be very appropriate in the present situation but in Legut (1990) this term has been introduced for games of this kind where the initial endowment $\{A_i\}_{i \in N}$ was the result of a fair division process. Since we are studying the same games it is not sensible to change the name.

well as the connection between the competitive equilibria of the economy $\mathscr{E}$ and the core allocations of the TU-game $(N, v_{\mathscr{E}})$.

Before we go on we introduce some functions derived from the utility densities $\{f_i\}_{i \in N}$.[2]

$$\bar{f}^S := \bigvee_{i \in S} f_i, \qquad \underline{f}^S := \bigwedge_{i \in S} f_i$$
$$\bar{f} := \bar{f}^N, \qquad \underline{f} := \bigwedge_{i \in N} \bar{f}^{N \setminus i}.$$

$\bar{f}^S$ and $\underline{f}^S$ are the maximum and the minimum of the functions $\{f_i\}_{i \in S}$ and $\underline{f}$ is the second-best function. Note that $\bar{f} = \bar{f}^N$ but that $\underline{f} \neq \underline{f}^N$. Using this notation we have

LEMMA 1. *For every coalition* $S \subset N$ $v_{\mathscr{E}}(S) = \int_{A(S)} \bar{f}^S \, d\nu$.

The proof is straightforward.

If we define, for every agent $i \in N$, the game $(N, v_{\mathscr{E},i})$ by

$$v_{\mathscr{E},i}(S) := \begin{cases} \int_{A_i} \bar{f}^S \, d\nu & \text{if } i \in S, \\ 0 & \text{if } i \notin S, \end{cases}$$

we have $v_{\mathscr{E}} = \Sigma_{i \in N} v_{\mathscr{E},i}$.

The following theorem reveals the structure of the set of fair division games. First we introduce a class of simple games. Let i be an agent of N and T a non-empty coalition which does not contain agent i. Then

$$u_i(S) := \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S, \end{cases} \qquad \text{and} \qquad u^*_{T,i}(S) := \begin{cases} 1 & \text{if } i \in S \text{ and } S \cap T \neq 0, \\ 0 & \text{else.} \end{cases}$$

THEOREM 2. *A TU-game* v *is a fair division game if and only if the game* v *is a non-negative linear combination of the games* $\{u_i\}_{i \in N}$ *and* $\{u^*_{T,i}\}_{i \in N, T \subset N \setminus i, T \neq \emptyset}$.

Proof. Let v be a fair division game generated by an economy with land $\mathscr{E} = \{N, \{A_i, f_i\}_{i \in N}\}$. Let

$$\lambda_{T,i} := \int_{A_i} (\underline{f}^T - \bar{f}^{N \setminus T})^+ \, d\nu \qquad \text{and} \qquad \lambda_i := \int_{A_i} f_i \, d\nu.$$

[2] The notations $\bigvee$ and $\bigwedge$ are used to denote the maximum and the minimum of two (or a finite set of) functions.

We use the notation $a^+ := \max(a, 0)$. Hence, the number $\lambda_{T,i}$ is only larger than zero if the set of points in A_i where the least value of $\{f_j\}_{j\in T}$ exceeds the largest value of $\{f_j\}_{j\in N\setminus T}$ has ν-measure greater than zero. We prove that $v = \Sigma_{i\in N} \lambda_i u_i + \Sigma_{i\in N, T\subset N, T\neq\emptyset} \lambda_{T,i} u^*_{T,i}$. Hence we have to prove that for every agent $i \in N$ and every coalition S with $i \in S$ the functions

$$\bar{f}^S \quad \text{and} \quad f_i + \sum_{T\subset N\setminus i, T\cap S\neq\emptyset} (\underline{f}^T - \bar{f}^{N\setminus T})^+ \tag{2.1}$$

have the same integral over A_i. We show in fact that these functions are equal. Take a point $x \in A_i$ and order the number $f_j(x)$ in a weakly decreasing order:

$$f_{a(1)}(x) \geq f_{a(2)}(x) \geq \ldots \geq f_{a(n)}(x).$$

Let $f_i(x)$ be the qth element of this sequence, i.e., $a(q) = i$, and let r be the smallest index with $a(r) \in S$. Then $(\underline{f}^T - \bar{f}^{N\setminus T})_+ (x) > 0$ implies $T := \{a(1), a(2), \ldots, a(p)\}$ for some p: $1 \leq p \leq n$. If moreover $T \subset N\setminus i$ and $S \cap T \neq \emptyset$, we have also $r \leq p < q$. From this fact follows that

$$\sum_{T\subset N\setminus i, S\cap T\neq\emptyset} (\underline{f}^T - \bar{f}^{N\setminus T})^+(x) = \sum_{k=r}^{q-1} (f_{a(k)}(x) - f_{a(k+1)}(x)) = f_{a(r)}(x) - f_{a(q)}(x).$$

