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Existence of general equilibria in economies with natural exhaustible resources and an infinite horizon

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This paper establishes the existence of a general equilibrium for economies with natural exhaustible resources and an infinite horizon. It is argued that the traditional methods for proving existence in economies with an infinite dimensional commodity space cannot be invoked here and an alternative proof is provided.

1. Introduction

The present paper deals with the existence of general equilibria in economies with an infinite dimensional commodity space. The infinite dimension is brought about by the fact that we employ an infinite horizon (discrete time) model. The economic context is given by the exploitation and use of exhaustible natural resources. It turns out that this framework gives rise to several serious problems if one tries to apply standard results from the vast literature on economies with an infinite dimensional commodity space. This introduction sketches these problems, thereby motivating the research reported in the sequel.

The first, and central, problem lies in the choice of the commodity space. When the economy is an economy with raw materials from exhaustible resources as the only commodities, it is quite natural to choose l_1 as the commodity space, as has rightly been put forward by Zame (1987) in his examples. One should in that case even be dissuaded from choosing l_∞ . This can be seen as follows.

Example 1. Consider an economy with one producer and one consumer. The production set is

*The authors are indebted to Donald Brown and Truman Bewley for their stimulating remarks. Any errors are the authors' sole responsibility.

$$Y = \left\{ y \in l_\infty \mid y(0) \leq 0, y(t) \geq 0 (t \geq 1), \sum_{t=0}^{\infty} y(t) \leq 0 \right\}.$$

The consumption set is

$$X = \{ x \in l_\infty \mid x(t) \geq 0 \}.$$

Preferences are induced by

$$U(x) := \sum_{t \geq 0} 2\beta^t x(t)^\frac{1}{2}, \quad 0 < \beta < 1.$$

The initial endowment vector ω is given by $((1/\beta^2(1-\beta^2), 0, 0, \dots))$.

Of course the only share equals unity. The consumer is the initial holder of the resource stock which amounts to $1/\beta^2(1-\beta^2)$. Part of it is sold to the producer ($y(0)$), who extracts $y(1), y(2), \dots$, at no cost.

Conditions (i)–(vi) of Theorem 3 in Bewley (1972) are satisfied. Hence, if there is an equilibrium with a price system in the topological dual of l_∞ (with norm topology), which seems to be a natural requirement, then there is an equilibrium, denoted by (π, \bar{x}, \bar{y}) where $\pi(>0)$ is an element of l_1 . So π (the price system) may be considered as a sequence $p(t)|_{t \geq 0}$, where $p(t) \geq 0$ for all $t \geq 0$. It is clear that $p(0) > 0, \pi \bar{y} = 0$ and that the consumer maximizes $U(x)$ over the set

$$\sum_{t=0}^{\infty} p(t)x(t) \leq p(0)\omega(0).$$

Hence $p(t) > 0$ for all $t, x(t) > 0$ for all $t, y(t) > 0$ for all $t \geq 1$ and, as a consequence, $p(t) = p > 0$ for all $t \geq 0$. So $\pi \notin l_1$ and the economy has no equilibrium (at least with l_∞ as the commodity space and prices in its dual).

However, when the raw material enters into the production sets less trivially and is not the only commodity, l_1 is no longer eligible as the commodity space, as is shown in the following example.

Example 2. Consider an economy whose (aggregate) production possibilities are described by a production function with capital (K) and resource commodities (R) as inputs. Gross output in period t is given by

$$K(t-1)^{\alpha_1} R(t)^{\alpha_2},$$

with $1 > \alpha_1 > \alpha_2 > 0$. Here $K(t-1)$ is the stock of capital available at the

beginning of period t . Gross output is used for consumption purposes (C) and (net) investments:

$$C(t) + K(t) - K(t-1) = K(t-1)^{\alpha_1} R(t)^{\alpha_2}, \quad t = 1, 2, \dots$$

The initial capital stock is given: K_0 . The total initial stock of the resource is denoted by S_0 . So the economy faces the constraint

$$\sum_{t=1}^{\infty} R(t) \leq S_0. \tag{*}$$

A typical element of the economy's production set is given by

$$y = (-K_0, -S_0, C(1), C(2), \dots).$$

Now take $K(t) = (t+1)K_0$, $C(t) = Bt^\gamma - K_0$ with B and γ yet to be determined. Then

$$R(t) = R(1)t^{(\gamma - \alpha_1)/\alpha_2}.$$

So there exist B and $\gamma (>0)$ such that the integral constraint (*) is satisfied. Therefore the production set is unbounded.

It is clear from this example that l_1 is not the appropriate commodity space and, moreover, that with l_∞ serious problems may arise. One could argue that, with a positive rate of time preference on the part of the consumers, equilibrium allocations will be in l_∞ , so that one could safely assume l_∞ to be the commodity space. This conjecture is false in general. It has been shown by Dasgupta and Heal (1974) that with constant returns to scale and a CES specification of the production function with an elasticity of substitution larger than unity, a centralized economy with a utilitarian social welfare objective, will enjoy unbounded future consumption, even with a positive rate of time preference. This of course also holds for a general equilibrium with one consumer. Admittedly, CES functions of the type described above are rather special in the sense that none of the inputs is necessary for production. When it is assumed that both inputs are necessary, as will be done in the sequel, other problems may arise.

These are related to an assumption which is commonly made in the literature and which is referred to as 'uniform properness' [Richard (1986)], 'boundedness of marginal efficiency' [Zame (1987)], 'universal technical substitutability' [Zame (1987)] and the like. It is not an easy task to check for a given production set whether this condition holds or not. But the

example below shows that for a technology description which is quite popular among economists the assumption does not hold.

Example 3. Consider again the Cobb–Douglas economy of the previous example. Denote the maximal sustainable constant rate of consumption by $C_{\max}(K_0, S_0)$. It is shown in Withagen (1990) that for the continuous-time analogue we have

$$C_{\max}(K_0, S_0) = (1 - \alpha_2) \{ (\alpha_1 - \alpha_2)^{\alpha_2} S_0^{\alpha_2} K_0^{\alpha_1 - \alpha_2} \}^{1/(1 - \alpha_2)}.$$

In discrete time C_{\max} will be of the same form. We adopt Zame’s boundedness definition.

