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THE LEARNING CURVE AND OPTIMAL PRODUCTION UNDER UNCERTAINTY

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ABSTRACT

This paper examines the implications of the learning curve in a world of uncertainty. We consider a competitive firm whose costs decline with cumulative output. Because the price of the firm's output evolves stochastically, future production and cumulative output are unknown, and are contingent on future prices and costs. We derive an optimal decision rule that maximizes the firm's market value: produce when price exceeds a critical level, which is a declining function of cumulative output. We show how the shadow value of cumulative production, as well as the total value of the firm, depend on the volatility of price and other parameters. Over the relevant range of prices, uncertainty reduces the shadow value of cumulative production, and therefore increases the critical price required for the firm to begin producing.

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1. Introduction.

Most students of business economics are taught early on about the learning curve and the role it plays in the cost structure of the firm. Since the early articles by Wright (1936) and Hirsch (1952) that showed how costs fall with cumulative output in the production of airframes and machine tools, a variety of studies have demonstrated the existence of learning curves in a wide range of industries, and management consultants have stressed the importance of learning for production planning.[1]

More recently, studies such as those of Spence (1981) and Kalish (1983) have shown how the presence of a learning curve can be taken into account when setting price and output levels.[2] These studies demonstrate that in setting output, a firm should produce at a point where current marginal cost exceeds marginal revenue. The reason is that an incremental unit of current production reduces future production costs by moving the firm down the learning curve, and therefore has a shadow value that partly offsets its cost. Current production should be such that current marginal cost exceeds marginal revenue by the amount of this shadow value.

Unfortunately, this point is of limited use as a guideline for production. First, this shadow value can be hard to calculate. Second, and more important, most firms face considerable uncertainty over future demand, and hence prices. This means that the firm cannot know how much it will produce in the future, or whether it will produce at all - its future production decisions are contingent on the evolution of demand. This complicates the valuation of the firm, the calculation of the current shadow value of cumulative production, and the optimal production decision.[3] And as we will see, it can reduce the importance of learning effects for production planning.

This paper re-examines the implications of the learning curve, but in a world of uncertainty. We consider a competitive firm whose production costs decline with cumulative output. However, the price of the firm's output evolves stochastically, so that all future production decisions are contingent on the evolution of price, and future cumulative output is unknown. We show how the firm's current decision to produce can be made in a way that is consistent with its financial objective - the maximization of its market value.[4]

If uncertainty over future output prices is spanned by the existing set of traded assets in the economy (in a manner to be made clear below), the firm's production problem can be treated as a problem in the valuation of a contingent claim. As McDonald and Siegel (1985) have shown, a firm facing a stochastic output price can be thought of as having a set of call options on future production at every instant of time. Each call option has an exercise price equal to production cost (which in our case declines with cumulative output), and a payoff equal to the output price. The value of the firm is therefore a contingent claim - in our case a function of both price and cumulative output to date - and will satisfy a partial differential equation that can be derived using the standard methods of contingent claims analysis, or equivalently by dynamic programming.

We solve this differential equation using numerical methods. The result is a simple optimal decision rule: produce when price exceeds a critical level, which is a declining function of cumulative output. The solution also yields the shadow value of a unit of cumulative output. We show how this shadow value, as well as the total value of the firm, depend on the volatility of price and other parameters. In particular, we show that over the relevant range of prices, uncertainty reduces the shadow value of cumulative production, and therefore increases the critical price required for the firm to begin producing. In terms of practical production decisions, this makes the learning curve less important than some have been led to believe.

In the next section we begin with a simple deterministic model of a firm facing a learning curve. This helps to illustrate the behavior of the shadow value of cumulative output and its dependence on price, and provides a benchmark for evaluating the effects of uncertainty. In Section 3 we extend the model to allow for a stochastically evolving output price, and we derive and solve an equation for the value of the firm and its optimal operating strategy. Section 4 presents numerical examples, and shows how the value of the firm, the shadow value of cumulative output, and the firm's optimal operating strategy depend on the variance of price. Section 5 contains some concluding remarks.

2. A Deterministic Learning Curve Model.

We consider a firm that owns a single factory, and sells its output in a competitive market at a price P. The firm's marginal production cost is constant with respect to the rate of output, up to a capacity constraint, which we arbitrarily set at 1. However, the firm faces a learning curve; marginal cost declines with cumulative output, Q, until it reaches a minimum level $\bar{c}$:

$$\begin{aligned} C(Q) &= ce^{-\gamma Q} \qquad , \quad Q < Q_m \\ &= ce^{-\gamma Q_m} = \bar{c} \qquad , \quad Q \geq Q_m \end{aligned} \tag{1}$$

Here, c is the initial marginal production cost, and Q_m is the level of cumulative output at which learning ceases, and cost reaches its constant minimum level $\bar{c}$.[5]

We will assume that the price of the firm's output grows over time at a known and constant rate ρ, i.e. $P(t) = P_0 e^{\rho t}$. (In Section 3 we expand the model to allow for a stochastically varying price.) The firm's problem is to choose a rate of output $x(t)$, $0 \leq x \leq 1$, to maximize:

$$\text{Max } V = \int_0^\infty [P(t) - C(Q)]x(t)e^{-rt}dt \tag{2}$$

subject to $dQ/dt = x$, $Q(0) = 0$, and $P(t) = P_0 e^{\rho t}$. Because there is no uncertainty in the problem, the discount rate is the risk-free rate, r.