As $f_{a(r)}(x) = \bar{f}^S(x)$ and $f_{a(q)}(x) = f_i(x)$, we find equality (2.1) and, therefore,

$$v_{\mathscr{E},i}(S) = \lambda_i u_i(S) + \sum_{T\subset N\setminus i, T\neq\emptyset} \lambda_{T,i} u^*_{T,i}(S)$$

for every agent $i \in N$ and every coalition $S \subset N$.

Next we prove that the set of fair division games is a cone which contains the simple games $u^*_{T,i}$ and u_i. If $\mathscr{E}$ is an economy with land and $v = v_\varepsilon$, then the game λv is generated by the economy $\lambda\mathscr{E}$ with the same set of agents N and the same initial endowments for all players $i \in N$ but with utility densities which are λ times the utility densities in $\mathscr{E}$. If (N, v) and (N, v') are fair division games generated by economies $\mathscr{E}$ and $\mathscr{E}'$, then the game $(N, v + v')$ is generated by the economy with the following data:

$\bar{L} := L \cup L'$ disjoint union,

$$\bar{\nu}(\cdot) := \nu(\cdot \cap L) + \nu'(\cdot \cap L'), \bar{A}_i := A_i \cup A'_i, \text{and} \bar{f}_i := \begin{cases} f_i & \text{on } L \\ f'_i & \text{on } L' \end{cases}.$$

Hence, the set of fair division games is a cone.

The game $u^*_{T,i}$ is generated by an economy with

$$\begin{cases} L := [0, 1], & \nu := \lambda \text{ (the Lebesgue measure)} \\ A_i := [0, 1], & A_j := \emptyset \text{ for } j \neq i \\ f_j := 1 \text{ if } j \in T, f_j := 0 \text{ if } j \notin T. \end{cases} \tag{2.2}$$

If we replace in this economy the utility densities by $f_j := 1$ if $j = i$ and $f_j := 0$ if $j \neq i$ we find an economy generating the game u_i. Note that all the functions $f_i = 1$ in (2.2) can be replaced by one and the same λ-measurable non-negative function g on [0, 1] with $\int_0^1 g(t)\, dt = 1$. In particular we can take for g a bell-shaped C^∞-function on [0, 1]. Q.E.D.

Comments. (1) Theorem 2 gives a characterization of fair division games. In Legut (1990) one can find another one. The present characterization, however, seems to be more accessible for computation.

(2) In Muto *et al.* (1989) *information markets with one initially informed player* have been studied. One of the results in that paper is that a game v is an information market game with initially informed player i if and only if the game v is a non-negative linear combination of the games $\{u^*_{T,i}\}_{T \neq \emptyset, T \subset N \setminus i}$ and u_i. If we compare this result with the statement in Theorem 2 we have

A TU-game v is a fair division game if and only if the game v is the sum of TU-games $\{v_i\}_{i \in N}$ wherein v_i is an information market game with initially informed player i.

In this paper and subsequent papers of the same authors (Muto *et al.*, 1989; b Potters *et al.*, 1989) the following results are obtained:

(a) The core of an information market game with initially informed player i consists of the efficient allocations x with $0 \leq x_j \leq M_j(v) := v(N) - v(N \setminus j)$, $\forall j \neq i$. The bargaining set coincides with the core.

(b) The nucleolus of an information market game is the center of the core, i.e., the point x with $x_j = \frac{1}{2} M_j(v)$ for all $j \neq i$ and $x(N) = v(N)$. This point is also the τ-value. Moreover, the kernel of such a game consists of the nucleolus only.

(c) The core is a stable set in the sense of von Neumann and Morgenstern if and only if the information market game is convex. An information market game $v = \Sigma_{T \subset N \setminus i} \lambda_T u^*_{T,i}$ is convex iff $\lambda_T = 0$ for all T with $|T| \geq 2$. Note that the decomposition of v for information market games is unique. For fair division games this is no longer true.

(d) The results (a, b, and c) also hold for the more general big boss games.

As fair division games arising from economies with land *with one initial owner* are information market games, these results hold for such games too.

