NBER WORKING PAPER SERIES

ON THE DETERMINANTS OF THE VALUE OF CALL OPTIONS ON DEFAULT-FREE BONDS

Stephen A. Buser

Patric H. Hendershott

Anthony B. Sanders

Working Paper No. 2529

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
March 1988

The research reported here is part of the NBER's research program in Financial Markets and Monetary Economics. Any opinions expressed are those of the authors and not those of the National Bureau of Economic Research.

NBER Working Paper #2529
March 1988

On the Determinants of the Value of Call Options on Default-Free Bonds

ABSTRACT

Models of interest-dependent claims that imply similar term structures and levels of interest rate volatility also produce similar estimates of bond option values. This result is established for simple option forms with known closed-form solutions as well as for more complex options that require numerical methods for evaluation. The finding is confirmed for a wide range of economic conditions, and it is robust with respect to the number and nature of factors that generate interest-rate movements.

Stephen A. Buser, Patric H. Hendershott and Anthony B. Sanders
Academic Faculty of Finance
The Ohio State University
1775 College Road
Columbus, Ohio 43210
614-292-5026

Early contributions to the theory of bond options suggest that techniques for pricing options on stock can be adapted with relatively minor refinements.[1] Subsequent research indicates potential difficulties. For example, Courtadon (1982) and Buser and Hendershott (1984) show that call values are sensitive to multiple unobservable parameters even in the simple case of a one-factor interest-rate process. Brennan and Schwartz (1983, 1985) and Dietrich-Campbell and Schwartz (1986) conclude that call values are sensitive to the number as well as the nature of interest rate factors. Difficulties in specifying and estimating numerous unobservable parameters of an uncertain interest rate process would thus appear to present a formidable barrier to the implementation of bond option pricing models.[2]

In this paper, we re-examine alternative models for pricing debt options and conclude that practical application is not substantially more difficult than in the case of stock options. The term structure and volatility of interest rates serve as summary characteristics of the economy for bond options, just as the stock price and volatility of stock returns are summary characteristics of the economy for stock options. Traditional comparative-static experiments that reportedly test for the specific effects of a given parameter (or model) are misleading in so far as the summary characteristics of the economy typically change in the experiment. We

---

1. See Cox, Ingersoll and Ross (1976/1985), Brennan and Schwartz (1977), and Dunn and McConnell (1981).

2. For additional attempts to estimate the factor structure of interest rates, see Oldfield and Rogalski (1981), Marsh and Rosenfeld (1983) and Gultekin and Rogalski (1985).

control for these fundamental determinants of bond option values and show that alternative models of the interest-rate process produce similar estimates of bond option values. Accordingly, even if researchers disagree about the determinants of either the term structure or the level of volatility of interest rates, they should still agree on the implications of a given term structure and level of volatility for bond option values.

Our results are robust with respect to the interest rate environment. Cases of high versus low interest-rate uncertainty are examined for three term structures: steeply upward sloping; gently upward sloping; and downward sloping. To guard against the chance that our results are unique to one particular contract, we examine three very different options including: (1) a 2-year call option on a 10-year zero-coupon bond, (2) a 2-year call option on a 30-year coupon bond, and (3) the borrower's option to prepay a 30-year fully-amortizing mortgage.

The remainder of our paper is divided into six sections. The first presents our formal assertions about the fundamental determinants of bond option values. Section II describes the alternative models of the interest rate process that we consider. Procedures for comparing alternative models are discussed in Section III, and simulation results are presented in Sections IV and V. A brief summary concludes the paper.

I. Properties of Bond-Option Values

Cox, Ingersoll and Ross (1976/85), hereafter CIR, show that the equilibrium value of any interest-dependent claim can be represented as a solution to a fundamental pricing equation subject to appropriate boundary conditions. We describe this equation in detail in the next section. Here we simply let θ denote the set of parameter values which, together with the current interest rate, r, describes the fundamental pricing equation. The primary contention of this paper is that as few as two characteristics of θ may be sufficient to value options on default-free debt. The measures that we propose are y, the slope of the term structure of yields on default-free zero-coupon bonds, and σ, the level of interest-rate volatility.

These summary characteristics allow us to clarify and extend two propositions that have been established for specific options and specific models of the interest-rate process. Our third proposition asserts that any remaining bond-option pricing effects are of secondary importance.

