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SHORT COMMUNICATION

NOTE ON THE PATH FOLLOWING APPROACH OF EQUILIBRIUM PROGRAMMING

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Recently Zangwill and Garcia introduced a general formulation of equilibrium problems. To prove the existence of an equilibrium they discussed a path following procedure. In this note we consider the application to the exchange economy problem. An economic equilibrium may be found by applying a simplicial variable dimension algorithm developed by Van der Laan and Talman.

We will show that when an appropriate triangulation and labelling rule is taken the limiting path of this algorithm coincides with the adjustment process induced by the procedure of Zangwill and Garcia. This process has a plausible economic interpretation and is an attractive alternative for the well-known tâtonnement process.

Key words: Equilibrium Programming, Exchange Economy, Simplicial Algorithms, Limiting Path.

1. The economic equilibrium problem

In this section we consider a path following procedure to obtain an economic equilibrium. This procedure has been given by Zangwill and Garcia [7] as an application to a general approach of equilibrium programming. In the next section we show that the procedure coincides with the variable dimension approach of Van der Laan and Talman (see [1–3] and [6]).

Consider an exchange economy of m agents with n commodities. Let $w^i = (w_1^i, \dots, w_n^i) > 0$ be the endowment of agent i, and let the utility function of agent i be given by $f^i : R^n \to R$. Let $w = \sum_{i=1}^m w^i$ be the total endowments.

Definition 1.1. A *competitive equilibrium* is a pair of vectors $(\bar{x}, \bar{p})$ where $\bar{x} = (\bar{x}^1, \dots, \bar{x}^m) \in R^{nm}$ and $\bar{p} \in R^n$ such that

(a) $$f^i(\bar{x}^i) = \max f^i(x^i),$$
$$\text{for} \quad \bar{p}x^i \le \bar{p}w^i \quad \text{and} \quad 0 \le x^i \le w, \quad i = 1, \dots, m,$$

(b) $$\sum_{i=1}^{m} \bar{x}^i \leq w, \tag{1.1}$$

(c) $$\bar{p} \geq 0 \quad \text{and} \quad \sum_{j=1}^{n} \bar{p}_j = 1.$$

To show the existence of a competitive equilibrium, Zangwill and Garcia [7] introduced the following equilibrium program.

(a) For $i = 1, \ldots, m$, given p,
$$\max f^i(x^i) \quad \text{for} \quad px^i \leq pw^i \quad \text{and} \quad 0 \leq x^i \leq w, \tag{1.2}$$
(b) Given x,
$$\max p\left(\sum_{i=1}^{m} x^i - w\right) \quad \text{for} \quad p \in S^{n-1}(t) = \{p \in S^{n-1} \mid p \geq \hat{p} - (t+\epsilon)e\},$$

where $S^{n-1} = \{p \in R^n_+ \mid \sum_{j=1}^n p_j = 1\}$, $\epsilon > 0$ is very small, e is a vector with all components 1, $0 \leq t \leq 1$, and $\hat{p}$ is some arbitrarily chosen initial price vector. For given p, let $z(p)$ be the excess demand, i.e. $z(p) = \sum_{i=1}^m x^i(p) - w$, where $x^i(p)$ solves (1.2a). It is assumed that $z : S^{n-1} \to R^n$ is a continuous function.

Without loss of generality we can assume that the excess demand at $\hat{p}$ has a maximum at a unique index, say k,. Clearly, for $t = -\epsilon$, (1.2) has a unique solution $(x(\hat{p}), \hat{p})$, where $x^i(\hat{p})$ solves (1.2a) given $p = \hat{p}$.

For $\epsilon > 0$ small enough we have that $z_k(p)$ is still the unique maximum excess demand for $t = 0$ and p satisfying the conditions of (1.2b). Therefore, given $x(p)$ with $p \in S^{n-1}(t)$, the solution of (1.2b) is

$$\mathring{p}_k = \hat{p}_k + (n-1)\epsilon,$$
$$\mathring{p}_j = \hat{p}_j - \epsilon, \quad j \neq k \tag{1.3}$$

when $t = 0$. Hence $(x(\mathring{p}), \mathring{p})$ is the unique solution of (1.2) at $t = 0$. In their paper Zangwill and Garcia prove the following theorem.

Theorem 1.2. *Suppose for all i, f^i is 3-differentiable and strictly concave. If the equilibrium program* (1.2) *is regular, then starting from* $(x, p, t) = (x(\mathring{p}), \mathring{p}, 0)$ *there is a path of solutions to* (1.2) *that reaches a solution to* (1.1) *at* $t = 1$.

As discussed by Zangwill and Garcia, the economic interpretation of this path is as follows. For small $\epsilon > 0$, when t is increased from zero, the price of good k having the largest excess demand is increased, whereas the other prices are decreased, each by the same amount. In general the price vector p and the variable t are adapted so as to try to decrease the usage of all goods with highest demand. So, at any p, the process works on the worst cases, i.e. on the markets with highest excess demand, until an equilibrium price vector $\bar{p}$ with $z(\bar{p}) \leq 0$ is reached.