(3) If we have a fair division game v which has been decomposed already into games $u^*_{T,i}$ and u_i, we can make, by using the proof of Theorem 2, an economy with $L = [0, M]$ where M is the number of games $u^*_{T,i}$ or u_i with nonzero coefficient; the initial endowment of every player is a segment. Further, we may assume that the utility functions f_i are C^∞-functions (see the last remark in the proof of Theorem 2).

3. Price Equilibria and Equilibrium Payoffs

Let $\mathscr{E} = \{N, \{A_i, f_i\}_{i\in N}\}$ be an economy with land. A *price equilibrium* of $\mathscr{E}$ consists of a $\mathscr{B}$-partition $X^* = (X^*_i)_{i\in N}$ of L and a bounded measurable function g on L (the *price density*) such that

$$\int_{X^*_i} (f_i - g)\, d\nu = \max_{B\in\mathscr{B}} \int_B (f_i - g)\, d\nu \qquad \text{for all } i \in N.$$

If (X^*, g) is a price equilibrium then X^* is called a *competitive equilibrium* and the vector $x^* \in R^N$ with coordinates

$$x^*_i := \int_{X^*_i} (f_i - g)\, d\nu + \int_{A_i} g\, d\nu$$

is called an *equilibrium payoff* (*vector*).

Remark. As we are in a TU-setting, each agent can purchase any piece of land he likes and will do so if his utility (density) is higher than the price (density) of a parcel of land. There is in fact no budget constraint (cf. the house market of Shapley and Shubik (1971) for a similar situation). This makes the existence of price equilibria easier to prove.

The following theorem gives besides the existence a characterization of price equilibria of an economy with land.

THEOREM 3. *A pair (X^*, g) where X^* is a $\mathscr{B}$-partition of L and g is a bounded ν-measurable function on L is a price equilibrium of an economy with land $\mathscr{E} = \{N, \{A_i, f_i\}_{i\in N}\}$ if and only if*

1. *$X^*_i \subset' \{x \in L \mid f_i(x) = \bar{f}(x)\}$ for all $i \in N$, and*
2. *$\underline{f} \leq g \leq \bar{f}$ almost everywhere on L.*

Remark. $A \subset' B$ means $\nu(A \cap B) = \nu(A)$.

Proof. Let (X^*, g) be a price equilibrium. If $E \subset X_i^*$ and $v(E) > 0$ then $f_i(x) \geq g(x)$ for almost every point $x \in E$. For otherwise there is a set E' $\subset E$ with $v(E') > 0$ and $f_i|_{E'} < g|_{E'}$. Then $X_i^* \backslash E'$ is strictly better for player i than X_i^*. For the same reason, if $E \subset X_j^*$, $j \neq i$, and $v(E) > 0$ we have $f_i(x) \leq g(x)$ for almost every point $x \in E$. Otherwise $X_i^* \cup E'$ is strictly better than X_i^* if $E' \subset X_j^*$ with $v(E') > 0$ and $f_i|_{E'} > g|_{E'}$. Hence, for almost every point $x \in X_i^*$ we have $f_i(x) \geq g(x) \geq \bigvee_{j \neq i} f_j(x)$. This gives Properties (1) and (2).

If, conversely, Properties (1) and (2) hold and there is a set $C \subset L$ with $\int_C (f_i - g)\, d\nu > \int_{X_i^*} (f_i - g)\, d\nu$ then there is a player $j \neq i$ such that $v(C \cap X_j^*) > 0$. Then we infer from Property (1) that $f_j(x) = \bar{f}(x)$ for almost every point $x \in C \cap X_j^*$ and from Property (2) that $\int_{C \cap X_j^*} (f_i - g)\, d\nu \leq 0$ for all $j \neq i$. Then $\int_{C \cap X_i^*} (f_i - g)\, d\nu \geq \int_C (f_i - g)\, d\nu > \int_{X_i^*} (f_i - g)\, d\nu$. This is not possible. Q.E.D.

Remark. Note that an equilibrium payoff is completely determined by the choice of the equilibrium price density g between $\underline{f}$ and $\bar{f}$. The $\mathcal{B}$-partition $\{X_i^*\}_{i \in N}$ is up to sets of ν-measure zero uniquely determined. For, in every competitive equilibrium X^* the set $Y_i := \{x \in L \mid f_i(x) = \bar{f}(x > \underline{f}(x)\} \subset' X_i^*$ and $\int_{X_i^*} (f_i - g)\, d\nu = \int_{Y_i} (f_i - g)\, d\nu$ since $f_i - g \neq 0$ on $X_i^* \backslash Y_i$.