The solution to this problem can be obtained by straightforward application of dynamic programming. The optimal production rule is to produce if:

$$P \geq C(0) - V_Q = c - V_Q \tag{3}$$

where V_Q denotes dV/dQ. First, suppose $\rho = 0$. In this case the firm either produces now and forever (at its maximum capacity of $x = 1$), or it never produces. Therefore the integral in (2) can be evaluated directly:

$$\begin{aligned} V &= \frac{P}{r} - \frac{ce^{-\gamma Q}}{r(\gamma+r)}[r + \gamma e^{-(\gamma+r)(Q_m-Q)}] \quad ; \quad P \geq c - V_Q,\ Q \leq Q_m \\ &= 0 \quad ; \quad P < c - V_Q \end{aligned} \tag{4}$$

The shadow value of an additional unit of cumulative output, V_Q, evaluated at $Q = 0$, is then:

$$V_Q(Q=0) = \frac{\gamma c}{(\gamma+r)}[1 - e^{-(\gamma+r)Q_m}] \tag{5}$$

V_Q is zero, however, if $c - P$ is greater than the right-hand side of (5), because in that case the firm will never produce. We will examine the behavior of V_Q shortly, but for the moment note that if the firm is producing, V_Q is independent of price. Although V depends on P, V_Q only reflects future cost savings from current production, and since the firm continues to produce forever, these savings are independent of price. Also note that if $r = 0$, $V_Q = c(1 - e^{-\gamma Q_m}) = c - \bar{c}$. This means that the firm should produce if P is at least as large as final marginal cost, $\bar{c}$. This is the result obtained by Spence (1981), in a slightly different form.

Now suppose $\rho > 0$. In this case if P is sufficiently high, the firm will produce immediately, and continue to produce forever. V_Q is then given by eqn. (5) above, and is again independent of price, so the critical price above which production takes place is the same as for $\rho = 0$.

If P is initially below the critical price, the firm will defer production until some future time T, when $P(T) = c - V_Q$. The lower limit of the integral in (2) is now T, which must be determined. At $t = 0$ and $Q = 0$, V_Q is therefore:

$$V_Q(Q{=}0) = \frac{\gamma c e^{-rT}}{(\gamma+r)}[1 - e^{-(\gamma+r)Q_m}] \tag{6}$$

Note that the smaller is P_0, the larger is T. By setting $P_0 e^{\rho T} = c - V_Q e^{rT}$, we find that T is given by:

$$T(P_0) = \frac{1}{\rho}\log\left[\frac{c}{P_0}\{1 - \frac{\gamma}{\gamma+r}[1 - e^{-(\gamma+r)Q_m}]\}\right] \tag{7}$$

when $P_0 < c - c\gamma[1 - e^{-(\gamma+r)Qm}]/(\gamma+r)$, and $T = 0$ otherwise. By substituting (7) into (6), T can be eliminated, and we can determine how V_Q depends on P.

Figure 1 illustrates the solution to the optimal production problem, and the dependence of V_Q on P, for the following set of parameter values: initial marginal cost $c = 40$, final marginal cost $\bar{c} = 10$, $Q_m = 20$ (so that $\gamma = -(1/Q_m)\log(\bar{c}/c) = .0693$), and $r = .05$. The graph shows $V_Q(P)$ for $\rho = 0$, .01, .03 and .05, as well as $c - P$. The firm produces when price equals or exceeds the critical price $P^* = \$19$; at this price, $V_Q = c - P = \$21$. If $P \geq P^*$, the firm produces forever, so V_Q is constant at \$21. If $P < P^*$, the firm will eventually produce (unless $\rho = 0$), because future cost savings are discounted. The smaller is ρ the smaller is V_Q because the longer it will take for P to rise to P^*. If $\rho = 0$, P does not rise at all, so $V_Q = 0$.

Although not shown in Figure 1, if r is smaller, V_Q will be larger, and for $r = 0$, $P^* = \bar{c} = 10$. Viewed in real terms, an r of .05 is large, so the rule of thumb "Produce when price is close to ultimate marginal cost" would seem reasonable. Yet the anecdotal evidence suggests that firms usually require price to be close to initial marginal cost. This may be because firms use too high a discount rate, or, as we will see, because of uncertainty over future prices.

3. A Stochastic Model.

To incorporate uncertainty over future prices, we assume that the output price, P, evolves according to the following stochastic process:

$$dP/P = \rho dt + \sigma dz \quad (8)$$

Here, ρ is the expected rate of change of P, σ the standard deviation of this rate of change, and dz is the increment of a Weiner process. Eqn. (8) says that the current value of P is known, but future values are lognormally distributed with a variance that grows linearly with the time horizon.