Boundedness of marginal efficiency says the following: there exists $M > 0$ such that for all $y \in (-K_0, -S_0, C(1), C(2), \dots) \in Y$ and all $a = (a_1, a_2)$ with $0 \leq a_1 \leq K_0, 0 \leq a_2 \leq S_0$ there exists a real number $\rho (0 < \rho < 1)$, a vector $b = (b(1), b(2), \dots) (0 \leq b(t) \leq C(t), t = 1, 2, \dots)$ and $\hat{y} \in Y$ such that

$$\hat{K}_0 = K_0 - \rho a_1,$$

$$\hat{S}_0 = S_0 - \rho a_2,$$

$$\hat{C}(t) = C(t) - b(t) \quad (t = 1, 2, \dots),$$

and $\|b\| \leq M \|\rho a\|$. Since in the economy under consideration both inputs are necessary, it seems plausible at first sight to take l_∞ as the commodity space, so that the norms in the definition are to be interpreted as l_∞ -norms.

Now take $y = (-K_0, -S_0, [C_{\max}(K_0, S_0)]_{t=1}^\infty)$, $a = (K_0, S_0), K_0 = S_0$.

Then

$$\begin{aligned} \hat{C}_{\max} &= C_{\max}((1 - \rho)K_0, (1 - \rho)S_0) \\ &= (1 - \rho)^{\alpha_1/(1 - \alpha_2)} C_{\max}(K_0, S_0). \end{aligned}$$

Now there exists $t = t_1$ such that $\hat{C}(t_1) \leq \hat{C}_{\max}$. Hence

$$\begin{aligned} b(t_1) &\geq (1 - (1 - \rho)^{\alpha_1/(1 - \alpha_2)}) C_{\max}(K_0, S_0) \\ &= (1 - (1 - \rho)^{\alpha_1/(1 - \alpha_2)}) A K_0^{\alpha_1/(1 - \alpha_2)}, \end{aligned}$$

with $A := (1 - \alpha_2)(\alpha_1 - \alpha_2)^{\alpha_2/(1 - \alpha_2)}$. We show that for any M there exists K_0 such that $\|b\|_\infty \geq M \rho K \|(1, 1)\|_\infty$. This boils down to showing that there exists K_0 such that

$$(1 - (1 - \rho)^{\alpha_1/(1 - \alpha_2)})AK_0^{(\alpha_1 + \alpha_2 - 1)/(1 - \alpha_2)} \geq M\rho.$$

This is an easy exercise: let K_0 go to zero. Note that $M\rho/[1 - (1 - \rho)^{\alpha_1/(1 - \alpha_2)}]$ is bounded.

This example suggests that there exists a large class of economies for which it is likely that l_∞ is the appropriate commodity space but where boundedness conditions such as Zame's fail to hold, implying that the standard results cannot be applied. It is even possible to construct non-pathological examples where l_1 is the appropriate commodity space and boundedness of marginal efficiency does not hold [in those examples there exists $q > 0$ such that $F(K, R) \leq qR$ for all $K > 0$, where F is the production function].

We have a major difficulty here. Dasgupta and Heal's (1974) work leads one to the conclusion that with bounded derivatives of the production function, consumption and production will be unbounded in equilibrium, so that l_∞ is not appropriate 'ex post', whereas with unbounded derivatives (or in the absence of bounded efficiency), l_∞ seems to be appropriate 'ex ante' but no standard results can be invoked in this case.

In our opinion, this conclusion and the fact that in applications involving production functions, the assumptions of the standard literature are difficult to check, provide a strong motivation for the development of another technique to solve the existence problem in general equilibrium models with exhaustible resources.

There are others who have dealt with the existence of general equilibria in resource economics. Dasgupta and Heal (1974) have already been mentioned. But their model is typically a one-sector model. Chiarella (1980) employs a two-country model with unilateral ownership of the resources and the non-resource technology and uses only Cobb-Douglas specifications. Finally, Mitra's model (1980) bears some resemblance to ours but he uses a homogeneous production function and a bounded utility function, which seems to be crucial in his existence proof along the lines set out by Koopmans (1965), Gale (1967), Brock (1970) and McKenzie (1968).

The sequel of the paper is organized as follows. Section 2 describes the model we employ and shows the existence of a general equilibrium for each finite horizon (or truncated) economy. Section 3 goes into the boundedness properties of the general equilibrium allocations in the truncated economy. Section 4 establishes the existence of a general equilibrium in the infinite horizon economy. Finally, section 5 contains the conclusions. As a final remark it should be stressed that we focus our attention here on an existence theorem and not on the description of possibly interesting features or characteristics of an equilibrium. For this we refer to Van Geldrop and Withagen (1988).

2. The model

In this section a description is given of a general equilibrium model of an economy with exhaustible resources and the existence of a general equilibrium of the finite horizon case is established.

Commodities

There is a non-resource commodity, which serves as the only desirable consumer good and as input and output of the production processes. This commodity can be interpreted as an aggregate, perfectly malleable, non-resource commodity. Furthermore there are $m (\geq 1)$ types of resource stocks from which a homogeneous raw material is extracted.

Hence we can describe the set M of commodities by $M = \{O_1, O_2, \dots, O_m\} \cup (\{C\} \times \mathbb{N}) \cup (\{R\} \times \mathbb{N})$, where

$\mathbb{N} = \{0, 1, 2, \dots\}$, the set of natural numbers,

O_τ = resource stock of type τ , $\tau = 1, \dots, m$,

C = non-resource commodity, frequently referred to as capital or consumption,

R = raw material, extracted from the resource stocks.

It should be noted that the resource stocks will be distinguished from each other by extraction costs, while the extracted raw material R is the same for all of them.

The commodity space is a subspace $L \subset \mathbb{R}^M$, the set of all functions $x: M \rightarrow \mathbb{R}$. Given $x \in \mathbb{R}^M$ we list the image as follows:

$$x_\tau := x(O_\tau), \quad \tau = 1, \dots, m,$$

$$x_C(t) := x(C, t), \quad t \in \mathbb{N},$$

$$x_R(t) := x(R, t), \quad t \in \mathbb{N}.$$

So, a commodity bundle will be represented as

$$x = \left[(x_1, \dots, x_m), \begin{pmatrix} x_C(0) \\ x_R(0) \end{pmatrix}, \dots, \begin{pmatrix} x_C(t) \\ x_R(t) \end{pmatrix}, \dots \right], \quad x \in L.$$

Production

There are $n+m$ production sectors. *The first $n (\geq 1)$ sectors produce the non-resource commodity, C , using extracted raw material, R , and the non-*

resource commodity as inputs, according to technology F_i ($i=1, \dots, n$). Production takes time. We define the production set $Y_c^{(i)}$ by

$$x \in Y_c^{(i)}: \Leftrightarrow C.1. \quad x_\tau \leq 0; \quad 1 \leq \tau \leq m.$$

C.2. There is a sequence $k(t)$ ($t \in \mathbb{N}$) such that

$$(i) \quad k(t) \geq 0, \quad \text{all } t.$$

$$(ii) \quad k(0) \leq -x_C(0).$$

$$(iii) \quad k(t) + x_C(t) \leq k(t-1) + F_i(k(t-1), -x_R(t-1)), \quad t \geq 1.$$

Here, resource stocks are considered formally as inputs. Moreover, $k(t)$ is introduced as an artefact. It represents the current stock of the non-resource commodity C as a capital input, without depreciation, in production. It is not lost during the production process.