Finally, we investigate the set of equilibrium payoffs of an economy $\mathcal{E}$ and find the following result.

THEOREM 4. *A payoff vector $x \in R^N$ is an equilibrium payoff of an economy with land $\mathcal{E}$ if and only if the vector x is an element of* $\Sigma_{i \in N}\, \mathcal{C}\,(v_{\mathcal{E},i})$.

Comment. Equilibrium payoffs are elements of the core $\mathcal{C}\,(v_{\mathcal{E}})$ of the game $(N, v_{\mathcal{E}})$ but in general they form only a part of the core.

Proof. Let (X^*, g) be a price equilibrium with equilibrium payoff vector x. Define n vectors $\{y_j\}_{j \in N}$ with coordinates

$$y_{jk} := \begin{cases} \int_{A_j \cap X_k^*} (f_k - g)\, d\nu & \text{for } k \neq j \\ v_{\mathcal{E},j}(N) - \Sigma_{k \neq j}\, y_{jk} & \text{for } k = j \end{cases}.$$

Then we can write for y_{jj}

$$\int_{A_j} \bar{f}\, dv - \sum_{k \neq j} \int_{A_j \cap X_k^*} (f_k - g)\, dv = \int_{A_j \cap X_j^*} \bar{f}\, dv + \sum_{k \neq j} \int_{A_j \cap X_k^*} g\, dv$$
$$= \int_{A_j \cap X_j^*} (\bar{f} - g)\, dv + \int_{A_j} g\, dv.$$

In the previously mentioned paper Muto *et al.* (1989) it is proved that core elements of games like $v_{\mathscr{E},j}$ (information market games with informed player j) are characterized by the (in)equalities $z(N) = v_{\mathscr{E},j}(N)$ and $0 \leq z_k \leq v_{\mathscr{E},j}(N) - v_{\mathscr{E},j}(N \backslash k)$ for all $k \neq j$.

So we must prove that $0 \leq y_{jk} \leq \int_{A_j} (\bar{f} - \bar{f}^{N \backslash k})\, dv$ for all $k \neq j$. It is easy to see that

$$\int_{A_j} (\bar{f} - f^{N \backslash k})\, dv = \int_{A_j \cap X_k^*} (f_k - \underline{f})\, dv \geq y_{jk} \geq 0.$$

The equality holds because $\bar{f} = \bar{f}^{N \backslash k}$ almost everywhere outside X_k^* and the inequalities follow from $\bar{f} \geq g \geq \underline{f}$ on L. Hence, y_j is a core element of $v_{\mathscr{E},j}$ and $x = \Sigma_{j \in N}\, y_j$.

Conversely, if a vector x is the sum of core elements y_j of $v_{\mathscr{E}j}$ we take a $\mathscr{B}$-partition X^* with $Y_j := \{x \in L \mid f_j(x) = \bar{f}(x) > \underline{f}\,(x)\} \subset' X_j^*$ for all $j \in N$. By Theorem 3.1 of Muto *et al.* (1989) we know that for $k \neq j$ the number y_{jk} is a fraction of the marginal of player k in the game $v_{\mathscr{E},j}$; i.e.,

$$y_{jk} = z_{jk}\,(v_{\mathscr{E},j}\,(N) - v_{\mathscr{E},j}\,(N \backslash \mathrm{k}))$$

for some real number $z_{jk} \in [0, 1]$. If the marginal of player k in the game $v_{\mathscr{E},j}$ vanishes we take $z_{jk} = 0$. For $j \neq k$ we define

$$g \mid_{A_j \cap X_k^*} := (1 - z_{jk})\bar{f} + z_{jk}\underline{f} \mid_{A_j \cap X_k^*}.$$

Further, we take for $g \mid_{A_j \cap X_j^*}$ an arbitrary function between $\underline{f}$ and $\bar{f}$. Then the pair (X^*, g) is a price equilibrium by Theorem 3. We prove that x is the equilibrium payoff of this price equilibrium. The payoff for player k is

$$\int_{X_k^*} (f_k - g)\, dv + \int_{A_k} g\, dv = \sum_{j \in N} \int_{X_k^* \cap A_j} (f_k - g)\, dv + \int_{A_k} g\, dv.$$

On X_k^* we have $f_k = \bar{f}$ (almost everywhere) and on $X_k^* \cap A_j$ with $j \neq k$ we have $f_k - g = z_{jk}(\bar{f} - \underline{f})$.