<u>Proposition #1</u>: For a given level of uncertainty about future interest rates, the value of a call option on a default-free bond is a decreasing function of the slope of the term structure. This result was illustrated by Courtadon (1982) and Buser, Hendershott and Sanders (1985) for specific contracts in the case of one-factor models. We extend and formalized the result. If θ_1 and θ_2 denote distinct models such that $\sigma(r,\theta_2) = \sigma(r,\theta_1)$ and $y(r,\theta_2) > y(r,\theta_1)$, then $C(r,\theta_2) < C(r,\theta_1)$.

Proposition #2: For a given term structure, the value of a call option on a default-free bond is an increasing function of the level of uncertainty about future interest rates. As in the case of proposition #1, this result was established by Courtadon (1982) and Buser, Hendershott and Sanders (1985) for specific contracts in the case of one-factor models. We extend and formalize the result. If θ_1 and θ_2 denote distinct models such that $y(r,\theta_2) = y(r,\theta_1)$ and $\sigma(r,\theta_2) > \sigma(r,\theta_1)$, then $C(r,\theta_2) > C(r,\theta_1)$.

Proposition #3: Given appropriate controls for the effects of the term structure and the level of uncertainty about future interest rates, call values are relatively insensitive to the number or nature of the factor processes that generate interest rates. If θ_1 and θ_2 denote distinct models such that $y(r,\theta_2) = y(r,\theta_1)$ and $\sigma(r,\theta_2) = \sigma(r,\theta_1)$, then $C(r,\theta_2) \approx C(r,\theta_1)$.

Propositions 1, 2 and 3 together imply that there are two and only two fundamental determinants of bond option values, namely the term structure and the volatility of interest rates. In the next section we describe the various pricing models used to substantiate this claim.

II. Models of the Interest Rate Process

We begin this section with a general model of the interest-rate process and then specify one-factor and two-factor versions that we use to identify the determinants of bond option values.

General Model

Cox, Ingersoll and Ross (1976/85), hereafter CIR, present a general theory of interest-dependent claims based on the premise that the state of the economy is fully described by a finite number (M) of state variables or factors, X_i, which follow a joint diffusion process:

$$dX_i = f_i dt + \sigma_i dZ_i \qquad i = 1,\ldots,M \tag{1}$$

where Z_i is a Wiener process with $E[dZ_i] = 0$, $dZ_i^2 = dt$ and $dZ_i dZ_j = \rho_{ij} dt$. The parameters of the process (f_i and σ_i) measure the drift and variance of the corresponding state variables and, in general, may vary with time (t) and the current values of the factors.

Because the factors describe the economy completely, security values are uniquely determined. Hence, CIR are able to apply Ito's lemma and represent the instantaneous change in value of an arbitrary security V (with given boundary conditions and cash flow) as a related diffusion process

$$dV = F dt + G dZ \,. \tag{2}$$

In equation (2), F is the drift in the value of the contract, and G is the stochastic component of the price path. By virtue of Ito's lemma, these parameters are, respectively,

$$F = V_t + \sum_{i=1}^{M} V_i f_i + 1/2 \sum_{i=1}^{M} \sum_{j=1}^{M} V_{ij} \rho_{ij} \sigma_i \sigma_j \,, \tag{3}$$

and

$$GdZ = \sum_{i=1}^{M} V_i \sigma_i dZ_i . \quad (4)$$

To preclude arbitrage profits, CIR impose Merton's (1973) risk/return equilibrium condition on the drift in (3) and variance in (4).

$$F + \delta = rV + \sum_{i=1}^{M} \lambda_i \sigma_i V_i . \quad (5)$$

In equation (5), δ is the instantaneous cash flow for the security, λ_i is the market-determined price of the ith source of risk in the economy, and r is the instantaneous rate of return on riskless investments, hereafter referred to as the spot rate. CIR use expression (3) to rewrite (5) as:

$$1/2 \sum_{i=1}^{M} \sum_{j=1}^{M} V_{ij} \rho_{ij} \sigma_i \sigma_j + \sum_{i=1}^{M} V_i (f_i - \sigma_i \lambda_i) + V_t + \delta - rV = 0. \quad (6)$$

Equation (6) uniquely determines the value of any security subject to appropriate boundary conditions. Based on this observation, CIR refer to (6) as the Fundamental Partial Differential Equation for Contingent Claims. CIR also show that subject to an invertablity restriction it is possible to use interest rates as instrumental variables in place of the true, but possibly unobservable, state variables or factors.

