General solutions for choice sets: The Generalized Optimal-Choice Axiom set

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GENERAL SOLUTIONS FOR CHOICE SETS: THE GENERALIZED OPTIMAL-CHOICE AXIOM SET

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Abstract. In this paper we characterize the existence of best choices of arbitrary binary relations over non finite sets of alternatives, according to the Generalized Optimal-Choice Axiom condition introduced by Schwartz. We focus not just in the best choices of a single set X, but rather in the best choices of all the members of a family $\mathcal{K}$ of subsets of X. Finally we generalize earlier known results concerning the existence (or the characterization) of maximal elements of binary relations on compact subsets of a given space of alternatives.

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1. Introduction

Individual or collective choice theories deal with the process of choice among a set of alternatives (a range of choice options) that are feasible or available. The choice process that may has the form of a rule, procedure, institutional mechanism, or a set of tastes may be represented by a function that is called choice function. The neoclassical economic theory assumes that each individual makes choices by selecting, from each feasible set of alternatives, those which maximize his own preference relation. This requirement links choice functions to underlying "preference"and "indifference"relations. More specifically, the individual's preference relation corresponding to a choice process is modeled as a binary relation on the set of alternatives. The choice set, from a given potential set, is the set of elements undominated under the relation of pairwise choice. This requires the relation to be acyclic. However, in collective choice problems (ranking of social preferences, voting in committees, etc) cyclicity is a plausible assumption as for example the Condorcet Paradox shows. In this case, no alternative appears to dominate all the others. We must then give up the requirement that choices are best. A number of theories of general solutions for choice sets deal with cyclic binary relations. These theories are called *solution theories*, and their main task is to specify what set of alternatives may be as reasonable solution of choice problems when the set of best alternatives does not exist. Example of such solution theories is the *Generalized Optimal-Choice Axiom* set ($\mathcal{GOCHA}$ set) introduced by Schwartz. The choice set from a given set specified by the $\mathcal{GOCHA}$ condition is the union of minimal sets the elements of which have the following property: No alternative outside this set is preferable to an alternative inside it.

Since the set of the best choices of a choice process corresponds to the maximal elements in the feasible set according to a binary relation, it is important to find conditions such that the set of maximal elements is non empty. To face this problem, there are two issues: (a) The type of the individual's preference relation and (b) the topological character of the feasible set over which the preferences are maximized. The standard approach is to assume acyclic and upper semi-continuous binary relations over compact sets of alternatives. In such case maximal elements exist. More precisely, when acyclicity holds, the classical results concerning maximality in binary relations are those of Sloss [16], Brown [6], Bergstrom [3] and Walker [24] (SBBW-Theorem in the sequel) which state that if an acyclic binary relation defined on a topological space X is upper continuous, then every compact subset of X contains a maximal element. Campbell and Walker in [7]: (i) Replace the assumption of lower continuity by the (weaker) assumption of weak lower continuity, (ii) assume stronger than acyclic binary relations (in fact pseudo-transitive binary relations) and obtain the existence of maximal elements on any compact subset of X. Peris and Subiza in [14] generalize the result of SBBW-Theorem for irreflexive (not necessarily acyclic) binary relations. Mehta in [13] and Subiza and Peris in [18] use the same framework as that considered in Bergstrom [3] and Walker in [24] and use a weaker than lower continuity condition to obtain maximal elements in compact sets. Alcantud in [1] relaxes the notion of compactness by introducing the concept of R-upper compactness of a topological space on which a binary relation R is defined. This allows him to give sufficient conditions that generalize the Peris and Subiza extension of the SBBW-Theorem for irreflexive binary relations. He also provides a characterization of the existence of maximal elements for acyclic binary relations. This characterization is refined by considering other sufficient conditions used by Mehta in [13, Theorem 3.7] and Subiza and Peris in [18].