Gross output: $k(t-1) + F_i(k(t-1), -x_R(t-1))$ is divided over production $x_C(t)$, leaving the sector, and new capital input $k(t)$. Note that $(C, 0)$ is a necessary input here.

About F_i the following assumptions are made (we omit the index i when there is no danger of confusion).

A.1. F is defined on \mathbb{R}_+^2 , is continuous, concave and weakly monotonically increasing.

$$A.2. \quad F(k, 0) = F(0, z) = 0.$$

$$A.3.1. \quad F(k, z) \leq qz \text{ for all } (k, z) \text{ for some given } q > 0.$$

or,

$$A.3.2. \quad \lim_{k \rightarrow \infty} \frac{F(k, z)}{k} = 0 \quad \text{for all } z, \quad \lim_{k \rightarrow 0} \frac{F(k, z)}{k} = \infty \quad \text{for all } z > 0.$$

A.1 is quite standard and needs no further comment. A.2 incorporates the necessity of both inputs. A.3.1 implies that the average product of the raw material is bounded. In case of A.3.2 the reader may recognize familiar elements from neoclassical growth models.

Extraction is carried out in the sectors $n+1, \dots, n+m$. It requires the input of the non-resource commodity, which is not lost during the production process. Production takes no time. The technology displays non-increasing returns to scale.

The production set $Y_e^{(j)}$ in sector $n+j$ is defined by

$$x \in Y_e^{(j)}: \Leftrightarrow E.1. \quad x_\tau \leq 0, \quad 1 \leq \tau \leq m,$$

$$E.2. \quad x_R(t) \geq 0, \quad t \in \mathbb{N},$$

$$E.3. \quad \sum_{t=0}^{\infty} x_R(t) + x_j \leq 0,$$

$$E.4. \quad G_j(x_R(t)) + \sum_{s=0}^t x_C(s) \leq 0.$$

Here $G_j(x_R)$ are the costs, in capital, necessary to extract an amount x_R from the resource stock of type j .

So we postulate that sector $n+j$ only exploits O_j and that for all t there must be available a non-negative (accumulated) amount of capital.

About G_j we make the following assumption:

A.4. G_j is defined on \mathbb{R}_+ , is continuous, convex and increasing, $G_j(0)=0$, $j=1, 2, \dots, m$.

Consumption

There are H consumers, indexed by $h=1, 2, \dots, H$, all of them having $X=L^+$ as the consumption set. The initial endowments are ω^h , where

$$I.1. \quad \omega_t^h \geq 0, \quad \sum_h \omega_t^h > 0, \quad 1 \leq t \leq m,$$

$$I.2. \quad \omega_C^h(0) > 0, \quad \omega_C^h(t) = 0, \quad t \geq 1,$$

$$I.3. \quad \omega_R^h(t) = 0, \quad t \in \mathbb{N}.$$

So, each consumer has at least a positive amount of capital at $t=0$, but no endowments in the future.

The consumers hold shares in the production sectors given by

$$\delta^h = (\delta_e^{h,1}, \dots, \delta_e^{h,m}, \delta_c^{h,1}, \dots, \delta_c^{h,n}) \geq 0,$$

$$\sum_h \delta^h = (1, 1, \dots, 1) \in \mathbb{R}^{n+m}.$$

The preference relations are described by

$$U^h(x) = \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho_h} \right)^t u_h(x_C(t)),$$

where $\rho_h > 0$ denotes the constant rate of time preference and u_h is the instantaneous utility function. u_h satisfies

A.6. u_h is defined on \mathbb{R}_+ , continuous, concave, strictly increasing with a continuous derivative on \mathbb{R}_{++} , $u'_h(0) = \infty$, and $u_h(0) = 0$.

Prices

Given $T \in \mathbb{N}$, we consider the subspace $L(T)$ of \mathbb{R}^M consisting of all x satisfying

$$\begin{pmatrix} X_c(t) \\ X_R(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{all } t > T.$$

We obtain the finite horizon version of the economy as a truncation of the infinite-horizon model in an obvious way. All truncations $L(T)$ are supposed to be in L .

For such a truncation we denote a price-system $\pi > 0$ by

$$\pi = \left[(\pi_1, \dots, \pi_m), \begin{pmatrix} p_C(0) \\ p_R(0) \end{pmatrix}, \dots, \begin{pmatrix} p_C(T) \\ p_R(T) \end{pmatrix} \right].$$

Note. Because, until now, we have no topology on $L \subset \mathbb{R}^M$, due to the fact that L is not specified, we are not committed yet to define the price-space, as a subspace of the dual L^* .

Remark. One can think of several economic frameworks for which the above model is an adequate description. Perhaps the most appealing setting is a world model with H countries, each possessing a stock of an exhaustible resource ($m=H$) and a technology to convert the raw material and capital into a commodity that can be used for consumption purposes and investment ($n=H$). Each country has to make costs to extract the raw material. The factors of production are perfectly mobile between countries. Each economy aims at the maximization of a utilitarian welfare functional. The current accounts are not required to equilibrate but of course total discounted expenditures are not allowed to exceed total discounted income; this condition constitutes a budget constraint for each country. In the present interpretation $\delta_e^{h,j} = \delta_c^{h,j} = 1$ for $h=j$.

Another framework is of course a closed economy with many consumers, producers and resource stocks.