We assume that the output price P is spanned by the set of existing traded assets in the economy (i.e., capital markets are sufficiently complete that there exists an asset, or a dynamic portfolio strategy using existing assets, that is perfectly correlated with P). This implies that the risk-adjusted discount rate for future output, which we denote by μ, is the expected return on the replicating portfolio. The expected rate of change of the output price, ρ, will in general be less than this risk-adjusted discount rate, i.e., $\mu = \rho + \delta$, $\delta > 0$. For example, if the output is a storable commodity, it may have a positive "convenience yield;" inventory holders are willing to accept an expected rate of price appreciation less than the expected return on assets with the same risk because of the flow of convenience benefits that inventories provide.[6]

With the assumption of spanning, we can value the firm and determine its optimal (value-maximizing) operating strategy using contingent claims analysis. Contingent claims analysis applied to options on traded securities usually relies on a replicating portfolio strategy to price the contingent claim relative to the underlying asset. However, it is not necessary that the securities in the replicating portfolio actually be traded. If the underlying asset is spanned by existing traded securities, any contingent claim on that asset is also spanned by the existing assets, and therefore can be valued using the same methodology.[7]

When P is stochastic, the firm may want to temporarily shut down (when P is low), and later resume production if and when P rises sufficiently. For simplicity, we will assume that there is no additional cost to stopping and later resuming production.[8] Of course, it may be optimal to produce even if the current cash flow is negative: the value of current production depends both on the current cash flow and on the amount by which future costs are lowered. In other words, there is a shadow value to current production which measures the benefit of moving down the learning curve.

The value of the firm will depend on P, and on how far it is along the learning curve (i.e., on cumulative output, Q). In turn, the firm's position on the learning curve depends on the production policy adopted: its choice of instantaneous output will determine the instantaneous cash flow as well as future production costs.

Under the assumptions described above, the firm's value-maximizing production decision can be characterized as a stochastic control problem. There are two state variables: the current market value of output, P, and the firm's cumulative output to date, Q. The control variable is the instantaneous rate of production, x(P,Q), subject to $0 \leq x \leq 1$. The optimal production policy is the production rule, $x^*(P,Q)$, that maximizes the current market value of the firm.

Because there are no adjustment costs or costs associated with changing the level of production, the optimal production rule has the same feature as in the deterministic model: the instantaneous level of production will be either 0 or 1, according to whether the market price of the output is below or above a critical price $P^*(Q)$. This critical price is determined endogenously with the (maximized) value of the firm. Using either the continuous time replicating approach of Merton (1977) or the risk-neutral valuation argument of Cox and Ross (1976) in conjunction with dynamic programming, one can derive a partial differential equation for the value of the firm, $V(P,Q)$, under the optimal production rule. In Appendix A we show that this equation is:

$$\tfrac{1}{2}\sigma^2 P^2 V_{PP} + (r-\delta)PV_P + V_Q - rV + [P - C(Q)] = 0 \; ; \quad P \geq P^* \qquad (9a)$$

$$\tfrac{1}{2}\sigma^2 P^2 V_{PP} + (r-\delta)PV_P - rV = 0 \; ; \quad P < P^* \qquad (9b)$$

This must be solved subject to the following boundary conditions:

$$V(0,Q) = 0 \qquad (10a)$$

$$\lim_{P\to\infty} V_P(P,Q) = 1/\delta \qquad (10b)$$

$$P^*(Q) - C(Q) + V_Q = 0 \qquad (10c)$$

$$V(P,Q_m) = \bar{V}(P) \qquad (10d)$$

Condition (10a) is implied by equation (8) for the dynamics of P; if P is ever zero, it will always remain zero, so the value of the firm will be zero. Condition (10b) follows from the fact that as the price becomes very large, the firm will almost surely always produce. In that case the incremental value of a \$1 increase in price is just the present value of \$1 per period paid forever, discounted at $\mu - \rho = \delta$. Condition (10c) is the free boundary, above which it is optimal to produce. It follows from the first-order condition for the optimization problem: The firm should produce whenever price equals or exceeds marginal cost less the shadow value of cumulative production, V_Q.

Finally, condition (10d) gives the value of the firm when it has produced to the point where production costs become constant (i.e. when $Q = Q_m$). At and beyond this level of cumulative production, the value of the firm is a function only of price. To calculate that value, which we denote by $\bar{V}(P)$, note that the firm faces a similar stochastic control problem as before. Using the same methods that led to equations (9a) and (9b), we get the following ordinary differential equation for the value of the firm when $Q \geq Q_m$:

$$\tfrac{1}{2}\sigma^2 P^2 \bar{V}_{PP} + (r-\delta)P\bar{V}_P - r\bar{V} + [P - \bar{c}] = 0 \quad ; \quad P \geq \bar{c} \tag{11a}$$

$$\tfrac{1}{2}\sigma^2 P^2 \bar{V}_{PP} + (r-\delta)P\bar{V}_P - r\bar{V} = 0 \quad ; \quad P < \bar{c} \tag{11b}$$