The existence of maximal elements for not acyclic binary relations over compact subsets of Hausdorff topological vector spaces have also been used in a general equilibrium theory context: Maximal theorems for preference correspondences in compact spaces are thus applied to obtain existence theorems for generalized games (=abstract economies). Debreu [8] and Arrow and Debreu [2] first proved the existence of equilibria for economies with finitely many agents, finite dimensional strategy space and quasi-concave utility functions. These results have been generalized in several directions. Sonnenschein in [17] and Shafer and Sonnenschein in [15] prove the existence of maximal elements on topological compact sets when certain assumptions of convexity and continuity are satisfied. Borglin and Keiding [5] extend the Debreu's result to abstract economies in which agents have preference correspondences with open graph or open lower sections. Yannelis and Prabhakar [25] and Yannelis [26] generalize the previous results and give a new existence proof for an equilibrium, in abstract economies with an infinite number of commodities and a countably infinite number of agents.

As the above properties (a) and (b) are inherited when one passes to any compact subset of the underlying space (if we use each subset's relative topology) maximal elements continue to exist. This is of particular interest in the theory of the consumer as we are interested in the family of all possible budget sets; and in the theory of production as we are interested in the family of all possible production sets.

In this paper, we identify conditions under which individual or collective choices, potentially cyclic, can be rationalized by binary relations according to the $\mathcal{GOCHA}$ condition. More precisely, we present a characterization of the existence of the $\mathcal{GOCHA}$ set, over non-finite sets. In particular, we focus in the case where the maximal elements continue to exist when one passes from whole space to any R-upper compact subspace. These results generalize the results mentioned above for acyclic binary relations. Finally, we show that if the $\mathcal{GOCHA}$ set is non-empty, then it is R-upper compact. This is analogous to the fact that the set of maximal elements is compact in the cases considered by Bergstrom and Walker and R-upper compact in the case considered by Alcantud.

2. Notation and definitions

We first we give some definitions that we use throughout the paper.[1]

Let X be a (finite or infinite) non-empty set of alternatives, and let $R \subseteq X \times X$ be a binary relation on X. By $P(R)$ and $I(R)$ denote, respectively, the *asymmetric part* of R and the *symmetric part* of R, which are defined, respectively, by $P(R) = \{(x,y) \in X \times X | (x,y) \in R \text{ and } (y,x) \notin R\}$ and $I(R) = \{(x,y) \in X \times X | (x,y) \in R \text{ and } (y,x) \in$

[1]These can also be found in [1], [18],[19], [20], [21].

$R\}$. We denote $I^*(R) = I(R) \cup \Delta$ where $\Delta = \{(x,x) | x \in X\}$ the diagonal of X. We sometimes abbreviate $(x,y) \in R$ as xRy. $\mathcal{M}(R)$ denotes the elements of X that are *R-maximal* in X, i.e., $\mathcal{M}(R) = \{x \in X|$ for all $y \in X, yRx$ implies $xRy\}$. $\mathcal{M}(R/Y)$ denotes the elements of Y that are R-maximal in Y. A subset A of X is *undominated* iff for no $x \in A$ there is a $y \in X \setminus A$ such that yRx. An undominated set is *minimal* if none of its proper subsets has this property. The *transitive closure* of a relation R is denoted by $\overline{R}$, that is for all $x, y \in X, (x,y) \in \overline{R}$ if there exists $k \in \mathbb{N}$ and $x_0, ..., x_K \in X$ such that $x = x_0, (x_{k-1}, x_k) \in R$ for all $k \in \{1, ..., K\}$ and $x_K = y$. Clearly, $\overline{R}$ is transitive and, because the case $K = 1$ is included, it follows that $R \subseteq \overline{R}$. A subset $Y \subseteq X$ is a cycle if, for all $x, y \in Y$, we have $(x,y) \in \overline{R}$ and $(y,x) \in \overline{R}$. We say that R is *acyclic* if there does not exist a cycle. The binary relation R is *consistent*, if for all $x, y \in X$, for all $k \in \mathbb{N}$, and for all $x_0, x_1, ..., x_K \in X$, if $x = x_0$, $(x_{k-1}, x_k) \in R$ for all $k \in \{1, ..., K\}$ and $x_K = y$, then $(y,x) \notin P(R)$. If a binary relation R is consistent then the asymmetric part of R is contained in the asymmetric part of the transitive closure of R, i.e., $P(R) \subseteq P(\overline{R})$ (Duggan [9]).