Hence we find that the payoff for player k is

$$\sum_{j\neq k} z_{jk} \int_{X_k^*\cap A_j} (\bar{f} - \underline{f})\, dv + \int_{X_k^*\cap A_k} f_k\, dv + \sum_{l\neq k} \int_{X_l^*\cap A_k} g\, dv.$$

Since $\int_{X_k^*\cap A_j} (\bar{f} - \underline{f})\, dv = \int_{X_k^*\cap A_j} (\bar{f} - \bar{f}^{N\setminus k})\, dv$ is the marginal of player k in the game $v_{\mathscr{E},j}$ for $k \neq j$ we find a payoff to player k equal to

$$\sum_{j\neq k} y_{jk} + \int_{X_k^*\cap A_k} f_k\, dv + \sum_{l\neq k} \int_{X_l^*\cap A_k} g\, dv. \tag{2.3}$$

Hence we only have to prove that the last two terms of (2.3) sum up to y_{kk}. As y_k is efficient in $v_{\mathscr{E},k}$ we have

$$y_{kk} = \int_{A_k} \bar{f}\, dv - \sum_{p\neq k} y_{kp} = \sum_{p} \int_{A_k\cap X_k^*} \bar{f}\, dv - \sum_{p\neq k} \int_{A_k\cap X_p^*} (\bar{f} - g)\, dv$$
$$= \int_{A_k\cap X_k^*} f_k\, dv + \sum_{p\neq k} \int_{A_k\cap X_p^*} g\, dv.$$

The equilibrium payoff associated with the price equilibrium (X^*, g) is the vector x. Q.E.D.

Let $\mathscr{E}$ be an economy with land with utility densities $\{f_i\}_{i\in N}$. We define another economy $\bar{\mathscr{E}}$ by replacing f_i with $f_i \vee \underline{f}$. We define as before $v_{\bar{\mathscr{E}},i}(S) = \int_{A_i} \underline{f}\, dv$ if $i \in S$ and 0 otherwise. The games $v_{\bar{\mathscr{E}},i}$ have the following properties.

LEMMA 5. 1. *$v_{\bar{\mathscr{E}},i}(S) \geq v_{\mathscr{E},i}(S)$ for all $S \subset N$ and $v_{\bar{\mathscr{E}},i}(S) = v_{\mathscr{E},i}(S)$ if* $|S| \geq |N| - 1$.

2. *The games $v_{\bar{\mathscr{E}},i}$ are convex games.*
3. *The cores of $v_{\bar{\mathscr{E}},i}$ and $v_{\mathscr{E},i}$ are the same.*
4. $\mathscr{C}(v_{\bar{\mathscr{E}}}) = \Sigma_{i\in N}\, \mathscr{C}(v_{\mathscr{E},i})$.

Proof. (1) Since $f_i \vee \underline{f} \geq f_i$ we have the inequalities immediately. Further, $\bar{f}^{N\setminus k} \vee \underline{f} = \bar{f}^{N\setminus k}$ gives the equalities for S with $|S| = n - 1$.

(2) We prove that the marginals of the players are weakly increasing in $v_{\bar{\mathscr{E}},i}$. Take $k \neq i$. Then the marginal

$$v_{\bar{\mathscr{E}},i}(S \cup \{k\}) - v_{\bar{\mathscr{E}},i}(S) = \int_{A_i} (\bar{f}^{S\cup\{k\}} \vee \underline{f} - \bar{f}^{S} \vee \underline{f})\, dv$$

is the same for all coalitions $S \ni i$. If $i \notin S$ then the marginal of player k vanishes.

For $k = i$ the marginal is $v_{\overline{\mathscr{E}},i}(S \cup \{i\})$ increasing by the monotonicity of the game $v_{\overline{\mathscr{E}},i}$.

(3) The core of an information market game is completely determined by the value of the coalitions N and $N\backslash k$, $k \in N$. Then Property (3) follows from Property (1).

(4) Since $v_{\overline{\mathscr{E}}} = \sum_{i \in N} v_{\overline{\mathscr{E}},i}$ and the terms $\{v_{\overline{\mathscr{E}},i}\}_{i \in N}$ are convex, Property (4) follows from Property (3) and the additivity of the core for convex games (cf. Dragan *et al.* (1989)). Q.E.D.

Corollary. A payoff vector $x \in R^N$ is an equilibrium payoff of an economy $\mathscr{E}$ if and only if the vector x is an element of the core of the game $v_{\overline{\mathscr{E}}}$.