subject to the boundary conditions 9a, b, and c, except with $\bar{c}$ substituted for P^*. These equations have the following analytical solution:

$$\bar{V}(P) = \begin{cases} b_1 P^{\beta_1} & ; \quad P < \bar{c} \quad (12a) \\ b_2 P^{\beta_2} + P/\delta - \bar{c}/r & ; \quad P \geq \bar{c} \quad (12b) \end{cases}$$

where:

$$\beta_1 = -\frac{(r-\delta-\sigma^2/2)}{\sigma^2} + \frac{1}{\sigma^2}[(r-\delta-\sigma^2/2)^2 + 2r\sigma^2]^{1/2} > 1$$

$$\beta_2 = -\frac{(r-\delta-\sigma^2/2)}{\sigma^2} - \frac{1}{\sigma^2}[(r-\delta-\sigma^2/2)^2 + 2r\sigma^2]^{1/2} < 0$$

$$b_1 = \frac{r - \beta_2(r-\delta)}{r\delta(\beta_1 - \beta_2)}(\bar{c})^{1-\beta_1} > 0$$

$$b_2 = \frac{r - \beta_1(r-\delta)}{r\delta(\beta_1 - \beta_2)}(\bar{c})^{1-\beta_2} > 0$$

This solution for $\tilde{V}(P)$ is interpreted as follows. When $P < \bar{c}$ the firm does not produce. Then, $b_1 P^{\beta_1}$ is the value of the firm's option to produce in the future, should P increase. When $P \geq \bar{c}$, the firm produces. If, irrespective of changes in P, the firm had no choice but to continue producing in the future, the present value of the expected flow of profits would be given by $P/\delta - \bar{c}/r$. (Costs are certain and so are discounted at the risk-free rate; future values of P are discounted at the risk-adjusted rate μ, but P is expected to grow at rate ρ, so the effective net discount rate is $\mu - \rho = \delta$.) However, should P fall, the firm can stop producing. The value of the option to stop producing is $b_2 P^{\beta_2}$.

Eqn. (9b) has the following closed form solution that satisfies condition (10a):

$$V(P,Q) = aP^{\beta_1} \tag{13}$$

where β_1 is the constant given above, and a must be chosen to satisfy the remaining boundary conditions. However, eqn. (9a) does not have an analytic solution and must be solved numerically. We employ a finite-difference technique, transforming the continuous variables P and Q into discrete variables, and the partial differential equation into a difference equation. This equation is solved algebraically, and the solution proceeds backwards as a dynamic program incorporating the optimal production decision at each point. Hence the cutoff level, $P^*(Q)$, is solved for simultaneously with the value of the firm.[9] (Details of this procedure are in an appendix that is available from the authors on request.)

4. Production Decisions and the Value of the Firm.

Table 1 shows a solution for the same parameter values as used in the deterministic case shown in Figure 1: initial marginal cost $c = 40$, final marginal cost $\bar{c} = 10$, $Q_m = 20$ (so that $\gamma = .0693$), and $r = .05$. We set the annual standard deviation $\sigma = .20$, which is a conservative number for a competitively produced commodity.[10] Finally, we set $\delta = .05$. (Recall that δ is the difference between the risk-adjusted discount rate μ and the expected rate of price growth, ρ. Thus if all price risk is diversifiable, $\mu = r$ so $\rho = 0$, but if there is systematic risk, $\mu > r$ so $\rho > 0$.)

The table shows, for various amounts of cumulative production, the value of the firm as a function of price, as well as the critical price required for the firm to produce (denoted by an asterisk). For example, when cumulative production is zero (so that current cost is $40), the firm should produce when price is $25.53 or more. At the $25.53 price, the value of the firm is $178.53. At prices below $25.53 the firm does not produce, but still has value because of the possibility that price will rise above $25.53 in the future. As cumulative production increases the value of the firm rises (because costs have been reduced), and the critical price falls. For example, when cumulative production is 4.0 (so cost is $30.32), the critical price is $20.09. The critical price falls to the long-run cost of $10 as cumulative production reaches 20.[11] At this point the firm has reached the bottom of the learning curve, and the shadow value of cumulative production is zero.

Note from Figure 1 that in the absence of uncertainty, the initial critical price is much lower - about $19. To see how uncertainty affects the firm's production decision, it is useful to examine the shadow value of cumulative production, V_Q, and its dependence on both price and σ. Figure 2 shows V_Q as a function of P for $\sigma = 0$, .05, .1, .2, .3, and .5, for zero cumulative production. Also shown is the line c - P. The critical price, P^*, satisfies $P = c - V_Q$, and so is given by the intersection of the V_Q curve with the line c - P. When $\sigma = 0$, $\rho = r - \delta = 0$, so V_Q is zero up to the critical price of \$19, and then is constant at \$21 (as in Figure 1). Note that the larger is σ, the larger is P^*. For example, when $\sigma = .5$, P^* is about \$31.