<u>Specific Models</u>

Initial efforts to derive explicit models of the term structure [CIR (1976), Vasicek (1977) and Dothan (1978)] focus on the simplest of cases: a single factor which, without loss of generality, is taken as the instantaneous riskless spot rate of interest (r). In terms of the notation of equation (1), the diffusion process is expressed as

$$dr = fdt + \sigma dZ. \tag{1'}$$

Various specifications of the drift and variance of this process have been employed such as a mean-reverting drift with constant elasticity of variance:

$$f = k(u-r) \tag{7}$$

and

$$\sigma = sr^{\alpha}. \tag{8}$$

In equation (7), k measures the expected rate of adjustment toward the long-run value u. In equation (8), s is the scale of the variation, and α is the elasticity of the variation in the process with respect to the level of the spot rate. Vasicek sets α equal to zero. CIR sets α equal to 0.5 (the "square-root" process), and Dothan sets α equal to 1. In each of the corresponding models, the risk premium in (5) is presumed to be a linear function of the level of interest rates:[3]

3. Arbitrage considerations rule out certain combinations of (8) and (9). In particular, for $\alpha \neq 0$, risk vanishes if r=0 which implies that the risk premium should be zero as well, i.e., a=0 in (9). The specific models we consider rule out the potentially inconsistent special cases.

$$\lambda\sigma = a + br. \tag{9}$$

Substituting equations (7), (8) and (9) into a one-factor variant of equation (6) yields a class of one-factor models of the following form:

$$1/2s^2r^{2\alpha}V_{11} + [(ku-a) - (k+b)r]V_1 + V_t + \delta - rV = 0. \tag{6'}$$

Two-factor models were initially proposed by CIR and Richard (1978). In these models, the spot rate is the sum of two factors:[4]

$$r = r_1 + r_2. \tag{10}$$

That is, $X_1 = r_1$ and $X_2 = r_2$. Both factors are presumed to exhibit mean reversion:

$$f_1 = k_1(u_1-r_1) \quad \text{and} \quad f_2 = k_2(u_2-r_2), \tag{7'}$$

where u_1 and u_2 are constants. We allow for a general elasticity of variation:

$$\sigma_1 = s_1r_1^{\alpha} \quad \text{and} \quad \sigma_2 = s_2r_2^{\alpha}. \tag{8'}$$

4. CIR and Richard (1978) interpret the additive factors as the real interest rate and inflation. Ayres and Barry (1979) take the factors to be a long rate and the spread between the short rate and the long rate.

We further presume that risk premia are linear in the corresponding factors.

$$\lambda_1\sigma_1 = a_1 + b_1r_1 \quad \text{and} \quad \lambda_2\sigma_2 = a_2 + b_2r_2. \tag{9'}$$

Substituting equations (7'), (8') and (9') into equation (6) with M=2 yields a class of two-factor models of the following form:

$$1/2[s_1^2r_1^{2\alpha}V_{11} + s_2^2r_2^{2\alpha}V_{22} + 2\rho s_1s_1r_1^{\alpha}r_2^{\alpha}V_{12}] + [(k_1u_1 - a_1)$$

$$- (k_1 + b_1)r_1]V_1 + [(k_2u_2 - a_2) - (k_2 + b_2)r_2]V_2 + V_t + \delta - rV = 0. \tag{6"}$$

Together equations (6') and (6") describe the various models examined in this paper. In the next section we discuss how to compare alternative models without contaminating the experiment with changes in the term structure or interest-rate volatility.

III. Procedures for Comparing Alternative Models

Risk-Neutral Analogs for Risk-Averse Models

CIR observe that it is always possible to construct a risk-neutral analog for any risk-averse specification of the fundamental equation (6). In the case of a single-factor mean-reverting model (6'), the mean and speed of adjustment for the matching risk-neutral process are:

$$u^0 = (ku-a)/(k+b), \tag{11}$$

and $$k^0 = k + b. \tag{12}$$

Because the risk-neutral analog replicates the pricing equation exactly, security values should be identical as well. Hereafter, we presume that the risk-neutral transformation has been made, so that we can omit the notation "0" without fear of ambiguity.

One-Factor Analogs for Two-Factor Models

To obtain a single-factor analog for a risk-neutral model with two additive mean-reverting factors, we choose a speed of adjustment for a single-factor model with a decay rate equal to a weighted average of the decay rates in the two-factor model. That is, we find k such that:

$$\exp(-kT) = \exp(-k_1T)r_1/r + \exp(-k_2T)r_2/r. \tag{13}$$

In mean-reverting models, the volatility of interest rates over a given period of time varies directly with the instantaneous variance of the process and inversely with the speed of adjustment which dampens shocks to the process. We capture both effects by choosing an instantaneous variance for a matching one-factor model such that:

$$\frac{s^2r^{2\alpha}}{k^2} = \frac{s_1^2r_1^{2\alpha}}{k_1^2} + \frac{2\rho s_1r_1^{\alpha}s_2r_2^{\alpha}}{k_1k_2} + \frac{s_2^2r_2^{2\alpha}}{k_2^2}. \tag{14}$$

