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A Quantitative Theory of Unsecured Consumer Credit With Risk of Default

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Abstract

We study, theoretically and quantitatively, the general equilibrium of an economy in which households smooth consumption by means of both a riskless asset and unsecured loans with the option to default. The default option resembles a bankruptcy filing under Chapter 7 of the U.S. Bankruptcy Code. Competitive financial intermediaries offer a menu of loan sizes and interest rates wherein each loan makes zero profits. We prove the existence of a steady-state equilibrium and characterize the circumstances under which a household defaults on its loans. We show that our model accounts for the main statistics regarding bankruptcy and unsecured credit while matching key macroeconomic aggregates and the earnings and wealth distributions. We use this model to address the implications of a recent policy change that introduces a form of "means-testing" for households contemplating a Chapter 7 bankruptcy filing. We find that this policy change yields large welfare gains.

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1 Introduction

In this paper we analyze a general equilibrium model with unsecured consumer credit that incorporates the main characteristics of U.S. consumer bankruptcy law and replicates the key empirical characteristics of unsecured consumer borrowing in the U.S. Specifically, we construct a model consistent with the following facts:

- Borrowers can default on their loans by filing for bankruptcy under the rules laid down in Chapter 7 of the U.S. Bankruptcy Code. In most cases, filing for bankruptcy results in seizure of all (non-exempt) assets and a full discharge of household debt. Importantly, filing for bankruptcy protects a household's current and future earnings from any collection actions by those to whom the debts were owed.
- Post-bankruptcy, a household has difficulty getting new (unsecured) loans for a period of about 10 years.[1]
- Households that default are typically in poor financial shape.[2]
- There is free entry into the consumer loan industry and the industry behaves competitively.[3]
- There is a large amount of unsecured consumer credit.[4]
- A large number of people who take out unsecured loans default each year.[5]

A key contribution of our paper is to establish a connection between the recent facts on household debt and bankruptcy filing rates with the theory of household behavior that macroeconomists routinely use. This connection is established by modifying the equilibrium models of Imrohoroğlu (1989), Huggett (1993), and Aiyagari (1994) to include default and by organizing the facts on consumer debt and bankruptcy in light of the model.

Turning first to the theory, we analyze an environment where households with infinitely long planning horizons choose how much to consume and how much to save or borrow. Households face uninsured idiosyncratic shocks to income, preferences, and asset position and therefore have a motive to accumulate assets and to sometimes borrow in order to

[1]This is documented in Musto (1999).

[2]This is documented in, for example, Flynn (1999).

[3]See Ch.10 of Evans and Schmalensee (2000) for a compelling defense of the view that the unsecured consumer credit industry in the U.S. is competitive.

[4]The Board of Governors of the Federal Reserve System constructs a measure of revolving consumer debt that excludes debt secured by real estate, as well as automobile loans, loans for mobile homes, trailers, or vacations. This measure is probably a subset of unsecured consumer debt which amounted to $692 billion in 2001, or almost 7 percent of the $10.2 trillion that constitutes U.S. GDP.

[5]In 2001, 1.45 million people filed for bankruptcy in the U.S., of which just over 1 million were under Chapter 7 (as reported by the American Bankruptcy Institute) and the rest filed under Chapter 13.

smooth consumption. We permit households to default on their loans. This default option resembles a Chapter 7 bankruptcy filing in which debts are discharged. We abstract from the out-of-pocket expenses of declaring bankruptcy but assume that a bankruptcy remains on a household's credit record for some (random) length of time that, on average, is compatible with the length of time mandated by law. We assume that while the bankruptcy filing remains on a household's record, it cannot borrow and incurs a (small) reduction in its earning capability.

It should be clear from this basic setup that an indebted household will weigh the benefit of maintaining access to the unsecured credit market against the benefit of declaring default and having its debt discharged. Accordingly, credit suppliers who make unsecured loans will have to price their loans taking into account the likelihood of default. We assume a market arrangement where credit suppliers can link the price of their loans to the observable total debt position of a household and to a household's type. The first theoretical contribution of the paper is to prove the existence of a general equilibrium in which the price charged on a loan of a given size made to a household with given characteristics exactly compensates lenders for the objective default frequency on loans of that size made to households with those characteristics. This proof is challenging because the default option may result in a discontinuity of the steady-state wealth distribution with respect to the rental rate on capital and wages.

A second theoretical contribution of the paper is a characterization of default behavior in terms of earnings for a given set of household characteristics. First we prove that for each level of debt, the set of earnings that triggers default is an interval. Specifically, an earnings-rich household (one with earnings above the upper threshold of the interval) is better off repaying its debt and saving while an earnings-poor household (one with earnings below the lower threshold of the interval) is better off repaying its debt and borrowing. This result is important because it reduces the task of computing default probabilities to that of computing default thresholds. Second, we prove that the default interval expands with increasing indebtedness.

A third theoretical contribution is to show that our equilibrium loan price schedules determine, endogenously, the borrowing limit facing each type of household. This is theoretically significant since borrowing constraints often play a key role in empirical work regarding consumer spending. Thus, we believe it is important to provide a theory of borrowing constraints that derives from the institutional and legal features of the U.S. unsecured consumer credit market.

Turning to our quantitative work, we first organize facts on consumer earnings, wealth, and indebtedness from the 2001 Survey of Consumer Finances (SCF) in light of the reasons cited for bankruptcy by Panel Study of Income Dynamics survey participants between 1984 and 1995. Our model successfully generates statistics that closely resemble these facts. To accomplish this, we model shocks that correspond to the reasons people give for filing for bankruptcy and which replicate the importance (for the filing frequency and debt) of each reason given. One of the three shocks is a standard earnings shock that captures the job-loss and credit-misuse reasons. A second shock is a preference shock that captures the effects of marital disruptions. The third shock is a liability shock that captures motives related to unpaid health-care bills and lawsuits. This last shock is important because it captures events that create liabilities without a person having actually borrowed from a financial intermediary – a fact that turns out to be important for simultaneously generating large amounts of debt and default. To incorporate this liability shock the model had to be expanded to incorporate a hospital sector.

We use our calibrated model to study the effects of a recent change in bankruptcy law that discourages above-median-income households from filing under Chapter 7. We find that the policy change has a substantial impact. There is a roughly two-fold increase in the level of debt extended without a significant increase in the total amount defaulted. We also find a significant welfare gain from this policy: households are willing (on average) to pay around 1.6 percent of annual consumption to implement this policy.

Our paper is related to several recent strands of literature on unsecured debt. One strand, starting with Kehoe and Levine (1993), and more recently Kocherlakota (1996), Kehoe and Levine (2001), and Alvarez and Jermann (2003), studies environments in which agents can write complete state-contingent contracts with the additional requirement that contracts satisfy a participation constraint – a constraint that comes from the option to permanently leave the economy (autarky) rather than meet one's contractual obligations. Since the participation constraint is always satisfied, there is no equilibrium default.[6] To model equilibrium default on contractual obligations that resembles a Chapter 7 filing, we depart from this literature in an important way. In our framework a loan contract between the lending institution and a household specifies the household's next-period obligation independent of any future shock but gives the household the option to default. The interest rate on the contract can, however, depend on such things as the household's current total debt, credit rating, and demographic characteristics that provide partial information on a household's earnings prospects (such as its zip code). This assumption is motivated by the typical credit card arrangement.[7] Because of the limited dependence of the loan contract on future shocks, our framework is closer to the literature on default with incomplete markets as in Dubey, Geanokoplos, and Shubik (2005), Zame (1993), and Zhang (1997). Zame's work is particularly relevant because he shows that with incomplete markets, it may be efficient to allow a bankruptcy option to debtors.

In innovative work, Athreya (2002) analyzes a model that includes a default option with stochastic punishment spells. But in his model financial intermediaries charge the

[6]Kehoe and Levine (2006) suggest that one can interpret the incentive constrained allocation as arising in an economy where individuals simultaneously lend and borrow from each other, where some agents default, and the bankruptcy penalty includes a mixture of a Chapter 7-style seizure of assets and a Chapter 13-style of garnishment of earnings.

[7]For more detail on the form of the standard credit card " "contract" see Section III of Gross and Souleles (2002).

same interest rate on loans of different sizes even though a large loan induces a higher probability of default than a small loan. As a result, small borrowers end up subsidizing large borrowers, a form of cross-subsidization that is not sustainable with free entry of intermediaries.[8] Enforcing zero profits on loans of varying sizes complicates our equilibrium analysis because there is now a *schedule* of loan prices to solve for rather than a single interest rate on loans.[9]

The paper is organized as follows. In Section 2 we describe our model economy and characterize the behavior of agents. We the prove existence of equilibrium in Section 3 and characterize properties of the equilibrium loan schedules. We describe and discuss our calibration targets in Section 4. We discuss the properties of the calibrated economies in Section 5. In Section 6, we pose and answer our policy question. All proofs are given in the Appendix.

2 The Model Economy

We begin by specifying the legal and physical environment of our model economy. Then we describe a market arrangement for the economy. This is followed by a statement of the decision problems of households, firms, financial intermediaries, and the hospital sector.

2.1 Legal Environment

We model the default option to resemble, in procedure and consequences, a Chapter 7 bankruptcy filing. Consider a household that starts the period with some unsecured debt. If the household files for bankruptcy (and we permit a household to do so irrespective of its current income or past consumption level) then the following things happen:

1. The household's beginning of period liabilities are set to zero (i.e., its debts are discharged) and the household is not permitted to save in the current period. The latter assumption is a simple way to recognize that U.S. bankruptcy law does *not* permit those invoking bankruptcy to simultaneously accumulate assets: a bankrupt must relinquish all (non-exempt) assets to creditors at the time that discharge of debt is granted by a bankruptcy court.[10]

[8] Lehnert and Maki (2000) have a model with competitive financial intermediaries who can precommit to long-term credit contracts in which a similar type of cross-subsidization is also permitted.

[9] Livshits, MacGee, and Tertilt (2003) follow our approach where the zero profit condition is applied to loans of varying size. However, they assume that creditors can garnish wages of a bankrupt person in the period in which that person files for bankruptcy and that a person has unrestricted access to unsecured credit in the period immediately following default.

[10] Some assets are exempt in a Chapter 7 bankruptcy filing and can be retained by the filer. However, the nature and value of exempt assets vary across US states, being quite low in some states and quite high in others. For simplicity, we ignore this variation and assume that no exemptions are permitted. We also do not address Chapter 13 bankruptcy filings in this paper. In a Chapter 13 filing, debtors can retain their assets in return for a promise to repay some portion of the total obligation from their future earnings.

2. The household begins next period with a record of bankruptcy. Let $h_t \in \{0,1\}$ denote the "bankruptcy flag" for a household in period t, where $h_t = 1$ indicates in period t a record of a bankruptcy filing in the past and $h_t = 0$ denotes the absence of any such record. In what follows, we will refer to h as simply the household's credit record, with the record being either clean ($h = 0$) or tarnished ($h = 1$). Thus, a household that declares bankruptcy in period t, starts period $t+1$ with $h_{t+1} = 1$.

3. A household that begins a period with a record of bankruptcy cannot get new loans.[11] Also, a household with a record of bankruptcy experiences a loss equal to a fraction $0 < \gamma < 1$ of earnings, a loss intended to capture the pecuniary costs of a bad credit record.[12]

4. There is an exogenous probability $1 - \lambda$ that a household with a record of bankruptcy will have its record expunged in the following period. That is, a household that starts period t with $h_t = 1$ will start period $t+1$ with $h_{t+1} = 0$ with probability $1 - \lambda$. This is a simple, albeit idealized, way of modeling the fact that a bankruptcy flag remains on an individual's credit history for only a finite number of years.

2.2 Preferences and Technologies

At any given time there is a unit mass of households. Each household is endowed with one unit of time. Households differ in their labor efficiency $e_t \in E = [e_{\min}, e_{\max}] \subset I\!R_{++}$ and in certain characteristics $s_t \in S$, where S is a finite set. There is a constant probability $(1-\rho)$ that any household will die at the end of each period. Households that do not survive are replaced by newborns who have a good credit rating ($h_t = 0$), zero assets ($\ell_t = 0$), and with labor efficiency and characteristics drawn independently from the probability measure space $(S \times E, \mathcal{B}(S \times E), \psi)$ where $\mathcal{B}(\cdot)$ denotes the Borel sigma algebra and ψ denotes the joint probability measure. Surviving households independently draw their labor efficiency and characteristics at time t from a stochastic process defined on the measurable space $(S \times E, \mathcal{B}(S \times E))$ with transition function $\Phi(e_t|s_t)\Gamma(s_{t-1}, s_t)$ where $\Phi(e_t|s_t)$ is a conditional density function and $\Gamma(s_{t-1}, s_t)$ is a Markov matrix. We assume that for all s_t, the probability measure defined by $\Phi(e_t|s_t)$ is atomless.

There is one composite good produced according to an aggregate production function $F(K_t, N_t)$ where K_t is the aggregate capital stock that depreciates at rate δ and N_t is

[11] We interpret the assumption that firms do not lend to households with a record of a bankruptcy filing in their credit history as a legal restraint on firm behavior. In the context of our model, financial intermediaries value this restriction because it prevents individual lenders from diluting the punishment from default. In Chatterjee, Corbae, and Rios-Rull (2004) we present a dynamic adverse selection model in which default provides an imperfect signal about a household's type and as a result defaulters may find it prohibitively costly to borrow.

[12] There are pecuniary costs of a bad credit rating – such as higher auto insurance premia. For an explicit analysis of pecuniary costs stemming from a loss of reputation in credit market, see Chatterjee, Corbae, and Rios-Rull (2007).

aggregate labor in efficiency units in period t. We make the following assumptions about technology:

Assumption 1. For all $K_t, N_t \geq 0$, F satisfies: (i) constant returns to scale; (ii) diminishing marginal returns with respect to the two factors; (iii) $\partial^2 F/\partial K_t \partial N_t > 0$; (iv) Inada conditions with respect to K_t, namely, $\lim_{K_t \to 0} \partial F/\partial K_t = \infty$ and $\lim_{K_t \to \infty} \partial F/\partial K_t = 0$; and (v) $\partial F/\partial N_t \geq b > 0$.

The composite good can be transformed one-for-one into consumption, investment, and medical services. As described in detail later, unforeseen medical expenditure is an oft-cited reason for Chapter 7 bankruptcy filing.

Taking into account the possibility of death, the preferences of a household are given by the expected value of a discounted sum of momentary utility functions:

$$E_0 \left\{ \sum_{t=0}^{\infty} (\beta\rho)^t \, U\left(c_t, \eta(s_t)\right) \right\}, \tag{1}$$

where $0 < \beta < 1$ is the discount factor, c_t is consumption and $\eta(s_t)$ is a preference shock in period t. We make the following assumptions on preferences.

Assumption 2. For any given s, $U\left(\cdot, \eta(s)\right)$ is strictly increasing, concave, and differentiable. Furthermore, there exist $\underline{s}$ and $\overline{s}$ in S such that for all c and s, $U(c, \eta(\underline{s})) \leq U(c, \eta(s)) \leq U(c, \eta(\overline{s}))$.

Consumption of medical services does not appear in the utility function because we treat this consumption as nondiscretionary.[13] Furthermore, we assume that consumption of medical services does not affect the productive efficiency of the household. When they occur, the household is presented with a hospital bill $\zeta(s_t)$. We assume that each surviving household has a strictly positive probability of experiencing a medical expense. Specifically, there exists $\widehat{s} \in S$ for which $\zeta(\widehat{s}) > 0$ and $\Gamma(s_{t-1}, \widehat{s}) > 0$ for all s_{t-1}.

2.3 Market Arrangements

We assume competitive factor markets. The real wage per efficiency unit is given by $w_t \in W = [w_{\min}, w_{\max}]$ with $w_{\min} > 0$. The rental rate on capital is given by r_t.

The addition of a default option necessitates a departure from the conventional modeling of borrowing and lending opportunities. In particular, we posit a market arrangement where

[13] Alternatively, we could assume that unless the medical expenditure is incurred the household receives $-\infty$ utility.

unsecured loans of different sizes for different types of households are treated as distinct financial assets. This expansion of the "asset space" is required to correctly handle the competitive pricing of default risk, a risk that will vary with the size of the loan and household characteristics. In our model a household with characteristics s_t can borrow or save by purchasing a single one-period pure discount bond with a face value in a finite set $L \subset I\!R$. The set L contains 0 and positive and negative elements. We will denote the largest and smallest elements of L by $\ell_{\max} > 0$ and $\ell_{\min} < 0$, respectively. We will assume that $F_K(\ell_{\max}, e_{\min}) - \delta > 0$.

A purchase of a discount bond in period t with a nonnegative face value ℓ_{t+1} means that the household has entered into a contract where it will receive $\ell_{t+1} \geq 0$ units of the consumption good in period $t+1$. The purchase of a discount bond with a negative face value ℓ_{t+1} and characteristics s_t means that the household receives $q_{\ell_{t+1},s_t} \cdot (-\ell_{t+1})$ units of the period-t consumption good and promises to deliver, conditional on not declaring bankruptcy, $-\ell_{t+1} > 0$ units of the consumption good in period $t+1$; if it declares bankruptcy, the household delivers nothing. The total number of financial assets available to be traded is $N_L \cdot N_S$, where N_X denotes the cardinality of the set X. Let the entire set of $N_L \cdot N_S$ prices in period t be denoted by the vector $q_t \in I\!R_+^{N_L \cdot N_S}$. We restrict q_t to lie in a compact set $Q \equiv [0, q_{\max}]^{N_L \cdot N_S}$ where $1 \geq q_{\max} \geq 0$. In the section on steady state equilibrium the upper bound on q will follow from assumptions on fundamentals.

Households purchase these bonds from financial intermediaries. We assume that both losses and gains resulting from death are absorbed by financial intermediaries. That is, a household that purchases a negative face value bond honors its obligation only if it survives and does not declare bankruptcy, and, symmetrically, an intermediary that sells a positive face value discount bond is released from its obligation if the household to which the contract was sold is not around to collect. We assume that there is a market where intermediaries can borrow or lend at the risk-free rate i_t. Also, without loss of generality, we assume that physical capital is owned by intermediaries who rent it to composite goods producers. There is free entry into financial intermediation and intermediaries can exit costlessly by selling all their capital.

The hospital sector takes in the composite good as an intermediate input and transforms it one-for-one into medical services. In our model, as in the real world, some households may default and not pay their medical bills $\zeta(s_t)$. We assume that if some proportion of aggregate medical bills is not paid back due to default, then hospitals supply medical services in the amount $\zeta(s_t)/m_t$ to households with characteristic s_t where the markup $m_t > 1$ is set to ensure zero profits.

2.4 Decision Problems

The timing of events in any period are: (i) idiosyncratic shocks s_t and e_t are drawn for survivors and newborns; (ii) capital and labor are rented and production of the composite good takes place; (iii) household default and borrowing/saving decisions are made, and consumption of goods and services takes place. In what follows, we will focus on steady-state equilibria where $w_t = w$, $r_t = r$, $i_t = i$, and $q_t = q$.

2.4.1 Households

We now turn to a recursive formulation of a household's decision problem. We denote any period t variable x_t by x and its period $t+1$ value by x'.

In addition to prices, the household's current period budget correspondence $B_{\ell,h,s,d}(e;q,w)$ depends on its exogenous state variables s and e, its beginning of period asset position ℓ, and its credit record h. It will also depend on the household's default decision $d \in \{0,1\}$, where $d=1$ indicates that the household is exercising its default option and $d=0$ indicates that it's not. Then $B_{\ell,h,s,d}(e;q,w)$ has the following form:

1. If a household with characteristics s has a good credit record ($h=0$) and does not exercise its default option ($d=0$) then

$$B_{\ell,0,s,0}(e;q,w) = \{c \in I\!R_+, \ell' \in L : c + q_{\ell',s}\, \ell' \leq e \cdot w + \ell - \zeta(s)\}. \tag{2}$$

We take into account the possibility that the budget correspondence may be empty in this case. In particular, if the household is deeply in debt, earnings are low, new loans are expensive, and/or medical bills are high, then the household may not be able to afford nonnegative consumption. As discussed below, allowing the budget correspondence to be empty permits us to analyze both voluntary and "involuntary" default.

2. If a household with characteristics s has a good credit record ($h=0$) and net liabilities ($\ell - \zeta < 0$) and exercises its default option ($d=1$), then

$$B_{\ell,0,s,1}(e;q,w) = \{c \in I\!R_+, \ell' = 0 : c \leq e \cdot w\}. \tag{3}$$

In this case, net liabilities disappear from the budget constraint and no saving is possible in the default period. That is we assume that during a bankruptcy proceeding a household cannot hide or divert funds owed to creditors.

3. If a household with characteristics s has a bad credit record ($h=1$) and net liabilities are nonnegative ($\ell - \zeta \geq 0$) then

$$B_{\ell,1,s,0}(e;q,w) = \{c \in I\!R_+, \ell' \in L^+ : c + q_{\ell',s}\, \ell' \leq (1-\gamma)e \cdot w + \ell - \zeta(s)\}, \tag{4}$$

where $L^+ = L \cap I\!R_+$. With a bad credit record, the household is not permitted to borrow and is subject to pecuniary costs of a bad credit record.

4. If a household with characteristics s has a bad credit record ($h=1$) and ($\ell - \zeta < 0$) then

$$B_{\ell,1,s,1}(e;q,w) = \{c \in I\!R_+, \ell' = 0 : c \leq (1-\gamma)\; e \cdot w\}. \tag{5}$$

A household with a bad credit record and a net medical liability pays only up to its assets, cannot accumulate new assets, and begins the next period with a bad credit record. When the budget set is empty, these assumptions correspond to giving the household another Chapter 7 discharge. For this reason we denote this case by setting $d = 1$.[14] For simplicity, we continue to make these assumptions even when the budget set is not empty.[15]

To set up the household's decision problem, define $\mathcal{L}$ to be all possible (ℓ, h, s)-tuples, given that only households with a good credit record can have debt and let $N_{\mathcal{L}}$ be the cardinality of $\mathcal{L}$. Then, $\mathcal{L} \equiv \{L^{--} \times \{0\} \times S\} \cup \{L^{+} \times \{0,1\} \times S\}$, where $L^{--} = L \backslash L^{+}$. Let $v_{\ell,h,s}(e; q, w)$ denote the expected lifetime utility of a household that starts with (ℓ, h, s) and e and faces the prices q and w and let $v(e; q, w)$ be the vector $\{v_{\ell,h,s}(e; q, w) : \{\ell, h, s\} \in \mathcal{L}\}$ in the set $\mathcal{V}$ of all continuous (vector-valued) functions $v : E \times Q \times W \to I\!R^{N_{\mathcal{L}}}$.

The household's optimization problem can be described in terms of a vector-valued operator $(\mathcal{T}v)(e; q, w) = \{(Tv)(\ell, h, s, e; q, w) : (\ell, h, s) \in \mathcal{L}\}$ which yields the maximum lifetime utility achievable if the household's future lifetime utility is assessed according to a given function $v(e; q, w)$.

Definition 1. For $v \in \mathcal{V}$, let $(Tv)(\ell, h, s, e; q, w)$ be defined as follows:

1. For $h = 0$ and $B_{\ell,0,s,0}(e; q, w) = \varnothing$:

$$(Tv)(\ell, 0, s, e; q, w) \ = U(e \cdot w, \eta(s)) \ + \ \beta\rho \int v_{0,1,s'}(e'; q, w) \ \Phi(e'|s') \ \Gamma(s, ds') \ de'.$$

2. For $h = 0$, $B_{\ell,0,s,0}(e; q, w) \neq \varnothing$, and $\ell - \zeta(s) < 0$:

$$(Tv)(\ell, 0, s, e; q, w) = \max \left\{ \begin{array}{c} \max_{c,\ell' \in B_{\ell,0,s,0}} U(c, \eta(s)) + \beta\rho \int v_{\ell',0,s'}(e'; q, w)\Phi(e'|s')\Gamma(s, ds')de', \\ U(e \cdot w, \eta(s)) \ + \ \beta\rho \int v_{0,1,s'}(e'; q, w)\Phi(e'|s')\Gamma(s, ds')de' \end{array} \right\}.$$

3. For $h = 0$, $B_{\ell,0,s,0}(e; q, w) \neq \varnothing$, and $\ell - \zeta(s) \geq 0$:

$$(Tv)(\ell, 0, s, e; q, w) = \max_{c,\ell' \in B_{\ell,0,s,0}} U(c, \eta(s)) + \beta\rho \int v_{\ell',0,s'}(e'; q, w)\Phi(e'|s')\Gamma(s, ds')de'.$$

4. For $h = 1$ and $\ell - \zeta(s) < 0$:

$$(Tv)(\ell, 1, s, e; q, w) = U(e \cdot w(1 - \gamma), \eta(s)) \ + \ \beta\rho \int v_{0,1,s'}(e'; q, w) \ \Phi(e'|s')\Gamma(s, ds')de'.$$

[14] U.S. law does not allow a household to file for Chapter 7 again within 6 years of having filed a Chapter 7 bankruptcy. Instead, such a household can only file for a Chapter 13. Since we do not consider Chapter 13 in this paper, we simply assume that these households receive another Chapter 7 discharge of net liabilities.

[15] In the computational work that follows this latter case almost never arises. That is, when $\ell - \zeta(s) < 0$ the budget set is invariably empty because when medical liabilities occur they are large relative to earnings of households with $h = 1$.

5. For $h = 1$ and $\ell - \zeta(s) \geq 0$:

$$(Tv)(\ell, 1, s, e; q, w) = \max_{c, \ell' \in B_{\ell,1,s,0}} U(c, \eta(s)) + \beta\rho \left[\begin{array}{c} \lambda \int v_{\ell',1,s'}(e'; q, w)\Phi(e'|s')\Gamma(s, ds')de' \\ +(1-\lambda) \int v_{\ell',0,s'}(e'; q, w)\Phi(e'|s')\Gamma(s, ds')de' \end{array} \right].$$

The first part of this definition says that if the household has debt and the budget set conditional on not defaulting is empty, the household must default. In this case, the expected lifetime utility of the household is simply the sum of the utility from consuming its current earnings and the discounted expected utility of starting next period with no assets and a bad credit record. The second part says that if the household has net liabilities and the budget set conditional on not defaulting is not empty, the household chooses whichever default option yields higher lifetime utility. In the case where both options yield the same utility, the household may choose either. The difference between default under part 1 and default under part 2 is the distinction between "involuntary" and "voluntary" default. In the first case, default is the *only* option, while in the second case it's the *best* option. The third part applies when a household with a good credit record has no net liabilities. In this case, the household does not have the default option and simply chooses how much to borrow/save.[16] The final two parts apply when the household has a bad credit record and hence no debt. It distinguishes between the case where it has some net liability (which arises from a large enough liability shock) and the case where it does not. In the first case, the household is permitted to partially default on its liabilities as described earlier. In the second case the household simply chooses how much to save.

Theorem 1 (Existence of a Recursive Solution to the Household Problem). There exists a unique $v^* \in \mathcal{V}$ such that $v^* = \mathcal{T}(v^*)$. Furthermore: (i) v^* is bounded and increasing in ℓ and e; (ii) a bad credit record reduces v^*; (iii) the optimal policy correspondence implied by $\mathcal{T}(v^*)$ is compact-valued and upper hemi-continuous; and (iv) provided $u(0, s)$ is sufficiently low, default is strictly preferable to zero consumption and optimal consumption is always strictly positive.

Because certain actions involve discrete choice, $\mathcal{T}(v^*)$ generally delivers an optimal policy *correspondence* instead of a function. Given property (iii) of Theorem 1, the Measurable Selection Theorem (Theorem 7.6 of Stokey, Lucas, and Prescott (1989)) guarantees the existence of measurable policy functions for consumption $c^*_{\ell,h,s}(e; q, w)$, asset holdings $\ell'^*_{\ell,h,s}(e; q, w)$, and the default decision $d^*_{\ell,h,s}(e; q, w)$.

The default decision rule along with the probabilistic erasure of a bankruptcy flag on the household's credit record implies a mapping $H^*_{(q,w)} : (\mathcal{L} \times E) \times \{0, 1\} \to [0, 1]$ which gives the probability that the household's credit record next period is h'. The mapping H^* is given

[16] We don't permit households to default on liabilities ζ when $\ell - \zeta \geq 0$. This is without loss of generality since all assets of a household can be seized during a bankruptcy filing (no exempt assets).

by:

$$H^*_{(q,w)}(\ell,h,s,e,h'=1) = \begin{cases} 1 & \text{if } d^*_{\ell,h,s}(e;q,w)=1 \\ \lambda & \text{if } d^*_{\ell,h,s}(e;q,w)=0 \text{ and } h=1 \\ 0 & \text{if } d^*_{\ell,h,s}(e;q,w)=0 \text{ and } h=0, \end{cases}$$

$$H^*_{(q,w)}(\ell,h,s,e,h'=0) = \begin{cases} 0 & \text{if } d^*_{\ell,h,s}(e;q,w)=1 \\ 1-\lambda & \text{if } d^*_{\ell,h,s}(e;q,w)=0 \text{ and } h=1 \\ 1 & \text{if } d^*_{\ell,h,s}(e;q,w)=0 \text{ and } h=0. \end{cases}$$

Then we can define a transition function $GS^*_{(q,w)} : (\mathcal{L} \times E) \times (2^{\mathcal{L}} \times \mathcal{B}(E)) \to [0,1]$ for a surviving household's state variables given by

$$\begin{aligned} &GS^*_{(q,w)}((\ell,h,s,e),Z) \\ &= \int_{Z_h \times Z_s \times Z_e} \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;q,w)\in Z_\ell\}}\, H^*_{(q,w)}(\ell,h,s,e,dh')\ \Phi(e'|s')de'\ \Gamma(s,ds') \end{aligned} \tag{6}$$

where $Z \in 2^{\mathcal{L}} \times \mathcal{B}(E)$ and Z_j denotes the projection of Z on $j \in \{\ell,h,s,e\}$ and where $\mathbf{1}_{.}$ is the indicator function. Since a household in state (ℓ,h,s,e) could die and be replaced with a newborn, we can define a transition function $GN : (\mathcal{L} \times E) \times (2^{\mathcal{L}} \times \mathcal{B}(E)) \to [0,1]$ to a newborn's initial conditions given by

$$GN((\ell,h,s,e),Z) = \int_{Z_s \times Z_e} \mathbf{1}_{\{(\ell',h')=(0,0)\}}\psi(ds',de'). \tag{7}$$

Combining these, we can define the transition function $G^*_{(q,w)} : (\mathcal{L}\times E)\times(2^{\mathcal{L}}\times\mathcal{B}(E)) \to [0,1]$ for the economy as a whole by

$$G^*_{(q,w)}((\ell,h,s,e),Z) = \rho GS^*_{(q,w)}((\ell,h,s,e),Z) + (1-\rho)GN((\ell,h,s,e),Z). \tag{8}$$

Finally, given the transition function G^*, we can describe the evolution of the distribution of households μ across their state variables (ℓ,h,s,e) for any given prices (q,w) by use of an operator Υ. Specifically, let $\mathcal{M}(\mathcal{L}\times E, 2^{\mathcal{L}}\times\mathcal{B}(E))$ denote the space of probability measures. For any probability measure $\mu \in \mathcal{M}(\mathcal{L}\times E, 2^{\mathcal{L}}\times\mathcal{B}(E))$ and any $Z \in 2^{\mathcal{L}}\times\mathcal{B}(E)$, define $(\Upsilon_{(q,w)}\mu)(Z)$ by

$$(\Upsilon_{(q,w)}\mu)(Z) = \int G^*_{(q,w)}(\ell,h,s,e,Z)d\mu. \tag{9}$$

Theorem 2 (Existence of a Unique Invariant Distribution). For any $(q,w) \in Q \times W$ and any measurable selection from the optimal policy correspondence, there exists a unique $\mu_{(q,w)} \in \mathcal{M}(\mathcal{L}\times E, 2^{\mathcal{L}}\times\mathcal{B}(E))$ such that $\mu_{(q,w)} = \Upsilon_{(q,w)}\mu_{(q,w)}$.