The effect of price uncertainty on the shadow value of cumulative production depends on the current level of price. The possibility of future increases in price raises V_Q, and the possibility of decreases reduces it. At prices well below the critical price for $\sigma = 0$, it is the possibility of increases in price that dominates, so price uncertainty <u>increases</u> V_Q. To see this, note that if $\sigma = 0$, price can never increase (because $\delta = r$), so future cost savings have no value if price is low. However if $\sigma > 0$, price may eventually rise sufficiently so that the firm produces, and thus reductions in future costs have some value. For low P, the greater is σ the greater is the probability that the firm will begin to produce during some finite horizon, and thus the greater is the present value of reductions in future costs.

At higher prices the net effect is the opposite. Consider prices equal to or exceeding the P^* for $\sigma = 0$. If $\sigma = 0$, production will continue indefinitely, but if $\sigma > 0$ price may fall in the future to the point at which the firm shuts down (temporarily). The higher is σ the sooner is this likely to occur, and the greater is the proportion of time that the firm can expect not to produce. Thus the higher is σ the lower is the present value of future cost savings, i.e. the lower is V_Q. Also, as Figure 2 shows, the higher is the critical price required for production.

Although an increase in uncertainty increases the critical price it also increases the value of the firm at every price. Figure 3 shows the value of the firm, V, as a function of P for $\sigma = 0$, .1, .2, .3, and .5, again for zero cumulative production. (For $\sigma = .2$ the curve corresponds to the second column in Table 1.) As McDonald and Siegel (1985) have shown, for every future time t the firm has an option to produce that is analogous to a European call option on a common stock. The exercise price of each option is the production cost $C(Q_t)$, so that the net payoff from exercising is $P_t - C(Q_t)$. (The exercise price is stochastic, because future production cost is contingent on the evolution of price.) The value of each call option is a convex function of price, and therefore is increasing in σ. The value of the firm is the total value of an infinite number of such call options, one for every future time t, and therefore also increases with σ.

Figure 4 shows the effects of changes in δ, the difference between the risk-adjusted discount rate, μ, and the expected rate of price growth, ρ. (In each case $r = .05$, and $\sigma = .20$.) Note that the higher is δ, the lower is V_Q, and the higher is the critical price P^* required for production. To see why, suppose first that the firm is not producing ($P < P^*$). Then the higher is δ the lower is ρ, so the greater is the time before production is expected to commence, and the lower is the present value of future cost savings. (This is much the same as the deterministic case shown in Figure 1.) Now suppose the firm is producing ($P \geq P^*$). Now the higher is δ, the shorter is the expected time before the firm will shut down (temporarily), which again makes future cost savings worth less. Although not shown, the total value of the firm V(P) is also smaller when δ is higher, because price is not expected to grow as fast.

5. Conclusions.

We have shown how a firm's optimal production decision can be derived when there is learning-by-doing and uncertainty over future prices. By assuming that the output price is spanned by existing assets, no assumptions were needed regarding risk aversion or risk-adjusted discount rates - our production rule always maximizes the market value of the firm. For simplicity we have assumed that marginal production cost is constant with respect to the rate of output, but this assumption can easily be relaxed. In addition, the model can be adapted to incorporate uncertainty over factor prices as well as the price of output.

As is now well known, when a firm can shut down and later resume production, its value is increased by uncertainty over future prices. For a firm facing a learning curve, however, the shadow value of cumulative production is reduced by price uncertainty over the relevant range of prices, so that a higher price is required for the firm to produce. For those industries in which the learning curve is an important determinant of cost, this has a curious implication: Other things equal, during periods of high volatility firms ought to be producing less, but are worth more.

Appendix - Derivation of Equation (9)

Our assumption that P is spanned by traded assets implies that there exists a dynamic portfolio strategy that "replicates" the total return on P; i.e., one can invest in an asset/portfolio with price dynamics $dP = (\mu-\delta)Pdt + \sigma Pdz$, which pays a dividend at rate δPdt.

Let F(P,Q) be the solution to equation (9),

$$\frac{1}{2}\sigma^2P^2F_{PP} + (r-\delta)PF_P + XF_Q - rF + x[P-c(Q)] = 0, \qquad (9')$$

with $0 \leq x \leq 1$, and boundary conditions 10(a) - 10(d). From Ito's Lemma:

$$dF = [\frac{1}{2}\sigma^2P^2F_{PP} + (\mu-\delta)PF_P + xF_Q]dt + \sigma PF_Pdz. \qquad (A.1)$$

Substituting from (A.1):

$$dF = F_PdP + [rF - (r-\delta)PF_P - x(P-c)]dt. \qquad (A.2)$$

The continuous-time portfolio strategy with a fraction $\lambda(t)$ invested in the asset that replicates P, a fraction $1-\lambda(t)$ invested in the riskless asset, and which makes net withdrawals at a rate x(P-C), has dynamics:

$$dY = \lambda Y\frac{[dP + \delta dt]}{P} + (1-\lambda)Yrdt - x(P-C)dt. \qquad (A.3)$$

Choosing $\lambda = F_PP/Y$, we have:

$$dY = F_P(dP+\delta dt) + (Y-F_{PP})rdt - x(P-C)dt \qquad (A.4)$$

Hence, $dF - dY = r(F-Y)dt$. If the initial value of the portfolio is chosen as Y = F, then the portfolio Y will <u>always</u> have the same value as F. Since portfolio Y has the same value as the contingent claim V(P,Q) at the boundaries, and receives the same net payments as the contingent claim, in a well-functioning capital market it must be the portfolio that replicates the contingent claim. But Y = F, so the value of the contingent claim is F, the solution to eqn.(9) and boundary conditions above.