Comment. If we compare the decomposition of the games $v_{\mathscr{E},i}$ and $v_{\overline{\mathscr{E}},i}$ we can observe (see the proof of Theorem 2) that the decomposition $v_{\mathscr{E},i} = \sum_{T \neq \emptyset, T \subset N \backslash i} \lambda_{T,i}\, u^*_{T,i} + \lambda_i\, u_i$ for the game $v_{\varepsilon,i}$ gives the decomposition $v_{\overline{\mathscr{E}},i} = (\lambda_i + \sum_{|T| \geq 2} \lambda_{T,i})u_i + \sum_{j \neq i} \lambda_{\{j\},i} u^*_{\{j\},i}$ for the game $v_{\overline{\mathscr{E}},i}$. Note that $u^*_{\{j\},i}(S) = 1$ iff $\{i, j\} \subset S$. As the games $\{u^*_{T,i}\}_{T \neq \emptyset, T \subset N \backslash i}$ together with the game u_i are linearly independent (see Muto *et al.* (1989)), the transition from $v_i = v_{\mathscr{E},i}$ to $\overline{v}_i = v_{\overline{\mathscr{E}},i}$ is a well-defined linear map on the cone of information market games with a fixed initially informed player i. The transition from v to $\overline{v}$ is not a well-defined map. In the next example we give two economies with land which define the same game $v = v_{\mathscr{E}}$ but *different* games $\overline{v} = v_{\overline{\mathscr{E}}}$.

EXAMPLE. Let $L = [0, 1]$ (with the Lebesgue measure) and let $\mathscr{B}$ be the σ-algebra of Borel sets in $[0, 1]$. We define $f_1(t) := 2t$, $f_2(t) = 2 - 2t$ and $f_3(t) = 1$ for all $t \in [0, 1]$. The initial endowments are $A_1 := [0, \frac{1}{2}]$, $A_2 := [\frac{1}{2}, \frac{3}{4}]$, and $A_3 := [\frac{3}{4}, 1]$. We find the following values for $v_{\mathscr{E}}(S)$ and $v_{\mathscr{E},i}(S)$:

	(1)	(2)	(3)	(12)	(13)	(23)	(123)
$v_{\mathscr{E},1}$	4/16	*	*	12/16	8/16	*	12/16
$v_{\mathscr{E},2}$	*	3/16	*	5/16	*	4/16	5/16
$v_{\mathscr{E},3}$	*	*	4/16	*	7/16	4/16	7/16
$v_{\mathscr{E}}$	4/16	3/16	4/16	17/16	15/16	8/16	24/16

There is only one $\mathscr{B}$-partition X^* which maximizes $\sum_{i=1}^{3} \mu_i(X_i^*)$: $X_1^* \simeq [\frac{1}{2}, 1]$, $X_2^* \simeq [0, \frac{1}{2}]$, and $X_3^* \simeq \emptyset$.