2.4.2 Characterization of the Default Decision

Since the option to default is the novel feature of this paper, it's useful to establish some results on the manner in which the decision to default varies with a household's level of earnings and with its level of debt. We will characterize the default decision in terms of the the *maximal default set* $\overline{D}^*_{\ell,h,s}(q,w)$. This set is defined as follows: for $h=0$ and $\ell-\zeta(s)<0$ it consists of the set of e's for which either the budget set $B_{\ell,0,s,0}(e;q,w)$ is empty or the value from not defaulting does not exceed the value from defaulting; for $h=1$ and $\ell-\zeta(s)<0$ it consists of the entire set E. The maximal default set will coincide with the set of e for which $d^*_{\ell,h,s}(e;q,w)=1$ if households that are indifferent between defaulting and not defaulting choose always to default.

Theorem 3 (The Maximal Default Set Is a Closed Interval). If $\overline{D}^*_{\ell,h,s}(q,w)$ is non-empty, it is a closed interval.

The intuition for this result can be seen in the following way. Suppose that there are two efficiency levels, say e_1 and e_2 with $e_1<e_2$, for which it is optimal for the household to default on its debt. Now consider an efficiency level $\widehat{e}$ that's intermediate between e_1 and e_2. Suppose that the household prefers to maintain access to the credit market at $\widehat{e}$ even though it defaults at a higher earnings level e_2. It seems intuitive then that the reason for not defaulting at the lower earnings level associated with $\widehat{e}$ must be that the household finds it optimal to consume more than its earnings and incur even more debt. On the other hand, the fact that the household defaults at the efficiency level e_1 but maintains access to the credit market at the higher efficiency level $\widehat{e}$ suggests that the reason for not defaulting at the earnings level associated $\widehat{e}$ must be that the household finds it optimal to consume less than its earnings and reduce its level of indebtedness. Since the household cannot simultaneously be consuming more and less than the earnings level associated with $\widehat{e}$, it follows that the household must default at the efficiency level $\widehat{e}$ as well.

Theorem 4 (Maximal Default Set Expands with Indebtedness). If $\ell^0>\ell^1$, then $\overline{D}^*_{\ell^0,h,s}(q,w)\subseteq\overline{D}^*_{\ell^1,h,s}(q,w)$.

The result follows from the property that $v^*_{\ell,0,s}(e;q,w)$ is increasing in ℓ and the utility from default is independent of the level of net liabilities. Figure 1 helps to visualize this.

2.4.3 Firms

Firms producing the composite good face a static optimization problem of choosing nonnegative quantities of labor and capital to maximize $F(K_t,N_t)-w\cdot N_t-r\cdot K_t$. The necessary conditions for profit maximization imply (with equality if the optimal N_t and K_t are strictly

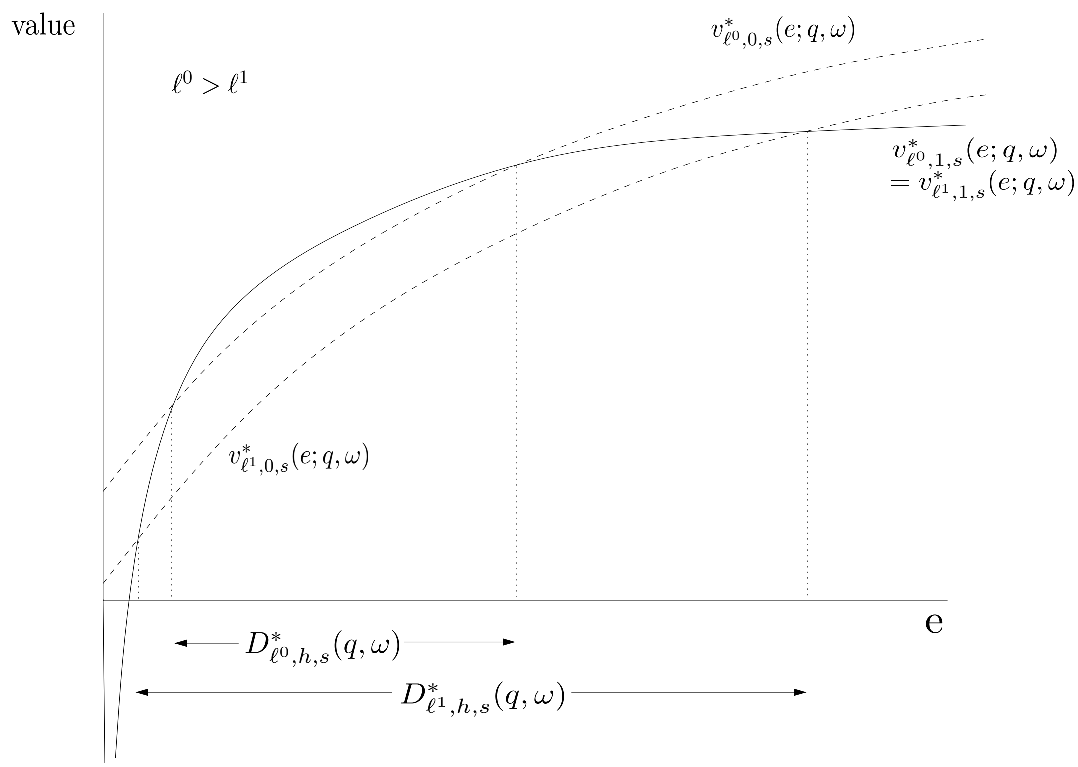

Figure 1: Typical Default Sets Conditional on Household Type

positive) that

$$w \geq \frac{\partial F(K_t, N_t)}{\partial N_t} \text{ and } r \geq \frac{\partial F(K_t, N_t)}{\partial K_t}. \tag{10}$$

2.4.4 Financial Intermediaries

The intermediary chooses the number $a_{\ell_{t+1},s_t} \geq 0$ of type (ℓ_{t+1}, s_t) contracts to sell and the quantity $K_{t+1} \geq 0$ of capital to own for each t to maximize the present discounted value of current and future cash flows

$$\sum_{t=0}^{\infty} (1+i)^{-t} \pi_t, \tag{11}$$

given K_0 and $a_{\ell_0,s_{-1}} = 0$. The period t cash flow is given by

$$\pi_t = \begin{bmatrix} (1-\delta+r)K_t - K_{t+1} \\ + \sum_{(\ell_t,s_{t-1})\in L\times S} \rho(1-p_{\ell_t,s_{t-1}}) a_{\ell_t,s_{t-1}} (-\ell_t) \\ - \sum_{(\ell_{t+1},s_t)\in L\times S} q_{\ell_{t+1},s_t} a_{\ell_{t+1},s_t} (-\ell_{t+1}) \end{bmatrix} \tag{12}$$

where p_{ℓ_{t+1},s_t} is the probability that a contract of type (ℓ_{t+1}, s_t), where $\ell_{t+1} < 0$, experiences default and it is understood that $p_{\ell_{t+1},s_t} = 0$ for $\ell_{t+1} \geq 0$.[17] Note that the calculation of cash flow takes into account that some borrowers will not survive to repay their loans and some depositors will not survive to collect on their deposits.[18]

If a solution to the intermediary's problem exists, then optimization implies

$$i \geq r - \delta, \tag{13}$$

$$q_{\ell_{t+1},s_t} \leq \frac{\rho}{1+i} \quad \text{if } \ell_{t+1} \geq 0, \tag{14}$$

and

$$q_{\ell_{t+1},s_t} \geq \frac{\rho}{1+i}(1 - p_{\ell_{t+1},s_t}) \quad \text{if } \ell_{t+1} < 0. \tag{15}$$

If the optimal K_{t+1} is strictly positive, then (13) holds with equality. Similarly, if any optimal a_{ℓ_{t+1},s_t} is strictly positive, the associated condition (14) or (15) holds with equality. Furthermore, any nonnegative sequence $\left\{K_{t+1}, a_{\ell_{t+1},s_t}\right\}_{t=0}^{\infty}$ implies a sequence of risk-free bond holdings $\{B_{t+1}\}_{t=0}^{\infty}$ by the intermediary given by the recursion

$$B_{t+1} = (1+i)B_t + \pi_t, \tag{16}$$

where $B_0 = 0$.

2.4.5 Hospital Sector

Hospital revenue received from a household in state $\ell, h, s,$and e, is given by

$$(1 - d^*_{\ell,h,s}(e; q, w))\zeta(s) + d^*_{\ell,h,s}(e; q, w)\max\{\ell, 0\}.$$

Observe that if a household has positive assets but negative net (after medical shock) liabilities and defaults, the hospital receives ℓ. If the household's assets are negative and it

[17] Note that households with $\ell_{t+1} \geq 0$ may still default on their medical liabilities if those liabilities are sufficiently high.

[18] Here, and in the household's decision problem, we assumed that a household enters into a single contract with some firm. This simplifies the situation in that a household's end-of-period asset holding is the same as ℓ', the size of the single contract entered into by the household. However, this is without loss of generality in the following sense. Let households write any collection of contracts $\{\ell'^k \in L\}$ as long as $\ell' = \sum_k \ell'^k \in L$. Consistent with the procedures of a Chapter 7 bankruptcy filing, assume that a household has the option to either (i) default on all negative face value subcontracts (i.e., loans) or (ii) not default on any of them. In the case of default, assume that creditor-firms can liquidate any positive face value subcontracts held by the household and use the proceeds to recover their loans in proportion to the size of each loan. With these bankruptcy rules in place, the price charged on any subcontract in the collection $\{\ell'^k \in L\}$ must be the price that applies to the single contract of size ℓ'. Consequently, as long as credit suppliers can condition their loan price on total end-of-period debt position of a household, there is a market arrangement in which the household is indifferent between writing a single contract or a collection of subcontracts with the same total value. Parlour and Rajan (2001) analyze equilibrium in a two-period model of unsecured consumer debt when such conditioning is not possible.

defaults, the hospital receives nothing. As noted before, for a bill of ζ, the hospital's resource cost is given by ζ/m. Thus, hospital profits in period t are given by

$$\int \left[(1 - d^*_{\ell,h,s}(e;q,w))\zeta(s) + d^*_{\ell,h,s}(e;q,w)\max\{\ell,0\} - \zeta(s)/m\right] d\mu_t \tag{17}$$

where μ_t is the distribution of households over $\mathcal{L} \times E$ at time t. In steady state, m must be consistent with zero profits for the hospital sector.

3 Steady-State Equilibrium

In this section we define and establish the existence of a steady-state equilibrium and characterize some properties of the equilibrium loan price schedule. The proof of existence uses Brouwer's FPT for a continuous function on a compact domain. Nevertheless, the proof is not straightforward. The nature of the difficulty – which is related to the possibility of default – is discussed later in this section.

Definition 2. A steady-state competitive equilibrium is a set of strictly positive prices w^*, r^*, i^*, a nonnegative loan-price vector q^*, a nonnegative default frequency vector $p^* = (p^*_{\ell',s})_{\ell' \in L, s \in S}$, a nonnegative hospital mark-up m^*, strictly positive quantities of aggregate labor and capital N^*, K^*, a nonnegative vector of quantities of contracts $a^* = (a^*_{\ell',s})_{\ell' \in L, s \in S}$, bond holdings by the intermediary B^*, decision rules $\ell'^*_{\ell,h,s}(e;q^*,w^*), d^*_{\ell,h,s}(e;q^*,w^*), c^*_{\ell,h,s}(e;q^*,w^*)$ and a probability measure μ^* such that:

(i) $\ell'^*_{\ell,h,s}(e;q^*,w^*)$, $d^*_{\ell,h,s}(e;q^*,w^*)$ and $c^*_{\ell,h,s}(e;q^*,w^*)$ solve the household's optimization problem;

(ii) N^*, K^* solve the firm's static optimization problem;

(iii) K^*, a^* solve the intermediary's optimization problem;

(iv) $p^*_{\ell',s} = \int d^*_{\ell',0,s'}(e';q^*,w^*)\Phi(e'|s')\Gamma(s;ds')de'$ for $\ell' < 0$ and $p^*_{\ell',s} = 0$ for $\ell' \geq 0$ (intermediary consistency);

(v) $\int \left[(1 - d^*_{\ell,h,s}(e;q^*,w^*))\zeta(s) + \ d^*_{\ell,h,s}(e;q^*,w^*)\max\{\ell,0\} - \zeta(s)/m^*\right] d\mu^* = 0$ (zero profits for the hospital sector);

(vi) $\int e \ \ d\mu^* = N^*$ (the labor market clears);

(vii) $\int \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;q^*,w^*)=\ell'\}}\mu^*(d\ell,dh,s,de) = a^*_{\ell',s}$, $\forall(\ell',s) \in L \times S$ (each loan market clears);

(viii) $B^* = 0$ (the bond market clears);

(ix) $\int c^*_{\ell,h,s}(e;q^*,w^*)d\mu^* + \int \frac{\zeta(s)}{m^*}d\mu^* + \delta K^* = F(K^*,N^*) - \gamma w^* \int e\mu^*(d\ell,1,ds,de)$ (the goods market clears);

(x) $\mu^* = \mu_{(q^*,w^*)}$ where $\mu_{(q^*,w^*)} = \Upsilon_{(q^*,w^*)}\mu_{(q^*,w^*)}$ (μ^* is an invariant probability measure).

We will use the above definition to derive a set of price equations whose solution implies the existence of a steady state. Conditions (ii) and (iii) in the definition imply the following equations. Since N^* and K^* are strictly positive, the first order conditions for the firm (10) and the intermediary (13) imply:

$$w^* = \frac{\partial F(K^*, N^*)}{\partial N^*}, \quad r^* = \frac{\partial F(K^*, N^*)}{\partial K^*}, \quad i^* = r^* - \delta. \tag{18}$$

For $a^*_{\ell',s} > 0$, the intermediary first order conditions (14) or (15) imply

$$q^*_{\ell',s} = \frac{\rho(1 - p^*_{\ell',s})}{1 + i^*}. \tag{19}$$

For $a^*_{\ell',s} = 0$ we will look for an equilibrium where the intermediary is indifferent between selling and not selling the associated (ℓ', s) contract. Then (19) holds for these contracts as well.

Condition (viii) implies the following equation. From the recursion (16), bond market clearing (viii) implies cash flow (12) can be written

$$\left[(1-\delta+r^*)K^* - \sum_{(\ell,s_{-1})\in L\times S} \rho(1-p^*_{\ell,s_{-1}})a^*_{\ell,s_{-1}}\ell\right] - \left[K^* - \sum_{(\ell',s)\in L\times S} q^*_{\ell',s}a^*_{\ell',s}\ell'\right] = 0$$

or using (18) and (19)

$$(1-\delta+r^*)\left[K^* - \sum_{(\ell,s_{-1})\in L\times S} q^*_{\ell,s_{-1}}a^*_{\ell,s_{-1}}\ell\right] - \left[K^* - \sum_{(\ell',s)\in L\times S} q^*_{\ell',s}a^*_{\ell',s}\ell'\right] = 0$$

where (ℓ, s_{-1}) denotes a loan size and characteristics pair from the previous period. Therefore, bond market clearing in steady state implies

$$K^* = \sum_{(\ell',s)\in L\times S} q^*_{\ell',s}a^*_{\ell',s}\ell'. \tag{20}$$

It can be shown that the goods market clearing condition (ix) is implied by the other conditions for an equilibrium.[19] Thus, we can summarize an equilibrium by the following set

[19] This is a nontrivial accounting exercise given that our environment admits default on loans and medical bills. For reasons of space we omit a proof here. The proof is available in the supplementary material section for this paper on the Econometrica website.

of four equations. The first two are price equations that incorporate household optimization (i), intermediary consistency (iv), labor market clearing (vi), loan market clearing (vii), and (20) into (18) and (19) to yield:

$$w^* = F_N\left(\sum_{(\ell',s)\in L\times S} \ell' q^*_{\ell',s} \int \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;q^*,w^*)=\ell'\}}\mu^*(d\ell,dh,s,de), \int e\, d\mu^*\right) \tag{21}$$

$$q^*_{\ell',s} = \begin{cases} \dfrac{\rho}{1+F_K\left(\sum_{(\ell',s)\in L\times S}\ell' q^*_{\ell',s}\int \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;q^*,w^*)=\ell'\}}\mu^*(d\ell,dh,s,de),\int e\, d\mu^*\right)-\delta} & \text{for } \ell' \geq 0 \\ \dfrac{\rho\left(1-\int d^*_{\ell',0,s'}(e';q^*,w^*)\Phi(e'|s')\Gamma(s;ds')de'\right)}{1+F_K\left(\sum_{(\ell',s)\in L\times S}\ell' q^*_{\ell',s}\int \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;q^*,w^*)=\ell'\}}\mu^*(d\ell,dh,s,de),\int e d\mu^*\right)-\delta} & \text{for } \ell' < 0 \end{cases} \tag{22}$$

The other two equations are given by (v) and (x).

Proving the existence of a steady-state equilibrium reduces to proving that there is a fixed point to equations (21) and (22) where the invariant distribution μ^* is itself a fixed point of a Markov process whose transition probabilities depend on the price vector. Provided the aggregate production function has continuous first derivatives (and these derivatives satisfy certain boundary conditions) a solution to this nested fixed point problem will exist (as a simple consequence of the Brouwer's FPT) if μ^* is continuous with respect to the price vector. Given a continuum of households, a sufficient condition for the continuity of μ^* is that the set of households that are indifferent between any two courses of action be of (probability) measure zero. The assumption that the efficiency shock e is drawn from a distribution with a continuous cdf goes a long way toward ensuring this but, surprisingly, not *all* the way. Even with this assumption we cannot rule out that a continuum of households may be indifferent between defaulting and paying back.[20] To work around this problem, in the appendix we first establish the existence of a steady-state equilibrium for an environment in which there is an additional bankruptcy cost that is paid in the filing period. The form of this cost ensures that the set of households that are indifferent between defaulting and paying back is finite and thereby restores the continuity of the invariant distribution with respect to the price vector.[21] We then take a sequence of steady-state equilibria in which the filing-period bankruptcy cost converges to zero and establish that the limit of this sequence is a steady-state equilibrium for the environment of this paper.

The equilibrium loan price vector has the property that all positive face-value loans (household deposits) bear the risk-free rate and negative face-value loans (household borrowings) bear a rate that reflects the risk-free rate and a premium for the objective default probability on the loan. Given the risk-free rate, which in equilibrium will depend on μ^*,

[20] This case will occur if a household that is indifferent between defaulting and paying back finds it optimal to consume its endowment when paying back. Then, *ceteris paribus*, households with slightly higher or slightly lower e's will also be indifferent between defaulting and paying back.

[21] This form of additional bankruptcy cost is $\alpha \cdot (e - e_{\min}) \cdot w$ where $\alpha < 1$.

default probabilities (and hence loan prices) do not depend on μ^*. This is because free entry into financial intermediation implies that cross-subsidization across loans of different sizes is not possible; i.e., it's not possible for intermediaries to charge more than the cost of funds on small low-risk loans in order to offset losses on large higher-risk loans. For if there were positive profits in some contracts that were offsetting the losses in others, intermediaries could enter the market for those profitable loans.

Theorem 5 (Existence) A steady-state competitive equilibrium exists.

For a finite r^*, it is possible that there are contracts $(\ell' < 0, s)$ whose equilibrium price $q^*_{\ell',s} = 0$. Even in this case, intermediaries are indifferent as to how many loans of type (ℓ', s) they "sell"; "selling" these loans doesn't cost the intermediary anything (since the price is zero) and it (rationally) expects the loans to generate no payoff in the following period. From the perspective of a household, taking out one of these free loans buys nothing in the current period but saddles the household with a liability. Since the household can do better by choosing $\ell' = 0$ in the current period, there is no demand for such loans either.

We now deal with the limits of the set L, for a given s. Models of precautionary savings have the property that when $\beta\rho(1 + r^* - \delta) < 1$ there is an upper bound on the amount of assets a household will accumulate. This upper bound arises because as wealth gets larger, the coefficient of variation of income goes to zero, and hence the role of consumption smoothing vanishes.[22] Since ours is also a model of precautionary saving, the same argument applies and $\ell_{\max}$ exists. With respect to the debt limit, $\ell_{\min}$, it can be set to any value less than or equal to $[e_{\max} \cdot w_{\max}]\,[(1 + r^* - \delta)/(1 - \rho + r^* - \delta)]$. This expression is the largest debt level that could be paid back by the luckiest household facing the lowest possible interest rate and is the polar opposite of the one in Huggett (1993), Aiyagari (1994), and Athreya (2002). As we show in the next theorem, for any s, a loan of this size or larger would have a price of zero in any equilibrium. Hence, as long as the lower limit is at least as low as $-[e_{\max} \cdot w_{\max}]\,[(1 + r^* - \delta)/(1 - \rho + r^* - \delta)]$, it will not have any effect on the equilibrium price schedule. We now turn to characterizing the equilibrium price schedule.

Theorem 6 (Characterization of Equilibrium Prices) In any steady-state competitive equilibrium: (i) $q^*_{\ell',s} = \rho(1 + r^* - \delta)^{-1}$ for $\ell' \geq 0$; (ii) if the grid for L is sufficiently fine, there exists $\ell^0 < 0$ such that $q^*_{\ell^0,s} = \rho(1 + r^* - \delta)^{-1}$; (iii) if the set of efficiency levels for which a household is indifferent between defaulting and not defaulting is of measure zero, $0 > \ell^1 > \ell^2$ implies $q^*_{\ell^1,s} \geq q^*_{\ell^2,s}$; (iv) when $\ell_{\min} \leq -[e_{\max} \cdot w_{\max}]\,[(1 + r^* - \delta)/(1 - \rho + r^* - \delta)]$, $q^*_{\ell_{\min},s} = 0$.

The first property simply says that firms charge the risk-free rate on deposits. The second property says that if the grid is taken to be fine enough, there is always a level of debt for

[22] See Huggett (1993) and Aiyagari (1994) for a detailed argument.

which it is never optimal for any household to default. As a result, competition leads firms to charge the risk-free rate on these loans as well. The third property says that the price on loans falls with the size of loans, i.e., the implied interest rate on loans rises with the size of the loan. The final property says that the price on loans eventually become zero; i.e., for any household the price on a loan of size $[e_{\max} \cdot w_{\max}] [(1 + r^* - \delta)/(1 - \rho + r^* - \delta)]$ or larger is always zero in every equilibrium. In other words, the equilibrium delivers an endogenous credit limit for each household with characteristics s.

4 Mapping the Model to U.S. Data

We now establish that our framework gives a plausible account of the overall facts on bankruptcy and credit. The challenging part is to account simultaneously for the high frequency of default *and* for significant levels of unsecured debt – high frequency of default makes unsecured debt very expensive, which deters consumer borrowing. We found that two realistic features account for aggregate default and credit statistics. First, not all unsecured consumer debt is a result of borrowing from financial intermediaries – some of it is in the nature of an "involuntary" loan resulting from reasons such as large medical bills. Second, marital disruption is often cited as a reason for filing – not necessarily related to earnings shocks. In our model we take into account the possibility of "involuntary" loans through our modeling of nondiscretionary medical expenses and we take into account private life-events (such as divorce) as a possible trigger for default through the preference shock. There is a third feature of the real world that we believe to be important as well but have not modeled in the interests of keeping the dimensionality of the state space lower: that many U.S. households hold both unsecured debt *and* (non-exempt) assets – a fact that no doubt lowers the default premium on unsecured loans and makes them less expensive. We skirt this issue by focusing on the net asset positions of households but (as explained below) this impairs our ability to explain some aspects of the data.

We map two versions of our model economy to the data differing by which idiosyncratic shocks are included. In both versions, the household characteristic s is simply the triplet (ξ, η, ζ), where ξ denotes a shock that controls the probability distribution of labor efficiency e, η is a multiplicative preference shock, and ζ is the medical expense shock. We think of ξ as socioeconomic status (or occupation) upon which the distribution of household labor efficiencies depend. In the first version, which we label Baseline model economy, we restrict $s = (\xi, 1, 0)$, so that the only idiosyncratic shocks are to socioeconomic status, efficiency, and death. We use the Baseline model for illustration purposes because it is simpler and in the vein of other incomplete market macro models like that of Aiyagari (1994). In the second version, labeled Extended model economy, we include the idiosyncratic shocks to preferences and medical liabilities. We use the Extended model economy for the quantitative analysis. We use the reasons for bankruptcy cited by the Panel Study of Income Dynamics (PSID) survey participants to determine targets for the fraction of consumer debt, the fraction of indebted households, and the fraction of people filing for bankruptcy that should plausibly be accounted for by the two versions. Plausibility in this context means that the model should explain the debt and default statistics without generating counterfactual predictions for macroeconomic aggregates and for earnings and wealth distributions.

4.1 Model Specification

4.1.1 The Baseline Model Economy (Earnings shocks only)

The Baseline model economy has 17 parameters. These parameters are listed below in separate categories with the number of parameters in each category appearing in parentheses.

Demographics (1) The probability of survival is ρ (which implies that the mass of new entrants is $1 - \rho$).

Preferences (2) We assume standard time-separable constant relative risk aversion preferences that are characterized by two parameters, the discount rate, β, and the risk aversion coefficient, σ.

Technology (3) There are two parameters that determine the properties of the production function: the exponent on labor in the Cobb-Douglas production function, θ, and the depreciation rate, δ. We also place in this category the fraction of lost earnings while a household has a bankruptcy on its credit record, γ.

Legal system (1) The legal system is characterized by the average length of the exclusion from access to credit, λ.

Earnings process (10) The process for earnings requires the specification of a Markov chain for $s = \xi$ and of the distribution of e conditional on ξ. We use a three-state Markov chain Γ that we loosely identify with "super-rich" (ξ_1), "white-collar" (ξ_2), and "blue-collar" (ξ_3). The persistence of the latter two states ensures that earnings display a sizable positive autocorrelation. The first state provides the opportunity and incentive for a high concentration of earnings and wealth (see Castañeda, Díaz-Giménez, and Ríos-Rull (2003)). This specification requires 6 parameters in the Markov transition matrix but we reduce it to 4 by setting the probability of moving from blue-collar to super-rich and *vice versa* to zero. For the distribution of labor efficiency shocks we need 6 more parameters, 5 of which pertain to the upper and lower limits of the range of e for each type (units do not matter and that frees up one parameter) and one additional parameter to specify the shape of the cdf of e. We assume the following one-parameter functional form for the distribution function:

$$\int_{e^{\xi}_{\min}}^{y} \Phi(e|\xi) = P[e \leq y|\xi] = \left[\frac{y - e^{\xi}_{\min}}{e^{\xi}_{\max} - e^{\xi}_{\min}} \right]^{\varphi}. \tag{23}$$

4.1.2 The Extended Model Economy (Earnings, Preference, and Liability Shocks)

In this economy, we keep the Baseline model shocks while adding a multiplicative shock to the utility function and a medical liability shock. The preference shock takes two values $\eta \in \{1, \overline{\eta} > 1\}$ and we assume that $\eta = \overline{\eta}$ cannot happen twice in a row. This implies the need to specify two parameters. The liability shock $\zeta \in \{0, \overline{\zeta} > 0\}$ can take on only two values independent of all other shocks and is $i.i.d$ over time. Therefore, the Markov process Γ for $s = (\xi, \eta, \zeta)$ has $3 \times 2 \times 2 = 12$ states.

4.2 Data Targets

We select model parameters to match three sets of statistics: aggregate statistics, earnings and wealth distribution-related statistics, and statistics on debt and bankruptcy. The targets – which appear in Tables 3 and 4 – contain relatively standard targets for macroeconomic variables and statistics of the earnings and wealth distribution obtained from the 2001 SCF (selected points of Lorenz curves, the Gini indices, and the mean to median ratios). We target the autocorrelation of earnings of all agents except those in the highest earnings group to 0.75.[23]

We now turn to the debt and bankruptcy targets and discuss them in more detail since they are novel. First, according to the Fair Credit Reporting Act, a bankruptcy filing stays on a household's credit record for 10 years. This fact is used in our model to calibrate the length of exclusion from the credit market. Second, according to the Administrative Office of the U.S. Courts, the total number of filers for personal bankruptcy under Chapter 7 was 1.087 million in 2002. According to the Census Bureau, the total population above age 20 in 2002 was 201 million. Therefore, the ratio of people who file to total population over age 20 is 0.54%. Third, since in our model households can only save or borrow, we use the 2001 Survey of Consumer Finances to obtain the consolidated asset position of households. Only people with negative net worth are considered to be debtors. We exclude households with negative net worth larger than 120% of average income since the debts are likely to be due to entrepreneurial activity that our model abstracts from. These households are a very small fraction of the SCF (comprising only 0.13% of the original sample of SCF 2001).[24] The average net negative wealth of the remaining households is \$631.46, which divided by per household GDP of \$94,077 is 0.0067. Thus, we take the target debt-to-income ratio to be 0.67%. Fourth, after excluding the few households with debt more than 120% of average income, 6.7% of the households in the 2001 SCF had negative net worth.[25]

The last three statistics are the relevant bankruptcy and debt targets for the Extended

[23] The PSID data set that can be used to compute autocorrelation of earnings does not include the highest earners.

[24] The average amount of debt for this group is \$100,817, or 145% of the average income and their income is relatively high.