Let Ω be a family of non-empty subsets of X that represents the different feasible sets presented for choice. A choice function is a mapping that assigns to each choice situation a subset of it:

$$C : \Omega \to X \text{ such that for all } A \in \Omega,\ C(A) \subseteq A.$$

For each A, $C(A)$ may represent the set of alternatives that are chosen by the given process, when A is presented to the individual or group. In sort, $C(A)$ is called the *choice set* of A. The problem of finding choice sets can be expressed as the maximization of the individuals's preferences over a set of alternatives. That is, for every $A \in \Omega$, $C(A) = \mathcal{M}(R/A)$.[2] To deal with the case where the set of maximal choices $C(A)$ is empty, Schwartz among others has proposed the following general solution:
*The Generalized Optimal-Choice Axiom $\mathcal{GOCHA}$ (*Schwartz*):* For each $A \in \Omega$, $C(A)$ is equivalent to the union of minimum undominated subsets of A. The *$\mathcal{GOCHA}$ set* is the choice set from a given set specified by the $\mathcal{GOCHA}$ condition. In what follows, $\mathcal{GOCHA}(R/Y)$ denotes the $\mathcal{GOCHA}$ set of Y under R. If $Y = X$, then we put $\mathcal{GOCHA}(R/Y) = \mathcal{GOCHA}(R)$.

Let R be a consistent binary relation defined on a topological space (X, τ). The relation R is: (a) *upper-semicontinuous* if for all $x \in X$ the set $\{y \in X | xP(R)y\}$ is open; (b) *upper tc-semicontinuous* if the set

[2] In the neoclassical economic theory the link between preference relations and choice functions is formed by Rational Choice Functions. This is because Rational Choice Functions choose the most preferred alternatives according to the individual's preferences.

$\{y \in X | xP(\overline{R})y\}$ is open. In consistent binary relations, upper semicontinuity implies upper tc-semicontinuity trivially, and these definitions are equivalent for partial orders. In acyclic binary relations, the two notions we have defined above coincide with the usual ones. A subset A of X is *lower* (*upper*) *set* if $x \in A$ and xRy (yRx) imply $y \in A$. The space (X, τ) is *R-upper compact* if for each collection of lower open sets which cover X there exists a finite subcollection that also covers X. If R is an irreflexive binary relation then the notion of R-upper compactness coincides with that of Alcantud in [1]. If $\approx$ is an equivalence relation on a topological space (X, τ), then the quotient set by this equivalence relation $\approx$ will be denoted by $\frac{X}{\approx}$, and its elements (equivalence classes) by $[x]$. Let the projection map $\pi : X \longrightarrow \frac{X}{\approx}$ which carries each point of X to the element of $\frac{X}{\approx}$ that contains it. In the quotient topology induced by π, a subset U of $\frac{X}{\approx}$ is open in $\frac{X}{\approx}$ if and only if $\pi^{-1}(U)$ is open in X. Thus, the typical open set in $\frac{X}{\approx}$ is a collection of equivalence classes whose union is an open set in X. If the set of alternatives is finite, we always assume the discrete topology. It is well known that if a finite set is endowed with the discrete topology, then every set is open. Thus, the topological conditions and continuity assumptions posed for the infinite case also hold in the finite case.

3. The Generalized Optimal-Choice Axiom set

3.1. Characterization of the existence of the $\mathcal{GOCHA}$ set. As we pointed out in the introduction, we are generally interested not in the $\mathcal{GOCHA}(R)$ set of just a single set X, but rather in the whole family $\mathcal{K}$ of R-upper compact subsets of some underlying set X, and in whether each member of $\mathcal{K}$ has non empty Generalized Optimal-Choice Axiom set.

We first give a general theorem which ensures the existence of maximal elements in every R- upper compact subset of a given topological space.

Proposition 1. Let R be a consistent binary relation on a topological space (X, τ). If R is upper semicontinuous then every R-upper compact subset of X has a maximal element.