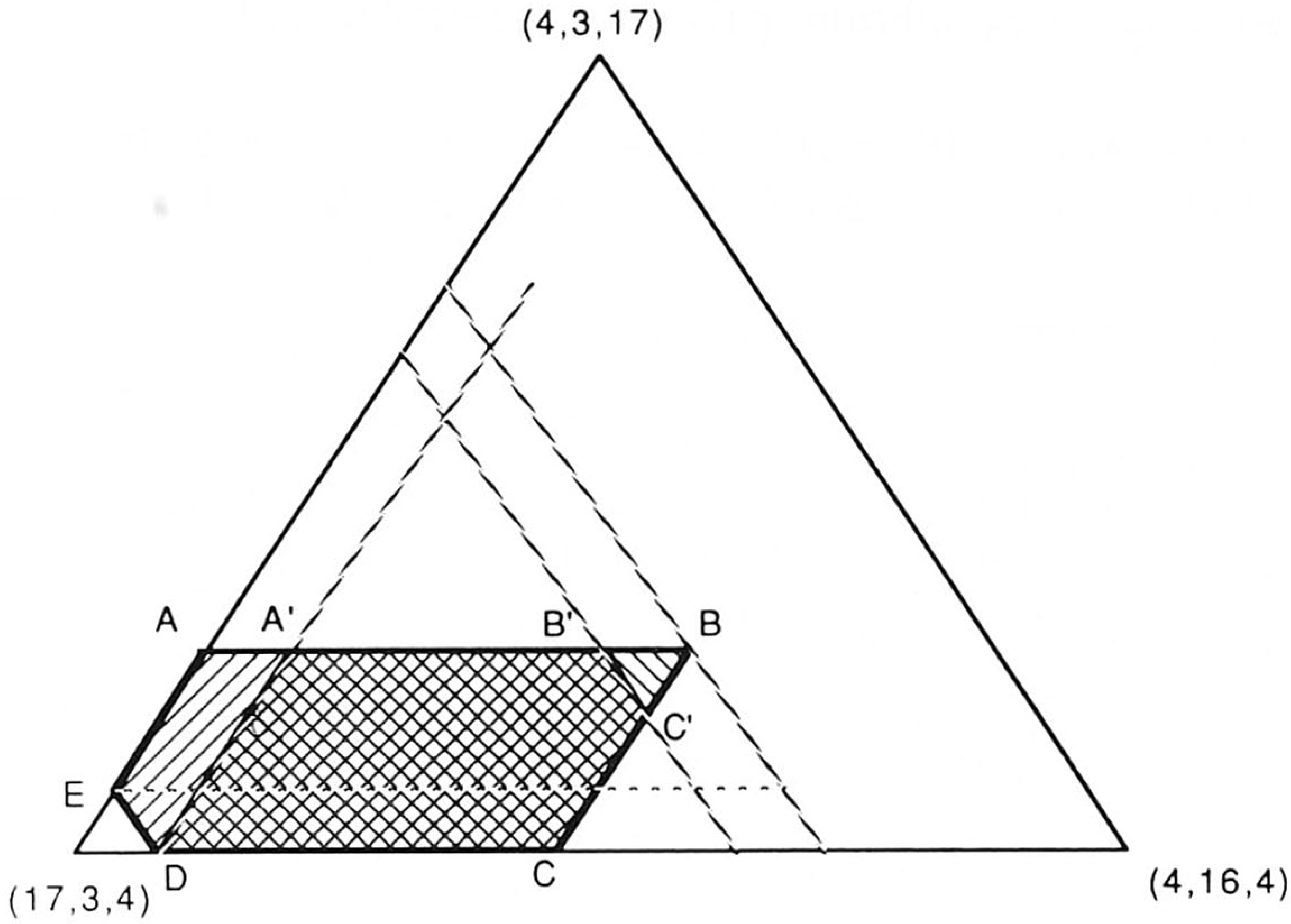

FIG. 1. Core and sets of equilibrium payoffs of the games of the example.

The core of $v_{\mathcal{E}}$ is the convex hull of the points

$$A := \tfrac{1}{16}(14, 3, 7), B := \tfrac{1}{16}(8, 9, 7), C := \tfrac{1}{16}(11, 9, 4),$$
$$D := \tfrac{1}{16}(16, 4, 4), \text{ and } VE := \tfrac{1}{16}(16, 3, 5).$$

The set of equilibrium payoffs (= the core of $v_{\overline{\mathcal{E}}}$) is the convex hull of

$$A' := \tfrac{1}{16}(13, 4, 7), B = \tfrac{1}{16}(8, 9, 7),$$
$$C = \tfrac{1}{16}(11, 9, 4), \text{ and } \dot{V}D = \tfrac{1}{16}(16, 4, 4)$$

(see Fig. 1).

There is another way to write the game $v = v_{\mathcal{E}}$ as a sum of information market games:

	(1)	(2)	(3)	(12)	(13)	(23)	(123)
$v'_{\mathcal{E},1}$	4/16	*	*	14/16	9/16	*	14/16
$v'_{\mathcal{E},2}$	*	3/16	*	3/16	*	3/16	3/16
$v'_{\mathcal{E},3}$	*	*	4/16	*	6/16	5/16	7/16
$v = v'_{\mathcal{E}}$	4/16	3/16	4/16	17/16	15/16	8/16	24/16

Now the set of equilibrium payoffs is the convex hull of

A = $\frac{1}{16}$ (14, 3, 7), B′ := $\frac{1}{16}$ (9, 8, 7), C′ := $\frac{1}{16}$ (9, 9, 6),
C = $\frac{1}{16}$ (11, 9, 4), D = $\frac{1}{16}$ (16, 4, 4) and E = $\frac{1}{16}$ (16, 3, 5) (see Fig. 1).

The games $v_{\overline{\mathscr{E}}}$ and $v'_{\overline{\mathscr{E}}}$ have the values

	(1)	(2)	(3)	(12)	(13)	(23)	(123)
$v_{\overline{\mathscr{E}}}$	8/16	4/16	4/16	17/16	15/16	8/16	24/16
$v'_{\overline{\mathscr{E}}}$	9/16	3/16	4/16	17/16	15/16	8/16	24/16

In this example every core element is an equilibrium payoff in at least one representation of v as a fair division game.