[25] We also note that 2.6% of the households had zero wealth in the 2001 SCF.

Table 1: Reasons for Filing for Bankruptcy (PSID, 1984-95)

1	Job loss	12.2%
2	Credit misuse	41.3%
3	Marital disruption	14.3%
4	Health-care bills	16.4%
5	Lawsuit/Harassment	15.9%

Source: Chakravarty and Rhee (1999)

Table 2: Debt and Bankruptcy Targets for Each Model Economy

Economy	U.S.	Baseline	Extended
Reasons covered	1,2,3,4,5	1,2	1,2,3,4,5
Percent of all bankruptcies	100%	53.5%	100%
Percent of households filing	0.54%	0.29%	0.54%
Debt-to-income ratio	0.67%	0.36%	0.67%
Percent of households in debt	6.7%	3.6%	6.7%

Note: The numbers in the "Reasons covered" row are associated to the number in the previous table.

model since it includes all important motives for bankruptcy. The Baseline model does not include all motives; consequently its appropriate target values are a fraction of their data values. According to Chakravarty and Rhee (1999) the PSID classified the reasons for bankruptcy filings into five categories and we report their findings in Table 1. Among the five reasons listed, we associate the first two (job loss and credit misuse) with earnings shocks; marital disruption with preference shocks; and the final two (health-care bills and lawsuits/harassment) with liability shocks.[26] Given these associations, we allocate the total volume of debt, the fraction of households in debt, and the fraction of filings according to the fraction of people citing the above reasons. For instance, given that 53.5% of households cited reasons we associate with earnings shocks, we assume that the fraction of filings corresponding to this reason is 0.29% (i.e., $0.535 \times 0.0054 = 0.0029$). We report these targets specific to each of the model economies in Table 2.

[26] We think of marital disruption as leading to higher nondiscretionary spending on the part of each partner, which in turn increases the marginal utility of discretionary spending. A high value for the multiplicative shock to preferences is meant to capture this effect.

Table 3: Baseline Model Statistics and Parameter Values

Statistic	Target	Model	Parameter	Value
Targets determined independently				
Average years of life	40	40	ρ	0.975
Coefficient of risk aversion	2.0	2.0	σ	2.000
Labor share of income	0.64	0.64	θ	0.640
Depreciation rate of capital	0.10	0.10	δ	0.100
Average years of punishment	10	10	λ	0.100
Targets determined jointly: Inequality				
Earnings Gini index	0.61	0.61	$\Gamma_{2,3}$	0.219
Earnings mean/median	1.57	1.95	$\Gamma_{3,3}$	0.964
Percentage of earnings of 2nd quintile	4.0	4.3	φ	0.470
Percentage of earnings of 3rd quintile	13.0	10.5	$e^1_{\max}/e^3_{\min}$	21263.9
Percentage of earnings of 4th quintile	22.9	20.3	$e^1_{\min}/e^3_{\min}$	14335.7
Percentage of earnings of 5th quintile	60.2	63.5	$e^2_{\max}/e^3_{\min}$	116.3
Percentage of earnings of top 2-5%	15.8	17.3	$e^2_{\min}/e^3_{\min}$	39.2
Percentage of earnings of top 1%	15.3	15.3	$e^3_{\max}/e^3_{\min}$	30.5
Autocorrelation of earnings	0.75	0.74		
Wealth Gini index	0.80	0.73		
Wealth mean/median	4.03	3.30		
Percentage of wealth of 2nd quintile	1.3	3.0		
Percentage of wealth of 3rd quintile	5.0	6.3		
Percentage of wealth of 4th quintile	12.2	15.2		
Percentage of wealth of 5th quintile	81.7	75.1		
Percentage of wealth of top 2-5%	23.1	13.6		
Percentage of wealth of top 1%	34.7	34.2		
Targets determined jointly: Savings, Debt and Default				
Prorated percentage of defaulters	0.29	0.29	β	0.917
Prorated percentage in debt	3.6	4.6	γ	0.019
Capital-output ratio	3.08	3.08	$\Gamma_{1,1}$	0.020
Prorated debt-output ratio	0.0036	0.0036	$\Gamma_{2,1}$	0.001

4.3 Moments Matching Procedure

The Baseline model economy has 17 parameters, which we classify into two groups. The first group consists of 5 parameters, each of which can be pinned down independently of all other parameters by one target. The survival rate ρ is determined to match the average length of adult life, which we take to be 40 years, a compromise for an economy without population growth or changes in marital status. The labor share of income is 0.64, which determines θ. The depreciation rate δ is 0.10, which is consistent with a wealth to output ratio of 3.08 (its value according to the 2001 Survey of Consumer Finances) and the consumption to output ratio of 0.70. The transition probability λ, which governs the average length of time that a bankruptcy remains on a person's credit record is set to 0.1, consistent with the Fair Credit Reporting Act. The coefficient of relative risk aversion σ is fixed at 2.

The 12 parameters in the second group – including the discount rate β, the cost of having a bad credit record γ, 4 parameters governing the transition of type characteristics Γ, 5 parameters defining the bounds of efficiency levels for the type characteristics, and 1 parameter characterizing the shape of the distribution function of the efficiency shocks in (23) φ – are determined jointly by minimizing the weighted sum of squared errors between the targets and the corresponding statistics generated by the model. The targets that we choose are listed in Table 3 and include the main macroeconomic aggregates, the properties of earnings and wealth inequality and the main statistics of debt and default (the percentage of defaulters, the percentage in debt, and the debt to output ratio). Our weighting matrix puts more emphasis on matching the debt and bankruptcy filing targets than on earnings and wealth distribution targets. The Extended model economy has 16 parameters; the same 12 as the Baseline economy plus 2 preference shock parameters (size and probability) and 2 liability shock parameters (again size and probability).

4.4 Computation of the Steady State

Computation of the equilibrium requires three steps: an inner loop, where the decision problem of households given parameter values and prices (including loan prices) is solved; a middle loop, where market-clearing prices are obtained; and an outside loop – or estimation loop – where parameter values that yield equilibrium allocations with the desired (target) properties are determined. We use a variety of (almost) off-the-shelf techniques, powerful software (Fortran 90) and hardware (26 processors Beowulf cluster) to accomplish our task. Space considerations preclude a more detailed discussion of the computational "tricks" employed to improve the speed and accuracy of calculations. We confirmed our findings by recomputing equilibria with standard methods that do not speed the process. The computational task of simulating equilibrium model moments is extremely burdensome: each equilibrium requires computing thousands of equilibrium loan prices – recall that we have to solve for equilibrium loan price *schedules* – and it has taken thousands of computed equilibria to find satisfactory configurations of parameter values.

4.5 Results

Table 3 reports the target statistics and their counterparts in the Baseline model economy as well as the implied parameter values while Table 4 reports the same information for the Extended model economy. Focusing on the the Extended economy, note that it successfully replicates the macro and distributional targets. The capital-output ratio is exactly as targeted and so is the earnings Gini. The wealth Gini is somewhat lower than in the data but as Figures 2 and 3 show, the overall fit of the model along these dimensions is quite good. Turning to the debt and bankruptcy targets, the Extended model economy replicates successfully the percentage of defaulters and the debt-to-income ratio. It also successfully replicates the relative importance of the reasons cited for default. However, the percentage of households in debt is lower than the target. This discrepancy is hard to eliminate because in the model households that borrow are very prone to default implying a high default premium on loans. This increases interest rates on loans and reduces the participation of households in the credit market.[27] One important difference between the model and the U.S. economy is that a typical indebted household has both liabilities *and* assets. The presence of non-exempt assets reduces the incentives to default and lowers interest rates and thereby increases borrowing.[28] The Baseline model economy also closely replicates the relevant targets with the exception of the percentage of households in debt.[29]

5 Properties of the Model Economies

5.1 Distributional Properties

Figure 4 shows the histogram of the wealth distribution in the Extended model, excluding the long right tail which comprises about 18% of the population. For households with a good credit record, the model generates a pattern of the wealth distribution that is typical of overlapping generations models. There is a significant fraction of households with zero wealth, many of whom are newborns. After that, the measure of households with each level of assets grows for a while (most households accumulate some savings) before starting to slowly come down for much larger levels of wealth. There is also a relatively large fraction of households with small amounts of debt relative to average income and there are some households with debt in the neighborhood of one-half of average income. There are no

[27] There are also some other factors that may account for this discrepancy. There are 3.2% of households with exactly zero assets due in part to the discrete nature of periods (all newborns have zero wealth); this makes the number of indebted people in the model lower than it ought to be. Also, upon being hit by either a liability or a preference shock, households default immediately, while in the data it takes longer.

[28] This difference is also the reason we do not target interest rate statistics. In the model all borrowers have negative net worth paying very high interest rates. In the real world many borrowers also own non-exempt assets and the interest rate they pay presumably reflects this fact. To target interest rates in a meaningful way would require a model in which households hold both assets and liabilities.

[29] In this case the percentage is somewhat higher in the model relative to the target. But this discrepancy could reflect an inaccurate target.

Table 4: Extended Model Economy: Statistics and Parameter Values

Statistic	Target	Model	Parameter	Value
Targets determined independently				
Average years of life	40	40	ρ	0.975
Coefficient of risk aversion	2.0	2.0	σ	2.000
Labor share of income	0.64	0.64	θ	0.640
Depreciation rate of capital	0.10	0.10	δ	0.100
Average years of punishment	10	10	λ	0.100
Targets determined jointly: Inequality				
Earnings Gini index	0.61	0.61	$\Gamma_{1,1}$	0.019
Earnings mean/median	1.57	2.12	$\Gamma_{2,1}$	0.001
Percentage of earnings in 2nd quintile	4.0	4.1	$\Gamma_{2,3}$	0.222
Percentage of earnings in 3rd quintile	13.0	9.7	$\Gamma_{3,3}$	0.969
Percentage of earnings in 4th quintile	22.9	20.2	φ	0.387
Percentage of earnings in 5th quintile	60.2	64.0	$e^1_{\max}/e^3_{\min}$	14599.2
Percentage of earnings in top 2-5%	15.8	18.0	$e^1_{\min}/e^3_{\min}$	7661.5
Percentage of earnings in top 1%	15.3	15.3	$e^2_{\max}/e^3_{\min}$	70.9
Autocorrelation of earnings	0.75	0.74	$e^2_{\min}/e^3_{\min}$	23.8
Wealth Gini index	0.80	0.73	$e^3_{\max}/e^3_{\min}$	18.0
Wealth mean/median	4.03	3.22		
Percentage of wealth in 2nd quintile	1.3	3.0		
Percentage of wealth in 3rd quintile	5.0	6.3		
Percentage of wealth in 4th quintile	12.2	15.0		
Percentage of wealth in 5th quintile	81.7	75.4		
Percentage of wealth in top 2-5%	23.1	14.6		
Percentage of wealth in top 1%	34.7	32.3		
Targets determined jointly: Savings, Debt and Default				
Percentage of defaulters	0.54	0.54	β	0.913
Percentage in debt	6.7	5.0	γ	0.035
Percentage of defaults due to divorces	0.077	0.074	Prob of $\bar{\eta}$	0.012
Percentage of defaults due to hospital bills	0.17	0.17	$\bar{\eta}$	16.80
Capital-output ratio	3.08	3.08	Prob of $\bar{\zeta}$	0.003
Debt-output ratio	0.0067	0.0068	$\bar{\zeta}$	1.150

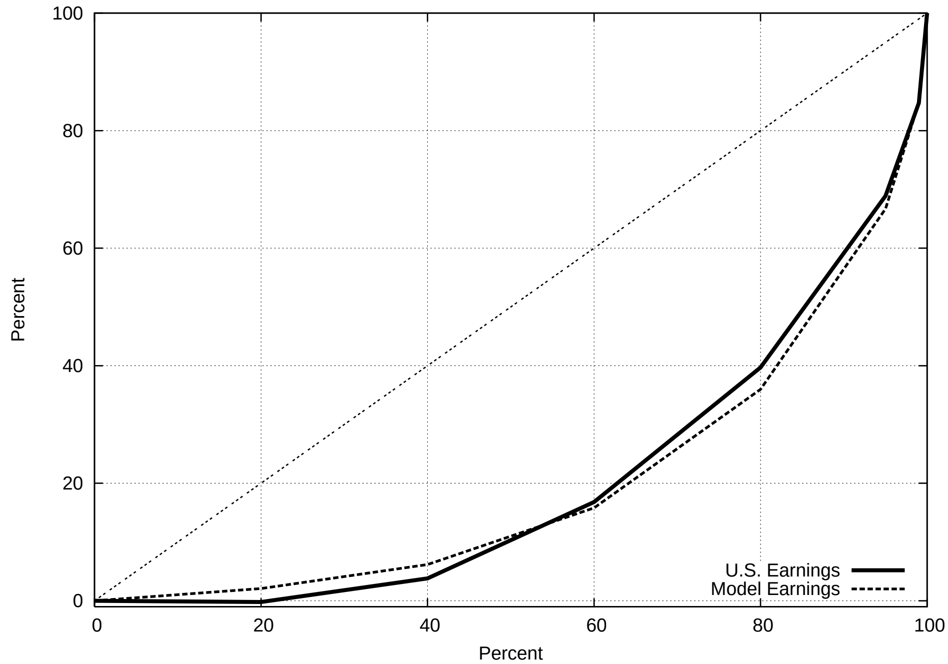

Figure 2: Earnings Distributions for the U.S. and Extended Model

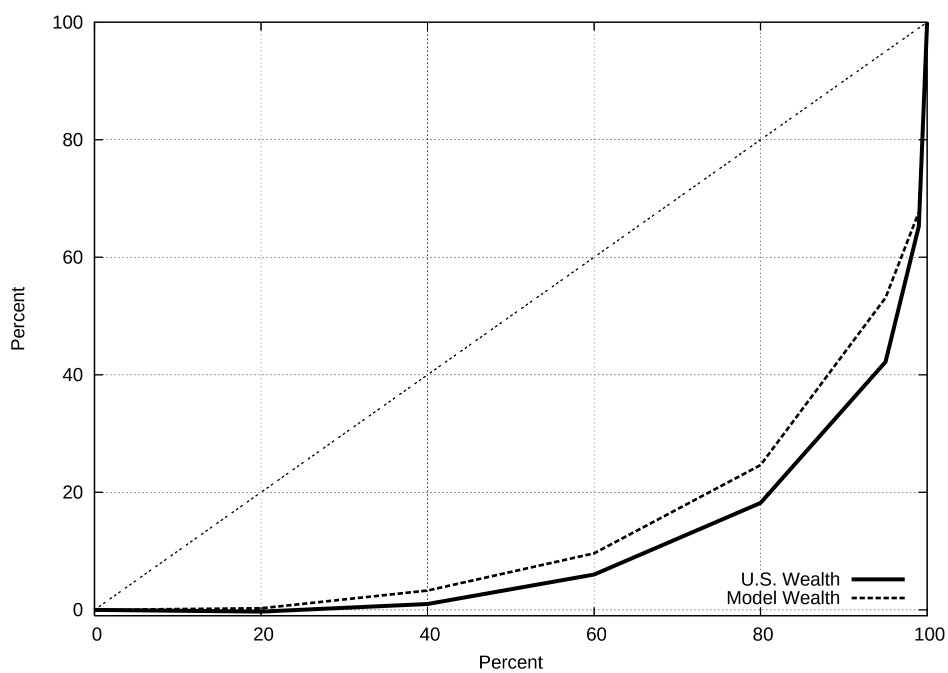

Figure 3: Wealth Distributions for the U.S. and Extended Model

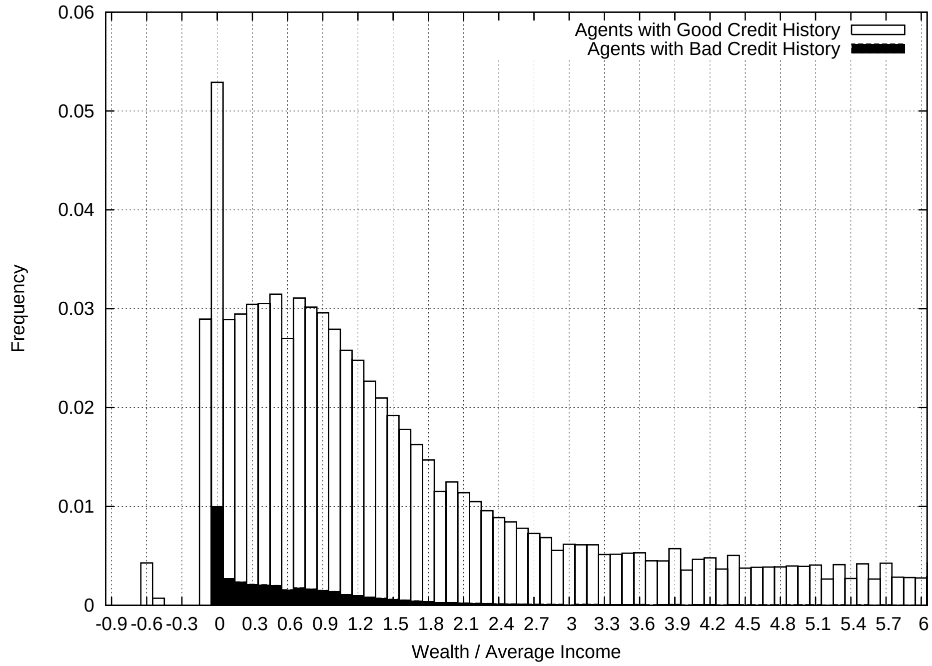

Figure 4: Wealth Histogram in the Extended Model

households borrowing more than 60% because the amount of current consumption derived from borrowing declines beyond this level of debt due to steeply rising default premia.[30] Households with a bad credit record consist mostly of households with very few assets. No one in this group has debt because these households are precluded from borrowing. The right tail of this distribution is relatively long, indicating that some households remain with a bad credit record for many periods and have relatively high earnings realizations.

5.2 Bankruptcy Filing Properties

Figure 5 shows default probabilities in the Extended model on loans taken out in the current period, conditional on whether households are blue collar or white collar and on whether they are hit by the preference or the liability shock in the next period.[31] We wish to make four points. First, the probability of filing for bankruptcy is higher for blue collar than white collar households for every level of debt. This is a natural consequence of white collar households receiving higher earnings on average than blue collar households. For instance, at a debt level of average income no white collar worker is expected to default while *all* blue collar workers are expected to default. Second, the default probabilities for both types of households are rising in the level of debt, which is consistent with Theorem 4. Third, no one is expected to file for bankruptcy with a level of debt near zero, which is consistent with Theorem 6.ii. In particular, even the blue collar households are not expected to default if

[30] This point is discussed in more detail in Section 5.5.

[31] For instance, in Figure 5 the line associated with "Blue Collar Agents" plots the value of $\int d^*_{\ell',0,s'}(e';q^*,w^*)\Phi(e'|s')de'$ where $s' = \{\xi'_3, 1, 0\}$; the line associated with "White Collar Agents: Liability Shock" is the same expression with $s' = \{\xi'_2, 1, \bar{\zeta}\}$.

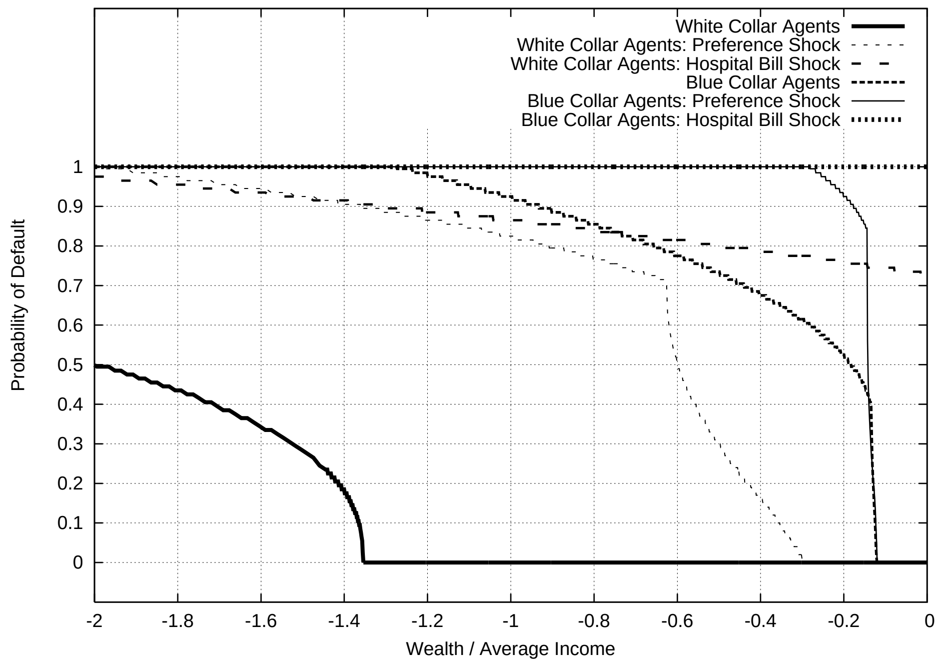

Figure 5: Default Probabilities by Household Types in the Extended Economy

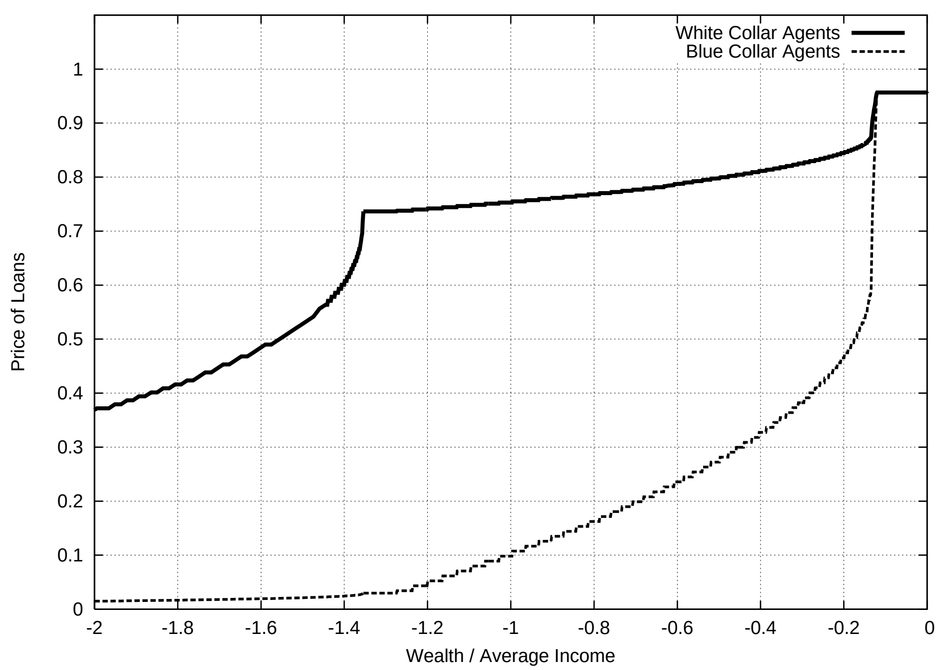

Figure 6: Loan Prices for Blue- and White-Collar Households in the Extended Economy

their debt is less than about one-tenth of average income unless they are hit by the liability shock. Fourth, the threshold debt level below which there is no default for white collar households that are not hit by the liability or preference shocks is around four-thirds of average annual income, and a fraction of white collar households hit by the liability shock do not default.

Table 5: Earnings and Bankruptcies: Fractions of Agents that Default

	Over total population		**Over population in debt**		
Economy	Baseline	Extended	Baseline	Extended w/o Hosp.	Extended with Hosp.
1st quintile	0.48%	0.75%	9.2%	10.7%	13.7%
2nd quintile	0.48%	0.75%	9.2%	10.7%	13.7%
3rd quintile	0.48%	0.75%	9.2%	10.7%	13.6%
4th quintile	0.03%	0.36%	4.2%	7.0%	10.0%
5th quintile	0.00%	0.09%	0.0%	0.4%	3.2 %
Total	0.29%	0.54%	6.4%	7.9%	10.8%

Table 5 shows the number of people filing for bankruptcy by earning quintiles as a fraction of the entire population and as a fraction of those in debt. Across the two economies, the conditional probability of bankruptcy for households in the lowest three earnings quintiles is very similar but declines sharply in the fourth quintile, and there are few defaulters in the top quintile (nobody defaults in the top quintile of the Baseline economy while some do in the Extended economy – recall that the liability shock is large and can hit all agents). The last two columns of the Table show the difference made by the liability shocks by comparing the fraction of those in debt due to past debts alone that default (fourth column) or the fraction of those in debt due to all reasons including this period's liability shocks. In the Extended model economy 0.27% of households get hit by the liability shock of which 0.17% default. The aggregate size of the liability shock (or aggregate medical services) is 0.58% of output, while actual medical expenditure is 0.31% of output (implying via equation (17) that the markup m is 87%). An additional aspect of default behavior that is not evident in these tables is that in every case households below some earnings threshold default. Although the theory allows for a second (lower) threshold below which people pay back, that does not happen in the equilibrium of these calibrated economies.

5.3 Properties of Loan Prices

Since household type is quite persistent, the lower probabilities of bankruptcy of white collar households translate into their having a lower default premium (higher q) than blue collar households. Figure 6 shows the price of loans conditional on the the loan size for white- and blue collar households who have not been hit with either the preference or liability shocks.[32] For a debt level of less than one-tenth of average income, the price schedule appears flat. As shown in Figure 5, for these levels of debt blue collar households default with probability one only if hit by a liability shock. Similarly, white collar households default only if hit by a liability shock, but they do not default with probability one. Instead, the probability of default rises as the loan size increases from zero. Because the probability of liability shocks is very small, the slight decline in loan prices implied by these default patterns is not evident in Figure 6. For higher levels of debt the loan price schedule for a white collar household is above that of blue collar households. This is because type shocks are persistent and, as evident in Figure 5 white collar workers are less likely to default than blue collar workers. For white collar households, the kink at a level of debt of approximately 1.4 of average income results from the fact that for this and lower debt levels, white collar households default only if hit by the preference or liability shock. For higher levels of debt, white collar households default also if earnings realizations are low. The average interest rate on loans (weighted by the number of households in debt) is 30.96%, implying an average default premium of 29.27%.[33]

5.4 Accounting for Debt and Default

These properties of default and loan price schedules indicate different roles of blue collar and white collar households in accounting for aggregate filing frequency and consumer debt. Blue-collar households receive (on average) lower earnings every period and frequently borrow in order to smooth consumption. On the other hand, if they receive a sequence of bad earnings shocks they find it beneficial to file for bankruptcy and erase their debt. Since they are more likely to default, blue collar households have to pay a relatively high default premium and the premium soars as the size of the loan increases. As a result blue collar households borrow relatively frequently in small amounts and constitute the majority of those who go bankrupt. But because they borrow small amounts they account for only a small portion of aggregate consumer debt. In contrast, white collar households face a lower default premium on their loans because they earn more on average. Therefore they borrow a lot more than blue collar households when they suffer a series of bad earnings shocks. The households with large amounts of debt in our Extended model consist of these white collar households. As long

[32] For instance, the line for "White Collar Agents" is $q_{\ell,s}$ where $s = \{\xi_2, 1, 0\}$.

[33] If we weight by the amount of debt for each debtor, the average loan interest rate is 55.97%, substantially higher than the average rate paid per household because there are a small number of households who borrow a large amount at very high interest rates. This is consistent with the histogram of household wealth shown in Figure 4.

Table 6: The Extended Model versus the Bewley and Aiyagari Economies

Economy	Extended	Bewley economy	Aiyagari economy
Availability of loans	Yes	No	Yes
Default premium	Yes	–	No
Output (normalization)	1.00	1.00	1.00
Total asset	3.08	3.12	2.90
Total debt	0.0068	–	0.13
Percentage that file	0.54%	–	–
Percentage with bad credit record	4.23%	–	–
Percentage in debt	4.99%	–	26.84%
Rate of return of capital	1.69%	1.55%	2.41%
Avg loans rate (persons-weighted)	30.96%	–	2.41%

as these households remain white collar they maintain access to credit markets. But they file for bankruptcy if their employment status changes to blue collar because they then face an extremely high default premium on their debt. This story resembles the plight of some members of the American middle class who borrowed a lot because they were considered to be earning a sufficient amount but filed for bankruptcy following a big persistent adverse shock to their earning stream. To summarize, in our model blue collar households account for a large fraction of bankruptcies, and a large fraction of households in debt while white collar households account for the large level of aggregate consumer debt.

5.5 A Comparison with Standard Exogenous Borrowing Limits

We conclude this section by comparing our results with the two extremes typically assumed in general equilibrium economies with heterogeneous agents: either agents are completely prevented from borrowing (the Bewley (1983) economy) or there is full commitment and hence agents can borrow up to the amount they can repay with probability one (the Aiyagari (1994) economy). Table 6 compares the steady states of the Bewley and Aiyagari economies with our Extended model economy. A critical difference between these three models is the form of the borrowing limit. The Bewley borrowing limit is exogenously set at zero. For the Aiyagari economy we assume that only those households who are hit with a liability shock and cannot consume a positive amount without default are permitted to default.[34]

As is apparent from the table, aggregate asset holdings in our economy are closer to

[34]The results are essentially the same under this assumption as in the case of the Baseline model, where no one is permitted to default. In this alternative model, the wealth-to-output ratio is 2.88.

the Bewley model than the Aiyagari model. In terms of aggregate wealth-to-output ratio, the Extended model accounts for (or economizes on) 18%(=(3.12-3.08)/(3.12-2.90)) of the difference in the wealth-to-income ratios between the Bewley and Aiyagari economies. In terms of debt, there is only about 5% of the debt of the Aiyagari economy and in terms of the percentage of households in debt, the Extended model economy has 19% of the number of borrowing households in the Aiyagari economy.