An equilibrium of the truncated economy $E(T) \in L(T)$ is a tuple $[x^1, \dots, x^H, \pi, y_c^{(1)}, \dots, y_c^{(m)}, y_e^{(1)}, \dots, y_e^{(m)}]$ satisfying

(i)
$$\sum_h x^h \leq \sum_h \omega^h + \sum_i y_c^{(i)} + \sum_j y_e^{(j)},$$

- (ii) x^h maximizes $U^h(x)$ over the set $\left\{x \mid \pi \cdot x \leq \pi \cdot \omega^h + \sum_i \delta_c^{h,i} \pi y_c^{(i)} + \sum_j \delta_e^{h,j} \pi y_e^{(j)}\right\}$,
- (iii) $y_c^{(i)}$ maximizes πy over $Y_c^{(i)}$, $i = 1, 2, \dots, n$,
 $y_e^{(j)}$ maximizes πy over $Y_e^{(j)}$, $j = 1, 2, \dots, m$,
- (iv) $\pi \left(\sum_h x^h - \sum_i y_c^{(i)} - \sum_j y_e^{(j)} - \sum_h \omega^h \right) = 0$.

It should be clear that we employ here the Arrow–Debreu dated-commodity framework. In such a world the assumption that trade in resource stocks only takes place at the outset is obviously innocuous. It is even convenient since no explicit account has to be taken of savings and investments: interest rates follow from the trajectories of the consumer prices.

Aggregation

It is our aim to investigate the behaviour of the finite-horizon equilibria when T goes to infinity. It will be convenient to consider an aggregate version of the production set, taking into account the fact that consumers are interested only in the commodity (C, t) , which is an output of the sectors $Y_c^{(i)}$, using the extraction output of the sectors $Y_e^{(j)}$.

Hence we redefine M as follows:

$$M := \{O_1, \dots, O_m\} \cup (\{C\} \times \mathbb{N}).$$

The commodity-space is a subspace L of \mathbb{R}^M .

Given $x \in \mathbb{R}^M$ we list the image as follows:

$$x = [x(-m), x(-m+1), \dots, x(-1), x(0), x(1), \dots],$$

where, obviously, $\{O_1, \dots, O_m\}$ is identified with

$$\{-1, -2, \dots, -m\}, \text{ and } (C, t) \text{ with } t; \quad t \in \mathbb{N}.$$

So $M \equiv \{t \in \mathbb{Z} \mid t \geq -m\}$.

Once again it is *assumed* that each truncation $L(T)$ is a subspace of L .

The aggregate production set $Y \subset L$ is now defined by: $y \in Y: \Leftrightarrow$

- (a) $y(t) \leq 0$ all $t \leq 0$,
 $y(t) \geq 0$ all $t \geq 1$
 (resource stocks and initial capital are inputs);
- (b) The system of inequalities,

$$(1) \sum_{t=1}^{\infty} e_j(t-1) \leq -y(-j), \quad j=1 \dots m$$

(extraction in sector j cannot exceed the resource-stock);

$$(2) \sum_{i=1}^n r_i(t-1) \leq \sum_{j=1}^m e_j(t-1), \quad t \geq 1$$

(total amount of raw material, used in production sectors, cannot exceed total extraction, in every period);

$$(3) \begin{cases} K(t) + y(t) \leq K(t-1) + \sum_{i=1}^n F_i(k_i(t-1), r_i(t-1)), \quad t \geq 1 \\ \text{(gross output is divided over consumption and future capital);} \\ K(0) + y(0) \leq 0 \\ \text{(initial capital } K \text{ in production is bounded by input of capital);} \end{cases}$$

$$(4) \sum_{j=1}^m G_j(e_j(t-1)) + \sum_{i=1}^n k_i(t-1) \leq K(t-1), \quad t \geq 1$$

(total capital inputs cannot exceed available capital),

has a non-negative solution

$$(K(t), k(t), r(t), e(t)), \quad t \in \mathbb{N},$$

where

$$r(t) = (r_1(t), \dots, r_n(t)),$$

$$e(t) = (e_1(t), \dots, e_m(t)),$$

$$k(t) = (k_1(t), \dots, k_n(t)).$$

The consumption sets are defined by

$$X^h = X \quad \text{all } h, \quad \text{where}$$

$$x \in X := \begin{cases} x(t) = 0, & t < 0 \\ x(t) \geq 0, & t \geq 0 \end{cases}$$

Preferences are described by

$$U^h(x) := \sum_{t=0}^{\infty} \left(\frac{1}{1 + \rho_h} \right)^t u_h(x(t)).$$

Initial endowments are

$$\omega^h(t) \geq 0, \quad \sum_{h=1}^H \omega^h(t) > 0, \quad t < 0,$$

$$\omega^h(0) > 0,$$

$$\omega^h(t) = 0, \quad t > 0.$$

Shares $\delta^1, \dots, \delta^H$, all ≥ 0 , and $\sum_{h=1}^H \delta^h = 1$.

Remark. One caveat is in order here. If for some horizon T the disaggregated economy $E(T)$ has a general equilibrium, then obviously there exists a general equilibrium for the aggregate economy $\tilde{E}(T)$ with the aggregate shares δ^h properly chosen, namely as the proportion of the total profits in the disaggregated economy each consumer is entitled to. Conversely, if for any distribution of shares in aggregated profits there exists an equilibrium in the aggregate economy, the disaggregated economy has an equilibrium for the initial distribution of shares. In the sequel we therefore confine ourselves to showing the existence of a general equilibrium in $\tilde{E}(T)$ and without loss of generality it will be assumed that the shares δ^h are fixed.

In the sequel of this paper we will show that equilibrium allocations of the finite-horizon aggregated economies are uniformly (with respect to T) bounded. Then the same holds for the disaggregated version.

Theorem 2.1. For all $T \geq 1$ the truncated, aggregated economy, henceforth denoted by $\tilde{E}(T)$, has a general competitive equilibrium $(x_T^h|_{h=1}^H, p_T, y_T)$, where

$$\begin{cases} x_T^h = [0, \dots, 0, x_T^h(0), \dots, x_T^h(T), 0 \dots], & h = 1 \dots H, \\ p_T = [p_T(-m), \dots, p_T(-1), p_T(0), p_T(T), 0 \dots], \\ y_T = [y_T(-m), \dots, y_T(-1), y_T(0), \dots, y_T(T), 0 \dots]. \end{cases}$$

Hence

$$(1) \quad \sum_{h=1}^H x_T^h \leq \sum_{h=1}^H \omega^h + y_T,$$

(2) x_T^h maximizes $\sum_{t=0}^T (1/(1 + \rho_h))^t u_h(x(t))$ over the set

$$\left\{ [x(0) \dots x(T), 0 \dots] \mid p_T \cdot x = \sum_{t=0}^T p_T(t)x(t) \leq p_T \cdot \omega^h + \delta^h p_T \cdot y_T \right\},$$