Looking at the process in more detail, we first note that obviously t is not necessarily increasing monotonically during the process. In [4], Scarf provides some examples for which the classical tâtonnement does not converge. It can easily be checked that for these examples t does not increase monotonically in the adjustment process. When we consider the projection of the path on the set of prices S^{n-1}, the adjustment process behaves as follows. Define, for $T \subset I_n = \{1, \dots, n\}$,

$$C(T) = \{p \in S^{n-1} \mid z_k(p) = \max_j z_j(p),\ k \in T\}$$

and, for $0 \le t \le 1$,

$$P(T,t) = \left\{p \in S^{n-1}(t) \mid p = \sum_{j \in T} \alpha_j p^j(t),\ \alpha_j \ge 0,\ \sum_{j \in T} \alpha_j = 1\right\},$$

where $p^j(t)$ is the vertex of $S^{n-1}(t)$ such that the jth component is maximal. We assume that for $|T| \ne 0, n$ $C(T)$ is an $(n-|T|)$-dimensional set, where $|T|$ denotes the cardinality of T.

Then, a point (x, p, t) on the path of solutions to (1.2) has the property that for some $T \subset I_n$, $p \in C(T) \cap P(T, t)$. The projection of this path on the price space S^{n-1} is illustrated in Fig. 1. In Fig. 1a the procedure starts in $C(\{2\})$ and hence p_2 is increased until $C(\{3\})$ is reached. Then the 1-manifold $C(\{2,3\}) = C(\{2\}) \cap C(\{3\})$ is followed until the equilibrium price $\bar{p}$ is obtained. Observe that t increases monotonically. In Fig. 1b again we have that the procedure starts in $C(\{2\})$, however in a subset of $C(\{2\})$ which is surrounded by $C(\{3\})$. Now t increases on the path from $\hat{p}$ ($\mathring{p}$) to p^1, decreases from p^1 to p^2 and increases again on the path from p^2 to the equilibrium price $\bar{p}$.

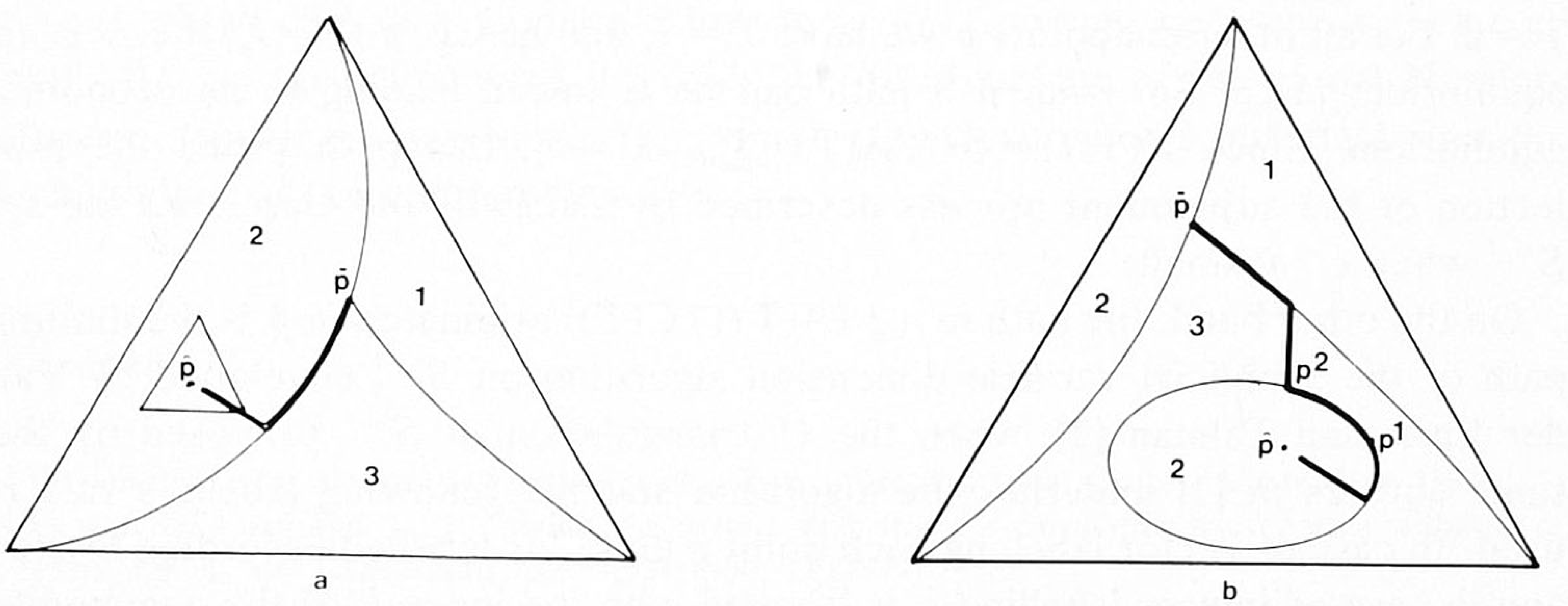

Fig. 1. The path of solution points projected on S^{n-1} have been drawn heavily. $C(\{j\})$ is indicated by j.