Table 7 shows the endogenous borrowing limits for each of the three earnings types (super-rich, white collar, and blue collar) for the model economies. In the Extended model there

Table 7: Comparison of Borrowing Limits

Earnings type	1 (Super-rich)	2 (White-collar)	3 (Blue-collar)
Bewley economy	0.00	0.00	0.00
Aiyagari economy	3.56	1.70	1.70
Extended model $B_1(s)$	1280	1280	5.35
Extended model $B_2(s)$	12.08	1.00	0.14

[1] Unit is proportion to the average income of the respective economy.
[1] For all economies, borrowing limits for agents that are not hit by either the preference shock or the hospital bills shock are shown.

are two kinds of borrowing limits. One is the smallest loan size for which the corresponding price q is zero, conditional on the type of household. We denote this borrowing limit $B_1(s)$. Formally, $B_1(s) = -\max\ \{\ell \in L^- : q_{\ell,s} = 0\}$. The other borrowing limit, $B_2(s)$, is the level of debt for which $\ell \cdot q_{\ell,s}$ is maximum. Formally, $B_2(s) = -\text{argmax}_{\ell \in L^-}\{\ell \cdot q_{\ell,s}\}$. No one would want to agree to pay back more than this amount because they would be receive less for such a commitment today: this happens because the default premium on the loan rises rapidly enough to actually lower the product $\ell \cdot q_{\ell,s}$. This borrowing limit for blue collar households (0.14 of average income) is less than one-tenth of Aiyagari's limit (1.70 of average income). From the histogram of the wealth distribution presented earlier we know there is a mass of borrowers at this debt level. This mass of households is constrained by the $B_2(s)$ limit. Since blue collar households are the ones most likely to be in need of loans, the Extended economy imposes a stricter borrowing constraint for a subset of the population than the Aiyagari economy.

5.6 Borrowing Constraints and Consumption Inequality

Borrowing constraints have important implications for consumption inequality. Table 8 shows the earnings and consumption inequality of the Extended economy compared to the Bewley and Aiyagari economies. In all economies the degree of consumption inequality is substantially lower than the degree of earnings inequality since the households use savings to smooth consumption fluctuations. However, there are some differences in consumption inequality across the three economies. The standard deviation of log consumption of the Extended economy is about 2% higher than for the Aiyagari economy. On the other hand, the standard deviation of log consumption in the Extended economy is 2% lower than in the Bewley economy. This shows that while the Extended economy may look more like the Bewley economy in terms of its borrowing characteristics it still manages to reduce half of the difference between the Bewley and the Aiyagari economy in terms of consumption inequality.

Table 8: Consumption and Earnings Inequality

	Std Dev	Quintiles				
	of log	1st	2nd	3rd	4th	5th
Extended model						
Earnings	1.159	2.1%	4.1%	9.7%	20.2%	64.0%
Consumption	0.677	6.5%	10.4%	13.5%	20.1%	49.5%
Bewley economy						
Earnings	1.159	2.1%	4.1%	9.7%	20.2%	64.0%
Consumption	0.691	6.4%	10.5%	13.6%	20.1%	49.4%
Aiyagari economy						
Earnings	1.159	2.1%	4.1%	9.7%	20.2%	64.0%
Consumption	0.658	7.2%	9.8%	12.8%	19.9%	50.3%

6 Policy Experiment

Given that our model matches the relevant U.S. statistics on consumer debt and bankruptcy, it is possible to examine the consequences of a change in regulation that affects unsecured consumer credit. Here we evaluate a recent change to the bankruptcy law, which limits "above-median-income" households from filing under Chapter 7. We assume that agents cannot file for bankruptcy if their income in the model economy is above the median (around 34% of average output) and repaying the loan would not force consumption to be negative.[35] In our Extended economy this means that white collar agents cannot default as their income is above the median. Table 9 reports the changes in the model statistics with this policy for two cases. In the first case, the real return on capital is held fixed at the pre-policy-change level (which is what we mean by "No" general equilibrium effect), while in the second case the real return is consistent with a post-policy-change general equilibrium. We focus on the numbers with the full general equilibrium effects but note that such effects are very small.

6.1 Effects on Allocations

The first thing to note is that the aggregate implications of the policy are very small in terms of total savings and the number of filers, but quite substantial in terms of total borrowing (which almost doubles) and the average interest rate charged on loans which is reduced by more than one-half. The first panel of Figure 7 shows the default probabilities in the Extended model for blue and white collar workers. Notice that the units in the horizontal axis is much larger than the units in Figure 5, indicating the strong limits imposed on default by the means-testing policy. The second panel compares the default probabilities for those that were not hit by either the liability or the preference shock in the Extended Economy with and without the means-testing policy. The default probabilities for blue collar workers are reduced substantially and for a certain range it is reduced to zero. The default probabilities for white collar households fall only for very large volumes of debt. As a result, the loan price schedules shift up (the default premium schedules shift down) for both types of households, as shown in Figure 8. Even though the change in *default probability* for the white collar households is not substantial, the *default premia* on loans to both white- and blue collar households drop substantially. This is because for both types of households there is a positive probability of being blue collar in the next period.

Table 9 presents the changes in aggregate statistics resulting from this policy. Most interestingly, the number of bankruptcy filings barely changes even though the default probability schedule conditional on type shifts down for each type of household. This occurs because the percentage of households in debt increases dramatically in response to lower interest rates on loans. Specifically, the percentage of households in debt goes up by more than 60%, from 5.0% to 8.2%. Total debt almost doubles, implying that on average households take on

[35]The law is more complicated than our experiment. A person cannot file under Chapter 7 (and effectively would have to pursue Chapter 13) if all of the following three conditions are met: (1) The filer's income is at least 100 percent of the national median income for families of the same size up to four members; larger families use median income for a family of four plus an extra $583 for each additional member over four. (2) The minimum percentage of unsecured debt that could be repaid over 5 years is 25 percent or $5000, whichever is less. (3) The minimum dollar amount of unsecured debt that could be repaid over 5 years is $5000 or 25%, whichever is less. We summarize these criteria by restricting filing to those with lower than median earnings as long as not doing so results in negative consumption. The alternative would be to keep increasing the liabilities for a few more periods. Our choices are consistent with the law which allows, for instance, a household with high medical liabilities to file even if their income is above median.

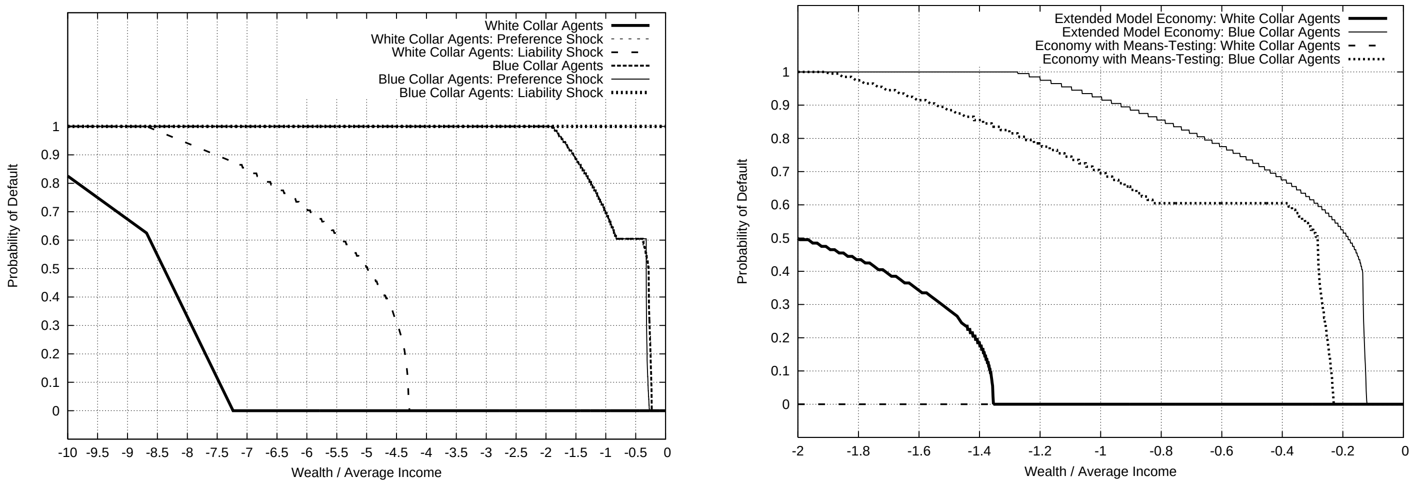

Figure 7: Comparison of Default Probabilities in the Economy With & Without Means Testing

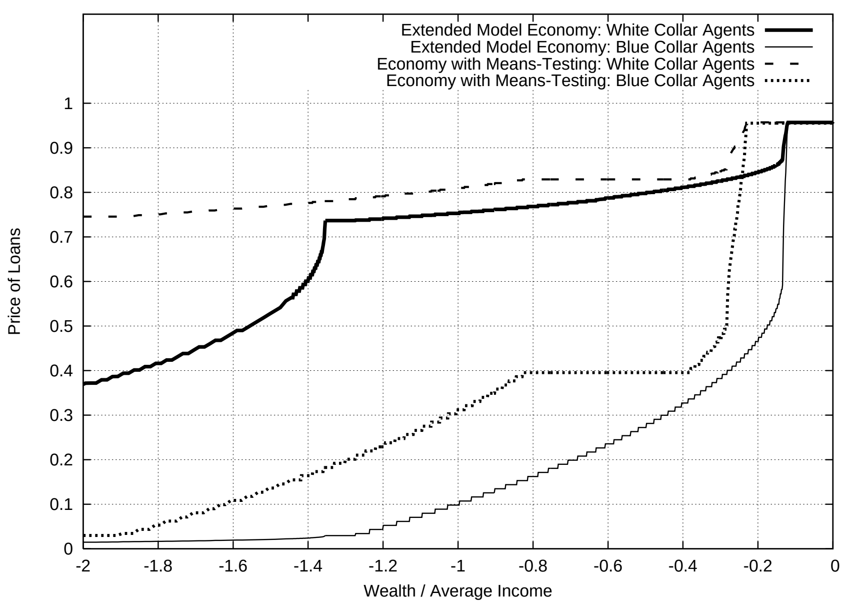

Figure 8: Loan Prices in the Extended Model With & Without Means Testing

Table 9: Allocation Effects of Means-Testing in the Extended Model

Economy	Extended	Bankruptcy restriction	
Max earnings for filing	∞	Median income	
General equilibrium effect	–	No	Yes
Output (normalization)	1.00	1.00	1.00
Total asset	3.08	3.03	3.05
Total debt	0.0068	0.0129	0.0128
Percentage that file	0.54%	0.53%	0.53%
Percentage with bad credit record	4.23%	4.14%	4.13%
Percentage in debt	4.99%	8.24%	8.17%
Rate of return of capital	1.69%	1.69%	1.80%
Avg loans rate (persons-weighted)	30.96%	12.90%	13.04%

bigger loans.

6.2 Effects on Welfare

In assessing the welfare effects of any policy change one must take into account the transition path to the new steady state. This is a daunting computational task when taking into account the general equilibrium effects on the rate of return and on wages (the closed economy assumption). Our findings that the long-run effects of either the small open economy assumption or the closed economy assumption reported in Table 9 indicate that the welfare calculation based on the small open economy assumption is not only interesting in itself but also quite close to what would prevail in a closed economy. Obviously, our welfare analysis compares allocations with and without the implementation of the means-testing policy under the same initial conditions – the wealth and credit record distribution in the steady state of the Extended model economy without means-testing.

An important additional consideration arises in our environment when conducting welfare analysis: There are multiple agent types and, therefore, there will not be agreement among types as to the desirability of a policy change. Consequently, some form of aggregation is necessary. We use two aggregation criteria. The first criterion is the percentage of households that are made better off by the policy change and thus support it. The second criterion is the average gain as measured by the average of the percentage increase in consumption each household would be willing to pay in all future periods and contingencies so that the expected utility from the current period in the initial steady state equals that of the equilibrium associated with the new policy. Because of our assumption on the functional form of the momentary utility function, the consumption equivalent welfare gain for a household of type

(ℓ, h, s) can be computed as:

$$100 \left(\left[\frac{\int \widetilde{v}_{\ell,h,s}(e; q, w)\Phi(e|s)de}{\int v_{\ell,h,s}(e; q, w)\Phi(e|s)de} \right]^{\frac{1}{1-\sigma}} - 1 \right), \tag{24}$$

where $\widetilde{v}_{\ell,h,s}$ is the value function in the equilibrium associated with the new policy.

Table 10: Welfare Effect of Means-Testing

Shock	Preference Shock		Liability Shock		Total
	Hit	Not	Hit	Not	
Proportion of households	0.012	0.988	0.003	0.997	1.000
Average % gain in flow consumption					
With bad credit record	0.52	0.80	0.75	0.80	0.80
With good credit record and debt	32.35	6.64	0.35	6.95	6.93
With good credit record and no debt	3.56	1.37	0.76	1.40	1.39
Total	4.75	1.60	0.74	1.64	**1.64**
% of households in favor of reform					
With bad credit record	100.0	100.0	100.0	100.0	100.0
With good credit record and debt	91.4	100.0	96.8	99.9	99.9
With good credit record and no debt	100.0	100.0	98.0	100.0	100.0
Total	99.6	100.0	98.0	100.0	**100.0**

Table 10 reports the desirability of the policy change for the two aggregation criteria. First of all, we see that the welfare benefits of the policy reform are large – about 1.6% of average consumption – as measured by the utilitarian average of consumption-equivalent gains. We also see that the policy reform receives almost unanimous support: around 0.01% of households oppose it. The largest gains accrue to those with a good credit record and debt. A possible explanation of why this is such a good policy is that means-testing takes away the right to default in situations where the size of the utility gain from defaulting are positive but not very high. Therefore, most people prefer to give this option up in exchange for lower interest rates (of course, people who oppose it are those with above-median earnings and a lot of debt – i.e., individuals for whom the policy binds strongly – but there are not many of them).[36]

[36] In an earlier version of this paper, which employed a somewhat different calibration, we explored another policy experiment where we reduced the mean exclusion time from borrowing following default from 10 to 5 years. For details, see Appendix B of http://www.phil.frb.org/files/wps/2005/wp05-18.pdf

7 Conclusions

In this paper we accomplished four goals. First, we developed a theory of default that is consistent with U.S. bankruptcy law. In the process we characterized some theoretical properties of the household's decision problem and proved the existence of a steady-state competitive equilibrium. A key feature of the model is that it treated different-sized consumer loans taken out by households with observably different characteristics as distinct financial assets with distinct prices. Second, we showed that the theory is quantitatively sound in that it is capable of accounting for the main facts regarding unsecured consumer debt and bankruptcy in the U.S. along with U.S. facts on macroeconomic aggregates and facts on inequality characteristics of U.S. earnings and wealth distributions. Third, we explored the implications of an important recent change in the bankruptcy law that limits the Chapter 7 bankruptcy option to households with below-median earnings. We showed that the likely outcome of this change will be a decrease in interest rates charged on unsecured loans, an increase in both the volume of debt and the number of borrowers without having necessarily having an increase in the number of bankruptcies. Furthermore, our measurements indicated that the changes will be big – for instance, the volume of net unsecured debt may almost double. Finally, we constructed measures of the welfare effects of the policy change. From the point of view of average consumption, our calculations indicate that the benefits of the change are large: on the order of 1.5 percent of average consumption. From the point of view of public support, we found that almost all households support the change. In terms of future research, two issues seem important. First, analyzing environments in which households have some motive for simultaneously holding both assets and liabilities is likely to improve our understanding of the unsecured consumer credit market. Second, incorporating unobserved differences among households with regard to willingness to default is also likely to improve our understanding of what happens to a household's credit opportunities after bankruptcy and, therefore, to the costs of default, especially if we take into account that individuals with bad credit scores can still have access to credit.

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A Appendix

For reasons given in the text, the appendix generalizes the environment in the paper to include a bankruptcy cost $\alpha \cdot (e - e_{\min}) \cdot w$ with $\alpha \in [0, \overline{\alpha}] = A$ where $\overline{\alpha} < 1$ to be paid only at the time of default. This requires us to expand the space on which the operator $\mathcal{T}$ is defined to include A and modify the operator $\mathcal{T}$ for case 2 (where the household chooses whether to default or not) in Definition 1 to be:

$$
\begin{aligned}
&(Tv)(\ell, 0, s, e; \alpha, q, w) = \\
&\qquad \max \left\{ \begin{array}{c} \max_{c, \ell' \in B_{\ell,0,s,0}} u(c, s) + \beta\rho \int v_{\ell',0,s'}(e', \alpha; q, w)\Phi(e'|s')\Gamma(s, ds')de', \\ u([e - \alpha \cdot (e - e_{\min})] \cdot w, s) \ + \ \beta\rho \int v_{0,1,s'}(e', \alpha; q, w)\Phi(e'|s')\Gamma(s, ds')de' \end{array} \right\},
\end{aligned}
$$

where to conserve on notation we let $u(c, s)$ denote $U(c, \eta(s))$.

A.1 Results for Theorems 1 and 2

The following restriction formalizes the assumption concerning $u(0, s)$ in part (iv) of Theorem 1.

Assumption A1. For every $s \in S$,

$$
\begin{aligned}
&u\left((1-\gamma)e_{\min} \cdot w_{\min}, s\right) - u(0, s) \\
&> \left(\frac{\beta\rho}{1-\beta\rho}\right) \left[u\left(e_{\max} \cdot w_{\max} + \ell_{\max} - \ell_{\min}, \overline{s}\right) - u\left((1-\gamma)e_{\min} \cdot w_{\min}, \underline{s}\right)\right].
\end{aligned}
$$

Definition A1. Let $\mathcal{V}$ be the set of all continuous (vector-valued) functions $v : E \times A \times Q \times W \to$

$R^{N_{\mathcal{L}}}$ such that:

$$v_{\ell,h,s}(e;\alpha,q,w) \in \left[\frac{u[e_{\min}\cdot w_{\min}(1-\gamma),\underline{s}]}{(1-\beta\rho)}, \frac{u(e_{\max}\cdot w_{\max}+\ell_{\max}-\ell_{\min},\overline{s})}{(1-\beta\rho)}\right], \tag{25}$$

$$\ell^0 \geq \ell^1 \Rightarrow v_{\ell^0,h,s}(e;\alpha,q,w) \geq v_{\ell^1,h,s}(e;\alpha,q,w), \tag{26}$$

$$e^0 \geq e^1 \Rightarrow v_{\ell,h,s}(e^0;\alpha,q,w) \geq v_{\ell,h,s}(e^1;\alpha,q,w) \tag{27}$$

$$v_{\ell,0,s}(e;\alpha,q,w) \geq v_{\ell,1,s}(e;\alpha,q,w), \tag{28}$$

$$\begin{aligned} &u(e_{\min}\cdot w_{\min}(1-\gamma),s)+\beta\rho\int v_{0,1,s'}(e',\alpha;q,w)\Phi(e'|s')\Gamma(s,ds')de' \\ &> u(0,s)+\beta\rho\int v_{\ell_{\max},0,s'}(e',\alpha;q,w)\Phi(e'|s')\Gamma(s,ds')de'. \end{aligned} \tag{29}$$

Lemma A1. $\mathcal{V}$ is non-empty. With $\|v\| = \max_{\ell,h,s}\left\{\sup_{e,\alpha,q,w\in E\times A\times Q\times W}|v_{\ell,h,s}(e;\alpha,q,w)|\right\}$ as the norm, $(\mathcal{V},\|\cdot\|)$ is a complete metric space.

Proof. To prove $\mathcal{V}$ is non-empty pick any constant vector-valued function satisfying (25). Such a function is continuous and clearly satisfies (26)-(28). Since it is a constant function, (29) reduces to the requirement that $u(e_{\min}(1-\gamma)\cdot w_{\min},s)-u(0,s)>0$ which is satisfied by virtue of $e_{\min}(1-\gamma)\cdot w_{\min}>0$ and the strict monotonicity of $u(\cdot,s)$. To prove $(\mathcal{V},\|\cdot\|)$ is complete, let $\mathcal{C}$ be the set of all continuous (vector-valued) functions from $E\times A\times Q\times W\rightarrow R^{N_{\mathcal{L}}}$. Then, $(\mathcal{C},\|.\|)$ is a complete metric space. Since any closed subset of a complete metric space is also a complete metric space it is sufficient to show that $\mathcal{V}\subset\mathcal{C}$ is closed in the norm $\|.\|$. Let $\{v_n\}$ be a sequence of functions in $\mathcal{V}$ converging to v, i.e., $\lim_{n\to\infty}\|v_n-v^*\|=0$. If v^* violates any of the range and monotonicity properties of $\mathcal{V}$, there must be some v_n, for n sufficiently large, that violates those properties as well. But that would contradict the assertion that v_n belongs to $\mathcal{V}$ for all n. Hence, v^* must satisfy all the range and monotonicity properties (25)-(28). To prove that $v^*(e;\alpha,q,w)$ is continuous simply adapt the final part of the proof of Theorem 3.1 in Stokey, Lucas, and Prescott (1989) to a vector-valued function. ■

We now turn to properties of the operator $\mathcal{T}$. It is convenient to have notation for the value of consumption for any action $\{\ell',d\}\in L\times\{0,1\}$ (including actions that imply negative consumption). We will denote consumption for a household with ℓ,h,s who takes actions ℓ',d by $c^{\ell',d}_{\ell,h,s}(e;\alpha,q,w)$. Then, $c^{0,1}_{\ell,0,s}(e;\alpha,q,w)\equiv[e_{\min}+(1-\alpha)(e-e_{\min})]\cdot w>0$, $c^{0,1}_{\ell,1,s}(e;\alpha,q,w)\equiv e(1-\gamma)\cdot w>0$, $c^{\ell',0}_{\ell,h,s}(e;\alpha,q,w)\equiv w\cdot e(1-\gamma h)+\ell-\zeta(s)-q_{\ell',s}\ell'$. Observe that the value of consumption for $\{\ell',0\}$ can be negative.

It is also convenient to define the expected utility of a person who starts next period with ℓ',h',s': $\omega_{\ell',h',s}(v)\equiv\int v_{\ell',h',s'}(e';\alpha,q,w)\Phi(e'|s')\Gamma(s,ds')de'$. Observe that ω is defined for a given v and so depends on α,q,w. In what follows, we will sometimes make this dependence explicit.

Using similar notation, for any given pair of discrete actions $\{\ell', d\} \in L \times \{0,1\}$, we define lifetime utility as follows: (i) For $h=0$, $\ell - \zeta(s) \geq 0$, $\phi^{\ell',0}_{\ell,0,s}(e,\alpha,q,w;\omega(v)) \equiv u\left(\max\left\{c^{\ell',0}_{\ell,0,s}(e;\alpha,q,w),0\right\},s\right) + \beta\rho\omega_{\ell',0,s}(v)$; (ii) For $h=0$, $\ell-\zeta(s)<0$, $\phi^{0,1}_{\ell,0,s}(e,\alpha,q,w;\omega(v)) \equiv u\left(c^{0,1}_{\ell,0,s}(e;\alpha,q,w),s\right)+\beta\rho\omega_{0,1,s}(v)$ and $\phi^{\ell',0}_{\ell,0,s}(e,\alpha,q,w;\omega(v)) \equiv u\left(\max\left\{c^{\ell',0}_{\ell,0,s}(e;\alpha,q,w),0\right\},s\right)+\beta\rho\omega_{\ell',0,s}(v)$; (iii) For $h=1$, $\ell-\zeta(s) \geq 0$, $\phi^{\ell',0}_{\ell,1,s}(e,\alpha,q,w;\omega(v)) \equiv u\left(\max\left\{c^{\ell',0}_{\ell,1,s}(e;\alpha,q,w),0\right\},s\right)+\beta\rho \times \left[\lambda\omega_{\ell',1,s}(\alpha,q,w;v)+(1-\lambda)\omega_{\ell',0,s}(v)\right]$; (iv) Finally, for $h=1$, $\ell-\zeta(s)<0$, $\phi^{0,1}_{\ell,1,s}(e,\alpha,q,w;\omega(v)) \equiv u\left(c^{0,1}_{\ell,1,s}(e;\alpha,q,w),s\right)+\beta\rho\omega_{0,1,s}(v)$.

Then we have:

Lemma A2. For any (ℓ',d), $\phi^{\ell',d}_{\ell,h,s}(e,\alpha,q,w;\omega(v))$ is continuous in e,α,q, and w.

Proof. Observe that $c^{\ell',d}_{\ell,h,s}(e;\alpha,q,w)$ are each continuous functions of e,α,q, and w and u is a continuous function in its first argument. Further, $\omega_{\ell',h',s}(v)$ is continuous in α,q and w because $v \in \mathcal{V}$ and integration preserves continuity. ■

Lemma A3. For $v \in \mathcal{V}$, $(\mathcal{T}v)(e;\alpha,q,w)$ is continuous in e,α,q, and w.

Proof. By Lemma A2, $\phi^{\ell',d}_{\ell,h,s}(e,\alpha,q,w;\omega(v))$ is continuous. Hence, $\max_{\ell',d}\phi^{\ell',d}_{\ell,h,s}(e,\alpha,q,w;\omega(v))$ is also continuous in e,α,q, and w. Then it is sufficient to establish that $\forall \ell,h,s \in \mathcal{L}$,

$$(Tv)(\ell,h,s,e;\alpha,q,w)=\max_{\ell',d}\phi^{\ell',d}_{\ell,h,s}(e,\alpha,q,w;\omega(v)).$$

If the maximum is over feasible (ℓ',d), $(Tv)(\ell,h,s,e;\alpha,q,w)=\max_{\ell',d}\phi^{\ell',d}_{\ell,h,s}(e,\alpha,q,w;\omega(v))$. Furthermore, for infeasible (ℓ',d) the payoff $\phi^{\ell',d}_{\ell,h,s}(e,\alpha,q,w;\omega(v))$ is assigned a value that is always (weakly) dominated by some feasible ℓ',d. This follows because by property (29) the utility from consuming nothing today and starting next period with $\ell_{\max}$ and a good credit record (the highest utility possible with an infeasible action) is less than the utility from consuming $e_{\min}\cdot w_{\min}(1-\gamma)$ today and starting next period with zero assets and a bad credit record (the lowest utility possible with a feasible action). ■

Corollary to Lemma A3. For any $v \in \mathcal{V}$, the consumption implied by $(Tv)(\ell,h,s,e;\alpha,q,w)$ is strictly positive.

Proof. The exact same argument as in Lemma A3 can be used to establish that a *feasible* choice involving zero consumption is always strictly dominated by a feasible choice involving positive consumption. ■

Lemma A4. Given Assumption A1, $\mathcal{T}$ is a contraction mapping with modulus $\beta\rho$.

Proof. We first establish that $\mathcal{T}(\mathcal{V}) \subset \mathcal{V}$.

For $v \in \mathcal{V}$, $\mathcal{T}$ is continuous by Lemma A3.

To establish that $\mathcal{T}$ preserves the boundedness property (25), note that since $q_{\ell_{\min},s} \in [0,1]$, consumption can never exceed $e_{\max}\cdot w_{\max}+\ell_{\max}-\ell_{\min}$. Therefore, $(Tv)(\ell,h,s,e;\alpha,q,w) \leq$

$(1-\beta\rho)^{-1}\ u(e_{\max}\cdot w_{\max}+\ell_{\max}-\ell_{\min},\overline{s})$. And, since $\alpha<1$, $c=(1-\gamma)e_{\min}{\cdot}w_{\min}$ is feasible for all ℓ, h,s,$e, \alpha,\ q$,and w. Therefore $(Tv)(\ell,h,s,e;\alpha,q,w)\geq \frac{1}{(1-\beta\rho)}\ u\left((1-\gamma)e_{\min}\cdot w_{\min},\underline{s}\right)$. Hence

$$
\begin{aligned}
&(\mathcal{T}v)(e;\alpha,q,w)\\
&\in\left[\frac{1}{(1-\beta\rho)}\ u\left((1-\gamma)e_{\min}\cdot w_{\min},\underline{s}\right),\ \frac{1}{(1-\beta\rho)}\ u\left(e_{\max}\cdot w_{\max}+\ell_{\max}-\ell_{\min},\overline{s}\right)\right]^{N_{\mathcal{L}}}.
\end{aligned}
$$

To establish that $\mathcal{T}$ preserves the monotonicity property (26), consider a household with a given h,s and two different asset holdings $\ell^0>\ell^1$. (i) If $0>\ell^0$ then for $d\in\{0,1\}$, $B_{\ell^1,0,s,d}(e;\alpha,q,w)\subseteq B_{\ell^0,0,s,d}(e;\alpha,q,w)$ and the result follows; (ii) if $\ell^0\geq 0>\ell^1$ and $\ell^0\geq\zeta(s)$ then $B_{\ell^1,0,s,d}(e;\alpha,q,w)\subseteq B_{\ell^0,0,s,0}(e;\alpha,q,w)$ and the result follows from using (28); (iii) if $\ell^0\geq 0>\ell^1$ and $\ell^0<\zeta(s)$ then $B_{\ell^1,0,s,d}(e;\alpha,q,w)\subseteq B_{\ell^0,0,s,d}(e;\alpha,q,w)$ and the result follows; (iv) if $\ell^1\geq 0$ and $\ell^1<\zeta(s)\leq\ell^0$ then $B_{\ell^1,h,s,d}(e;\alpha,q,w)\subseteq B_{\ell^0,h,s,0}(e;\alpha,q,w)$ and the result follows from using (28); and (v) if $\ell^1\geq 0$ and $\ell^1\geq\zeta(s)$ then $B_{\ell^1,h,s,0}(e;\alpha,q,w)\subseteq B_{\ell^0,h,s,0}(e;\alpha,q,w)$ and the result follows.

To establish that $\mathcal{T}$ preserves the monotonicity property (27), consider a household with a given $\ell,\ h,s$ and two different efficiency levels $e^0>e^1$. Since $\alpha<1$, $B_{\ell,h,s,d}(e^1;\alpha,q,w)\subseteq B_{\ell,h,s,d}(e^0;\alpha,q,w)$ and the result follows.

To establish that $\mathcal{T}$ preserves the monotonicity property (28), consider a household with a given $\ell,\ s$ and two different credit records. (i) If $\ell-\zeta(s)<0$, $B_{\ell,1,s,1}(e;\alpha,q,w)\subseteq B_{\ell,0,s,d}(e;\alpha,q,w)$ and the result follows; (ii) if $\ell-\zeta(s)\geq 0$, $B_{\ell,1,s,0}(e;\alpha,q,w)\subseteq B_{\ell,0,s,0}(e;\alpha,q,w)$ the result follows from using (28).