(3) y_T maximizes

$$p_T \cdot y = \sum_{t=-m}^T p_T(t)y_T(t) \text{ over } Y \cap L(T).$$

Proof. This theorem can be proved using fairly standard techniques [see, e.g., Arrow and Hahn (1972)]. The economy satisfies all the usual conditions sufficient for the existence of a general equilibrium, except for $\omega^h \in \text{int } X^h$ ($h=1, 2, \dots, H$). But there obviously exists a compensated or quasi-equilibrium without this assumption. In order to prove the theorem then it is sufficient to show that in such an equilibrium the incomes imputed to the consumers are positive. This is a simple exercise: it makes use of the fact that in a compensated equilibrium not all prices are zero and that such an equilibrium is Pareto-efficient. \square

3. Properties of the finite horizon general equilibrium

In this section some properties of the general equilibria described in the previous section will be established. The focus is on the uniform boundedness of equilibrium prices and quantities.

We derive, for fixed T , several properties of the equilibria of $\tilde{E}(T)$. While doing this, we omit the subscript T in all symbols.

Theorem 3.1

$$p(t) > 0, \quad 0 \leq t \leq T, \tag{3.1}$$

$$\sum_{t=0}^T p(t)x^h(t) = \delta^h \cdot p \cdot y + p \cdot \omega^h. \tag{3.2}$$

There are constants $\phi^h > 0$ such that for all h :

$$\left(\frac{1}{1 + \rho_h} \right)^t u'_h(x^h(t)) = \phi^h p(t), \quad 0 \leq t \leq T, \quad \text{while } x^h(t) > 0 \text{ all } h \text{ and } t. \tag{3.3}$$

For all t with $1 \leq t \leq T$ there are $\alpha(t) \geq 0$ and $\beta(t) \geq 0$ such that, in equilibrium, for all t and i ($k_i(t-1), r_i(t-1)$) maximizes

$$F_i(k, r) - \alpha(t)k - \beta(t)r \quad \text{over } k \geq 0, r \geq 0, \tag{3.4}$$

$$\sum_j e_j(t-1) - \sum_i r_i(t-1) \geq 0, \quad \text{all } t, \tag{3.5}$$

$$\beta(t) \left(\sum_j e_j(t-1) - \sum_i r_i(t-1) \right) = 0, \quad \text{all } t,$$

$$K(t-1) - \sum_i k_i(t-1) - \sum_j G_j(e_j(t-1)) \geq 0, \quad \text{all } t, \tag{3.6}$$

$$\alpha(t) \left(K(t-1) - \sum_i k_i(t-1) - \sum_j G_j(e_j(t-1)) \right) = 0, \quad \text{all } t,$$

$$K(t) + y(t) = K(t-1) + \sum_i F_i(k_i(t-1), r_i(t-1)), \quad \text{all } t, \tag{3.7}$$

$$K(0) + y(0) = 0, \tag{3.8}$$

$$p(t) = \frac{p(t-1)}{1 + \alpha(t)}, \quad t \geq 1, \quad \text{so } p(t) \leq p(0), \quad \text{all } t, \tag{3.9}$$

$$p_j(-j) \left(y(-j) - \sum_1^T e_j(t-1) \right) = 0, \quad j = 1, \dots, m, \tag{3.10}$$

$$\sum_{h=1}^H x^h(t) = y(t) + \sum_h \omega^h(t), \quad t = 0, 1, \dots, T. \tag{3.11}$$

Proof. This is a standard, but tedious exercise in finite-dimensional optimization. The sequences α and β have their roots in Lagrange-multipliers. \square

Now assume that $\alpha(\hat{t}) = 0$ for some \hat{t} . Then we have from (3.4) that for all i

$$F_i(k, r) - \beta(\hat{t})r \quad \text{is maximized by } (k_i(\hat{t}-1), r_i(\hat{t}-1)).$$

So $\beta(\hat{t}) > 0$ and $r_i(\hat{t}-1) = 0$ for all i , and hence

$$F_i(k, r) - \beta(\hat{t})r \leq 0 \quad \text{for all } i \text{ and all } (k, r) \geq (0, 0).$$

Then it follows from (3.7) in Theorem 3.1 and the definition of general equilibrium that

$$\begin{aligned} \sum_{t=1}^T y(t) &\leq k(0) + \sum_{t=1}^T \sum_i \beta(t) r_i(t-1) \leq k(0) + \beta(t) \sum_{t=1}^T \sum_j e_j(t-1) \\ &\leq k(0) + \beta(t) \sum_h \sum_j \omega^h(-j). \end{aligned}$$

In other words, if for some T there is some t with $\alpha_T(t)=0$, then there is a constant $\hat{\beta}$ such that for all i

$$F_i(k, r) \leq \hat{\beta}r, \quad \text{all } k \geq 0, r \geq 0.$$

Or, otherwise stated, all functions F_i satisfy assumption A.3.1, and there is a uniform upper bound for all allocations in the equilibria.

Remark. $L=l_1$ will then be an appropriate choice.

If $\alpha(t) > 0$ for all $t \geq 1$, we observe the following:

- (i) there is a uniform upper bound for $-y(0)$ and $y(1)$,
- (ii) if $\alpha(t) < \rho := \min \rho_h$, then it follows from (3.3) and (3.9) that for all h

$$\frac{u'_h(x^h(t))}{u'_h(x^h(t-1))} = \frac{1 + \rho_h}{1 + \alpha(t)} > 1,$$

and then $x^h(t) < x^h(t-1)$ for all h which, together with (3.11), yields:

$$y(t) < y(t-1), \quad t \geq 2,$$

- (iii) if $\alpha(t) \geq \rho$, then

$$F_i(k_i(t-1), r_i(t-1)) - \alpha(t)k_i(t-1) - \beta(t)r_i(t-1) \geq 0$$

implies $F_i(k_i(t-1), r_i(t-1)) \geq \rho k_i(t-1)$.

Let $S > 0$ be the total initial endowment of all resource-stocks of the consumers. Then $F_i(k_i(t-1), S) \geq \rho k_i(t-1)$, and $k_i(t-1) \leq \bar{k}_i$ where $\bar{k}_i > 0$ satisfies $F_i(\bar{k}_i, S) = \rho \bar{k}_i$. Moreover, it follows from (3.6) that $K(t-1) = \sum_i k_i(t-1) + \sum_j G_j(e_j(t-1)) \leq \sum_i \bar{k}_i + \sum_j G_j(S)$.