Table 1 - Value of Firm and Optimal Production Rule

	Cumulative Production (and Current Cost)					
	0.00 (40.00)	4.00 (30.32)	8.00 (22.98)	12.00 (17.41)	16.00 (13.20)	20.00 (10.00)
Price						
42.95	495.30	546.50	613.86	646.28	664.38	670.12
41.26	462.72	531.49	580.74	613.13	631.23	636.96
39.65	431.58	499.84	548.96	581.33	599.42	605.16
38.09	401.85	469.52	518.49	550.82	568.91	574.65
36.60	373.49	440.49	489.27	526.56	539.65	545.38
35.16	346.49	412.69	461.26	493.51	511.59	517.32
33.78	320.83	386.09	434.41	466.60	484.68	490.41
32.46	296.49	360.67	408.69	440.82	458.88	464.61
31.19	273.47	336.38	384.06	416.10	434.16	439.89
29.96	251.77	313.20	360.48	392.43	410.47	416.20
28.79	231.40	291.11	337.91	369.75	387.78	393.50
27.66	212.39	270.08	316.33	348.03	366.04	371.76
26.58	194.75	250.10	295.70	327.24	345.23	350.95
25.53	178.53*	231.15	276.00	307.36	325.31	331.03
24.53	163.76	213.24	257.21	288.33	306.25	311.98
23.57	150.22	196.35	239.29	270.15	288.03	293.75
22.65	137.79	180.50	222.24	252.78	270.61	276.32
21.76	126.40	165.70	206.03	236.20	253.96	259.67
20.91	115.94	151.97	190.65	220.39	238.06	243.77
20.09	106.35	139.32*	176.09	205.32	222.89	228.60
19.30	97.56	127.80	162.35	190.97	208.42	214.12
18.54	89.49	117.23	149.42	177.33	194.62	200.32
17.81	82.09	107.53	137.31	164.38	181.49	187.18
17.12	75.30	98.64	126.01	152.11	169.00	174.67
16.44	69.07	90.48	115.54	140.50	157.13	162.69
15.80	63.36	83.00	105.92*	129.55	145.86	151.50
15.18	58.12	76.13	97.16	119.26	135.19	140.79
14.59	53.31	69.83	89.12	109.61	125.09	130.65
14.01	48.90	64.06	81.75	100.62	115.55	121.06
13.46	44.86	58.76	74.99	92.29	106.57	112.00
12.94	41.15	53.90	68.79	84.61*	98.13	103.47
12.43	37.74	49.44	63.10	77.62	90.22	95.45
11.94	34.62	45.35	57.88	71.20	82.84	87.93
11.47	31.76	41.60	53.09	65.31	76.00	80.89
11.02	29.13	38.16	48.70	59.91	69.68*	74.33
10.59	26.72	35.00	44.67	54.95	63.92	68.24
10.18	24.51	32.11	40.98	50.41	58.63	62.61*
9.78	22.48	29.45	37.59	46.24	53.78	57.43
9.39	20.62	27.02	34.48	42.41	49.33	52.68
9.03	18.92	24.78	31.63	38.90	45.25	48.33
8.67	17.35	22.73	29.01	35.69	41.51	44.33
8.33	15.92	20.85	26.61	32.74	38.08	40.66
8.00	14.60	19.13	24.41	30.03	34.93	37.30

Note: Entries show the value of the firm as a function of price and cumulative production. Asterisks denote critical prices at which production occurs. Solution is for $r = .05$, $\delta = .05$, $c = 40$, $\underline{c} = 10$, $Q_{max} = 20$, and $\sigma = .20$

FOOTNOTES

1. For examples, see Alchian (1963), Rapping (1965), Baloff (1971), and Lieberman (1984). For a number of years, the Boston Consulting Group (1972) made the explanation of the existence and implications of the learning curve a central focus of their corporate consulting practice.

2. The learning curve also has implications for strategic behavior, which we will not address in this paper. Strategic considerations are discussed by Spence (1981) and Fudenberg and Tirole (1983). We also ignore learning spillovers from one firm to another; see Zimmerman (1982) for estimates of learning externalities in the construction of nuclear power plants.

3. As Spence (1979) has shown, if the discount rate is zero and there is no uncertainty over future demand or cost, current marginal revenue can be set equal to the ultimate marginal cost that will prevail when the firm has reached the bottom of the learning curve. However, there is no reason to for the discount rate to be zero, and there is every reason for future demand to be uncertain.