To establish that $\mathcal{T}$ preserves the "default at zero consumption" property (29), by Assumption A1 and the fact that $\mathcal{T}$ satisfies the boundedness property it follows that

$$u\left((1-\gamma)e_{\min}\cdot w_{\min},s\right)-u(0,s)>\beta\,\rho\ \left[(Tv)(\ell_{\max},0,s,e;\alpha,q,w)-(Tv)(0,1,s,e;\alpha,q,w)\right].$$

Re-arranging gives:

$$u\left((1-\gamma)e_{\min}\cdot w_{\min},s\right)+\beta\,\rho\ (Tv)(0,1,s,e;\alpha,q,w)>u(0,s)+\beta\,\rho\ (Tv)(\ell_{\max},0,s,e;\alpha,q,w).$$

Next we establish that $\mathcal{T}$ is a contraction with modulus $\beta\rho$. The first step is to establish the analogue of the Blackwell monotonicity and discounting properties. Monotonicity: Let v, $v'\in\mathcal{V}$ and $v(e;\alpha,q,w)\leq v'(e;\alpha,q,w)$ for all $e,\alpha,\ q,w$. From the definition of the $\mathcal{T}$ operator it's clear that $(\mathcal{T}\ v)\leq(\mathcal{T}v')$. Discounting: It's also clear that for any $\kappa\in R_{+}^{N_{\mathcal{L}}}$, $[\mathcal{T}(v+\kappa)]$ $(e;\alpha,q,w)=(\mathcal{T}v)(e;\alpha,q,w)+\beta\,\rho\,\kappa$. To prove that $\mathcal{T}$ is a contraction mapping, simply adapt the final part of the proof of Theorem 3.3 in Stokey, Lucas, and Prescott (1989) to a vector-valued function. This establishes that $\mathcal{T}$ is a contraction mapping with modulus $\beta\rho$. ■

Theorem 1 (Existence of a Recursive Solution to the Household Problem). There exists a unique $v^*\in\mathcal{V}$ such that $v^*=\mathcal{T}(v^*)$. Furthermore: (i) v^* is bounded and increasing in ℓ and e; (ii) a bad credit record reduces v^*; (iii), the optimal policy correspondence implied by $\mathcal{T}(v^*)$ is compact-valued and upper hemi-continuous; and (iv) provided $u(0,s)$ is sufficiently low, default is strictly preferable to zero consumption and consumption is strictly positive.

Proof. Existence and uniqueness of v^*, as well as properties (i), (ii), and (iv) follow directly from Lemmas A1, A3, A4, and the Corollary to Lemma A3. To prove part (iii) define the *optimal policy correspondence* to be

$$\chi_{\ell,h,s}(e;\alpha,q,w)=\{(c,\ell',d)\in B_{\ell,h,s,d}(e;\alpha,q,w):(c,\ell',d)\text{ attains }v^*_{\ell,h,s}(e;\alpha,q,w)\}.$$

To establish the first part of (iii), note that the correspondence $\chi_{\ell,h,s}(e;\alpha,q,w)$ is bounded because c is bounded between 0 and $e_{\max}\cdot w_{\max}+\ell_{\max}-\ell_{\min}$, and $(\ell',d)\in L\times\{0,1\}$. To prove that $\chi_{\ell,h,s}(e;\alpha,q,w)$ is closed, let $\{c_n,\ell'_n,d_n\}$ be a sequence in $\chi_{\ell,h,s}(e;\alpha,q,w)$ converging to $(\overline{c},\overline{\ell}',\overline{d})$. Since (ℓ',d) are elements of finite sets, $\exists\eta$ such that $\forall n>\eta$, $(\ell'_n,d_n)=(\overline{\ell}',\overline{d})$. Given that $(c_n,\overline{\ell}',\overline{d})$ attains $v^*_{\ell,h,s}(e;\alpha,q,w)$, $\forall n>\eta$ we have $c_n=c^{\overline{\ell}',\overline{d}}_{\ell,h,s}(e;\alpha,q,w)$. Therefore, $\overline{c}=c^{\overline{\ell}',\overline{d}}_{\ell,h,s}(e;\alpha,q,w)$ and $(\overline{c},\overline{\ell}',\overline{d})\in\chi_{\ell,h,s}(e;\alpha,q,w)$. To establish the second part of property (iii), let $\{\ell_n,h_n,s_n,e_n,\alpha_n,q_n,w_n\}\to(\overline{\ell},\overline{h},\overline{s},\overline{e},\overline{\alpha},\overline{q},\overline{w})$. Since $\mathcal{L}$ is finite we can fix $(\ell_n,h_n,s_n)=(\overline{\ell},\overline{h},\overline{s})$ and simply consider $e_n,\alpha_n,q_n,w_n\to\overline{e},\overline{\alpha},\overline{q},\overline{w}$. Let $\{c_n,\ell'_n,d_n\}\in\chi_{\overline{\ell},\overline{h},\overline{s}}(e_n;\alpha_n,q_n,w_n)$. Since the correspondence is compact-valued, there must exist a subsequence $\{c_{n_k},\ell'_{n_k},d_{n_k}\}$ converging to $(\overline{c},\overline{\ell}',\overline{d})$. Furthermore, since ℓ' and d take on only a finite number of values, $\exists\,\eta$ such that $\forall n_k>\eta$ we have $(c_{n_k},\ell'_{n_k},d_{n_k})=\left(c^{\overline{\ell}',\overline{d}}_{\overline{\ell},\overline{h},\overline{s}}(e_{n_k};\alpha_{n_k},q_{n_k},w_{n_k}),\overline{\ell}',\overline{d}\right)$. Then by optimality $\phi^{\overline{\ell}',\overline{d}}_{\overline{\ell},\overline{h},\overline{s}}(e_{n_k},\alpha_{n_k},q_{n_k},w_{n_k};\omega^*(\alpha_{n_k},q_{n_k},w_{n_k}))=v^*_{\overline{\ell},\overline{h},\overline{s}}(e_{n_k};\alpha_{n_k},q_{n_k},w_{n_k})$ and by continuity of $\phi^{\overline{\ell}',\overline{d}}_{\overline{\ell},\overline{h},\overline{s}}$, $v^*_{\overline{\ell},\overline{h},\overline{s}}$ and ω^* w.r.t. e,α,q and w we have $\phi^{\overline{\ell}',\overline{d}}_{\overline{\ell},\overline{h},\overline{s}}(\overline{e},\overline{\alpha},\overline{q},\overline{w};\omega^*(\overline{\alpha},\overline{q},\overline{w}))=v^*_{\overline{\ell},\overline{h},\overline{s}}(\overline{e};\overline{\alpha},\overline{q},\overline{w})$. Therefore $\left(\overline{c}=c^{\overline{\ell}',\overline{d}}_{\overline{\ell},\overline{h},\overline{s}}(\overline{e};\overline{\alpha},\overline{q},\overline{w}),\overline{\ell}',\overline{d}\right)\in\chi_{\overline{\ell},\overline{h},\overline{s}}(\overline{e};\overline{\alpha},\overline{q},\overline{w})$ and the correspondence is u.h.c. ■

Theorem 2 (Existence of a Unique Invariant Distribution). For $(\alpha,q,w)\in A\times Q\times W$ and any measurable selection from the optimal policy correspondence, there exists a unique $\mu_{(\alpha,q,w)}\in\mathcal{M}(\mathcal{L}\times E,2^{\mathcal{L}}\times\mathcal{B}(E))$ such that $\mu_{(\alpha,q,w)}=\Upsilon_{(\alpha,q,w)}\mu_{(\alpha,q,w)}$.

Proof. By the Measurable Selection Theorem, there exists an optimal policy rule that is measurable with respect to any measure in $\mathcal{M}(\mathcal{L}\times E,2^{\mathcal{L}}\times\mathcal{B}(E))$. Therefore, $G^*_{(\alpha,q,w)}$ is well-defined. To establish this lemma we then simply need to verify that $G^*_{(\alpha,q,w)}$ satisfies the conditions stipulated in Theorem 11.10 of Stokey, Lucas, and Prescott (1989). The first condition is that $G^*_{(\alpha,q,w)}$ satisfies Doeblin's condition (which states that there is a finite measure φ on $(\mathcal{L}\times E,2^{\mathcal{L}}\times\mathcal{B}(E))$, an integer $I\geq 1$, and a number $\varepsilon>0$, such that if $\varphi(Z)\leq\varepsilon$, then $G^{*I}_{(\alpha,q,w)}((\ell,h,s,e),Z)\leq 1-\varepsilon$, for all (ℓ,h,s,e)). It is sufficient to show that GN satisfies the Doeblin condition (see Exercise 11.4.g of Stokey, Lucas, and Prescott (1989)). Observe that since GN is independent of (ℓ,h,s,e), we can pick $\varphi(Z)=GN((\ell,h,s,e),Z)$. Then GN satisfies the Doeblin condition for $I=1$ and $\varepsilon<\frac{1}{2}$. Second, we need to show that if Z is any set of positive φ -measure, then for each (ℓ,h,s,e), there exists $n\geq 1$ such that $G^{*n}_{(\alpha,q,w)}((\ell,h,s,e),Z)>0$. To see this, observe that if $\varphi(Z)>0$, then $GN((\ell,h,s,e),Z)>0$ for any (ℓ,h,s,e). Therefore, $G^{*1}_{(\alpha,q,w)}((\ell,h,s,e),Z)>0$. ■

A.2 Results for Theorems 3 and 4

We turn now to the proof of Theorem 3. We give a formal definition of the maximal default set and then establish two key lemmas. The maximal default $\overline{D}^*_{\ell,h,s}(\alpha,q,w) = \{e : v^*_{\ell,h,s}(e;\alpha,q,w) = \phi^{0,1}_{\ell,h,s}(e,\alpha,q;\omega^*)\}$, where ω^* is $\omega(v^*)$.

Lemma A5. Let $\widehat{e} \in E \backslash \overline{D}^*_{\ell,0,s}(0,q,w)$, $e > \widehat{e}$, and $\ell - \zeta(s) < 0$. If $e \in \overline{D}^*_{\ell,0,s}(0,q,w)$, then $c^*_{\ell,0,s}(\widehat{e};0,q,w) > \widehat{e} \cdot w$.

Proof. Since $\widehat{e} \in E \backslash \overline{D}^*_{\ell,0,s}(0,q,w)$,

$$u\left(c^*_{\ell,0,s}(\widehat{e};0,q,w),s\right) + \beta\rho\omega^*_{\ell'^*_{\ell,0,s}(\widehat{e};0,q,w),0,s}(0,q,w) > u(\widehat{e}\cdot w,s) + \beta\rho\omega^*_{0,1,s}(0,q,w). \tag{30}$$

Let $\Delta = (e-\widehat{e})\cdot w > 0$. The pair $\{\underline{c} = c^*_{\ell,0,s}(\widehat{e};0,q,w) + \Delta, \underline{\ell'} = \ell'^*_{\ell,0,s}(\widehat{e};0,q,w)\}$ clearly belongs in $B_{\ell,0,s,0}(e;0,q,w)$. Then by optimality, utility obtained by not defaulting when labor efficiency is e must satisfy the inequality

$$u\left(\tilde{c}_{\ell,0,s,0}(e;0,q,w),s\right) + \beta\rho\omega^*_{\tilde{\ell}'_{\ell,0,s,0}(e;0,q,w),0,s}(0,q,w) \geq u(\underline{c},s) + \beta\rho\omega^*_{\underline{\ell'},0,s}(0,q,w), \tag{31}$$

where $\tilde{c}_{\ell,0,s,0}(e;0,q,w)$ and $\tilde{\ell}'_{\ell,0,s,0}(e;0,q,w)$ are the optimal choices of c and ℓ' conditional on not defaulting. Since $e \in \overline{D}^*_{\ell,h,s}(0,q,w)$,

$$u\left(\tilde{c}_{\ell,0,s,0}(e;0,q,w),s\right) + \beta\rho\omega_{\tilde{\ell}'_{\ell,0,s,0}(e;0,q,w),0,s}(0,q,w) \leq u(e\cdot w,s) + \beta\rho\omega^*_{0,1,s}(0,q,w). \tag{32}$$

By (31) and the fact that $\widehat{e}\cdot w + \Delta = e\cdot w$, (32) can be rewritten

$$u(\underline{c},s) + \beta\rho\omega^*_{\underline{\ell'},0,s}(0,q,w) \leq u(\widehat{e}\cdot w + \Delta,\mu) + \beta\rho\omega^*_{0,1,s}(0,q,w). \tag{33}$$

Then (33) minus (30) implies

$$\begin{aligned} u(\underline{c},s) + \beta\rho\omega^*_{\underline{\ell'},0,s}(0,q,w) - u\left(c^*_{\ell,0,s}(\widehat{e};0,q,w),s\right) - \beta\rho\omega^*_{\ell'^*_{\ell,0,s}(\widehat{e};0,q,w),0,s}(0,q,w) \\ < u(\widehat{e}\cdot w + \Delta,s) + \beta\rho\omega^*_{0,1,s}(0,q,w) - u(\widehat{e}\cdot w,s) - \beta\rho\omega^*_{0,1,s}(0,q,w). \end{aligned} \tag{34}$$

Or, by definition of $(\underline{c}, \underline{\ell'})$,

$$u\left(c^*_{\ell,0,s}(\widehat{e};0,q,w)+\Delta,s\right) - u\left(c^*_{\ell,0,s}(\widehat{e};0,q,w),s\right) < u(\widehat{e}\cdot w+\Delta,s) - u(\widehat{e}\cdot w,s).$$

Since $u(\cdot,s)$ is strictly concave, the last inequality implies $c^*_{\ell,0,s}(\widehat{e};0,q,w) > \widehat{e}\cdot w$. ■

Lemma A6. Let $\widehat{e} \in E \backslash \overline{D}^*_{\ell,0,s}(0,q,w)$, $e < \widehat{e}$, and $\ell - \zeta(s) < 0$. If $e \in \overline{D}^*_{\ell,0,s}(0,q,w)$, then $c^*_{\ell,0,s}(\widehat{e};0,q,w) < \widehat{e}\cdot w$.

Proof. Since $\widehat{e} \in E \backslash \overline{D}^*_{\ell,0,s}(0,q,w)$,

$$u\left(c^*_{\ell,0,s}(\widehat{e};0,q,w),s\right) + \beta\rho\omega^*_{\ell'^*_{\ell,0,s}(\widehat{e};0,q,w),0,s}(0,q,w) > u(\widehat{e}\cdot w,s) + \beta\rho\omega^*_{0,1,s}(0,q,w). \tag{35}$$

Let $\Delta = (\widehat{e}-e)\cdot w > 0$. Consider the quantity $c^*_{\ell,0,s}(\widehat{e};0,q,w) - \Delta$. If $c^*_{\ell,0,s}(\widehat{e};0q,w) - \Delta \leq 0$ then it must be the case that $c^*_{\ell,0,s}(\widehat{e};0,q,w) < \widehat{e}\cdot w$ because $\widehat{e}\cdot w - \Delta = e\cdot w > 0$. So, we only need to consider the case where $c^*_{\ell,0,s}(\widehat{e};0,q,w) - \Delta > 0$. The pair $\{\underline{c} = c^*_{\ell,0,s}(\widehat{e};0,q,w) - \Delta,$

$\underline{\ell'} = \ell'^*_{\ell,0,s}(\widehat{e};0,q,w)\}$ clearly belongs in $B_{\ell,0,s,0}(0,q,w)$. Then by optimality, utility obtained by not defaulting when labor efficiency is e must satisfy the inequality

$$u\left(\tilde{c}_{\ell,0,s,0}(e;0,q,w),s\right) + \beta\rho\omega^*_{\tilde{\ell}'_{\ell,0,s,0}(e;0,q,w),0,s}(0,q,w) \geq u(\underline{c},s) + \beta\rho\omega^*_{\underline{\ell'},0,s}(0,q,w), \tag{36}$$

where, once again, $\tilde{c}_{\ell,0,s,0}(e;0,q,w)$ and $\tilde{\ell}'_{\ell,0,s,0}(e;0,q,w)$ are the optimal choices of c and ℓ' conditional on not defaulting. Since $e \in \overline{D}^*_{\ell,0,s}(0,q,w)$,

$$u\left(\tilde{c}_{\ell,0,s,0}(e;0,q,w),s\right) + \beta\rho\omega^*_{\tilde{\ell}'_{\ell,0,s,0}(e;0,q,w),0,s}(0,q,w) \leq u(e\cdot w,s) + \beta\rho\omega^*_{0,1,s}(0,q,w). \tag{37}$$

By (36) and the fact that $\widehat{e}\cdot w - \Delta = e\cdot w$, (37) can be rewritten

$$u(\underline{c},s) + \beta\rho\omega^*_{\underline{\ell'},0,s}(0,q,w) \leq u(\widehat{e}\cdot w - \Delta,s) + \beta\rho\omega^*_{0,1,s}(0,q,w). \tag{38}$$

Then (38) minus (35) implies

$$\begin{aligned} u(\underline{c},s) + \beta\rho\omega^*_{\underline{\ell'},0,s}(0,q,w) - u\left(c^*_{\ell,0,s}(\widehat{e};0,q,w),s\right) - \beta\rho\omega^*_{\ell'^*_{\ell,0,s}(\widehat{e};0,q,w),0,s}(0,q,w) \\ < u(\widehat{e}\cdot w - \Delta,s) + \beta\rho\omega^*_{0,1,s}(0,q,w) - u(\widehat{e}\cdot w,s) - \beta\rho\omega^*_{0,1,s}(0,q,w). \end{aligned}$$

Or, by definition of $(\underline{c},\, \underline{\ell'})$,

$$u\left(c^*_{\ell,0,s}(\widehat{e};0,q,w),s\right) - u\left(c^*_{\ell,0,s}(\widehat{e};0,q,w) - \Delta,s\right) > u(\widehat{e}\cdot w,s) - u(\widehat{e}\cdot x - \Delta,s). \tag{39}$$

Since $u(\cdot,s)$ is strictly concave, the last inequality implies $c^*_{\ell,0,s}(\widehat{e};0,q,w) - \Delta < \widehat{e}\cdot w - \Delta$, or, $c^*_{\ell,0,s}(\widehat{e};0,q,w) < \widehat{e}\cdot w$. ■

Theorem 3 (The Maximal Default Set Is a Closed Interval). If $\overline{D}^*_{\ell,0,s}(0,q,w)$ is non-empty, it is a closed interval.

Proof. First, consider the case $h=0$. If $\ell - \zeta(s) \geq 0$, then $\overline{D}^*_{\ell,0,s}(0,q,w) = \varnothing$. If $\ell - \zeta(s) < 0$, let $e_L = \inf \overline{D}^*_{\ell,0,s}(0,q,w)$ and $e_U = \sup \overline{D}^*_{\ell,0,s}(0,q,w)$. Since $\overline{D}^*_{\ell,0,s}(0,q,w) \subset E$, which is bounded, both e_L and e_U exist by the Completeness Property of R . If $e_L = e_U$, the default set contains only one element $e = e_L = e_U$ and the result is trivially true. Suppose, then, that $e_L < e_U$. Let $\widehat{e} \in (e_L, e_U)$ and assume that $\widehat{e} \notin \overline{D}^*_{\ell,0,s}(0,q,w)$. Then there is an $e \in \overline{D}^*_{\ell,0,s}(0,q,w)$ such that $e > \widehat{e}$ (if not, then $e_U = \widehat{e}$ which contradicts the assertion that $\widehat{e} \in (e_L,e_U)$). Then, by Lemma A5, $c^*_{\ell,0,s}(\widehat{e};0,q,w) > \widehat{e}\cdot w$. Similarly, there is an $e \in \overline{D}^*_{\ell,0,s}(0,q,w)$ such that $e < \widehat{e}$. Then, by Lemma A6, $c^*_{\ell,0,s}(\widehat{e};0,q,w) < \widehat{e}\cdot w$. But $c^*_{\ell,0,s}(\widehat{e};0,q,w)$ cannot be both greater and less than $\widehat{e}\cdot w$. Hence, the assertion $\widehat{e} \notin \overline{D}^*_{\ell,0,s}(0,q,w)$ must be false and $(e_L,e_U) \subset \overline{D}^*_{\ell,0,s}(0,q,w)$. To show that $e_U \in \overline{D}^*_{\ell,0,s}(0,q,w)$, pick a sequence $\{e^n\} \subset (e_L,e_U)$ converging to e_U. Then, $v^*_{\ell,0,s}(e_n;0,q,w) - u(e_n\cdot w,s) = \beta\rho\omega^*_{0,1,s}(0,q,w)$ for all n. Since e_U is clearly in E, by the continuity of $v^*_{\ell,0,s}(e;0,q,w)$ and u , it follows that $\lim_{n\to\infty}\{v^*_{\ell,0,s}(e_n;0,q,w) - u(e_n\cdot w,s)\} = v^*_{\ell,0,s}(e_U;0,q,w) - u(e_U\cdot w,s)$. Since every element of the sequence $\{v^*_{\ell,0,s}(e_n;0,q,w) - u(e_n\cdot w,s)\}$ is equal to $\beta\rho\omega^*_{0,1,s}(0,q,w)$, it must be the case that $v^*_{\ell,0,s}(e_U;0,q,w) - u(e_U\cdot w,s) = \beta\rho\omega^*_{0,1,s}(0,q,w)$. Hence, $e_U \in \overline{D}^*_{\ell,0,s}(0,q,w)$. By analogous reasoning, $e_L \in \overline{D}^*_{\ell,0,s}(0,q,w)$. Hence, $[e_L,e_U] \subseteq \overline{D}^*_{\ell,0,s}(0,q,w)$. But by the definition of e_L and e_U, $\overline{D}^*_{\ell,0,s}(0,q,w) \subset [e_L,e_U]$. Hence $[e_L,e_U] = \overline{D}^*_{\ell,0,s}(0,q,w)$. Next consider the case $h=1$. If $\ell - \zeta(s) \geq 0$, then $\overline{D}^*_{\ell,0,s}(0,q,w) = \varnothing$. If $\ell - \zeta(s) < 0$, then $\overline{D}^*_{\ell,0,s}(0,q,w) = E$. ■

Theorem 4 (Maximal Default Set Expands with Indebtedness). If $\ell^0 > \ell^1$, then $\overline{D}^*_{\ell^0,0,s}(0,q,w) \subseteq \overline{D}^*_{\ell^1,h,s}(\alpha,q,w)$.

Proof. Suppose $e \in \overline{D}^*_{\ell^0,0,s}(0,q,w)$. Since $v^*_{\ell,0,s}(e;\alpha,q,w)$ is increasing in ℓ, $v^*_{\ell^0,0,s}(e;\alpha,q,w) \geq v^*_{\ell^1,0,s}(e;\alpha,q,w)$. But $v^*_{\ell^0,0,s}(e;\alpha,q,w) = u(e \cdot w, s) + \beta\rho\omega^*_{0,1,s}(\alpha,q,w)$. Since default is also an option at ℓ^1, it must be the case that $v^*_{\ell^1,0,s}(e;\alpha,q,w) = u(e \cdot w, s) + \beta\rho\omega^*_{0,1,s}(\alpha,q,w)$. Hence any e in $\overline{D}^*_{\ell^0,0,s}(0,q,w)$ is also in $\overline{D}^*_{\ell^1,0,s}(0,q,w)$. ■

A.3 Results for Theorems 5 and 6

We now turn to the proof of existence of equilibrium. For the environment with $\alpha > 0$, all conditions in Definition 2 remain the same except for the goods market clearing condition (ix) which we now call (ixA)

$$\begin{aligned}
&\int c^*_{\ell,h,s}(e;\alpha,q^*,w^*)d\mu^* + K^* + \int \frac{\zeta(s)}{m^*}d\mu^* \\
&= F(N^*,K^*) + (1-\delta)\,K^* - \gamma w^* \int e\mu^*(d\ell,1,ds,de) \\
&- \alpha w^* \int (e - e_{\min}) \cdot d^*_{\ell,0,s}(e;\alpha,q^*,w^*)\mu^*(d\ell,0,ds,de).
\end{aligned}$$

We state without proof the following (the proof is available in the supplementary material section of this article on the *Econometrica* website):

Lemma A7. The goods market clearing condition (ixA) is implied by the other conditions for an equilibrium in Definition 2.

Next, we establish the important result that for $\alpha > 0$ the set of (ℓ,h,s,e) for which a household is indifferent between two courses of action is finite. Since the probability measure associated with a finite set is zero this result allows us to ignore behavior of households at "indifferent points" and simplifies the proof of existence of an equilibrium.

Given (ℓ,h,s), define the set of e for which a household is indifferent between any two distinct feasible actions (ℓ',d) and $(\overline{\ell}',\overline{d})$ as $I^{(\ell',d),(\overline{\ell}',\overline{d})}_{\ell,h,s}(\alpha,q,w) \equiv \{e \in E : \phi^{(\ell',d)}_{\ell,h,s}(e;\alpha,q,w,\omega^*) = \phi^{(\overline{\ell}',\overline{d})}_{\ell,h,s}(e;\alpha,q,w,\omega^*)\} \cap \{e \in E : c^{(\ell',d)}_{\ell,h,s}(e;\alpha,q,w) \geq 0, c^{(\overline{\ell}',\overline{d})}_{\ell,h,s}(e;\alpha,q,w) \geq 0\}$.

Lemma A8. (i) $I^{(\ell',0),(\overline{\ell}',0)}_{\ell,h,s}(\alpha,q,w;\omega^*)$ contains at most one element and (ii) if $\alpha > 0$, then $I^{(\ell',0),(0,1)}_{\ell,0,s}(\alpha,q,w;\omega^*)$ contains at most two elements.

Proof. (i) Let $e \in I^{(\ell',0),(\overline{\ell}',0)}_{\ell,h,s}(\alpha,q,w)$. Since $\omega^*_{\ell',0,s}(\alpha,q,w) \neq \omega^*_{\overline{\ell}',0,s}(\alpha,q,w)$, it follows that $\Delta \equiv u(w \cdot e(1-\gamma h) + \ell - \zeta(s) - q_{\ell',s}\ell', s) - u(w \cdot e(1-\gamma h) + \ell - \zeta(s) - q_{\overline{\ell}',s}\overline{\ell}', s) \neq 0$. Therefore, consumption under each of the two actions must be different. Since $u(\cdot)$ is strictly concave, an equal change in consumption from these two different levels must lead to unequal changes in utility. Therefore, for

$y \neq 0$ we must have that $u(w \cdot [e+y](1-\gamma h)+\ell-\zeta(s)-q_{\ell',s}\ell', s)-u(w \cdot [e+y](1-\gamma h)+\ell-\zeta(s)-q_{\overline{\ell}',s}\overline{\ell}', s) \neq \Delta$. Hence there can be at most one e for which $\phi^{\ell',0}_{\ell,0,s}(e,q,w;\omega^*) = \phi^{\overline{\ell}',0}_{\ell,0,s}(e,q,w;\omega^*)$.

(ii) Let $e \in I^{(\ell',0),(0,1)}_{\ell,0,s}(\alpha,q,w)$ and let $y > 0$.

(a): Suppose that $u(w \cdot e+\ell-\zeta(s)-q_{\ell',s}\ell', s)-u(w \cdot [e_{\min}+(1-\alpha)(e-e_{\min})], s) = \Delta \geq 0$. Given $\alpha > 0$, it follows that $u'(w \cdot [e+y]+\ell-\zeta(s)-q_{\ell',s}\ell', s) < u'(w \cdot [e_{\min}+(1-\alpha)(e+y-e_{\min})], s)$. Now observe that $u(w \cdot [e+y]+\ell-\zeta(s)-q_{\ell',s}\ell', s)$ is $u(w \cdot e+\ell-\zeta(s)-q_{\ell',s}\ell', s)+\int_0^y u'(w \cdot [e+x]+\ell-\zeta(s)-q_{\ell',s}\ell', s)dx$ and $u(w \cdot [e_{\min}+(1-\alpha)(e+y-e_{\min})], s)$ is $u(w \cdot [e_{\min}+(1-\alpha)(e-\underline{e})], s)+\int_0^y u'(w \cdot [e_{\min}+(1-\alpha)(e+x-e_{\min})], s)dx$. Therefore, $u(w \cdot [e+y]+\ell-\zeta(s)-q_{\ell',s}\ell', s)-u(w \cdot [e_{\min}+(1-\alpha)(e+y-e_{\min})], s) < \Delta$. Hence $e+y \notin I^{(\ell',0),(0,1)}_{\ell,0,s}(\alpha,q,w)$. On the other hand, it's possible that there is a $z > 0$ such that $e-z \in I^{(\ell',0),(0,1)}_{\ell,0,s}(\alpha,q,w)$. If so, $\int_0^z u'(w \cdot [e-x]+\ell-\zeta(s)-q_{\ell',s}\ell', s)dx = \int_0^z u'(w \cdot [e_{\min}+(1-\alpha)(e-x-e_{\min})], s)dx$. Since $\Delta \geq 0$, we have $u'(w \cdot e+\ell-\zeta(s)-q_{\ell',s}\ell', s) < u'(w \cdot [e_{\min}+(1-\alpha)(e-e_{\min})], s)$. Therefore, $w \cdot [e-z]+\ell-\zeta(s)-q_{\ell',s}\ell' < w \cdot e_{\min}+(1-\alpha)(e-z-e_{\min})$. Then, given $\alpha > 0$ $u(w \cdot [e-z-y]+\ell-\zeta(s)-q_{\ell',s}\ell', s)-u(w \cdot [e_{\min}+(1-\alpha)(e-z-y-e_{\min})], s) \neq \Delta$ because would be taking *more* consumption away from the l.h.s. than from the r.h.s. when the l.h.s. already has less. Therefore, $I^{(\ell',0),(0,1)}_{\ell,0,s}(\alpha,q,w)$ can have at most two elements.

(b) Suppose that $u(w \cdot e+\ell-\zeta(s)-q_{\ell',s}\ell', s)-u(w \cdot [e_{\min}+(1-\alpha)(e-e_{\min})], s) = \Delta < 0$. Then, given $\alpha > 0$, $u'(w \cdot [e-y]+\ell-\zeta(s)-q_{\ell',s}\ell', s) < u'(w \cdot [e_{\min}+(1-\alpha)(e-y-e_{\min})], s)$. By an argument analogous to the first part of (a) we can establish that $e-y \notin I^{(\ell',0),(0,1)}_{\ell,0,s}(\alpha,q,w)$. On the other hand, it is possible that there is a $z > 0$ such that $e+z \in I^{(\ell',0),(0,1)}_{\ell,0,s}(\alpha,q,w)$. If so, then $u'(w \cdot [e+x]+\ell-\zeta(s)-q_{\ell',s}\ell', s)dx = \int_0^z u'(w \cdot [e_{\min}+(1-\alpha)(e+x-e_{\min})], s)dx$. By an argument analogous to the second part of (a) we can establish that $w \cdot [e+z]+\ell-\zeta(s)-q_{\ell',s}\ell' > w \cdot [e_{\min}+(1-\alpha)(e+z-e_{\min})]$. Therefore, given $\alpha > 0$, $u(w \cdot [e+z+y]+\ell-\zeta(s)-q_{\ell',s}\ell', s)-u(w \cdot [e_{\min}+(1-\alpha)(e+z+y-e_{\min})], s) \neq \Delta$ because we would be giving more consumption to the l.h.s than to the r.h.s. when the l.h.s. already has more. Therefore, $I^{(\ell',0),(0,1)}_{\ell,0,s}(\alpha,q,w)$ can have at most two elements. ■

Define $E^{\ell',d}_{\ell,h,s}(\alpha,q,w) \equiv \{e \in E : \ell'^*_{\ell,h,s}(e;q,w) = \ell', d^*_{\ell,h,s}(e;q,w) = d\}$ to be set of E that returns (ℓ',d) as the optimal decision (when the household has ℓ,h,s). Define $ES^{\ell',d}_{\ell,h,s}(\alpha,q,w) \equiv \left\{e \in E : \left[\phi^{\ell',d}_{\ell,h,s}(e,q,w;\omega^*) - \max_{(\widetilde{\ell}',\widetilde{d}) \neq (\ell',d)} \phi^{\widetilde{\ell}',\widetilde{d}}_{\ell,h,s}(e,q,w;\omega^*)\right] > 0\right\}$ to be the set of e for which (ℓ',d) is strictly better than any other action.