When considering models with positive interest-rate elasticity, we set volatility at the current spot rate equal to volatility in the inelastic case.

$$sr^{\alpha}/k = s^{0}/k. \qquad (15)$$

Equations (13), (14) and (15) are not based on precise analytic rules. Nevertheless, these ad hoc controls reflect our intuition about models of the interest rate process, and, as we show in the remainder of the paper, they provide an adequate summary of interest-rate volatility for the purpose of pricing bond options.

IV. Call Options on Zero-Coupon Bonds

Closed-form solutions for the value of a call option on a zero-coupon bond have been developed by CIR (equation 32, p.396), Ball and Torous (1983, equation 8, p.528), and Jamshidian (1987, equation 7a, p.7). In all three formulas, the price of the option (C) is a weighted difference between the values of two default-free bonds: B_1, the underlying zero-coupon bond; and B_2, a zero-coupon bond with maturity equal to that of the option:

$$C = B_1N_1 - B_2KN_2. \qquad (16)$$

In equation (16), K is the exercise price of the option, and the weights are determined by parameters of the respective models. CIR employ the square-root form of elastic volatility (α=1/2), and use the chi-square distribution to define N_1 and N_2. Volatility is inelastic in the Jamshidian model (α=0) and N_1 and N_2 are determined by the normal distribution.

Equation (16) bears a striking resemblance to the familiar Black-Scholes (1972) equation for a stock option; the long bond (B_1) corresponds to the stock, and the short bond (B_2) corresponds to the present value factor (exp-rT). Ball and Torous draw the analogy even more tightly by observing that their version of (16) is identical to Merton's (1973) extension of the Black-Scholes equation that incorporates stochastic interest rates.

The closed-form solutions of CIR and Jamshidian correspond to special cases of the one-factor model represented by equation (6'). By focusing on these cases, we can examine our basic propositions without relying on numerical methods. We can then use the closed-form results to verify the numerical procedures that are required under more general conditions.

Economic Environments

Panel A of Table 1 reports option values for four specifications of one-factor models that have been fit to nine very different economic environments; three term structure slopes are examined for each of three different levels of interest rate volatility. The slopes cover the gamut from a downward sloping yield curve (minus 100 basis points) to a steeply upward sloping yield curve (plus 300 basis points). The intermediate case corresponds to a "normal" slope of plus 100 basis points. The choice of volatility levels ranges from a value that is low by historical standards (s^0=0.015) to one that is high (s^0=0.045). An intermediate case was chosen as well (s^0=0.03).

Each of the models was fit to the prespecified economic environments in two. In a "risk neutral" approach, we set the risk aversion parameters equal to zero and varied the mean to generate each of the term structures. In a "risk averse" approach, we set the mean equal to the initial spot rate in order to minimize the role played by expectations. We then used equation (12) to solve for the risk-aversion parameter. As noted previously, models paired in this way should produce identical value estimates for interest-dependent claims. In all cases, the initial spot rate is 0.10 and the speed of mean reversion is 0.25. Specific parameter values for each case examined are identified in Panel B of Table 1.

Results

The rows of Panel A are organized into four groups each of which corresponds to a distinct version of the one-factor model. In turn, each group of rows contains three price estimates: (1) a value based on numerical integration with 12 changes in interest rates per year; (2) a value based on numerical integration with 96 changes in interest rates per year; and (3) a value based on the closed-form solution (which presumes that interest rates change continuously).

Values computed by numerical methods differ little from values computed by closed-form solutions. The error based on 96 intervals per year is never more than one cent per hundred dollar face amount of bonds. With 12 intervals, the numerical solutions are systematically higher (by 2 to 7 cents). Yet even these estimates are within 10 percent of the closed-form solutions and serve as a verification of our numerical procedures. As previously noted, we must rely exclusively on these procedures for the balance of the paper because closed-form solutions do not exist for the additional contracts examined.

The results in Panel A clearly support proposition 1; call values fall sharply as the term structure is increased. Moreover, the percentage effect is roughly uniform across term structures. At the low level of volatility call values fall by approximately 50% when the slope of the term structure is changed from downward to normal. Call values fall by 50% again when the slope of the term structure is changed from normal to steeply upward. At the high level of volatility the percentage decline is roughly 40% in the analogous experiments.