4. Dierkens (1984) examines the decision to invest in a production technology with uncertainty, in which costs decline with cumulative output. However in her model the only allowed change in production is a once and for all abandonment.

5. Note that different learning curves can be characterized by different values for the parameters c and γ. For example, the firm may have a choice between two production technologies, one with a high level of initial unit cost (c) but with a steeper learning curve (i.e., higher γ).

6. If there is a futures market for the output, the convenience yield can be estimated from the spot and futures prices. See Brennan and Schwartz (1985) and McDonald and Siegel (1984).

7. See Merton (1977), and the discussion on pages 7-8 in Majd and Myers (1986).

8. It is not difficult to include costs of stopping and restarting production. See Brennan and Schwartz (1985) for a model of a mining firm that includes these costs.

9. We employ this method in Majd and Pindyck (1987). For a useful discussion of finite difference methods, see Brennan and Schwartz (1978).

10. Bodie and Rosansky (1980) report the following annual standard deviations of percentage price changes over 1950 - 1976: wheat, 30.7 percent; corn, 26.3; oats, 19.5; eggs, 27.9; broilers, 39.2; cattle, 21.6; hogs, 36.6; wool, 37.0; cotton, 36.2; orange juice, 31.8; copper, 47.2; silver, 25.6; and lumber, 34.7.

11. In Table 1 the critical price is $10.18 when cumulative production is 20. This is an artifact of the discretization used to obtain a solution.

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FIGURE 1 - SHADOW VALUE OF CUMULATIVE PRODUCTION

($r = 0.05$, $\rho = 0.01, 0.03, 0.05$)

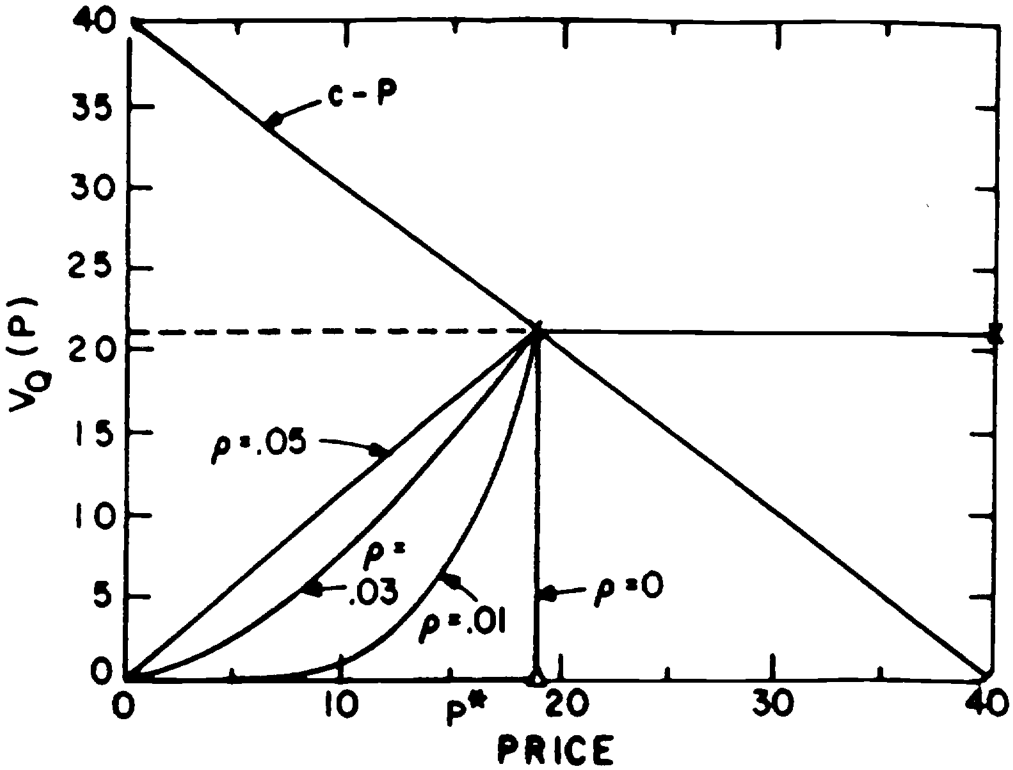

FIGURE 2 - SHADOW VALUE OF PRODUCTION

($\delta = .05$, $r = 0.05$, $\sigma = .05, .01, .2, .3, .5$)