Then $y(t) \leq \sum_i \bar{k}_i + \sum_j G_j(S) + \sum_i \rho \bar{k}_i$.

The whole argument boils down to the following:

Lemma 3.1. *There is a constant $B > 0$ such that for all T and all h : $\|x_T^h\|_\infty \leq B$ and $\|y_T\| \leq B$.*

In showing the existence of a general equilibrium for the infinite horizon we shall also use boundedness of the initial marginal utility and of the prices. To prove these boundedness properties we define

$$Q := \sum_{t=0}^{\infty} \sum_{h=1}^H \left(\frac{1}{1 + \rho_h} \right)^t u_h(B) < \infty, \quad \gamma := \min_h u'_h(B).$$

Then

$$\begin{aligned} Q &\geq \sum_{t=0}^T \sum_{h=1}^H \left(\frac{1}{1 + \rho_h} \right)^t u_h(x^h(t)) \geq \sum_{h=1}^H \sum_{t=0}^T \left(\frac{1}{1 + \rho_h} \right)^t u'_h(x^h(t)) x^h(t) \\ &\quad \text{(Lemma 3.1)} \qquad \qquad \qquad \text{(concavity of } u_h) \\ &\geq \sum_{h=1}^H \sum_{t=0}^T \phi^h p(t) x^h(t) \geq \sum_{h=1}^H \phi^h (p \cdot \omega^h + \delta^h p \cdot y) \\ &\quad \text{(3.3)} \qquad \qquad \qquad \text{(3.2)} \\ &\geq \sum_{h=1}^H \phi^h \left(p(0) \omega^h(0) + \sum_{j=1}^m p(-j) \omega^h(-j) + \delta^h p \cdot y \right) \\ &\quad \text{(since } \omega(t) = 0, t > 0) \\ &\geq \sum_{h=1}^H \frac{u'_h(x^h(0))}{p(0)} \left(p(0) \omega^h(0) + \sum_{j=1}^m p(-j) \omega^h(-j) + \delta^h p \cdot y \right) \\ &\quad \text{(3.3)} \\ &\geq \sum_{h=1}^H u'_h(x^h(0)) \left(\omega^h(0) + \frac{1}{p(0)} \sum_{j=1}^m p(-j) \omega^h(-j) + \frac{1}{p(0)} \delta^h \cdot p \cdot y \right) \\ &\geq \sum_{h=1}^H u'_h(x^h(0)) \omega^h(0) + \sum_{h=1}^H \frac{u'_h(x^h(0))}{p(0)} \sum_{j=1}^m p(-j) \omega^h(-j) \\ &\quad + \sum_{h=1}^H \frac{u'_h(x^h(0))}{p(0)} \delta^h p \cdot y \\ &\geq \sum_{h=1}^H u'_h(x^h(0)) \omega^h(0) + \sum_{h=1}^H \frac{u'_h(B)}{p(0)} \sum_{j=1}^m p(-j) \omega^h(-j) + \sum_{h=1}^H \frac{u'_h(x^h(0))}{p(0)} \delta^h p \cdot y. \\ &\quad \text{(concavity of } u_h) \end{aligned}$$

So,

$$Q \geq \sum_{h=1}^H u'_h(x^h(0))\omega^h(0) + \frac{\gamma}{p(0)} \sum_{j=1}^m p(-j) \left(\sum_{h=1}^H \omega^h(-j) \right) + \frac{\gamma}{p(0)} p \cdot y.$$

We can now easily derive the following:

Lemma 3.2. *There are $\beta > 0$ and $V > 0$ such that for all T :*

- (i) $u'_h(x^h_T(0)) \leq V; \quad h = 1, \dots, H.$
- (ii) $p_T(t) \leq \beta p_T(0); \quad -m \leq t \leq T.$
- (iii) $p_T \cdot y_T \leq V p_T(0).$
- (iv) $p_T(t)x^h_T(t) \leq \frac{u_h(B)}{\gamma(1 + \rho_h)^t}$ for all h and all t .

Proof.

$$V := \max_h \left(\frac{Q}{\omega^h(0)}, \frac{Q}{\gamma} \right),$$

$$\beta := \max \left[1, \frac{Q}{\gamma \sum_h \omega^h(-j)} \right]. \quad \square$$

Summarizing, we state the following:

Theorem 3.2. *Each finite-horizon economy $\tilde{E}(T)$ has an equilibrium (x^h_T, p_T, y_T) . There are $B > 0, V > 0, \beta > 0$ such that*

- (i) $\|x^h_T\|_\infty \leq B; \quad \|y_T\|_\infty \leq B; \quad \|p_T\|_\infty \leq \beta p_T(0); \quad \text{all } h, T.$
- (ii) $u'_h(x^h_T(0)) \leq V; \quad \text{all } h, T.$
- (iii) $p_T \cdot y_T \leq V \cdot p_T(0).$
- (iv) $\frac{u'_h(x^h_T(t+1))}{u'_h(x^h_T(t))} = (1 + \rho_h) \frac{p_T(t+1)}{p_T(t)}.$

We normalize prices by setting $p_T(0) = 1$. This is certainly the most appropriate choice if all F_t are of type A.3.1, where $L = l_1$. But, in general, when normalizing prices by $\sum_{t=-m}^T p_T(t) = 1$, where $L = l_\infty$, it may turn out that, in the limit, $p(t) = \lim_{T \rightarrow \infty} p_T(t) = 0$. This implies that the l_1 -part of a limit-price will be zero. So, in that case the limit-price system is purely-

finitely additive and each consumer's income equals zero. This will happen, for example, in the situation where $p_T = p_T(0) \cdot [1, 1, \dots, 1]$.

4. Existence of an infinite horizon equilibrium

In this section we prove that there exists a general equilibrium for the infinite horizon economy. The method of proof is to show that the limit of the finite horizon equilibria exists and satisfies the definition of a general equilibrium.