2. The variable dimension approach

In this section we show that the projection on the price space S^{n-1} of the adjustment process of Zangwill and Garcia coincides with the limiting path of the simplicial variable dimension algorithm developed by Van der Laan and Talman, provided an appropriate labelling rule and triangulation underlying the algorithm.

To do so, for $T \subset I_n$, let the sets $A(T)$ be defined by

$$A(T) = \left\{ p \in S^{n-1} \,\middle|\, p = \hat{p} + \sum_{j \in T} \lambda_j u(j),\ \lambda_j \geq 0 \right\}, \tag{2.1}$$

where $u(j)$, $j = 1, \ldots, n$, is the jth column of the $n \times n$ matrix

$$U = \begin{bmatrix} (n-1) & -1 & \cdots & -1 \\ -1 & (n-1) & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & & -1 \\ -1 & \cdots & -1 & (n-1) \end{bmatrix}$$

(see Van der Laan and Talman [3, p. 282]). For $\hat{p} \in \operatorname{int} S^{n-1}$, $A(T)$ is a $|T|$-dimensional subset of S^{n-1}. So, under some regularity conditions, we have that the set of points p with

$$p \in \bigcup_{T \subset I_n} (A(T) \cap C(T))$$

is a collection of paths and loops. The set of endpoints of the paths is the set of points $p \in \bigcup_{|T|=0,n} (A(T) \cap C(T)$. Clearly, the point $\hat{p}$ is the unique endpoint for $T = \emptyset$. For all other endpoints p we have $T = I_n$ and hence, $p \in C(I_n)$, i.e. p is an equilibrium price. So, from $\hat{p}$ a path can be followed leading to an economic equilibrium. Since $P(T, t) = \{p \in A(T) \mid \sum_{j \in T} \lambda_j = t\}$, this path yields the projection of the adjustment process described by Zangwill and Garcia on the set S^{n-1} when $\epsilon > 0$ small.

On the other hand, the path in $\bigcup (A(T) \cap C(T))$ originated in $\hat{p}$ is the limiting path of the simplicial variable dimension algorithm on S^{n-1} developed by Van der Laan and Talman [2], when the U triangulation of S^{n-1} proposed by the same authors in [3] underlies the algorithm and the following labelling rule is used. In case of vector labelling each point $p \in S^{n-1}$ is labelled according to $z(p)$ and in case of integer labelling p is labelled with the index k of the commodity with the highest excess demand. In both cases the same limiting path in $\bigcup (A(T) \cap C(T)$ is obtained, as discussed in Van der Laan [1, pp. 72, 73 and 83, 84]. Observe that the computational results in [1] were obtained for these labelling rules and in both [1] and [6] also for the U triangulation.

Therefore the adjustment path of Zangwill and Garcia can be traced by the variable dimension algorithm. Starting in an arbitrarily chosen price vector $\hat{p} \in S^{n-1}$ with $T = \emptyset$ the algorithm generates for varying T a path of prices p in $A(T)$ such that all goods j have highest excess demand $z_j(p)$, for $j \in T$. As soon as good i,

$i \notin T$, has an excess demand $z_i(p)$ equal to the highest one, the algorithm continues in $A(T \cup \{i\})$ with prices such that $z_h(p) = \max_j z_j(p)$ for all $h \in T \cup \{i\}$. If, however, the algorithm generates a price p in $A(T \setminus \{k\})$ for some $k \in T$, it continues in $A(T \setminus \{k\})$ with prices p such that $z_h(p) = \max_j z_j(p)$ for all $k \in T \setminus \{k\}$. The latter step happens when λ_k in (2.1) becomes equal to zero. So, for (x, p, t) a solution generated by the adjustment process and letting T, $|T| < n$, be the unique index set such that

$$p = \hat{p} + \sum_{j \in T} \lambda_j u(j), \quad \lambda_j > 0,$$

we have, $p \in A(T) \cap C(T)$, i.e. the complementarity conditions

$$\lambda_j = 0 \quad \text{and} \quad z_j(p) < \max_i z_i(p), \quad \text{if } j \notin T$$

and

$$\lambda_j > 0 \quad \text{and} \quad z_j(p) = \max_i z_i(p), \quad \text{if } j \in T.$$

hold.

Concluding, the variable dimension algorithm can be utilized as a simplicial path following scheme for the projection on S^{n-1} of the path of solutions to (1.2). Moreover, the limiting path of the algorithm generates the path of points of the adjustment process proposed by Zangwill and Garcia. This adjustment process both converges to a price equilibrium and can start anywhere. Moreover, it has an plausible economic interpretation. Therefore the process is rather attractive. Firstly it avoids the disadvantage of the classical tâtonnement process which can start anywhere but does not converge necessarily (see e.g. Scarf [4]). On the other hand it avoids the disadvantage of the global Newton adjustment process, (see Smale [5]), which always converges but must start on the boundary of the unit simplex.

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