Lemma A9. For $\alpha > 0$, $E^{\ell',d}_{\ell,h,s}(\alpha,q,w) \backslash ES^{\ell',d}_{\ell,h,s}(\alpha,q,w)$ is a finite set.

Proof. Observe that $\left\{E^{\ell',d}_{\ell,h,s}(\alpha,q,w) \backslash ES^{\ell',d}_{\ell,h,s}(\alpha,q,w)\right\} \subseteq \cup_{(\widetilde{\ell}',\widetilde{d}) \neq (\ell',d)} I^{(\ell',d),(\widetilde{\ell}',\widetilde{d})}_{\ell,h,s}(\alpha,q,w)$. Since the sets $I^{(\ell',d),(\widetilde{\ell}',\widetilde{d})}_{\ell,h,s}(\alpha,q,w)$ are finite by Lemma A8, the result follows. ■

Lemma A10. Let $Z \in 2^{\mathcal{L}} \times \mathcal{B}(E)$ and $(\ell_n, h_n, s_n, e_n, \alpha_n, q_n, w_n) \to (\ell, h, s, e, \alpha, q, w)$. If $\alpha > 0$ then for all but a finite set of (ℓ, h, s, e): (i) $\ell'^*_{\ell,h,s}(e_n;\alpha_n,q_n,w_n) \to \ell'^*_{\ell,h,s}(e;\alpha,q,w)$ and $d^*_{\ell,h,s}(e_n;\alpha_n,q_n,w_n) \to d^*_{\ell,h,s}(e;\alpha,q,w)$, and (ii) $\lim_{n\to\infty} G^*_{(\alpha_n,q_n,w_n)}(\ell_n,h_n,s_n,e_n,Z) = G^*_{(\alpha,q,w)}(\ell,h,s,e,Z)$.

Proof. Since $\mathcal{L}$ is finite, it follows that there is some η such that for all $n \geq \eta$, $(\ell_n, h_n, s_n) = (\ell, h, s)$. Without loss of generality, we simply consider the sequence $(e_n, \alpha_n, q_n, w_n) \to (e, \alpha, q, w)$. (i) Consider the set of efficiency levels $ES_{\ell,h,s}(\alpha, q, w)$ for which the household strictly prefers some action (ℓ', d) i.e., $ES_{\ell,h,s}(\alpha, q, w) = \left\{ e : e \in \cup_{(\ell',d)} ES^{\ell',d}_{\ell,h,s}(\alpha, q, w) \right\}$. For $e \in ES_{\ell,h,s}(\alpha, q, w)$ consider the sequence $\left\{ \ell'^{*}_{\ell,h,s}(e_n; \alpha_n, q_n, w_n),\ d^{*}_{\ell,h,s}(e_n; \alpha_n, q_n, w_n) \right\}$. The sequence lies in a compact and finite subset of R^2 so we can extract a subsequence $\{n_k\}$ converging to $(\overline{\ell}', \overline{d})$. Furthermore, there is an η such that for $n_k \geq \eta$, $(\ell'^{*}_{\ell,h,s}(e_{n_k}; \alpha_{n_k}, q_{n_k}, w_{n_k}),\ d^{*}_{\ell,h,s}(e_{n_k}; \alpha_{n_k}, q_{n_k}, w_{n_k})) = (\overline{\ell}', \overline{d})$. Therefore, for $n_k \geq \eta$, $\phi^{\overline{\ell}',\overline{d}}_{\ell,h,s}(e_{n_k}; \alpha_{n_k}, q_{n_k}, w_{n_k}, \omega^*) =$ $v^{*}_{\ell,h,s}(e_{n_k}; \alpha_{n_k}, q_{n_k}, w_{n_k})$. Taking limits of both sides and using continuity of ϕ and v^* established in Lemma A2 and Theorem 1 respectively, $\phi^{\overline{\ell}',\overline{d}}_{\ell,h,s}(e; \alpha, q, w, \omega^*) = v^{*}_{\ell,h,s}(e; \alpha, q, w)$. Since $e \in ES_{\ell,h,s}(\alpha, q, w)$, $(\overline{\ell}', \overline{d}) = (\ell'^{*}_{\ell,h,s}(e; \alpha, q, w), d^{*}_{\ell,h,s}(e; \alpha, q, w))$. Since the set of efficiency levels for which there is indifference can be expressed as $\cup_{(\ell',d)} \left\{ E^{\ell',d}_{\ell,h,s}(\alpha, q, w) \backslash ES^{\ell',d}_{\ell,h,s}(\alpha, q, w) \right\}$, by Lemma A9 this set is finite and the result follows. (ii) Follows from the definition of $G^{*}_{(\alpha,q,w)}(\ell, h, s, e, Z)$ and part (i). ■

The next step is to establish the weak convergence of the invariant distribution $\mu_{(\alpha,q,w)}$ w.r.t α, q and w. Theorem 12.13 of Stokey, Lucas, and Prescott (1989) provide sufficient conditions under which this holds. However, if the household is indifferent between two courses of action at (ℓ, h, s, e), the probability measure $G^{*}_{(\alpha,q,w)}((\ell, h, s, e_n), \cdot)$ need not converge weakly to $G^{*}_{(\alpha,q,w)}((\ell, h, s, e), \cdot)$ as $e_n \to e$ so condition (b) of the Theorem is not satisfied. To get around this problem, we use Theorem 12.13 to establish the weak convergence of an invariant distribution $\pi_{(\alpha,q,w)}(\ell, h, s)$ with the property that $\mu_{(\alpha,q,w)}(\ell, h, s, e) = \pi_{(\alpha,q,w)}(\ell, h, s)\Phi(e|s)$. Since $\Phi(e|s)$ is independent of (α, q, w), the weak convergence of $\mu_{(\alpha,q,w)}$ w.r.t (α, q, w) follows.

We begin by constructing a finite state Markov chain over the space (ℓ, h, s). Let

$$P^{*}_{(\alpha,q,w)}\left[(\ell, h, s), (\ell', h', s')\right] \equiv \int_E G^{*}_{(\alpha,q,w)}((\ell, h, s, e), \left(\ell', h', s', E\right))\Phi(e|s)de. \tag{40}$$

Then by (8), (6), (7), and the fact that $\int_E \Phi(e'|s')de' = 1 = \int_E \Phi(e|s)de$, we have

$$\begin{aligned} &P^{*}_{(\alpha,q,w)}\left[(\ell, h, s), (\ell', h', s')\right] \\ &= \left[\begin{array}{c} \rho \int_E \mathbf{1}_{\{\ell'^{*}_{\ell,h,s}(e;\alpha,q,w)=\ell'\}} H^{*}_{(\alpha,q,w)}(\ell, h, s, e, h')\Gamma(s, s')\Phi(e|s)de \\ +(1-\rho) \int_E \mathbf{1}_{\{(\ell',h')=(0,0)\}} \psi(s', de') \end{array} \right]. \end{aligned} \tag{41}$$

It is easy to verify that $P^{*}_{(\alpha,q,w)}$ is a transition matrix and therefore defines a Markov chain on the space (ℓ, h, s).

Lemma A11. $P^{*}_{(\alpha,q,w)}$ induces a unique invariant distribution $\pi_{(\alpha,q,w)}$ on $(\mathcal{L}, 2^{\mathcal{L}})$.

Proof. The proof follows by applying Theorem 11.4 in Stokey, Lucas, and Prescott (1989). Let $\widehat{s} \in S$ be such that $\psi(\widehat{s}, E) > 0$. Since newborns must be of *some* type, such an $\widehat{s}$ exists. Then $P^{*}_{(\alpha,q,w)}\left[(\ell, h, s), (0, 0, \widehat{s})\right] \geq (1-\rho)\psi(\widehat{s}, E) > 0$, $\forall \ell, h, s$. Therefore $\varepsilon =$ $\sum_{(\ell',h',s')} \left\{ \min_{(\ell,h,s)} P\left[(\ell, h, s), (\ell', h', s')\right] \right\} \geq (1-\rho)\psi(\widehat{s}, E) > 0$ which satisfies the requirement of Theorem 11.4 (for $N = 1$). ■

Lemma A12. If $(\alpha_n, q_n, w_n) \in A \times Q \times W$ is a sequence converging to $(\alpha, q, w) \in A \times Q \times W$ where $\alpha_n, \alpha > 0$, then the sequence $\pi_{(\alpha_n, q_n, w_n)}$ converges weakly to $\pi_{(\alpha, q, w)}$.

Proof. The proof follows by applying Theorem 12.13 in Stokey, Lucas, and Prescott (1989). Part a of the requirements follows since $\mathcal{L}$ is compact. Part b requires that $P^*_{(\alpha_n, q_n, w_n)}[(\ell_n, h_n, s_n), \cdot]$ converges weakly to $P^*_{(\alpha,q,w)}[(\ell, h, s), \cdot]$ as $(\ell_n, h_n, s_n, \alpha_n, q_n, w_n) \to (\ell, h, s, \alpha, q, w)$. By Theorem 12.3d of Stokey, Lucas, and Prescott (1989) it is sufficient to show that for any (ℓ', h', s'), $\lim_{n\to\infty} P^*_{(\alpha_n, q_n, w_n)}[(\ell_n, h_n, s_n), (\ell', h', s')] = P^*_{(\alpha,q,w)}[(\ell, h, s), (\ell', h', s')]$. Since $\mathcal{L}$ is finite, without loss of generality consider the sequence $(\alpha_n, q_n, w_n) \to (\alpha, q, w)$. But from (40), $\lim_n P^*(\ell', h', s', E))$ of e converges almost everywhere to the measurable function $G^*_{(\alpha,q,w)}((\ell, h, s, e), (\ell', h', s', E))$ of e. Since $G^*_{(\alpha_n, q_n, w_n)}((\ell, h, s, e), (\ell', h', s', E)) \leq 1$, requirement (b) follows from the Lebesgue Dominated Convergence Theorem (Theorem 7.10 in Stokey, Lucas, and Prescott (1989)). Part c requires that for each (α, q, w), $P^*_{(\alpha,q,w)}$ induce a unique invariant measure; this follows from Lemma A11.

Lemma A13. $\mu_{(\alpha,q,w)}(\ell, h, s, e) = \pi_{(\alpha,q,w)}(\ell, h, s)\Phi(e|s)$.

Proof. Define $m_{(\alpha,q,w)}(\ell, h, s)$ by $\mu_{(\alpha,q,w)}(\ell, h, s, e) \equiv m_{(\alpha,q,w)}(\ell, h, s)\Phi(e|s)$and let $Z' = \ell' \times h' \times s' \times J'$. Let

$$\mu_{(\alpha,q,w)}(\ell, h, s, e) \equiv m_{(\alpha,q,w)}(\ell, h, s)\Phi(e|s). \tag{42}$$

Then

$$\begin{aligned}\mu_{(\alpha,q,w)}(Z') &= (\Upsilon_{(q,w)}\mu_{(q,w)})(Z') \\ &= \int \left[\begin{array}{c} \rho \int_E \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha,q,w)=\ell'\}} H^*_{(\alpha,q,w)}(\ell, h, s, e, h')\Phi(de|s)\Gamma(s, s') \int_{J'} \Phi(e'|s')de' \\ +(1-\rho)\mathbf{1}_{\{(\ell',h')=(0,0)\}} \int_{J'} \psi(s', de') \end{array} \right] m_{(\alpha,q,w)}(d\ell, dh, ds)\end{aligned} \tag{43}$$

where the first equality follows as a consequence of μ^* being a fixed point and the second equality follows from the definitions in (9), (42), and from recognizing $\int_E \Phi(de|s) = 1$.

Letting $J' = E$ in (43)

$$\begin{aligned}&\mu_{(\alpha,q,w)}(\ell', h', s', E) \\ &= \int \left[\begin{array}{c} \left\{ \rho \int_E \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha,q,w)=\ell'\}} H^*_{(\alpha,q,w)}(\ell, h, s, e, h')\Phi(de|s)\Gamma(s, s') \right\} \\ + \left\{ (1-\rho) \int_E \mathbf{1}_{\{(\ell',h')=(0,0)\}} \psi(s', de') \right\} \end{array} \right] m_{(\alpha,q,w)}(d\ell, dh, ds) \\ &= \int P^*_{(\alpha,q,w)}\left[(\ell, h, s), (\ell', h', s')\right] m_{(\alpha,q,w)}(d\ell, dh, ds)\end{aligned}$$

where the first equality uses $\int_E \Phi(e'|s')de' = 1$ and the second follows by definition (41).

But by (42)

$$\mu_{(\alpha,q,w)}(\ell', h', s', E) \equiv m_{(\alpha,q,w)}(\ell', h', s') \int_E \Phi(e'|s')de' = m_{(\alpha,q,w)}(\ell', h', s').$$

Therefore,

$$m_{(\alpha,q,w)}(\ell',h',s') = \int P^*_{(\alpha,q,w)}\left[(\ell,h,s),(\ell',h',s')\right] m_{(\alpha,q,w)}(d\ell,dh,ds).$$

which implies that $m_{(\alpha,q,w)}(\ell',h',s')$ is the fixed point of the Markov chain whose transition function is $P^*_{(\alpha,q,w)}$. Hence $m_{(\alpha,q,w)}(\ell,h,s) = \pi_{(\alpha,q,w)}(\ell,h,s)$ and the result follows. ■

Lemma A14. If $(\alpha_n,q_n,w_n) \in A \times Q \times W$ is a sequence converging to $(\alpha,q,w) \in A \times Q \times W$ where $\alpha_n, \alpha > 0$, then the sequence $\mu_{(\alpha_n,q_n,w_n)}$ converges weakly to $\mu_{(\alpha,q,w)}$.

Proof. Since $\Phi(e|s)$ is independent of (α,q,w), the result follows from Lemmas A12 and A13. ■

Lemma A15. Let $\alpha > 0$, $K_{(\alpha,q,w)} \equiv \sum_{(\ell',s)\in L\times S} \ell' q_{\ell',s} \int \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha,q,w)=\ell'\}}\mu_{(\alpha,q,w)}(d\ell,dh,s,de)$, $N_{(\alpha,q,w)} \equiv \int e d\mu_{(\alpha,q,w)}$, and $p_{(\alpha,q,w)}(\ell',s) \equiv \int d^*_{\ell',0,s'}(e';\alpha,q,w)\Phi(e'|s')\Gamma(s;ds')de'$. Then (i) $K_{(\alpha,q,w)}$, (ii) $N_{(\alpha,q,w)}$, (iii) $p_{(\alpha,q,w)}(\ell',s)$ are continuous with respect to (α,q,w).

Proof. To prove (i) note that by Lemma A13, $\int_{L\times H\times E} \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha_n,q_n,w_n)=\ell'\}}\mu_{(\alpha_n,q_n,w_n)}(d\ell,dh,s,de) = \sum_{\ell,h}\int_E \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha_n,q_n,w_n)=\ell'\}}\Phi(de|s)\pi_{(\alpha_n,q_n,w_n)}(\ell,h,s)$. By Lemma A12, $\lim_n \pi_{(\alpha_n,q_n,w_n)}(\ell,h,s) = \pi_{(\alpha,q,w)}(\ell,h,s)$. By Lemma A10, $\mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha_n,q_n,w_n)=\ell'\}} \to \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha,q,w)=\ell'\}}$ except possibly for a finite number of points in E. By the Lebesgue Dominated Convergence Theorem (Stokey, Lucas, and Prescott (1989) Theorem 7.10), $\lim_n \int_E \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha_n,q_n,w_n)=\ell'\}}\Phi(de|s) = \int_E \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha,q,w)=\ell'\}}\Phi(de|s)$. Then, since $K_{(\alpha_n,q_n,w_n)}$ is the sum of a finite number of products each of which converge, the sum converges as well. To prove (ii) simply apply Lemma A14. To prove (iii) note that by Lemma A10 $d^*_{\ell,h,s}(e;\alpha_n,q_n,w_n) \to d^*_{\ell,h,s}(e;\alpha,q,w)$ except possibly for a finite number of points in E. By LDCT, $\lim_n \int_{E\times S} d^*_{\ell',0,s'}(e';\alpha_n,q_n,w_n)\Phi(e'|s')\Gamma(s;ds')de' = \int_{E\times S} d^*_{\ell',0,s'}(e';\alpha,q,w)\Phi(e'|s')\Gamma(s;ds')de'$. ■

With these lemmas in hand we are ready to prove the existence of a steady-state equilibrium for the case where $\alpha > 0$. Once this is done, the existence of equilibrium for the $\alpha = 0$ case will be accomplished via a limiting argument. Define the vector-valued function whose fixed point gives us a candidate equilibrium price vector. At this point, we need to be explicit about the upper and lower bounds of the sets W and the upper bound of the set Q.

Assumption A2. Assume that $q_{\max} = \rho\left(1 + F_K(\ell_{\max}, e_{\min}) - \delta\right)^{-1}$, $w_{\min} = b$ and $w_{\max} = F_N(\ell_{\max}, e_{\min})$.

Note that our earlier assumption that $\ell_{\max}$ is such that $F_K(\ell_{\max}, e_{\min}) > \delta$ guarantees that $q_{\max}$ is strictly positive.

Let $\Omega^\alpha : Q \times W \to R^{N_L\cdot N_S+1}$ be given by [37]

$$\Omega^\alpha(q,w) \equiv \begin{bmatrix} \Omega^\alpha_{\ell'\geq 0,s}(q,w) \\ \Omega^\alpha_{\ell'<0,s}(q,w) \\ \Omega^\alpha_w(q,w) \end{bmatrix} \tag{44}$$

where

$$\Omega^\alpha_{\ell'\geq 0,s}(q,w) = \begin{cases} \rho\left(1 + F_K\left(K_{(\alpha,q,w)}, N_{(\alpha,q,w)}\right) - \delta\right)^{-1} & \text{for } K_{(\alpha,q,w)} > 0 \\ 0 & \text{for } K_{(\alpha,q,w)} \leq 0 \end{cases},$$

[37] $\overline{w}$ can always be made to exceed $\underline{w}$ by placing assumptions on the production technology.

$$\Omega^{\alpha}_{\ell'<0,s}(q,w) = \begin{cases} \rho\left(1-p_{(\alpha,q,w)}(\ell',s)\right)\left(1+F_K\left(K_{(\alpha,q,w)},N_{(\alpha,q,w)}\right)-\delta\right)^{-1} & \text{for } K_{(\alpha,q,w)}>0 \\ 0 & \text{for } K_{(\alpha,q,w)}\le 0 \end{cases},$$

and

$$\Omega^{\alpha}_{w}(q,w) = \begin{cases} F_N\left(K_{(\alpha,q,w)},N_{(\alpha,q,w)}\right) & \text{for } K_{(\alpha,q,w)}>0 \\ F_N\left(0,N_{(\alpha,q,w)}\right) & \text{for } K_{(\alpha,q,w)}\le 0 \end{cases}.$$

A fixed point of this function is an equilibrium price vector provided an $m^* \geq 1$ can be found for which condition (v) in Definition 2 is satisfied.

Lemma A16. For $\alpha > 0$, there exists $(q^*, w^*) \in Q \times W$ such that $(q^*, w^*) = \Omega^{\alpha}(q^*, w^*)$.

Proof. The set $Q \times W$ is compact. By Assumption A2, $\Omega^{\alpha}(q,w) \subset Q \times W$. To see this, observe that by Assumption 1(iii) $F_K(\ell_{\max}, e_{\min})$ is the lowest marginal product of capital possible in this economy and therefore, $q_{\max}$ is the highest price on deposits possible. The lower bound on wages is the lower bound on the marginal product of labor in Assumption 1(v) and the upper bound on wages is, by Assumption 1(iii) again, the highest marginal product of labor possible in this economy.

Next we need to establish that $\Omega^{\alpha}(q,w)$ is continuous in q and w. Note that $N_{(\alpha,q,w)}$ is always strictly positive since it is bounded below by $e_{\min}$. First, consider (α,q,w) such that $K_{(\alpha,q,w)} > 0$ and let $(q_n, w_n) \to (q,w)$. By Lemma A15 and continuity of F_K and F_N, it follows that $\Omega^{\alpha}(q_n,w_n) \to \Omega^{\alpha}(q,w)$. Second, consider (α,q,w) such that $K_{(\alpha,q,w)} < 0$. Then for any $\varepsilon > 0$ there exists η such that for all $n \geq \eta$,

$$\rho\left(1+F_K\left(K_{(\alpha,q_n,w_n)},N_{(\alpha,q_n,w_n)}\right)-\delta\right)^{-1} \leq \rho\left(1+F_K\left(K_{(\alpha,q_n,w_n)},e_{\min}\right)-\delta\right)^{-1} < \varepsilon$$

Therefore, since ε can be made arbitrarily small, $\Omega^{\alpha}_{\ell'<0,s}(q_n,w_n) \to 0 = \Omega^{\alpha}_{\ell'<0,s}(q,w)$ and $\Omega^{\alpha}_{\ell'\geq 0,s}(q_n,w_n) \to 0 = \Omega^{\alpha}_{\ell'\geq 0,s}(q,w)$. Furthermore, there exists η such that for all $n \geq \eta$, $\Omega^{\alpha}_{w}(q_n,w_n) = F_N(0, N_{(\alpha,q_n,w_n)})$. Therefore, by Lemma A15 and continuity of F_N, it follows that $\Omega^{\alpha}_{w}(q_n,w_n) \to \Omega^{\alpha}_{w}(q,w)$. Third, consider (α,q,w) such that $K_{(\alpha,q,w)} = 0$. Then $\Omega^{\alpha}_{\ell'<0,s}(q_n,w_n) \to 0 = \Omega^{\alpha}_{\ell'<0,s}(q,w)$ and $\Omega^{\alpha}_{\ell'\geq 0,s}(q_n,w_n) \to 0 = \Omega^{\alpha}_{\ell'\geq 0,s}(q,w)$ by an argument similar to the above case where $K_{(\alpha,q,w)} < 0$. Furthermore, for any $\varepsilon > 0$ there exists η such that for all $n \geq \eta$, $K_{(\alpha,q_n,w_n)} < \varepsilon$ and hence

$$F_N\left(\varepsilon, N_{(\alpha,q_n,w_n)}\right) \geq \Omega^{\alpha}_{w}(q_n,w_n) \geq F_N\left(0, N_{(\alpha,q_n,w_n)}\right).$$

Therefore, by Lemma A15 and continuity of F_N, it follows that

$$F_N\left(\varepsilon, N_{(\alpha,q,w)}\right) \geq \lim_{n\to\infty} \Omega^{\alpha}_{w}(q_n,w_n) \geq F_N\left(0, N_{(\alpha,q,w)}\right).$$

Since ε can be arbitrarily small, it follows that $\Omega^{\alpha}_{w}(q_n,w_n) \to F_N\left(0, N_{(\alpha,q,w)}\right) = \Omega^{\alpha}_{w}(q,w)$.

The result follows from Brouwer's Fixed Point Theorem. ■

Lemma A17. $\ell_{\max} \geq K_{(\alpha,q^*,w^*)} > 0$.

Proof. If $K_{(\alpha,q^*,w^*)} = 0$, then $q^*_{\ell',s} = 0$ for all ℓ' by (44). Hence, the optimal decision for households with $\ell \geq 0$ is to choose $\ell' = \ell_{\max}$ and the optimal decision for households with $\ell < 0$ is either to choose default today and choose $\ell_{\max}$ tomorrow or to pay back and choose $\ell_{\max}$

today. Therefore, within at most one period the invariant distribution will have all its mass on points with $(\ell_{\max}, h, s, e)$. Hence $K_{(\alpha,q^*,w^*)} = \ell_{\max}$. But this implies that $\Omega^{\alpha}_{\ell' \geq 0,s}(0, w^*) = \rho\left(1 + F_K\left(\ell_{\max}, N_{(0,w^*)}\right) - \delta\right)^{-1} > 0$, which yields a contradiction. Hence $K_{(\alpha,q^*,w^*)} > 0$. Since the asset holding of each household is bounded above by $\ell_{\max}$, it follows that $\ell_{\max} \geq K_{(\alpha,q^*,w^*)}$. ■

Lemma A18. There exists a steady-state competitive equilibrium with $\alpha > 0$.

Proof. For $\alpha > 0$, we know there exists $(q^*, w^*) = \Omega^{\alpha}(q^*, w^*)$ by Lemma A16. Then provided (v) is satisfied, all the conditions for a competitive steady-state equilibrium in Definition 2 are satisfied by construction of Ω^{α}. Observe that if the hospital sector has strictly positive revenue in the steady state, that is

$$\int \left[(1 - d^*_{\ell,h,s}(e; \alpha, q^*, w^*))\zeta(s) + d^*_{\ell,h,s}(e; \alpha, q^*, w^*) \max\{\ell, 0\}\right] d\mu^* > 0, \tag{45}$$

then we can always choose $m^* \geq 1$ to satisfy condition (v). Since we have assumed that every surviving household has a strictly positive probability of experiencing a medical expense and $K_{(\alpha,q^*,w^*)} > 0$ by Lemma A17, (45) is satisfied. ■

We now turn to the proof of existence of equilibrium when $\alpha = 0$. This proof is constructive. We take a sequence of equilibrium steady states for strictly positive but vanishing cost α and from this sequence construct equilibrium prices and decision rules that work for the $\alpha = 0$ case.

To do this, we will need the following definitions. For a given pair of optimal decision rules $(\ell'^*_{\ell,h,s}(e; \alpha, q, w), d^*_{\ell,h,s}(e; \alpha, q, w))$ define the (optimal) probability of choosing (ℓ', d) given (ℓ, h, s) and (α, q, w) as $x^{(\ell',d)}_{\ell,h,s}(\alpha, q, w) \equiv \int_E \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha,q,w)=\ell', d^*_{\ell,h,s}(e;\alpha,q,w)=d\}} \Phi(de|s)$. Further, define the $2 \cdot N_L \cdot N_{\mathcal{L}}$-element vector of choice probabilities by $x(\alpha, q, w) \equiv \{x^{(\ell',d)}_{\ell,h,s}(\alpha, q, w)\ \forall\ (\ell, h, s) \in \mathcal{L}$ and $(\ell', d) \in L \times \{0, 1\}\}$.

Let a sequence of costs $\alpha_n \to 0$ with $\alpha_n > 0$. For each α_n, let $(q^*_n, w^*_n) \in Q \times W$ be an equilibrium price vector whose existence is guaranteed by Lemma A18. Since $Q \times W$ is compact, we can extract a subsequence $(q^*_{n_k}, w^*_{n_k})$ converging to $(\overline{q}, \overline{w}) \in Q \times W$. Let the corresponding sequence of measurable optimal decision rules and the sequence of optimal choice probability vectors be $\ell'^*_{\ell,h,s}(e; \alpha_{n_k}, q^*_{n_k}, w^*_{n_k})$, $d^*_{\ell,h,s}(e; \alpha_{n_k}, q^*_{n_k}, w^*_{n_k})$ and $x(\alpha_{n_k}, q^*_{n_k}, w^*_{n_k})$. Since each term in the sequence $\{x(\alpha_{n_k}, q^*_{n_k}, w^*_{n_k})\}$ is in $[0, 1]^{2 \cdot N_L \cdot N_{\mathcal{L}}}$ we can extract another convergent subsequence converging to some $\overline{x} \in [0, 1]^{2 \cdot N_L \cdot N_{\mathcal{L}}}$. Denote this subsequence of $\{n_k\}$ by $\{m\}$.

Thus, we have a sequence $\{\alpha_m, q^*_m, w^*_m\}$, where q^*_m and w^*_m are equilibrium prices, converging to $(0, \overline{q}, \overline{w})$ and a corresponding sequence of optimal decision rules with choice probabilities $x^{(\ell',d)}_{\ell,h,s}(\alpha_m, q^*_m, w^*_m)$ converging to $\bar{x}^{(\ell',d)}_{\ell,h,s})$. In Lemma 20 we construct, using information on choice probabilities along the sequence, measurable decision rules that are optimal for $(0, \overline{q}, \overline{w})$ and which deliver the limiting choice probabilities $\overline{x}$.

Recall that the set of e for which (ℓ, d) is the optimal action given (α, q, w) is denoted $E^{(\ell',d)}_{\ell,h,s}(\alpha, q, w)$ and the set of e for which (ℓ', d) is the *strictly* optimal action given (α, q, w) is denoted $ES^{(\ell',d)}_{\ell,h,s}(\alpha, q, w)$. Let $I^{(\ell',d)}_{\ell,h,s}(\alpha, q, w)$ be the set $E^{(\ell',d)}_{\ell,h,s}(\alpha, q, w) \backslash (ES^{(\ell',d)}_{\ell,h,s}(\alpha, q, w))$, i.e., the set of e for which (ℓ', d) is an optimal action and for which there is also some other action that is equally good. Further, let

$ED^{(\ell',d)}_{\ell,h,s}(\alpha,q,w)$ to be the set $E\backslash\left(ES^{(\ell',d)}_{\ell,h,s}(\alpha,q,w)\cup I^{(\ell',d)}_{\ell,h,s}(\alpha,q,w)\right)$, i.e., the set of e for which the action (ℓ',d) is strictly dominated by some other action.