Proof. Let Y be an R-upper compact subset of X. Suppose that Y has no maximal elements. Then, for each $x \in Y$, there exists $y \in Y$ such that $(y, x) \in P(R) \subseteq P(\overline{R})$. Thus,

$$Y = \bigcup_{y \in Y} (\{x \in X | yP(\overline{R})x\} \cap Y).$$

Clearly, for every $y \in Y$, the sets $\{x \in X|yP(\overline{R})x\} \cap Y$ are lower sets. On the other hand, upper semicontinuity and consistency imply that the sets $\{y \in Y|yP(\overline{R})x\} \cap Y$ are open in the relative topology of Y. Since the space is R-upper compact, there exist $\{y_1, ..., y_n\}$ such that

$$Y = \bigcup_{y \in \{y_1, \ldots, y_n\}} (\{x \in X|yP(\overline{R})x\} \cap Y).$$

Consider the finite set $\{y_1, ..., y_n\}$. Since $y_1 \in Y$, then there exist $i \in \{1, ..., n\}$ such that $y_i P(\overline{R}) y_1$. If $i = 1$, then we have a contradiction. Otherwise, call this element y_2. We have $y_2 \overline{P(R)} y_1$. Similarly, $y_3 P(\overline{R}) y_2 P(\overline{R}) y_1$. As $\{y_1, ..., y_n\}$ is finite, by an induction argument based on this logic, we obtain the existence of a cycle for $P(\overline{R})$. This last conclusion contradicts the acyclicity of $P(\overline{R})$. □

Notice that acyclicity implies consistency and compactness implies R-upper compactness. Hence, as a result of Proposition 2, we have the corresponded results of Sloss [16], Brown [6], Bergstrom [3], Walker [24], Subiza and Peris[14] and Alcantud [1].[3]

For a binary relation R, we define a binary relation R^* on $\frac{X}{I^*(\overline{R})}$ (in a similar way as Peris and Subiza define relation P in [14] for irreflexive relations), as follows:

$[x]R^*[y]$ if there are $x' \in [x], y' \in [y]$ such that $x'P(\overline{R})y'$.

We call R^* the *quotient relation of* R. Clearly, R^* is an acyclic binary relation.

Proposition 2. Let R be a consistent binary relation defined on a set X and let R^* be the quotient relation of R on $\frac{X}{I^*(\overline{R})}$. If $[z]$ is a maximal element on $\frac{X}{I^*(\overline{R})}$, then each $t \in [z]$ belongs to $\mathcal{GOCHA}(R)$.

Proof. Let $[z]$ be a maximal element on $\frac{X}{I^*(\overline{R})}$. We prove that $[z]$ is a minimal undominated set in X. Suppose that there exists a $t \in X \setminus [z]$ such that $(t, s) \in R$ for some $s \in [z]$. We cannot have $(s, t) \in R$, since in that case $t \in [z]$ which is impossible. Hence, $(t, s) \in P(R) \subseteq P(\overline{R})$ which jointly with $sI(\overline{R})z$ would imply that $tP(\overline{R})z$. Hence, $[t]R^*[z]$ and $[t] \neq [z]$, but this contradicts the maximality of $[z]$. Therefore, $t \in \mathcal{GOCHA}(R)$. □

[3] Sloss, Brown, Bergstrom and Walker prove that if an acyclic binary relation defined on a topological space (X, τ) is upper semicontinuous, then every non-empty compact subset of the space contains a maximal element. Subiza and Peris derive the same result in the case where R is acyclic and lower quasi-continuous and Alcantud in the case where R is partial order, upper semicontinuous and the space is R-upper compact.

We now use Propositions 1 to show the existence theorem of $\mathcal{GOCHA}(R)$ set for every R-upper compact subset of the underling set.

Theorem 3. Let (X,τ) be a topological space, and let R be an upper tc-semicontinuous binary relation on X. Then every R-upper compact subset of X has a non empty $\mathcal{GOCHA}$ set.

Proof. Let Y be an R-upper compact subset of X. We first show that $(\frac{X}{I^\star(\overline{R})}, R^*)$ with the quotient topology satisfies the hypotheses of Proposition 1:
(a) R^* *is a consistent binary relation in* $\frac{X}{I^\star(\overline{R})}$. Suppose that $[x],[y]\in \frac{X}{I^\star(\overline{R})}$, $n\in\mathbb{N}$, $[x_0],[x_1],...,[x_n]\in\frac{X}{I^\star(\overline{R})}$ such that $[x]=[x_0]$, $([x_{k-1}],[x_k])\in R^*$ for all $k\in\{1,...,n\}$ and $[x_n]=[y]$. We must prove that $([y],[x])\notin P(R^*)=R^*$. Indeed, There are $x'\in[x]$, $x_1',x_1''\in[x_1]$, $x_2',x_2''\in[x_2]$,...,$x_{n-1}',x_{n-1}''\in[x_{n-1}]$, $y'\in[y]$ such that

$$x'P(\overline{R})x_1'I^\star(\overline{R})x_1''P(\overline{R})x_2'...P(\overline{R})x_{n-i}'I^\star(\overline{R})x_{n-1}''P(\overline{R})y'$$