The results in Panel A also support proposition 2; call values are substantially larger in cases of higher volatility. However, the effect is not uniform. The percentage change in call values declines with the level of volatility and increases with the slope of the term structure.

With respect to proposition 3 (model-specific effects), the value of the call option is clearly insensitive to whether risk aversion or risk neutrality is used to generate the term structure; values are identical in all cases as required by the theory. The effect of the elasticity parameter is more difficult to evaluate. Call values are reliably lower for the square-root process; the difference exceeds the numerical approximation error if volatility is moderate or high and the term structure is not steeply upward sloping. However, even in extreme cases, the value of the call option is only 12% less in the elastic model. These results suggest two possibilities: 1) there is a small but nontrivial negative relationship between bond call values and elasticity; or 2) our ad hoc procedures for standardizing the levels of volatility in the models are not sufficiently precise. In support of the second possibility, we note that models with different elasticities cannot provide the same level of volatility at all interest rates. Our procedures standardize volatility at the prevailing interest rate, but in so doing, we assure that a model with higher elasticity has a lower probability of very low interest rates (the range most relevant for call options). A lower risk-adjusted

V. Calls on Coupon Bonds and Amortizing Mortgages

The results reported in Table 1 could be contract specific. To guard against this possibility, we repeat our analysis of bond-option values for two additional types of contracts. These additional results are identified in Table 2. Panel A reports simulated values for a two-year option to call a 30-year bond with a semiannual coupon that is set such that the bond is initially priced at par. The exercise price for the option is the par value of the bond. Panel B provides estimates of the prepayment option in a fully amortizing 30-year fixed-rate mortgage with a coupon such that the mortgage value plus the option equals par. The exercise price for the option is the amortized value of the mortgage. Under this specification, the values reported in Panel B represent "points" charged at the origination of the mortgage.

The construction of Panels A and B differs from that for Panel A of Table 1 in three respects. First, we only report numerical results based on 12 changes in interest rates per year. Second, we only report results for risk-neutral specifications because, as in the case of Table 1, results for appropriately matched risk-averse models are identical. Third, we expand the class of models to include two-factor models as well as well as one-factor models, and we examine the unit-elastic case in addition to the inelastic and square-root specifications of the interest-rate process.

Economic Environments

To design appropriate tests of our claim that one-factor analogs can be found for two-factor price structures, we must take care to avoid trivial comparisons. For example, we should not focus on cases where the factors are highly correlated or where one factor clearly dominates the other. Even factors that are uncorrelated but which are similar in structure (symmetric) can be replicated by one-factor models with relative ease. These concerns suggest that the speeds of adjustment should differ for the individual factors. Yet neither speed of adjustment should be so large that it inhibits volatility nor so small that it enhances volatility to a point where its factor dominates the remaining factor. These considerations, in conjunction with reasonable empirical bounds on interest-rate movements, helped to shape the following experiments.

The option values shown in Panels A and B of Table 2 are computed for an interest rate of 0.10. The respective initial values for two-factor models are $r_1 = 0.07$ and $r_2 = 0.03$. We also set $k_1 = 0.25$ and $k_2 = 0.50$. These choices meet our objectives for nontrivial asymmetric processes. In particular, while the k values differ, neither factor is dominated in the uncertainty structure because it has a speed of mean reversion that is too large (which would inhibit volatility), nor does either factor dominate by virtue of too little mean reversion (which would enhance volatility). From (13), the value for a matching one-factor model is $k = 0.2619$ for $T = 30$. Scale parameters are chosen as follows. First, we assume that the two factors are uncorrelated ($\rho=0$) in order to maximize separation between the factors. This structure presents the greatest challenge for the task of finding a matching one-factor model. For a similar reason we choose different levels of variance for the individual factors. In the low volatility case, we set s_1^0 and s^0_2 equal to 0.02 and 0.01, respectively. Based on (14) the corresponding value for a matching one-factor model is $s^0=0.0216$. In the high volatility case, the respective volatility constants are 0.03 and 0.015 and 0.0324.

As in Table 1, option values are reported for three term structures which can be generated either by varying the mean-reverting rate (u) with $b = 0$ or by varying the risk-aversion parameter (b) with $u = r$. In the two-factor case, the ratios of the means to the corresponding spot rates (u_1/r_1 and u_2/r_2) are held constant. Parameter values for the various cases are listed in Panel C of Table 2.