The next lemma bounds the measure of the sets (why?) $ES^{(\ell,d)}_{\ell,h,s}(0,\bar{q},\bar{w})$ and $ED^{(\ell',d)}_{\ell,h,s}(0,\bar{q},\bar{w})$. For convenience, denote $ES^{(\ell,d)}_{\ell,h,s}(0,\bar{q},\bar{w})$ by $\overline{ES}^{(\ell,d)}$, $ED^{(\ell,d)}_{\ell,h,s}(0,\bar{q},\bar{w})$ by $\overline{ED}^{(\ell,d)}$, $ES^{(\ell,d)}_{\ell,h,s}(\alpha_m,q^*_m,w^*_m)$ by $ES^{(\ell,d)}_m$, $E^{(\ell,d)}_{\ell,h,s}(\alpha_m,q^*_m,w^*_m)$ by $E^{(\ell,d)}_m$ and $I^{(\ell,d)}_{\ell,h,s}(0,\bar{q},\bar{w})$ by $\overline{I}^{(\ell,d)}$. And denote $\int \mathbf{1}_{\{e\in A\}}(e)\Phi(de|s)$ by $\Phi_s(A)$. Then we have:

Lemma A19. For all $(\ell,h,s)\in\mathcal{L}$ the following hold (i) $\Phi_s(\overline{ES}^{(\ell',d)})\le\bar{x}^{(\ell',d)}_{\ell,h,s}$, (ii) $\Phi_s(\overline{ED}^{(\ell',d)})\le\left[1-\bar{x}^{(\ell',d)}_{\ell,h,s}\right]$, and (iii) $\sum_{\ell'\in L}\Phi_s(\overline{ES}^{(\ell',0)})+\Phi_s(\overline{ES}^{(0,1)})+\Phi_s(\overline{I}^{(0,1)})=1=\sum_{\ell'\in L}\bar{x}^{(\ell',0)}_{\ell,h,s}+\bar{x}^{(0,1)}_{\ell,h,s}$.

Proof. See the supplementary material section of this article on the *Econometrica* website.

Lemma A20. For all $(\ell,h,s)\in\mathcal{L}$ there exist measurable functions $c_{\ell,h,s}(e)$, $\ell'_{\ell,h,s}(e)$, and $d_{\ell,h,s}(e)$ for which the implied choice probabilities $\int_E \mathbf{1}_{\{\ell'_{\ell,h,s}(e)=\ell',d_{\ell,h,s}(e)=d\}}\Phi(de|s)=\overline{x}^{(\ell',d)}_{(\ell,h,s)}$ and the triplet $\left(c_{\ell,h,s}(e),\ell'_{\ell,h,s}(e),d_{\ell,h,s}(e)\right)\in\chi_{\ell,h,s}(e;0;\overline{q},\overline{w})$.

Proof. See the supplementary material section of this article on the *Econometrica* website.

We now establish the analogs of Lemma A12, A14, and A15 for the sequence $\{\alpha_m,q^*_m,w^*_m\}$ converging to $(0,\overline{q},\overline{w})$.

Lemma A21. Let $\overline{\pi}_{(0,\overline{q},\overline{w})}$ be the invariant distribution of the Markov chain $\overline{P}$ defined by the decision rules $(\ell'_{\ell,h,s}(e),d_{\ell,h,s}(e))$. Then the sequence $\pi_{(\alpha_m,q^*_m,w^*_m)}$ converges weakly to $\overline{\pi}_{(0,\overline{q},\overline{w})}$.

Proof. See the supplementary material section of this article on the *Econometrica* website.

Lemma A22. Let $\overline{\mu}_{(0,\overline{q},\overline{w})}$ be the invariant distribution corresponding to the decision rules $\ell'_{\ell,h,s}(e)$ and $d_{\ell,h,s}(e)$. Then, the sequence $\mu_{(\alpha_m,q^*_m,w^*_m)}$ converges weakly to $\overline{\mu}_{(0,\overline{q},\overline{w})}$.

Proof. Since $\Phi(e|s)$ is independent of (α,q,w), the result follows from Lemmas A13 and A21. ■

Lemma A23. Let $K_{(0,\overline{q},\overline{w})}\equiv\sum_{(\ell',s)\in L\times S}\ell'\overline{q}_{\ell',s}\int\mathbf{1}_{\{(\ell'_{\ell,h,s}(e)=\ell'\}}\overline{\mu}_{(0,\overline{q},\overline{w})}(d\ell,dh,s,de)$, $N_{(0,\overline{q},\overline{w})}\equiv\int ed\overline{\mu}_{(0,\overline{q},\overline{w})}$, and $p_{(0,\overline{q},\overline{w})}(\ell',s)\equiv\int d_{\ell',0,s'}(e')\Phi(e'|s')\Gamma(s;ds')de'$. Then (i) $\lim_m K(\alpha_m,q^*_m,w^*_m)=K_{(0,\overline{q},\overline{w})}$, (ii) $\lim_m N(\alpha_m,q^*_m,w^*_m)=N_{(0,\overline{q},\overline{w})}$, and (iii) $\lim_m p_{(\alpha_m,q^*_m,w^*_m)}(\ell',s)=p_{(0,\overline{q},\overline{w})}(\ell',s)$.

Proof. See the supplementary material section of this article on the *Econometrica* website.

Since the choice probabilities along the sequence satisfy all equilibrium conditions and the constructed decision rules imply the limiting choice probabilities, it is straightforward to establish that all equilibrium conditions are satisfied by the constructed decision rules as well. Therefore the pair $(\bar{q},\bar{w})$ is an equilibrium price vector when $\alpha=0$.

Theorem 5 (Existence). A steady-state competitive equilibrium exists.

Proof. For the sequence $\{q^*_m,w^*_m\}$ converging $(\overline{q},\overline{w})$, let $\left(\ell'_{\ell,h,s}(e),d_{\ell,h,s}(e),c_{\ell,h,s}(e)\right)$ be the decision rules whose existence is guaranteed in Lemma A20. Using $\overline{q}, \overline{w}, \ell'_{\ell,h,s}(e), d_{\ell,h,s}(e), c_{\ell,h,s}(e)$ we will construct a collection

$$\{\overline{q}, \overline{w}, \overline{\ell}'_{\ell,h,s}(e;\overline{q},\overline{w}), \overline{d}_{\ell,h,s}(e;\overline{q},\overline{w}), \overline{c}_{\ell,h,s}(e;\overline{q},\overline{w}), \overline{r}, \overline{i}, \overline{p}, \overline{m}, \overline{N}, \overline{K}, \overline{a}, \overline{B}, \overline{\mu}\}$$

that satisfies all the conditions of steady-state equilibrium in Definition 2.

Given $\overline{q}, \overline{w}$, the conditions we satisfy by construction are:

(i) $\overline{c}_{\ell,h,s}(e;\overline{q},\overline{w}) = c_{\ell,h,s}(e)$, $\overline{\ell}'_{\ell,h,s}(e;\overline{q},\overline{w}) = \ell'_{\ell,h,s}(e)$, and $\overline{d}_{\ell,h,s}(e;\overline{q},\overline{w}) = d_{\ell,h,s}(e)$. By Lemma A19 these decision rules solve the household's optimization problem for $\alpha = 0, q = \overline{q}$,and $w = \overline{w}$.

(x) $\overline{\mu} = \mu_{(\overline{q},\overline{w})} = \Upsilon_{(\overline{q},\overline{w})}\mu_{(\overline{q},\overline{w})}$ (where Υ is based on $(\overline{\ell}'_{\ell,h,s}(e;\overline{q},\overline{w}), \overline{d}_{\ell,h,s}(e;\overline{q},\overline{w}))$;

(vi) $\overline{N} = \int e d\overline{\mu}$;

(vii) $\overline{a}_{\ell',s} = \int \mathbf{1}_{(}\overline{\ell}'_{\ell,h,s}(e;\overline{q},\overline{w}) = \ell'\}\mu_{(\overline{q},\overline{w})}(d\ell, dh, s, de)$;

(viii) $\overline{K} = \sum_{(\ell',s)\in L\times S} \overline{q}_{\ell',s}\ell' \int \mathbf{1}_{(\overline{\ell}'_{\ell,h,s}(e;\overline{q},\overline{w})=\ell'\}}\mu_{(\overline{q},\overline{w})}(d\ell, dh, s, de)$;

(v) $\overline{m} = \left[\int \left[(1 - \overline{d}_{\ell,h,s}(e;\overline{q},\overline{w}))\zeta(s) + \ \overline{d}_{\ell,h,s}(e;\overline{q},\overline{w}) \max\{\ell, 0\}\right] d\mu_{(\overline{q},\overline{w})}\right]^{-1} \cdot \int \zeta(s) d\mu_{(\overline{q},\overline{w})}$;

(iib) $\overline{r} = \frac{\partial F(\overline{K},\overline{N})}{\partial \overline{K}}$.

(iv) $\overline{p}_{\ell',s} = \int \overline{d}_{\ell',0,s'}(e';\overline{q},\overline{w})\Phi(e'|s')\Gamma(s;ds')de'$ for $\ell' < 0$ and $\overline{p}_{\ell',s} = 0$ for $\ell' \geq 0$.

The conditions we must verify are:

(iia)

$$\overline{w} = \frac{\partial F(\overline{K},\overline{N})}{\partial \overline{N}}.$$

Since (α_m, q^*_m, w^*_m) are equilibrium prices and $K_{(\alpha_m,q^*_m,w^*_m)} > 0$ by Lemma A17, then for all m :

$$f(w^*_m, K_{(\alpha_m,q^*_m,w^*_m)}, N_{(\alpha_m,q^*_m,w^*_m)}) \equiv w^*_m - F_N\left(K_{(\alpha_m,q^*_m,w^*_m)}, N_{(\alpha_m,q^*_m,w^*_m)}\right) = 0.$$

Observe that f is continuous in all arguments because F_N is continuous. Therefore

$$\lim_{m\to\infty} f(w^*_m, K_{(\alpha_m,q^*_m,w^*_m)}, N_{(\alpha_m,q^*_m,w^*_m)}) = \overline{w} - F_N\left(\overline{K},\overline{N}\right) = 0$$

since by Lemma A23 we know $\lim_{m\to\infty}\left(K_{(\alpha_m,q^*_m,w^*_m)}, N_{(\alpha_m,q^*_m,w^*_m)}\right) = \left(K_{(0,\overline{q},\overline{w})}, N_{(0,\overline{q},\overline{w})}\right) = \left(\overline{K},\overline{N}\right)$ by construction.

(iii)

$$\overline{q}_{\ell',s} = \frac{\rho(1 - \overline{p}_{\ell',s})}{1 + \overline{r} - \delta}.$$

Since (α_m, q^*_m, w^*_m) are equilibrium prices and $K_{(\alpha_m,q^*_m,w^*_m)} > 0$ by Lemma A17, then for all m and $\ell' \geq 0$:

$$f\left(\left(q^*_{\ell'\geq 0,s}\right)_m, K_{(\alpha_m,q^*_m,w^*_m)}, N_{(\alpha_m,q^*_m,w^*_m)}\right)$$
$$\equiv \left(q^*_{\ell'\geq 0,s}\right)_m - \rho\left(1 + F_K\left(K_{(\alpha_m,q^*_m,w^*_m)}, N_{(\alpha_m,q^*_m,w^*_m)}\right) - \delta\right)^{-1} = 0.$$

Observe that f is continuous in all arguments because F_K is continuous. Therefore, by Lemma A23 again

$$\lim_{m\to\infty} f\left(\left(q^*_{\ell'\geq 0,s}\right)_m, K_{(\alpha_m,q^*_m,w^*_m)}, N_{(\alpha_m,q^*_m,w^*_m)}\right)$$
$$= \overline{q}_{\ell'\geq 0,s} - \rho\left(1+F_K\left(\overline{K},\overline{N}\right)-\delta\right)^{-1} = 0.$$

Similarly,

$$f\left(\left(q^*_{\ell'<0,s}\right)_m, K_{(\alpha_m,q^*_m,w^*_m)}, N_{(\alpha_m,q^*_m,w^*_m)}\right)$$
$$\equiv \left(q^*_{\ell'<0,s}\right)_m - \frac{\rho\left(1-\int d^*_{\ell',0,s'}(e',\alpha_m,q^*_m,w^*_m)\Phi(de'|s')\Gamma(s;ds')\right)}{1+F_K\left(K_{(\alpha_m,q^*_m,w^*_m)},N_{(\alpha_m,q^*_m,w^*_m)}\right)-\delta} = 0.$$

By the choice of $\overline{d}_{\ell,h,s}(e;\overline{q},\overline{w})$ and Lemma A20,

$$\lim_{m\to\infty}\int d^*_{\ell',0,s'}(e',\alpha_m,q^*_m,w^*_m)\Phi(de'|s') = \int \overline{d}_{\ell,h,s}(e;\overline{q},\overline{w})\Phi(de'|s').$$

Therefore by Lemma A23

$$\lim_{m\to\infty} f\left(\left(q^*_{\ell'<0,s}\right)_m, K_{(\alpha_m,q^*_m,w^*_m)}, N_{(\alpha_m,q^*_m,w^*_{n_k})}\right)$$
$$= \overline{q}_{\ell'<0,s} - \frac{\rho\left(1-\int \overline{d}_{\ell,h,s}(e;\overline{q},\overline{w})\Phi(de'|s')\Gamma(s;ds')\right)}{1+F_K\left(\overline{K},\overline{N}\right)-\delta} = 0.$$

Finally, since the collection satisfies all conditions for an equilibrium except condition (ixA), it follows from Lemma A7 that (ixA) is satisfied as well. ■

Theorem 6 (Characterization of Equilibrium Prices) In any steady-state competitive equilibrium: (i) $q^*_{\ell',s} = \rho(1+r^*-\delta)^{-1}$ for $\ell' \geq 0$; (ii) if the grid for L is sufficiently fine, there exists $\ell^0 < 0$ such that $q^*_{\ell^0,s} = \rho(1+r^*-\delta)^{-1}$; (iii) if the set of efficiency levels for which a household is indifferent between defaulting and not defaulting is of measure zero, $0 > \ell^1 > \ell^2$ implies $q^*_{\ell^1,s} \geq q^*_{\ell^2,s}$; (iv) when $\ell_{\min} \leq -\left[e_{\max}\cdot w_{\max}\right]\left[(1+r^*-\delta)/(1-\rho+r^*-\delta)\right]$, $q^*_{\ell_{\min},s} = 0$.

Proof. (i) Follows from condition (iii) in the definition of competitive equilibrium; (ii) Let the grid be fine enough so that there is at least one $\ell^0 < 0$ for which $w_{\min}\cdot e_{\min} + \ell^0 > 0$. For a household, the utility from defaulting on a loan of size ℓ^0 can be expressed as:

$$u(e\cdot w,s) \; + \; \beta\rho\int u\left(c^*_{0,1,s'}(e';q^*,w^*),s'\right)\Phi(de'|s')\Gamma(s,ds')$$
$$+ \; (\beta\rho)^2\int\left[\lambda\omega^*_{\ell'^*_{0,1,s'}(e';q^*,w^*),1,s'}(q^*,w^*) + (1-\lambda)\omega^*_{\ell'^*_{0,1,s'}(e';q^*,w^*),0,s'}(q^*,w^*)\right]\Phi(de'|s')\Gamma(s,ds')$$

Since $w_{\min}\cdot e_{\min} + \ell^0 > 0$, an alternative to not defaulting is to pay off the loan, consume the remaining endowment, and in the following period set consumption equal to

$$c^*_{0,1,s'}(e';q^*,w^*) + \gamma e'.$$

The utility from this course of action is:

$$u(e\cdot w+\ell_0,s)\ +\ \beta\rho\int u\left(c^*_{0,1,s'}(e';q^*,w^*)+\gamma e',s'\right)\Phi(de'|s')\Gamma(s,ds')$$
$$+\ (\beta\rho)^2\int \omega^*_{\ell'^*_{0,1,s'}(e';q^*,w^*),0,s'}(q^*,w^*)\Phi(de'|s')\Gamma(s,ds').$$

In view of (28), the utility-gain from not defaulting must be at least as large as

$$u(e\cdot w+\ell_0,s)-u(e\cdot w,s)$$
$$+\beta\rho\int\left[u\left(c^*_{0,1,s'}(e';q^*,w^*)+\gamma e',s'\right)-u\left(c^*_{0,1,s'}(e';q^*,w^*),s'\right)\right]\Phi(de'|s')\Gamma(s,ds'). \tag{46}$$

Since consumption is bounded above by $e_{\max}\cdot w_{\max}+\ell_{\max}-\ell_{\min}$ and the $u(\cdot,s)$ is strictly concave for each s , the integral in the above expression is bounded *below* by

$$\int[u(e_{\max}\cdot w_{\max}+\ell_{\max}-\ell_{\min}+\gamma e',s')-u(e_{\max}\cdot w_{\max}+\ell_{\max}-\ell_{\min},s')]\Phi(de'|s')\Gamma(s,ds').$$

Notice that the above integral is strictly positive and independent of the fineness of the grid for L. Therefore, since $u(\cdot,s)$ is continuous, the expression in (46) will be strictly positive if $\ell^0$is sufficiently close to zero. Hence, for a sufficiently fine grid there exists an $\ell^0<0$ for which defaulting is not optimal and $q^*_{\ell^0,s}=\rho(1+r^*-\delta)^{-1}$; (iii) If the set of efficiency levels for which a household is indifferent between defaulting and not defaulting is of measure zero, by Theorem 4 (the maximal default set expands with liabilities) it follows that $d^*_{\ell^2,0,s}(e,q^*,w^*)\geq d^*_{\ell^1,0,s}(e,q^*,w^*)$ for all e except, possibly, for those in a set of $\Phi(e|s)$-measure zero. Therefore

$$\int d^*_{\ell^2,0,s}(e,q^*,w^*)\Phi(de|s)=p_{\ell^2,s}\geq p_{\ell^1,s}=\int d^*_{\ell^1,0,s}(e,q^*,w^*)\Phi(de|s)$$

and the result follows; (iv) Set $\ell_{\min}\leq-\left[e_{\max}\cdot w_{\max}\right]\left[(1+r^*-\delta)/(1-\rho+r^*-\delta)\right]$. If a household has characteristics s, loan $\ell_{\min}$ and endowment $e\cdot w$ then its consumption, conditional on not defaulting, is bounded above by $e\cdot w+\ell_{\min}-\zeta(s)+\max_{\ell'\in L}\{-q^*_{\ell',s}\cdot\ell'\}$. Since $e\cdot w\leq e_{\max}\cdot w_{\max}$, $-\zeta(s)\leq 0$, and $\max_{\ell'\in L}\{-q_{\ell',s}\cdot\ell'\}\leq-\rho/(1+r^*-\delta)\cdot\ell_{\min}$, consumption conditional on not defaulting is bounded above by $e_{\max}\cdot w_{\max}+\ell_{\min}-\rho/(1+r^*-\delta)\cdot\ell_{\min}\leq 0$. This means either that the set $B_{\ell_{\min},0,\eta,0}(e,q)$ is either empty or that the only feasible consumption is zero consumption. In the first case default is the only option and in the second case it's the best option by (29). Therefore in any competitive equilibrium $q^*_{\ell_{\min},s}$ must be zero. ■

1 A Quantitative Theory of Unsecured Consumer Credit With Risk of Default, Supplementary Material: Proofs of Lemmas 7, 19-21, and 23

This document provides the proof of Lemma 7 and the proofs of Lemmas 19-21 and 23 which were omitted from the Appendix to the main article. The notation is the same as that used in the Appendix and equation numbers refer to equations in this document or to equations in the main article or in the Appendix to the main article.

Lemma A7. The goods market clearing condition (ixA) is implied by the other conditions for an equilibrium in Definition 2.

Proof: First note that the household budget sets (2)-(5) imply

$$
\begin{aligned}
& c^*_{\ell,h,s}(e;\alpha,q^*,w^*) + q^*_{\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*),s} \cdot \ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*) \cdot \left[1 - d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right] \\
&= \left[e\,(1-\gamma h) - \alpha(e-e_{\min})(1-h)\cdot d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\cdot w^* + (\ell - \zeta(s))\cdot\left[1 - d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right].
\end{aligned}
$$

Then aggregating over all households yields

$$
\begin{aligned}
& \int \left\{ c^*_{\ell,h,s}(e;\alpha,q^*,w^*) + q^*_{\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*),s}\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)\left[1 - d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right] d\mu^* \right\} \\
& + \int \left\{ \zeta(s)\left[1 - d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\right\} d\mu^* \\
& = \int \left\{\left[e\,(1-\gamma h) - \alpha(e-e_{\min})(1-h)\cdot d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\cdot w^* + \ell\cdot\left[1 - d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\right\} d\mu^*.
\end{aligned}
\tag{47}
$$

Condition (v) along with (47) imply

$$
\begin{aligned}
& \int \left\{ c^*_{\ell,h,s}(e;\alpha,q^*,w^*) + q^*_{\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*),s}\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)\left[1 - d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right] d\mu^* \right\} \\
& + \int \left\{ \zeta(s)\left[1 - d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\right\} d\mu^* \\
& = \int \left\{\left[e\,(1-\gamma h) - \alpha(e-e_{\min})(1-h)\cdot d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\cdot w^* + \ell\cdot\left[1 - d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\right\} d\mu^* \\
& + \int \left\{\left[1 - d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\zeta(s) + \ d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\max\{\ell,0\} - \zeta(s)/m^*\right\} d\mu^*
\end{aligned}
$$

or

$$
\begin{aligned}
& \int \left\{ c^*_{\ell,h,s}(e;\alpha,q^*,w^*) + q^*_{\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*),s}\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)\left[1 - d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\right\} d\mu^* + \int \frac{\zeta(s)}{m^*}d\mu^*. \\
& = \int \left\{\left[e\,(1-\gamma h) - \alpha(e-e_{\min})(1-h)\cdot d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\cdot w^* + \ell\cdot\left[1 - d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\right\} d\mu^* \\
& + \int \left\{ d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\max\{\ell,0\}\right\} d\mu^*
\end{aligned}
\tag{48}
$$

Since $d^*_{\ell,h,s}(e;\alpha,q^*,w^*) = 1$ implies $\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*) = 0$, it follows that the product of $\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)$ and $d^*_{\ell,h,s}(e;\alpha,q^*,w^*)$ is 0 for all ℓ,h,s,e. Hence, the left hand side of (48) can be written

$$\int c^*_{\ell,h,s}(e;\alpha,q^*,w^*)d\mu^* + \int q^*_{\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*),s}\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)d\mu^* + \int \frac{\zeta(s)}{m^*}d\mu^*.$$

Next the first term on the right hand side can be written

$$w^*\left[\int ed\mu^* - \gamma\int e\mu^*(d\ell,1,ds,de) - \alpha\int (e-e_{\min})\cdot d^*_{\ell,0,s}(e;\alpha,q^*,w^*)\mu^*(d\ell,0,ds,de)\right]$$

Finally, the remaining term on the right hand side of (48) can be written

$$\begin{aligned}
&\sum_{\ell,s}\ell\int(1-d^*_{\ell,h,s}(e;\alpha,q^*,w^*))\mu^*(\ell,dh,s,de) + \sum_{\ell\geq 0,s}\int d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\ell\mu^*(\ell,dh,s,de)\\
&= \sum_{\ell,s}\ell\int \mu^*(\ell,dh,s,de) - \sum_{\ell<0,s}\ell\int d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\mu^*(\ell,dh,s,de)\\
&= \sum_{\ell>0,s}\ell\int \mu^*(\ell,dh,s,de) + \sum_{\ell<0,s}\ell\int (1-d^*_{\ell,h,s}(e;\alpha,q^*,w^*))\mu^*(\ell,dh,s,de) \qquad (49)
\end{aligned}$$

Next, observe that for $x \neq 0$, we have from (x), (6), and (vii)

$$\begin{aligned}
&\int \mu^*(x,dh',\tilde{s},de';q^*,w^*)\\
&= \rho\int\left[\mathbf{1}_{\{(\ell,h,s,e):(\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)=x\}}\sum_{h'}H^*(\ell,h,s,e;h')\int_E \Phi(e'|\sigma)de'\Gamma(s;\sigma)\right]d\mu^*\\
&= \rho\int\left[\mathbf{1}_{\{(\ell,h,s,e):(\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)=x\}}\Gamma(s;\sigma)\right]\mu^*(d\ell,dh,ds,de)\\
&= \rho\sum_{s}a^*_{x,s}\Gamma(s;\tilde{s}),
\end{aligned}$$

where for ease of notation we have replaced s_{-1} with $\tilde{s}$. Hence, the first term in (49):

$$\begin{aligned}
\sum_{x>0,\tilde{s}}x\int\mu^*(x,dh,\tilde{s},de) &= \sum_{x>0,\tilde{s}}x\rho\sum_{s}a^*_{x,s}\Gamma(s;\tilde{s})\\
&= \rho\sum_{x>0,s}xa^*_{x,s}\sum_{\tilde{s}}\Gamma(s;\tilde{s})\\
&= \rho\sum_{x>0,s}xa^*_{x,s}
\end{aligned}$$

Now consider the second term in (49):

$$\begin{aligned}
&\int(1-d^*_{x,h,\tilde{s}}(e;\alpha,q^*,w^*))\mu^*(x,dh,\tilde{s},de)\\
&= \int\mu^*(x,dh,\tilde{s},de) - \int d^*_{x,h,\tilde{s}}(e;\alpha,q^*,w^*)\mu^*(x,dh,\tilde{s},de)
\end{aligned}$$

We can re-write the latter part of this expression as

$$\int d^*_{x,h,\tilde{s}}(e;\alpha,q^*,w^*)\mu^*(x,dh,\tilde{s},de;\alpha,q^*,w^*) =$$

$$\rho\int\left[\mathbf{1}_{\{(\ell,\eta,s,\varepsilon):(\ell'^*_{\ell,\eta,s}(\varepsilon;q^*,w^*)=x\}}\sum_h H(\ell,\eta,s,\varepsilon;h)\int_E d^*_{x,h,\tilde{s}}(e;\alpha,q^*,w^*)\Phi(e|\tilde{s})de\Gamma(s;\tilde{s})\right]\mu^*(d\ell,d\eta,ds,d\varepsilon).$$

Since $x<0$, it follows that $\eta=0$ and $h=0$ so that $H(\ell,0,s,\varepsilon;0)=1$ and $H(\ell,0,s,\varepsilon;1)=0,\forall\ell,s,\varepsilon$. Therefore

$$\begin{aligned}
&\int d^*_{x,h,\tilde{s}}(e;\alpha,q^*,w^*)\mu^*(x,dh,\tilde{s},de;\alpha,q^*,w^*)\\
&=\rho\int\left[\mathbf{1}_{\{(\ell,0,s,\varepsilon):(\ell'^*_{\ell,0,s}(\varepsilon;q^*,w^*)=x\}}\int_E\sum_h H(\ell,0,s,\varepsilon;h)d^*_{x,h,\tilde{s}}(e;\alpha,q^*,w^*)\Phi(e|\tilde{s})de\Gamma(s;\tilde{s})\right]\mu^*(d\ell,0,ds,d\varepsilon)\\
&=\rho\int\left[\mathbf{1}_{\{(\ell,0,s,\varepsilon):(\ell'^*_{\ell,0,s}(\varepsilon;q^*,w^*)=x\}}\int_E d^*_{x,0,\tilde{s}}(e;\alpha,q^*,w^*)\Phi(e|\tilde{s})de\Gamma(s;\tilde{s})\right]\mu^*(d\ell,0,ds,d\varepsilon).
\end{aligned}$$

Let $p^{*\tilde{s}}_x=\int_E d^*_{x,0,\tilde{s}}(e;\alpha,q^*,w^*)\Phi(e|\tilde{s})de$ be the probability of default on a loan of size x by households with characteristic $\tilde{s}$. Then

$$\begin{aligned}
&\int d^*_{x,h,\tilde{s}}(e;\alpha,q^*,w^*)\mu^*(x,dh,\tilde{s},de;\alpha,q^*,w^*)\\
&=\sum_s\rho\int\left[\mathbf{1}_{\{(\ell,0,s,e):(\ell'^*_{\ell,0,s}(e;\alpha,q^*,w^*)=x\}}p^{*\tilde{s}}_x\Gamma(s;\tilde{s})\right]\mu^*(d\ell,0,s,de;\alpha,q^*,w^*)\\
&=\rho\sum_s p^{*\tilde{s}}_x\Gamma(s;\tilde{s})a^*_{x,s}.
\end{aligned}$$

The second equality follows from (vii) recognizing that $\mu^*(Z)=0$ for all $Z\in L_{--}\times\{1\}\times S\times\mathcal{B}(E)$. Thus the second part of (49) can be written

$$\begin{aligned}
&\sum_{x>0,\tilde{s}}x\int\mu^*(x,dh,\tilde{s},de)+\sum_{x<0,\tilde{s}}x\int(1-d^*_{x,h,\tilde{s}}(e;\alpha,q^*,w^*))\mu^*(x,dh,\tilde{s},de)\\
&=\rho\sum_{x>0,s}xa^*_{x,s}+\rho\sum_{x<0,s}xa^*_{x,s}-\sum_{x<0,s}x\rho\sum_{\tilde{s}}p^{*\tilde{s}}_x\Gamma(s;\tilde{s})a^*_{x,s}\\
&=\rho\left[\sum_{x>0,s}xa^*_{x,s}+\sum_{x<0,s}xa^*_{x,s}(1-p^*_{x,s})\right].
\end{aligned}$$

Thus, re-writing (48) we have

$$\begin{aligned}
&\int c^*_{\ell,h,s}(e;\alpha,q^*,w^*)d\mu^*+\int q^*_{\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*),s}\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)d\mu^*+\int\frac{\zeta(s)}{m^*}d\mu^*\\
&=w^*\int ed\mu^*-\gamma w^*\int e\mu^*(d\ell,1,ds,de)-\alpha w^*\int(e-e_{\min})\cdot d^*_{\ell,0,s}(e;\alpha,q^*,w^*)\mu^*(d\ell,0,ds,de)\\
&+\rho\sum_{\ell,s}\ell a^*_{\ell,s}(1-p^*_{\ell,s}).
\end{aligned}$$

But

$$\begin{aligned}\int q^*_{\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*),s}\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)d\mu^* &= \sum_{\ell'}\int \mathbf{1}_{\{(\ell,h,s,e):(\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)=\ell'\}}q_{\ell',s}\ell'\mu^*(d\ell,dh,ds,de) \\ &= \sum_{\ell',s} q^*_{\ell',s}a^*_{\ell',s}\ell' \\ &= K^*\end{aligned}$$

where the last inequality follows from (20). Another implication of (20) is

$$(1+r^*-\delta)\,K^* = \rho \sum_{(\ell',s)\in L\times S}(1-p^*_{\ell',s})a^*_{\ell',s}\ell'.$$

Thus, we have

$$\begin{aligned}&\int c^*_{\ell,h,s}(e;\alpha,q^*,w^*)d\mu^* + K^* + \int \frac{\zeta(s)}{m^*}d\mu^* \\ &= w^*N^* - \gamma w^*\int e\mu^*(d\ell,1,ds,de) + (1+r^*-\delta)\,K^* \\ &= F(N^*,K^*) + (1-\delta)\,K^* - \gamma w^*\!\int\! e\mu^*(d\ell,1,ds,de) - \alpha w^*\!\int\!(e-e_{\min})\cdot d^*_{\ell,0,s}(e;\alpha,q^*,w^*)\mu^*(d\ell,0,ds,de).\end{aligned}$$

So that the goods market clears. ■

Lemma A19. (i) $\Phi_s(\overline{ES}^{(\ell',d)}) \le \bar{x}^{(\ell',d)}_{\ell,h,s}$ (ii) $\Phi_s(\overline{ED}^{(\ell',d)}) \le (1-\bar{x}^{(\ell',d)}_{\ell,h,s})$ and (iii) $\sum_{\ell'\in L}\Phi_s(\overline{ES}^{(\ell',0)}) + \Phi_s(\overline{ES}^{(0,1)}) + \Phi_s(\overline{I}^{(0,1)}) = 1 = \sum_{\ell'\in L}\bar{x}^{(\ell',0)}_{\ell,h,s} + \bar{x}^{(0,1)}_{\ell,h,s}$.