In view of Alaoglu's Theorem and the results of Theorem 3.2, we know that there are

$$\begin{aligned} x^h &= [0, \dots, 0, x^h(0), x^h(1), \dots] \in X, \quad h = 1, \dots, H, \\ y &= [y(-m), \dots, y(-1), y(0), y(1), \dots] \in Y, \\ p &= [p(-m), \dots, p(-1), p(0), p(1), \dots] \in l_\infty, \quad p(0) = 1, \end{aligned}$$

and a subsequence $T_k \rightarrow \infty$ such that

$$\left. \begin{aligned} x_{T_k}^h &\rightarrow x^h: h = 1 \dots H \\ y_{T_k} &\rightarrow y \\ p_{T_k} &\rightarrow p \\ \text{as } k &\rightarrow \infty \end{aligned} \right\} .$$

The convergence is pointwise. Without proof we used the fact that Y is closed with respect to pointwise-convergence.

Although we would be obliged, formally, to use the subsequence T_k , we suppress the index k and denote the convergence by $T \rightarrow \infty$.

It is our aim to show that (x^h, p, y) is an equilibrium for $\tilde{E}(\infty)$. First of all we observe that

$$\|x_h\|_\infty \leq B; \quad \|y\|_\infty \leq B; \quad \|p\|_\infty \leq \beta p(0); \quad \sum_h x^h(t) \leq \sum_h \omega^h(t) + y(t) \quad \text{all } t.$$

So

$$\begin{aligned} \sum_h x^h(t) &\leq y(t), \quad \text{for } t \geq 1, \quad \sum_h x^h(0) \leq \sum_h \omega^h(0) + y(0), \\ \sum_h \omega^h(t) + y(t) &\geq 0 \quad \text{for } -m \leq t \leq -1. \end{aligned}$$

Given $\bar{y} \in Y$, we define

$$p \cdot \bar{y} := \sum_{t=-m}^{\infty} p(t)\bar{y}(t) \leq \infty \quad [\text{note that } \bar{y}(t) \geq 0, t \geq 1].$$

Lemma 4.1. $p \cdot \bar{y} \leq p(0)V$ for all $\bar{y} \in Y$.

Proof. Let $p \cdot \bar{y} > V$ for some $\bar{y} \in Y$. Then $\sum_{t=-m}^{T^*} p(t)\bar{y}(t) > V$ for some $T^* > 0$. Hence $\sum_{t=-m}^{T^*} p_T(t)\bar{y}(t) > V$ for $T > T^*$ and T large enough and so $\sum_{t=-m}^T p_T(t)\bar{y}(t) > V$ for T large enough. But $\bar{y} \in Y$ implies that the truncation of \bar{y} is feasible in $\tilde{E}(T)$ and we have a contradiction. \square

Lemma 4.2.

- (i) $\lim_{T \rightarrow \infty} p_T \cdot x_T^h = p \cdot x^h, \quad h = 1, \dots, H.$
- (ii) $\lim_{T \rightarrow \infty} p_T \cdot y_T = p \cdot y.$
- (iii) $p \cdot x^h = p \cdot (\omega^h + \delta^h y); \quad h = 1, \dots, H.$
- (iv) $\sum_{h=1}^H x^h \leq y + \sum_{h=1}^H \omega^h.$

Proof. (i) $0 \leq x^h(t) \leq y(t); t \geq 1$ for so $p \cdot x^h < \infty$ by the previous lemma. Lemma 3.2(iv) says

$$p_T(t)x_T^h(t) \leq \frac{u_h(B)}{\gamma(1 + \rho_h)^t}; \quad \text{all } h, T.$$

Let $\varepsilon > 0$ be given. Fix T^* such that

$$\sum_{t=T^*+1}^{\infty} p(t)x^h(t) < \frac{\varepsilon}{3}$$

and

$$\sum_{t=T^*+1}^{\infty} \frac{u_h(B)}{\gamma(1 + \rho_h)^t} < \frac{\varepsilon}{3}.$$

For $T > T^*$ and T large enough, we have

$$\left| \sum_{t=0}^{T^*} (p(t)x^h(t) - p_T(t)x_T^h(t)) \right| < \frac{\varepsilon}{3}.$$

Now we use the identity

$$\begin{aligned} p \cdot x^h - p_T \cdot x_T^h &= \sum_{t=0}^{T^*} (p(t)x^h(t) - p_T(t)x_T^h(t)) + \sum_{t=T^*+1}^{\infty} p(t)x^h(t) \\ &\quad - \sum_{t=T^*+1}^T p_T(t)x_T^h(t) \end{aligned}$$

in order to obtain

$$|p \cdot x^h - p_T \cdot x_T^h| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sum_{t=T^*+1}^T p_T(t)x_T^h(t) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

for T large enough.

(ii) The proof goes along the same lines as in (i) making use of the fact that $y(t) = \sum_h x^h(t)$ for $t \geq 1$; $p \cdot y \leq V$ and

$$p_T(t)y_T(t) \leq \frac{u_h(B)}{\gamma(1 + \rho_h)^t}.$$

(iii) and (iv) are now trivial by (i), (ii) and the properties of the finite horizon equilibrium. \square

In the sequel P and R will denote strict preference and weak preference, respectively.

Lemma 4.3. For all h , if xR^hx^h and $\bar{y} \in Y$, then

$$p \cdot x \geq p \cdot (\omega^h + \delta^h \bar{y}).$$

Proof. The proof is given in three steps.

Step 1. For all $\varepsilon > 0$ and all $\bar{x} \in l_{\infty}^+$ there are $x' \in l_{\infty}^+$ and $T^* \in \mathbb{N}$ such that:

$$p(0)x'(0) + \dots + p(T^*)x'(T^*) < \varepsilon + p(0)\bar{x}(0) + \dots + p(T^*)\bar{x}(T^*),$$

$$x'(t) = 0 \quad \text{for } t > T^*,$$

$$\sum_{t=0}^{T^*} \frac{u_h(x'(t))}{(1 + \rho_h)^t} > \sum_{t=0}^{\infty} \frac{u_h(\bar{x}(t))}{(1 + \rho_h)^t} \quad (\text{so } x' P^h \bar{x}).$$

Proof of Step 1. Fix $0 < \eta < \varepsilon / \sum_{t=0}^{\infty} 2^{-t} p(t)$.

Define $x''(t) := \bar{x}(t) + \eta \cdot 2^{-t}$; $t \geq 0$. Then $x'' P^h \bar{x}$. So there exists T^* such that

$$\sum_{t=0}^T \frac{u_h(x''(t))}{(1 + \rho_h)^t} > \sum_{t=0}^{\infty} \frac{u_h(\bar{x}(t))}{(1 + \rho_h)^t} \quad \text{all } T \geq T^*.$$

Define $x'(t) := x''(t)$; $0 \leq t \leq T^*$

$$x'(t) := 0; \quad t > T^*.$$

This x' has the desired properties. \square

Step 2. For all $\varepsilon > 0$ and all $\bar{y} \in Y$ we have $p \cdot \bar{y} \leq \varepsilon + p_T \cdot \bar{y}$ for T large enough.