Results

The additional simulations indicate that our findings for propositions 1 and 2 in the case of options on zero-coupon bonds are not contract specific. In Panels A and B of Table 2, an increase in the slope of the term structure clearly lowers call values, and an increase in volatility clearly raises call values. In addition, the patterns for the effects appear to be similar to those established for call options on zero-coupon bonds. For either the coupon bond or the mortgage, an increase in the slope of the term structure (from downward to normal or from normal to steeply upward) cuts the value of the option by roughly one third in the case of low volatility and by roughly one fourth in the case of high volatility. The percentage effect of a change in volatility increases with the slope of the term structure but at a much lower rate than in the zero-coupon case; the increase is roughly 40% with the downward sloping term structure and 70% with a steeply upward term structure.

The additional simulations also confirm our earlier findings regarding model-specific effects. Option values are insensitive to whether risk aversion or expectations accounts for the slope of the term structure. (Results are omitted because they are identical.) Option values are also insensitive to the number of interest-rate factors used to generate the term structure. Only the elasticity parameter emerges as a potential model-specific determinant of simulated option values; call values are noticeably smaller when the elasticity parameter is increased. Patterns in the effect are similar to those established for options on zero-coupon bonds. The percentage effect of increasing elasticity is greatest when volatility is high and the yield curve is downward sloping; differences in value estimates are as large as 10% for the bond option and 15% for the mortgage option.

Part of the effect we have attributed to elasticity may in fact be due to our inability to match volatilities precisely at all interest rates.

Accordingly, even the apparent effect of elasticity could be due more to the general effect of volatility than to a truly model-specific effect. In support of this interpretation, we note that the apparent effects of elasticity are uniform in sign and magnitude. Specifically, call values decline at a uniform rate as elasticity is increased, and the magnitude of the effect declines as the slope of the term structure is increased. These patterns suggest that it may be possible to improve on our procedures for controlling for differences in volatility. For example, the apparent effect of elasticity might be reduced if interest-rate expectations were used in the volatility control.[5] Alternatively, we note that even if models are constructed to provide comparable levels of volatility at the prevailing interest rate, an increase in elasticity reduces the probability of very low interest rates (the range most relevant for call options). Thus it might be more appropriate to standardize models on the basis of the probability of interest rate less than some critical value.

Although we are guardedly optimistic that such improvements are possible, the pursuit of such refinements is beyond the intended scope of this paper. Our purpose was simply to show that differences in estimates between models are far less than is currently perceived, provided that

---

5. Alternatively, it might be possible to extract an implied variance measure for one model that equates the option estimate with a matching model for some benchmark case. Such an adjustment would be contract-specific, contrary to the spirit of our current investigation. Nevertheless the technique would be of interest if it could be shown to improve the fit between models for all contracts over the full range of economic environments.

comparable term structures are used and some attempt, however crude, is made to impose comparable levels of interest-rate volatility.

VI. Conclusions

The value of a given call option on a default-free bond is a decreasing function of the slope of the term structure and an increasing function of the volatility of interest rates. Little else seems to matter. In particular, bond option values are not sensitive to either the number of factors driving the interest-rate process or the reason that the term structure has a given slope. Bond option values appear to be moderately sensitive to the degree of interest-rate elasticity in volatility, but even this result may say more about our ad hoc controls for volatility than it does about the fundamental determinants of bond option values. Thus we conclude that even if elaborate models of the interest-rate process are required to estimate volatility, they are not needed to price bond options.

In the absence of analytic results, we can not claim confirmation for all possible specifications of multiple factor models. Nevertheless, we regard our simulations as sufficiently compelling to encourage, if not require, future advocates of multiple factor models of bond-option values to make relevant comparisons vis a vis a comparable one-factor models. As a minimum, the slope of the term structure and the level of interest-rate volatility must be the same for competing models.

References

Ayres, H.R. and Barry, J.Y., "The Equilibrium Yields Curve for Government Securities," Financial Analysts Journal, May/June 1979.

Ball, Clifford H. and Torous, Walter N., "Bond Price Dynamics and Options," Journal of Financial and Quantitative Analysis, December 1983.

Brennan, M.J. and Schwartz, E.S., "Savings Bonds, Retractable Bonds, and Callable Bonds," Journal of Financial Economics 5, 1977.

_____, "Alternative Methods for Valuing Debt Options," Finance, 4, 1983.

_____, "Determinants of GNMA Mortgage Prices," AREUEA Journal, Fall 1985.

Buser, S.A. and Hendershott, P.H., "Pricing Default-Free Fixed-Rate Mortgages," Housing Finance Review, 3, 1984.