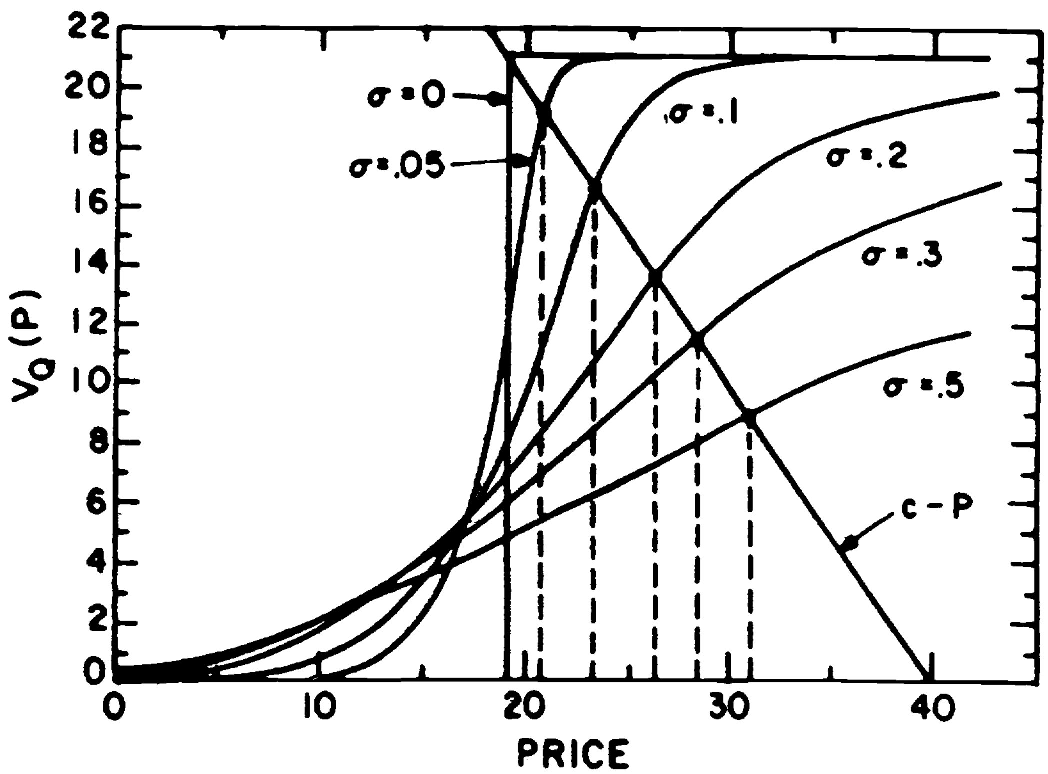

FIGURE 3 - VALUE OF FIRM

(σ = 0, .1, .2, .3, .5)

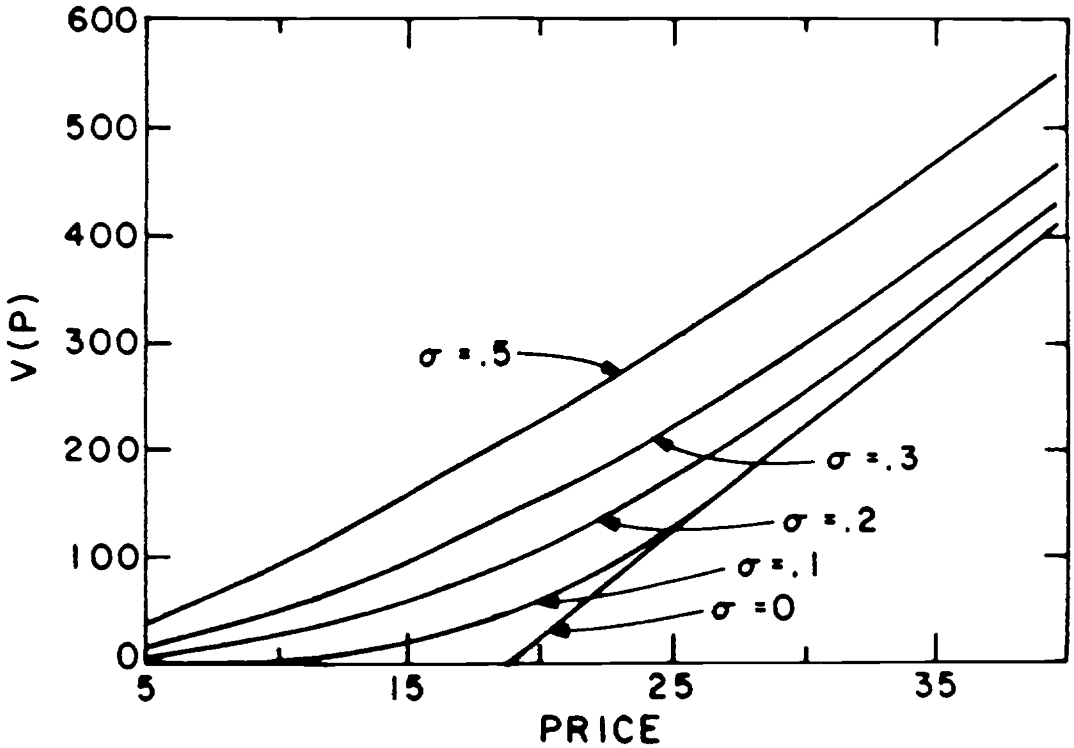

FIGURE 4 - SHADOW VALUE OF PRODUCTION

(r = .05, σ = .2, δ = .01, .05, .1)