Proof. To prove (i) we first establish that $\overline{ES}^{(\ell',d)} \subseteq \cup_{m=1}^{\infty}\left(\cap_{k\ge m}ES_k^{(\ell',d)}\right)$. Consider $\hat{e}\in\overline{ES}^{(\ell',d)}$. Then $\phi^{(\ell',d)}_{\ell,h,d)}(\hat{e};0,\bar{q},\bar{w}) - \max_{(\tilde{\ell}',\tilde{d})\neq(\ell',d)}\phi^{(\tilde{\ell}',\tilde{d})}_{\ell,h,d}(\hat{e};0,\bar{q},\bar{w}) > 0$. By Lemma A2 it follows that there exists $N(\hat{e})$ such that for all $m \ge N(\hat{e})$, $\phi^{(\ell',d)}_{\ell,h,d)}(\hat{e};\alpha_m,q_m,w_m) - \max_{(\tilde{\ell}',\tilde{d})\neq(\ell',d)}\phi^{(\tilde{\ell}',\tilde{d})}_{\ell,h,d}(\hat{e};\alpha_m,q_m,w_m) > 0$. Therefore, $\hat{e}\in\cap_{k\ge N(\hat{e})}ES_k^{(\ell',d)}$. Hence we must have $\hat{e}\in\cup_{m=1}^{\infty}\left(\cap_{k\ge m}ES_k^{(\ell',d)}\right)$. Next, observe that for each m, $\cap_{k\ge m}ES_k^{(\ell',d)}$ is Borel measurable since it is a countable intersection of Borel measurable sets. Therefore, $\Phi_s(\overline{ES}^{(\ell',d)}) \le \Phi_s\left(\cup_m\left(\cap_{k\ge m}ES_k^{(\ell',d)}\right)\right) = \lim_{m\to\infty}\Phi_s(\cap_{k\ge m}ES_k^{(\ell',d)})$. The last equality follows because the sets $\cap_{k\ge m}ES_k^{(\ell',d)}$ are increasing in m. Next, observe that $\Phi_s(\cap_{k\ge m}ES_k^{(\ell',d)}) \le \Phi_s(ES_m^{(\ell',d)}) = x^{(\ell'd)}_{\ell,h,s}(\alpha_m,q^*_m,w^*_m)$, where the last equality follows from Lemma A8 which implies the set $E_m^{(\ell',d)} \cap \left(ES_m^{(\ell',d)}\right)^c$ is finite and therefore of Φ_s-measure 0. Thus, $\lim_{m\to\infty}\Phi_s(\cap_{k\ge m}ES_k^{(\ell',d)}) \le \lim_{m\to\infty}x^{(\ell',d)}_{\ell,h,s}(\alpha_m,q^*_m,w^*_m) = \bar{x}^{(\ell',d)}_{\ell,h,s}$. Therefore $\Phi_s(\overline{ES}^{(\ell',d)}) \le \bar{x}^{(\ell',d)}_{\ell,h,s}$. This establishes (i).

To prove (ii) we first establish that $\overline{ED}^{(\ell',d)} \subseteq \cup_{m=1}^{\infty}\left(\cap_{k\ge m}ED_k^{(\ell',d)}\right)$. Consider $\hat{e}\in\overline{ED}^{(\ell',d)}$. Then $\phi^{(\ell',d)}_{\ell,h,d)}(\hat{e};0,\bar{q},\bar{w}) - \max_{(\tilde{\ell}',\tilde{d})\neq(\ell',d)}\phi^{(\tilde{\ell}',\tilde{d})}_{\ell,h,d}(\hat{e};0,\bar{q},\bar{w}) < 0$. By Lemma A2, there exists $N(\hat{e})$ such that for all $m \geq N(\hat{e})$, $\phi^{(\ell',d)}_{\ell,h,d)}(\hat{e};\alpha_m,q_m,w_m) - \max_{(\tilde{\ell}',\tilde{d})\neq(\ell',d)} \phi^{(\tilde{\ell}',\tilde{d})}_{\ell,h,d}(\hat{e};\alpha_m,q_m,w_m) < 0$. Therefore, $\hat{e} \in \cap_{k\geq N(\hat{e})} ED^{(\ell',d)}_k$. Hence we must have $\hat{e} \in \cup_{m=1}^{\infty}\left(\cap_{k\geq m} ED^{(\ell',d)}_k\right)$. Next, observe that for each m, $\cap_{k\geq m} ED^{(\ell',d)}_k$ is Borel measurable since it is a countable intersection of Borel measurable sets. Therefore, $\Phi_s(\overline{ED}^{(\ell',d)}) \leq \Phi_s\left(\cup_m\left(\cap_{k\geq m} ED^{(\ell',d)}_k\right)\right) = \lim_{m\to\infty}\Phi_s(\cap_{k\geq m} ED^{(\ell',d)}_k)$. The last equality follows because the sets $\cap_{k\geq m} ED^{(\ell',d)}_k$ are increasing in m. Next, observe that $\Phi_s(\cap_{k\geq m} ED^{(\ell',d)}_k) \leq \Phi_s(ED^{(\ell',d)}_m) = 1 - x^{(\ell'd)}_{\ell,h,s}(\alpha_m, q^*_m, w^*_m)$, where the last equality follows from Lemma A8 which implies $\left(E^{(\ell',d)}_m\right)^c \cap \left(ED^{(\ell',d)}_m\right)^c$ is a finite set and therefore of Φ_s-measure 0. Thus $\lim_{m\to\infty}\Phi_s(\cap_{k\geq m} ES^{(\ell',d)}_k) \leq \lim_{m\to\infty}[1 - x^{(\ell',d)}_{\ell,h,s}(\alpha_m, q^*_m, w^*_m)] = 1 - \bar{x}^{(\ell',d)}_{\ell,h,s}$. Therefore $\Phi_s(\overline{ED}^{(\ell',d)}) \leq 1 - \bar{x}^{(\ell',d)}_{\ell,h,s}$. This establishes (ii).

To prove (iii), consider the set $\left(\cup_{\ell'\in L}\overline{ES}^{(\ell',0)} \cup \overline{ES}^{(0,1)} \cup \overline{I}^{(0,1)}\right)^c$. A member of this set is any e for which there is more than one optimal action none of which involve default. By Lemma A8 this is a finite set and therefore of Φ_s-measure 0. Hence $\Phi_s\left(\cup_{\ell'\in L}\overline{ES}^{(\ell',0)} \cup \overline{ES}^{(0,1)} \cup \overline{I}^{(0,1)}\right) = 1$. Since any pair of sets in the union is disjoint, it follows that $\sum_{\ell'\in L}\Phi_s(\overline{ES}^{(\ell',0)}) + \Phi_s(\overline{ES}^{(0,1)}) + \Phi_s(\overline{I}^{(0,1)}) = 1$. Next, consider the set $\left(\cup_{\ell'\in L} ES^{(\ell',0)}_m \cup ES^{(0,1)}_m\right)^c$. A member of this set is any e for which there is more than one optimal action. By Lemma A8 again this is a finite set. Therefore $\Phi_s\left(\cup_{\ell'\in L} ES^{(\ell',0)}_m \cup ES^{(0,1)}_m\right) = 1$. Since any pair of sets in this union is disjoint, it follows that $\sum_{\ell'\in L}\Phi_s(ES^{(\ell',0)}_m) + \Phi_s(ES^{(0,1)}_m) = 1$. Since $ES^{(\ell',d)}_m$ and $E^{(\ell',d)}_m$ can differ by at most a finite set of points (by Lemma A8), it follows that $\sum_{\ell'\in L} x^{(\ell',0)}_{\ell,h,s}(\alpha_m, q^*_m, w^*_m) + x^{(0,1)}_{\ell,h,s}(\alpha_m, q^*_m, w^*_m) = 1$. Taking limits on both sides yields $\sum_{\ell'\in L}\bar{x}^{(\ell',0)}_{\ell,h,s} + \bar{x}^{(0,1)}_{\ell,h,s} = 1$. This establishes (iii). ■

Lemma A20. For all $(\ell,h,s)\in\mathcal{L}$ there exist measurable functions $c_{\ell,h,s}(e)$, $\ell'_{\ell,h,s}(e)$, and $d_{\ell,h,s}(e)$ for which the implied choice probabilities $\int_E \mathbf{1}_{\{\ell'_{\ell,h,s}(e)=\ell', d_{\ell,h,s}(e)=d\}}\Phi(de|s) = \overline{x}^{(\ell',d)}_{(\ell,h,s)}$ and the triplet $\left(c_{\ell,h,s}(e), \ell'_{\ell,h,s}(e), d_{\ell,h,s}(e)\right) \in \chi_{\ell,h,s}(e;0;\overline{q},\overline{w})$.

Proof. The decision rules are constructed for two mutually exclusive cases. First, consider the case where $\Phi_s(\bar{I}^{(0,1)}) = 0$. For this case construct the decision rules as follows. Assign to action (ℓ',d) all e such that $e\in\overline{ES}^{(\ell',d)}$. This step leaves unassigned the set $\overline{I}^{0,1} \cup \left(\cup_{\ell'\in L}\overline{I}^{(\ell',0)}\right)$. To complete the assignment, assign all elements of $\overline{I}^{0,1}$ to $(0,1)$ and assign any remaining elements to actions in any manner provided that each element is assigned to an action only once and an element is assigned to an action (ℓ',d) only if it belongs to $\overline{I}^{(\ell',d)}$. Since $\overline{ES}^{(\ell',d)}$ are disjoint, the assignment maps each e to exactly one action (ℓ',d). Let $\ell'_{\ell,h,s}(e)$ and $d_{\ell,h,s}(e)$ be the resulting decision rules for ℓ' and d and let $c_{\ell,h,s}(e)$ be the decision rule for c implied by the household budget constraint given $\ell'_{\ell,h,s}(e)$ and $d_{\ell,h,s}(e)$.

We will now establish that these decision rules are measurable, optimal and imply the limiting choice probability vector $\bar{x}$. To establish measurability it is sufficient to establish that for each action (ℓ',d) the set $\{e : \ell'_{\ell,h,s}(e) = \ell' \text{ and } d_{\ell,h,s}(e) = d\}$ is Borel measurable. For $(0,1)$, the corresponding set is the union of $\overline{ES}^{(0,1)}$ and $\overline{I}^{(0,1)}$ both of which are Borel measurable and therefore the union is Borel measurable. Furthermore, $\Phi_s\left(\{e : \ell'_{\ell,h,s}(e) = 0 \text{ and } d_{\ell,h,s}(e) = 1\}\right) = \Phi_s\left(\overline{ES}^{(0,1)}\right)$ since $\Phi_s(\overline{I}^{(0,1)}) = 0$. For $(\ell, 0)$, the corresponding set is the union of $\overline{ES}^{(\ell',0)}$, which is Borel measurable, and some subset of $\overline{I}^{(\ell',0)}$. By Lemma A8, $\overline{I}^{(\ell',0)}$ is a finite set and therefore any subset of it is Borel measurable. Hence $\{e : \ell'_{\ell,h,s}(e) = \ell' \text{ and } d_{\ell,h,s}(e) = 0\}$ is also a union of Borel measurable sets and therefore Borel measurable. Furthermore, $\Phi_s\left(\{e : \ell'_{\ell,h,s}(e) = \ell' \text{ and } d_{\ell,h,s}(e) = 0\}\right) = \Phi_s\left(\overline{ES}^{(\ell',0)}\right)$ since $\Phi_s(\overline{I}^{(\ell,0)}) = 0$ (being a finite set). The decision rules are optimal by construction. Finally, note that by Lemma A19(iii) we have $\sum_{\ell' \in L}[\Phi_s(\overline{ES}^{(\ell,0)}) - \bar{x}^{(\ell',0)}_{\ell,h,s}] + [\Phi_s(\overline{ES}^{(0,1)}) - \bar{x}^{(0,1)}_{\ell,h,s}] = 0$. By Lemma A19(i) each term in this sum is nonnegative. It follows immediately that $\Phi_s(\overline{ES}^{(\ell',d)}) = \bar{x}^{(\ell',d)}_{\ell,h,s}$. Hence, $\Phi_s\left(\{e : \ell'_{\ell,h,s}(e) = \ell' \text{ and } d_{\ell,h,s}(e) = d\}\right) = \Phi_s\left(\overline{ES}^{(\ell',d)}\right) = \bar{x}^{(\ell',d)}_{\ell,h,s}$.

Next, consider the case where $\Phi_s(\overline{I}^{(0,1)}) = \delta > 0$. The assignment has to distribute members $\overline{I}^{(0,1)}$ in such a way that choice probabilities induced by the assignment are the limiting choice probabilities $\bar{x}$. To begin, we first claim that there must exist exactly one action $(\hat{\ell}', 0)$ for which $\overline{I}^{(0,1)} = \overline{I}^{(\hat{\ell}',0)}$. Suppose there were two such actions $(\hat{\ell}', 0)$ and $(\tilde{\ell}', 0)$. Then, $I^{(\hat{\ell}',0),(\tilde{\ell}',0)}_{\ell,h,s}(0, \bar{q}, \bar{w}) \supseteq \overline{I}^{(0,1)}$ implying that $I^{(\hat{\ell}',0),(\tilde{\ell}',0)}_{\ell,h,s}(0, \bar{q}, \bar{w})$ has strictly positive measure which, by Lemma A8, is impossible.

Next, we claim that $\Phi_s(\overline{ES}^{(0,1)}) + \Phi_s(\overline{I}^{(0,1)}) + \Phi_s(\overline{ES}^{(\hat{\ell}',0)}) = \bar{x}^{(0,1)}_{\ell,h,s} + \bar{x}^{(\hat{\ell},0)}_{\ell,h,s}$. To see this, suppose that $\Phi_s(\overline{ES}^{(0,1)}) + \Phi_s(\overline{I}^{(0,1)}) + \Phi_s(\overline{ES}^{(\hat{\ell}',0)}) < \bar{x}^{(0,1)} + \bar{x}^{(\hat{\ell},0)}$. But by Lemma A19(iii) this implies that $\sum_{\ell' \neq \hat{\ell}'} \Phi_s(\overline{ES}^{(\ell',0)}) > \sum_{\ell' \neq \hat{\ell}'} \bar{x}^{(\ell',0)}_{\ell,h,s}$, which contradicts the bound in Lemma A19(i). Suppose then that $\Phi_s(\overline{ES}^{(0,1)}) + \Phi_s(\overline{I}^{(0,1)}) + \Phi_s(\overline{ES}^{(\hat{\ell}',0)}) > \bar{x}^{(0,1)} + \bar{x}^{(\hat{\ell},0)}$. By Lemma A19(iii) $\sum_{\ell' \neq \hat{\ell}'} \Phi_s(\overline{ES}^{(\ell',0)}) < \sum_{\ell' \neq \hat{\ell}'} \bar{x}^{(\ell',0)}_{\ell,h,s}$. But this implies $\sum_{\ell' \neq \hat{\ell}'}[1 - \Phi_s(\overline{ES}^{(\ell',0)})] > \sum_{\ell' \neq \hat{\ell}'}[1 - \bar{x}^{(\ell',0)}_{\ell,h,s}]$, which contradicts the bound in Lemma A19(ii). This establishes the claim.

We can now proceed with the assignment. To (ℓ', d) distinct from $(0, 1)$ or $(\hat{\ell}', 0)$, assign all e such that $e \in \overline{ES}^{(\ell',0)}$. Next, partition the set $\overline{I}^{(0,1)}$ into two disjoint (measurable) sets I_1 and I_2 such that $\Phi_s\left(\overline{ES}^{(\hat{\ell},0)} \cup I_1\right) = \bar{x}^{(\hat{\ell},0)}_{\ell,h,s}$ and $\Phi_s\left(\overline{ES}^{(0,1)} \cup I_2\right) = \bar{x}^{(0,1)}_{\ell,h,s}$ (since Φ_s is atomless such a partition exists). Finally, assign in any manner all remaining elements provided that each element is assigned to an action only once and an element is assigned to an action (ℓ', d) only if it belongs to $\overline{I}^{(\ell',d)}$.

These assignments assign each e to exactly one action (ℓ, d) and therefore imply decision rules $\ell'_{\ell,h,s}(e)$, $d_{\ell,h,s}(e)$ and, via the household budget constraint, $c_{\ell,h,s}(e)$. The measurability of these decision rules can be established by expressing the sets $\{e : \ell'_{\ell,h,s}(e) = \ell' \text{ and } d_{\ell,h,s}(e) = d\}$ as unions of measurable sets as was done for the first case. By construction, the decision rules are optimal. Finally, note that by our earlier claim $\sum_{\ell' \neq \hat{\ell}'}[\Phi_s(\overline{ES}^{(\ell',0)}) - \bar{x}^{(\ell',0)}_{\ell,h,s}] = 0$. By Lemma A19(i) each term in this sum is nonnegative and, therefore, $\Phi_s(\overline{ES}^{(\ell',0)}) = \bar{x}^{(\ell',0)}_{\ell,h,s}$ for $\ell' \neq \hat{\ell}'$. Hence,

$\Phi_s\left(\{e:\ell'_{\ell,h,s}(e)=\ell' \text{ and } d_{\ell,h,s}(e)=d\}\right)=\Phi_s\left(\overline{ES}^{(\ell',d)}\right)=\bar{x}^{(\ell',d)}_{\ell,h,s}$ where the first equality uses the fact that the set $\{e:\ell'_{\ell,h,s}(e)=\ell' \text{ and } d_{\ell,h,s}(e)=d\}$ differs from the set $\overline{ES}^{\ell',d)}$ by at most a finite set of points. Finally, by construction $\Phi_s\left(\{e:\ell'_{\ell,h,s}(e)=\hat{\ell}' \text{ and } d_{\ell,h,s}(e)=0\}\right)=\bar{x}^{(\hat{\ell}',0)}_{\ell,h,s}$ and $\Phi_s\left(\{e:\ell'_{\ell,h,s}(e)=0 \text{ and } d_{\ell,h,s}(e)=1\}\right)=\bar{x}^{(0,1)}_{\ell,h,s}$. ■

We now establish the analogs of Lemma A12 and A15 for the sequence $\{\alpha_m,q^*_m,w^*_m\}$ converging to $(0,\overline{q},\overline{w})$.

Lemma A21. Let $\overline{\pi}_{(0,\overline{q},\overline{w})}$ be the invariant distribution of the Markov chain $\overline{P}$ defined by the decision rules $(\ell'_{\ell,h,s}(e),d_{\ell,h,s}(e))$. Then the sequence $\pi_{(\alpha_m,q^*_m,w^*_m)}$ converges weakly to $\overline{\pi}_{(0,\overline{q},\overline{w})}$.

Proof. We apply Theorem 12.13 in Stokey, Lucas, and Prescott (1989). Part a of the requirements follows since $\mathcal{L}$ is compact. Part b requires that $P^*_{(\alpha_m,q^*_m,w^*_m)}[(\ell_n,h_n,s_n),\cdot]$ converge weakly to $\overline{P}_{(0;\overline{q},\overline{w})}[(\ell,h,s),\cdot]$ as $(\ell_n,h_n,s_n,\alpha_m,q^*_m,w^*_m)\to(\ell,h,s,0,\overline{q},\overline{w})$. By Theorem 12.3d of Stokey, Lucas, and Prescott (1989) it is sufficient to show that for any (ℓ',h',s'),

$$\lim_{k\to\infty}P^*_{(\alpha_m,q^*_m,w^*_m)}\left[(\ell_n,h_n,s_n),(\ell',h',s')\right]=\overline{P}_{(0;\overline{q},\overline{w})}\left[(\ell,h,s),(\ell',h',s')\right].$$

By definition

$$\begin{aligned}&P^*_{(\alpha_m,q^*_m,w^*_m)}\left[(\ell,h,s),(\ell',h',s')\right]\\&=\left[\begin{array}{c}\rho\int_E \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha_m,q^*_m,w^*_m)=\ell'\}}H^*_{(\alpha_m,q^*_m,w^*_m)}(\ell,h,s,e,h')\Phi(de|s)\Gamma(s,s')\\+(1-\rho)\int_E\mathbf{1}_{\{(\ell',h')=(0,0)\}}\psi(s',de')\end{array}\right]\end{aligned}$$

and

$$H^*_{(q,w)}(\ell,h,s,e,h'=1)=\begin{cases}1 & \text{if } d^*_{\ell,h,s}(e;q,w)=1\\ \lambda & \text{if } d^*_{\ell,h,s}(e;q,w)=0 \text{ and } h=1\\ 0 & \text{if } d^*_{\ell,h,s}(e;q,w)=0 \text{ and } h=0\end{cases},$$

$$H^*_{(q,w)}(\ell,h,s,e,h'=0)=\begin{cases}0 & \text{if } d^*_{\ell,h,s}(e;q,w)=1\\ 1-\lambda & \text{if } d^*_{\ell,h,s}(e;q,w)=0 \text{ and } h=1\\ 1 & \text{if } d^*_{\ell,h,s}(e;q,w)=0 \text{ and } h=0\end{cases}.$$

By construction, the Markov chain $\overline{P}$ is

$$\begin{aligned}&\overline{P}\left[(\ell,h,s),(\ell',h',s')\right]\\&=\left[\begin{array}{c}\rho\int_E \mathbf{1}_{\{\ell'_{\ell,h,s}(e)=\ell'\}}H^*_{(0,\overline{q},\overline{w})}(\ell,h,s,e,h')\Phi(de|s)\Gamma(s,s')\\+(1-\rho)\int_E\mathbf{1}_{\{(\ell',h')=(0,0)\}}\psi(s',de')\end{array}\right]\end{aligned}$$

where $H^*_{(0,\overline{q},\overline{w})}(\ell,h,s,e,h')$ is determined by $d_{\ell,h,s}(e)$.

Since $\mathcal{L}$ is finite, without loss of generality consider the sequence $(\alpha_m,q^*_m,w^*_m)\to(0,\overline{q},\overline{w})$. Since the second term on the r.h.s. is independent of (α,q,w), it is sufficient to consider the limiting behavior of the integral

$$\int_E\mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha_m,q^*_m,w^*_m)=\ell'\}}H^*_{(\alpha_m,q^*_m,w^*_m)}(\ell,h,s,e,h')\Phi(de|s).$$

For $h=0$ and $h'=0$, this integral in P^* is

$$\int_E \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha_m,q^*_m,w^*_m)=\ell',d^*_{\ell,h,s}(e;\alpha_m,q^*_m,w^*_m)=0\}}\Phi(de|s)=x^{(\ell',0)}_{(\ell,0,s)}(\alpha_m,q^*_m,w^*_m)$$

and in $\overline{P}$ it is

$$\int_E \mathbf{1}_{\{\ell'_{\ell,0,s}(e)=\ell'\}}H^*_{(0,\overline{q},\overline{w})}(\ell,0,s,e,0)\Phi(de|s)=\int_E \mathbf{1}_{\{\ell'_{\ell,0,s}(e)=\ell',d_{\ell,0,s}(e)=0\}}\Phi(de|s).$$

By Lemma A20 we have

$$\lim_{k\to\infty} x^{(\ell',0)}_{(\ell,0,s)}(\alpha_m,q^*_m,w^*_m)=\overline{x}^{(\ell',0)}_{(\ell,0,s)}=\int_E \mathbf{1}_{\{\ell'_{\ell,h,s}(e)=\ell',d_{\ell,h,s}(e)=0\}}\Phi(de|s).$$

Hence

$$\lim_{k\to\infty} P^*_{(\alpha_m,q^*_m,w^*_m)}\left[(\ell,0,s),(\ell',0,s')\right]=\overline{P}_{(0;\overline{q},\overline{w})}\left[(\ell,0,s),(\ell',0,s')\right].$$

The remaining cases can be dealt with in exactly the same way. We simply note here which choice probabilities are involved in each case and omit the details.

For $h=0$ and $h'=1$, the integral in P^* is

$$\int_E \mathbf{1}_{\{\ell'^*_{\ell,0,s}(e;\alpha_m,q^*_m,w^*_m)=\ell',d^*_{\ell,0,s}(e;\alpha_m,q^*_m,w^*_m)=1\}}\Phi(de|s)=x^{(\ell',1)}_{(\ell,0,s)}(\alpha_m,q^*_m,w^*_m).$$

For $h=1$ and $h'=0$, the integral in P^* is

$$(1-\lambda)\int_E \mathbf{1}_{\{\ell'^*_{\ell,1,s}(e;\alpha_m,q^*_m,w^*_m)=\ell',d^*_{\ell,1,s}(e;\alpha_m,q^*_m,w^*_m)=0\}}\Phi(de|s)=x^{(\ell',0)}_{(\ell,0,s)}(\alpha_m,q^*_m,w^*_m).$$

For $h=1$ and $h'=1$, the integral in P^* is

$$\int_E \begin{bmatrix} \mathbf{1}_{\{\ell'^*_{\ell,1,s}(e;\alpha_m,q^*_m,w^*_m)=\ell',d^*_{\ell,1,s}(e;\alpha_m,q^*_m,w^*_m)=1\}} \\ +\lambda\mathbf{1}_{\{\ell'^*_{\ell,1,s}(e;\alpha_m,q^*_m,w^*_m)=\ell',d^*_{\ell,1,s}(e;\alpha_m,q^*_m,w^*_m)=0\}} \end{bmatrix}\Phi(de|s)=\begin{bmatrix} x^{(\ell',1)}_{(\ell,1,s)}(\alpha_m,q^*_m,w^*_m) \\ +\lambda x^{(\ell',0)}_{(\ell,1,s)}(\alpha_m,q^*_m,w^*_m)\end{bmatrix}.$$

■

Lemma A23. Let $K_{(0,\overline{q},\overline{w})}\equiv\sum_{(\ell',s)\in L\times S}\ell'\overline{q}_{\ell',s}\int\mathbf{1}_{\{(\ell'_{\ell,h,s}(e)=\ell'\}}\overline{\mu}_{(0,\overline{q},\overline{w})}(d\ell,dh,s,de)$, $N_{(0,\overline{q},\overline{w})}\equiv\int ed\overline{\mu}_{(0,\overline{q},\overline{w})}$, and $p_{(0,\overline{q},\overline{w})}(\ell',s)\equiv\int d_{\ell',0,s'}(e')\Phi(e'|s')\Gamma(s;ds')de'$. Then (i) $\lim_m K(\alpha_m,q^*_m,w^*_m)=K_{(0,\overline{q},\overline{w})}$, (ii) $\lim_m N(\alpha_m,q^*_m,w^*_m)=N_{(0,\overline{q},\overline{w})}$, and (iii) $\lim_m p_{(\alpha_m,q^*_m,w^*_m)}(\ell',s)=p_{(0,\overline{q},\overline{w})}(\ell',s)$.

Proof. To prove (i), note that we know by Lemma A13,

$$\begin{aligned}&\int_{L\times H\times E}\mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha_m,q^*_m,w^*_m)=\ell'\}}\mu_{(\alpha_m,q^*_m,w^*_m)}(d\ell,dh,s,de)\\ &=\sum_{\ell,h}\int_E\mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha_m,q^*_m,w^*_m)=\ell'\}}\Phi(de|s)\pi_{(\alpha_m,q^*_m,w^*_m)}(\ell,h,s)\end{aligned}$$

By Lemma A20

$$\lim_{n_k \to \infty} \int_E \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha_m,q^*_m,w^*_m)=\ell', d^*_{\ell,h,s}(e;\alpha_m,q^*_m,w^*_m)=d\}} \Phi(de|s)$$
$$= \int_E \mathbf{1}_{\{(\ell'_{\ell,h,s}(e)=\ell', d_{\ell,h,s}(e)=d\}} \Phi(de|s).$$

Since

$$\int_E \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha_m,q^*_m,w^*_m)=\ell'\}} \Phi(de|s)$$
$$= \sum_{d\in\{0,1\}} \int_E \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha_m,q^*_m,w^*_m)=\ell', d^*_{\ell,h,s}(e;\alpha_m,q^*_m,w^*_m)=d\}} \Phi(de|s),$$

then

$$\lim_{n_k \to \infty} \int_E \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha_m,q^*_m,w^*_m)=\ell'\}} \Phi(de|s)$$
$$= \sum_{d\in\{0,1\}} \int_E \mathbf{1}_{\{(\ell'_{\ell,h,s}(e)=\ell', d_{\ell,h,s}(e)=d\}} \Phi(de|s) = \int_E \mathbf{1}_{\{(\ell'_{\ell,h,s}(e)=\ell'\}} \Phi(de|s).$$

Next, by Lemma A21,

$$\lim_{n\to\infty} \pi_{(\alpha_m,q^*_m,w^*_m)}(\ell,h,s) = \pi_{(0,\overline{q},\overline{w})}(\ell,h,s).$$

Therefore $\lim_{n_k\to\infty} K_{(\alpha_m,q^*_m,w^*_m)} = K_{(0,\overline{q},\overline{w})}$. To prove (ii) simply apply Lemma A21. To prove (iii), note that by Lemma A20

$$\lim_{n_k \to \infty} \int_E d^*_{\ell',0,s'}(e';\alpha_m,q^*_m,w^*_m)\Phi(de'|s') = \int_E d_{\ell',0,s'}(e')\Phi(de'|s').$$

Thus,

$$\lim_{m \to \infty} \int_{E\times S} d^*_{\ell',0,s'}(e';\alpha_m,q^*_m,w^*_m)\Phi(de'|s')\Gamma(s;ds')$$
$$= \int_{E\times S} d_{\ell',0,s'}(e')\Phi(de'|s')\Gamma(s;ds').$$

■

References

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