4. Population Monotonic Allocation Schemes

In Sprumont (1990) the concept of a *population monotonic allocation scheme* has been defined as a kind of extension of a core allocation. A population monotonic allocation scheme (PMAS for short) gives a core allocation x_S for every subgame $v|_S$ of a game v such that every player i gets a higher (at least not a lower) payoff in larger coalitions:

$$\sum_{i \in S} x_{S,i} = v(S) \text{ for every non-empty coalition } S \subset N \text{ and}$$

$$x_{S,i} \leq x_{T,i} \text{ whenever } i \in S \subset T.$$

As Sprumont proves that every game which can be written as a positive combination of zero-monotonic simple games with veto-control has a PMAS and the games u_{T,i^*} and u_i have veto player i, every fair division game has a PMAS by Theorem 2.

In fact, $x_{S,i} := v_{\mathscr{E},i}(S) = \int_{A_i} \bar{f}^S \, dv$, $i \in S \subset N$ gives a PMAS. We can prove a more general result.

THEOREM 6. *If $v_{\mathscr{E}}$ is a fair division game every equilibrium payoff can be extended to a population monotonic allocation scheme.*

Proof. If x is an equilibrium payoff it can be written as sum $\Sigma_{i \in N} y_i$ where $y_i \in \mathscr{C}(v_{\mathscr{E},i})$ for all $i \in N$. If we prove that every core allocation y of an information market game with initially informed player i can be extended to a PMAS, then the theorem follows from the superadditivity of the set of population monotonic allocation schemes; i.e.,

$$x \in \text{PMAS}(v), \quad y \in \text{PMAS}(w) \Rightarrow \quad x + y \in \text{PMAS}(v + w).$$

We define

$$y_{S,j} := \begin{cases} 0 & \text{if } i \notin S \text{ and } j \in S \\ y_j & \text{if } i \in S \text{ and } j \in S \backslash i \\ v(S) - \Sigma_{j \in S, j \neq i} y_j & \text{if } j = i \in S. \end{cases}$$

We must prove that $0 \leq y_{S,i} \leq y_{T,i}$ for $i \in S \subset T$. In Muto *et al.* (1989)[b]) we find that an information market game satisfies the inequalities:

$$v(T) - v(S) \geq \sum_{j \varepsilon T \backslash S} (v(N) - v(N \backslash j)) \qquad \text{for } i \in S \subset T$$

(an information market game is a strong big boss game). As $y \in \mathscr{C}(v)$, we have $y_j \leq v(N) - v(N \backslash j)$ for all $j \in N \backslash \{i\}$ and therefore,

$$v(T) - v(S) \geq \sum_{j \varepsilon T \backslash S} y_j \qquad \text{for all } i \in S \subset T.$$

This gives the desired inequalities. Q.E.D.

Remark. In the previous example we have seen that two different economies generating the same TU-game may have different equilibrium payoff sets. Theorem 6 implies that every core element of a fair division game which is an equilibrium payoff for *any* economy generating the game v can be extended to a PMAS. Hence, if (N, v) is a fair division game, we have the following inclusions:

$$X_1 := \{x \in R^N \mid x \text{ is an equilibrium payoff of an economy generating } v\} \quad \subset$$
$$X_2 := \{x \in R^N \mid x \text{ can be extended to a PMAS}\} \quad \subset \mathscr{C}(v).$$

In the previous example every core element was an equilibrium payoff in *some* representation of v as fair division game. The next example shows that this is not the general situation, in other words the inclusion of X_2 in the core is in general a strict inclusion.

EXAMPLE. Let N be a four-person set and let v be defined by $v(i) = 0$ for $i \in N$ and further

S:	(12)	(13)	(14)	(23)	(24)	(34)	(123)	(124)	(134)	(234)	N
$v(S)$	2	0	4	3	2	3	5	6	5	5	8

This is a fair division game with decomposition

$$v = u^*_{1,2} + u^*_{1,4} + u^*_{2,3} + u^*_{13,4} + u^*_{24,1} + u^*_{24,3} + u^*_{34,2} + u^*_{123,4}.$$

The vector $x := (3, 2, 2, 1)$ is a core element but cannot be extended to a PMAS. If x could be extended, then for $S = (1, 4)$ the vector x_S would be $(3, *, *, 1)$ and for $S = (3, 4)$ we would have $x_S = (*, *, 2, 1)$. Then for $S = (1, 3, 4)$ the vector $x_S \geq (3, *, 2, 1)$ but $v\ (134) = 5$.

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