Buser, S.A. and Hendershott, P.H. and Sanders, A.B., "Pricing Life-of-Loan Rate Caps on Default-Free Adjustable-Rate Mortgages," AREUEA Journal, Fall 1985.

Courtadon, Georges, "The Pricing of Options on Default-Free Bonds," Journal of Financial and Quantitative Analysis, March 1982.

Cox, J.C., and Ingersoll, J.E. and Ross, S.A., "A Theory of the Term Structure of Interest Rates," Research Paper No. 468, Graduate School of Business, Stanford University, 1976. Subsequently published in Econometrica, 53, (1985), 385-407.

Dietrich-Campbell, Bruce and Schwartz, Eduardo, "Valuing Debt Options: Empirical Evidence," Journal of Financial Economics, July 1986.

Dothan, L. Uri, "On the Term Structure of Interest Rates," Journal of Financial Economics, 6, 1978.

Dunn, K.B. and McConnell, J.J., "Valuation of GNMA Mortgage-Backed Securities," Journal of Finance, June 1981.

Gultekin, N. Bulent and Rogalski, Richard J., "Government Bond Returns, Measurement of Interest Rate Risk, and the Arbitrage Pricing Theory," Journal of Finance, March 1985.

Jamshidian, Farshid, "An Exact Bond Option Formula," unpublished manuscript, April 1987.

Marsh, Terry A. and Rosenfeld, Eric R., "Stochastic Processes for Interest Rates and Equilibrium Bond Prices," Journal of Finance, May 1983.

Merton, R.C., "Theory of Rational Option Pricing," Bell Journal of Economics and Management Science 4, 1973.

Oldfield, George S., Jr. and Rogalski, Richard J., "Treasury Bill Factors and Common Stock Returns," Journal of Finance, May 1981.

Richard, Scott F., "An Arbitrage Model of the Term Structure of Interest Rates," Journal of Financial Economics, March 1978.

Vasicek, Oldrich, "An Equilibrium Characterization of the Term Structure," Journal of Financial Economics, 5, 1977.

Table 1

Two-Year Call Option on a Ten-Year Zero-Coupon Bond

Panel A

Option Value Per Hundred Dollar Face Amount of Bonds for Alternative Models and Economic Environments

	Slope of the Term Structure								
	Downward			Normal			Steeply Upward		
	Volatility			Volatility			Volatility		
Model	Low	Mod	High	Low	Mod	High	Low	Mod	High
(1) Inelastic Risk Neutral									
Numerical(12)	1.28	2.21	3.17	0.64	1.42	2.21	0.27	0.89	1.53
Numerical(96)	1.25	2.19	3.16	0.59	1.37	2.17	0.24	0.83	1.47
Closed Form	1.24	2.19	3.16	0.58	1.37	2.17	0.24	0.83	1.47
(2) Inelastic Risk Averse									
Numerical(12)	1.28	2.21	3.17	0.64	1.42	2.21	0.27	0.89	1.53
Numerical(96)	1.25	2.19	3.16	0.59	1.37	2.17	0.24	0.83	1.48
Closed Form	1.24	2.19	3.16	0.58	1.37	2.17	0.24	0.83	1.47
(3) Square Root Risk Neutral									
Numerical(12)	1.26	2.10	2.81	0.64	1.35	1.96	0.28	0.86	1.37
Numerical(96)	1.23	2.07	2.80	0.58	1.30	1.92	0.24	0.80	1.31
Closed Form	1.22	2.06	2.79	0.58	1.29	1.91	0.25	0.79	1.30
(4) Square Root Risk Averse									
Numerical(12)	1.26	2.10	2.81	0.64	1.35	1.96	0.28	0.86	1.37
Numerical(96)	1.23	2.07	2.80	0.58	1.30	1.92	0.24	0.80	1.31
Closed Form	1.22	2.06	2.79	0.58	1.29	1.91	0.25	0.79	1.30

Contract assumptions:
The exercise price for the option at time t is $100\exp[-R(T-t)]$, where R is the original yield and T-t is the remaining term to maturity.

Parameters for case-by-case results are shown in Panel B.

Table 1

Panel B:

Parameter Values

Exogenous Parameters:

r=0.10 and k=0.25 in all models.

Model (1) Inelastic Risk Neutral: α=a=b=0; u endogenous.