Proof of Step 2. Given $\varepsilon > 0$ and $\bar{y} \in Y$, fix T^* such that

$$\sum_{t=T^*+1}^{\infty} p(t) \bar{y}(t) < \frac{1}{2} \varepsilon.$$

Take $T > T^*$ such that

$$\sum_{t=-m}^{T^*} (p(t) - p_T(t)) \bar{y}(t) < \frac{1}{2} \varepsilon.$$

Then

$$\begin{aligned} p \cdot \bar{y} - p_T \cdot \bar{y} &= \sum_{t=-m}^{T^*} (p(t) - p_T(t)) \bar{y}(t) - \sum_{t=T^*+1}^T p_T(t) \bar{y}(t) \\ &\quad + \sum_{t=T^*+1}^{\infty} p(t) \bar{y}(t) < \frac{1}{2} \varepsilon + 0 + \frac{1}{2} \varepsilon = \varepsilon. \quad \square \end{aligned}$$

Step 3. Given $\varepsilon > 0$ and x and $\bar{y} \in Y$ satisfying the assumptions of the lemma we choose T^* and $x' P^h x$ such that

$$p(0)x'(0) + \dots + p(T^*)x'(T^*) < \varepsilon + p(0)x(0) + \dots + p(T^*)x(T^*),$$

$$x'(t) = 0 \quad \text{for } t > T^*.$$

So, for $T > T^*$ and T large enough, we have $x' P^h x_T^h$, because $x R^h x^h$. Hence $p_T \cdot x' > p_T \cdot x_T^h = p_T \cdot (\omega^h + \delta^h y_T) \geq p_T \cdot (\omega^h + \delta^h \bar{y})$. The (strict) first inequality occurs from the fact that consumer h could choose x' in the economy with horizon T . On the other hand,

$$\begin{aligned}
 p_T \cdot x' &= \sum_{t=0}^{T^*} p_T(t) x'(t) < \sum_{t=0}^T p(t) x'(t) + \varepsilon < 2\varepsilon + \sum_{t=0}^{T^*} p(t) x(t) \\
 &\leq 2\varepsilon + \sum_{t=0}^{\infty} p(t) x(t).
 \end{aligned}$$

Moreover $p_T \cdot \bar{y} \geq p \cdot \bar{y} - \varepsilon$. Hence $p \cdot \omega^h + \delta^h p \cdot \bar{y} < 3\varepsilon + p \cdot x$ for all $\varepsilon > 0$ or $p \cdot x \geq p \cdot \omega^h + \delta^h p \cdot \bar{y}$. \square

Lemma 4.4. We have $p \cdot \bar{y} \leq p \cdot y$ for all $\bar{y} \in Y$.

Proof. $p \cdot x^h \geq p \cdot \omega^h + \delta^h p \cdot \bar{y}$ for all h (see the previous lemma and take $x = x^h$). Then also

$$\sum_h p \cdot x^h \geq \sum_h p \cdot \omega^h + p \cdot \bar{y}.$$

But $p \cdot x^h = p \cdot \omega^h + p \cdot \delta^h y$ for all h (see Lemma 4.2). \square

Lemma 4.5.

$$\frac{u'_h(x^h(t))}{(1 + \rho_h)^t} = u'_h(x^h(0)) \cdot p(t) \quad \text{for } t \geq 0 \text{ and all } h.$$

Proof. It follows from Lemma 3.2(i) that there exists $c > 0$ such that $x^h_T(0) \geq c$ for all h , and all T . From

$$\frac{u'_h(x^h_T(t+1))}{u'_h(x^h_T(t))} \leq 1 + \rho_h$$

it follows by induction that $x^h(t) > 0$ all t . Moreover, since $\|x_T^h\|_\infty \leq B$ we have $\alpha(t) := \lim_{T \rightarrow \infty} \alpha_T(t)$ exists where $\alpha_T(t)$ is defined by

$$p_T(t) = \frac{p_T(t-1)}{1 + \alpha_T(t)} \text{ in Theorem 3.1.}$$

So

$$u'_h(x^h(t+1)) = \frac{1 + \rho_h}{1 + \alpha(t+1)} u'_h(x^h(t)), \quad t \geq 0,$$

$$p(t+1) = \frac{p(t)}{1 + \alpha(t+1)}, \quad t \geq 0,$$

$$p(0) = 1. \quad \square$$

Lemma 4.6. $p \cdot \bar{x} \leq p \cdot (\omega^h + \delta^h y) \Rightarrow x^h R^h \bar{x}$.

Proof. $u_h(\bar{x}(t)) - u_h(x^h(t)) \leq u'_h(x^h(t+1))(\bar{x}(t) - x^h(t))$.

$$\frac{u_h(\bar{x}(t))}{(1 + \rho_h)^t} \leq \frac{u_h(x^h(t))}{(1 + \rho_h)^t} + u'_h(x^h(0))(p(t)\bar{x}(t) - p(t)x^h(t)),$$

$$\begin{aligned} \sum_{t=0}^{\infty} \frac{u_h(\bar{x}(t))}{(1 + \rho_h)^t} &\leq \sum_{t=0}^{\infty} \frac{u_h(x^h(t))}{(1 + \rho_h)^t} + u'_h(x^h(0)) \left(\sum_{t=0}^{\infty} p(t)\bar{x}(t) - p(t)x^h(t) \right) \\ &\leq \sum_{t=0}^{\infty} \frac{u_h(x^h(t))}{(1 + \rho_h)^t} \end{aligned}$$

since $p \cdot \bar{x} \leq p \cdot (\omega^h + \delta^h y) \leq p \cdot x^h$. \square

Theorem 4.1. (x^h, p, y) is a general competitive equilibrium for the infinite horizon economy.

Proof. This is a combination of the previous lemmata. \square

5. Conclusions

The central issue of the present paper has been the existence of general competitive equilibria in a model with exhaustible resources. It is argued that for obtaining existence one cannot rely on the results derived in the theory of infinite dimensional commodity spaces. Our method of proof is to show the

uniform boundedness of equilibrium allocations in finite horizon economies and to prove subsequently that the limit of the truncated equilibria constitutes a general equilibrium for the infinite horizon economy.

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