Model (2) Inelastic Risk Averse: α=0; b=0; u=r; a endogenous

Model (3) Square Root Risk Neutral: α=1/2; a=b=0; u endogenous

Model (4) Square Root Risk Averse: α=1/2; a=0; u=r; b endogenous

Values of the Endogenous Parameters That Make 10-Year Slope of the Term Structure:

		Downward (-100 bp)			Normal (100 bp)			Steeply Upward (300 bp)		
		For Volatility			For Volatility			For Volatility		
		Low	Mod	High	Low	Mod	High	Low	Mod	High
Model	s^0 =	0.015	0.03	0.045	0.015	0.03	0.045	0.015	0.03	0.045
(1)	u =	.0855	.0895	.0961	.1171	.1211	.1277	.1487	.1527	.1593
(2)	a =	-.0036	-.0026	-.0010	.0043	.0053	.0069	.0122	.0132	.0148
(3)	u =	.0854	.0889	.0944	.1172	.1213	.1278	.1490	.1537	.1612
(4)	k =	.2135	.2223	.2360	.2930	.3033	.3195	.3726	.3844	.4030
	b =	-.0365	-.0277	-.0140	.0430	.0533	.0695	.1226	.1344	.1530

Table 2

Calls on Coupon Instruments

Panel A

Value of a Two-Year Call Option
on a Thirty-Year Coupon Bond

	Slope of the Term Structure					
	Downward		Normal		Steeply Upward	
	Volatility		Volatility		Volatility	
Model	Low	High	Low	High	Low	High
Inelastic:						
Two Factors	3.30	4.71	2.32	3.64	1.61	2.84
One Factor	3.30	4.72	2.31	3.64	1.60	2.83
Square Root:						
Two Factors	3.20	4.42	2.26	3.43	1.56	2.67
One Factor	3.21	4.47	2.27	3.48	1.56	2.70
Unit Elastic:						
Two Factors	3.11	4.19	2.20	3.25	1.52	2.53
One Factor	3.12	4.26	2.22	3.32	1.53	2.60

Contract assumptions:
The initial face value of the bond is $100, and the coupon is set to initially price the bond at par. The exercise price for the option is $100.

Model parameters for case-by-case results are shown in Panel C.

Table 2

Panel B

Value of a Homeowner's Option to Prepay a Thirty-Year Fixed-Rate Mortgage

	Slope of the Term Structure					
	Downward		Normal		Steeply Upward	
	Volatility		Volatility		Volatility	
Model	Low	High	Low	High	Low	High
Inelastic:						
Two Factors	5.32	7.49	3.56	5.58	2.28	4.07
One Factor	5.38	7.52	3.53	5.54	2.24	3.99
Square Root:						
Two Factors	4.95	6.66	3.38	5.08	2.24	3.79
One Factor	4.99	6.83	3.38	5.19	2.25	3.88
Unit Elastic:						
Two Factors	4.68	6.12	3.25	4.70	2.19	3.58
One Factor	4.81	6.31	3.25	4.77	2.23	3.70

Contract assumptions:
The face amount of the mortgage is $100, and the coupon is set such that a noncallable mortgage would initially be priced at par. That is, the value of the prepayment option represents the "points" or initial discount from par for the mortgage. The exercise price for the option at time t is the outstanding loan balance.

Model parameters for case-by-case results are shown in Panel C.

Table 2

Panel C

Parameter Values

Exogenous parameters:

$a_1 = a_2 = b_1 = b_2 = 0;$

$k_1 = 0.25;\ k_2 = 0.50;\ k = 0.2619;$

$r_1 = 0.07;\ r_2 = 0.03;\ r = 0.10;$

$u_1 = ur_1/r;\ u_2 = ur_2/r;$

$s^o_{1L} = 0.02;\ s^o_{2L} = 0.01;\ s^o{}_L = 0.0216;$

$s^o_{1H} = 0.03;\ s^o_{2H} = 0.015;\ s^o_H = 0.0324.$

		Values of the Mean That Make the 30-Year Slope of the Term Structure					
		Downward (-100 bp)		Normal (100 bp)		Steeply Upward (300 bp)	
		Volatility		Volatility		Volatility	
		Low	High	Low	High	Low	High
Model	s^o =	.0126	.0324	.0126	.0324	.0126	.0324
Inelastic (α=0):							
Two Factors	u =	.0918	.0957	.1144	.1182	.1369	.1408
One Factor	u =	.0917	.0956	.1146	.1186	.1375	.1415
Square Root (α=1/2):							
Two Factors	u =	.0914	.0945	.1145	.1183	.1376	.1420
One Factor	u =	.0913	.0946	.1148	.1188	.1383	.1430
Unit Elastic (α=1):							
Two Factors	u =	.0914	.0949	.1152	.1197	.1390	.1450
One Factor	u =	.0913	.0946	.1152	.1197	.1395	.1455