Hence, $x'P(\overline{R})y'$. Then for each $t\in[x]$ and each $s\in[y]$ we have $tI^\star(\overline{R})x'P(\overline{R})y'I^\star(\overline{R})s$ which implies that $tP(\overline{R})s$. This last conclusion implies that $([y],[x])\notin R^*$.
(b) R^* *is upper semicontinuous.* Indeed, since $\pi^{-1}\{[y]|[x]R^*[y]\}=\{y\in X|xP(\overline{R})y\}\in\tau$, we conclude that the set $\{[y]|[x]R^*[y]\}$ is open in the quotient topology for each $[x]\in\frac{X}{I(\overline{R})}$. Hence, R^* is upper semicontinuous. It remains to prove that:
(c) $\frac{Y}{I^*(\overline{R})}$ *is R^*-upper compact.* Indeed, let $\frac{Y}{I^*(\overline{R})}=\bigcup_{a\in A}(U_a\cap\dfrac{Y}{I^*(\overline{R})})$ with U_a R^*-lower open set. Then,

$$Y=\bigcup_{a\in A}\pi^{-1}(U_a\cap\frac{Y}{I^*(\overline{R})}).$$

To show that $\pi^{-1}(U_a\cap\frac{Y}{I^*(\overline{R})})$ is an R-lower set, suppose that $x\in\pi^{-1}(U_a\cap\frac{Y}{I^*(\overline{R})})$ and $y\in X$ such that $(x,y)\in R$. There are two cases to consider: (i) $(y,x)\in\overline{R}$; (ii) $(y,x)\notin\overline{R}$. In the first case, $[x]=[y]$ and $[x]\in U_a\cap\frac{Y}{I^*(\overline{R})}$ imply that $y\in\pi^{-1}(U_a\cap\frac{Y}{I^*(\overline{R})})\subseteq Y$. In case (ii) it follows that $[x]\in U_a\cap\frac{Y}{I^*(\overline{R})}$, $[x]\neq[y]$ and $([x],[y])\in R^*$. Thus, $[y]\in U_a\cap\frac{Y}{I^*(\overline{R})}$ which implies that $y\in\pi^{-1}(U_a\cap\frac{Y}{I^*(\overline{R})})$. Therefore, since Y is R-upper compact we have $Y=\bigcup_{i\in 1,...,n}\pi^{-1}(U_{a_i}\cap\dfrac{Y}{I^*(\overline{R})})$ for some $a_i\in A$, and thus $\frac{Y}{I^*(\overline{R})} = \bigcup_{i\in 1,\ldots,n} (U_{a_i} \cap \dfrac{Y}{I^*(\overline{R})})$. Hence, $\frac{Y}{I^*(\overline{R})}$ is R^*-upper compact. By Proposition 1 we conclude that $\frac{Y}{I^*(\overline{R})}$ has a maximal element. Hence, Proposition 2 implies that the $\mathcal{GOCHA}(R/Y)$ set is not empty. □

Proposition 4. Let X be a non empty set of alternatives and let R be an acyclic binary relation over X. Then, $\mathcal{GOCHA}(R) = \mathcal{M}(R)$.

Proof. Suppose that $x \in \mathcal{GOCHA}(R)$. Then, there exists an undominated subset D of X such that $x \in D$. We prove that $x \in \mathcal{M}(R)$. Suppose to the contrary that x is not a maximal element of R. Then, there exists a $y \in X$ such that $(y, x) \in R$. Since for each $y \in X \setminus D$, we have $(y, x) \notin R$, we conclude that $y \in D$. Hence, the acyclicity of R implies that $(x, y) \notin \overline{R}$.

Let

$$D^* = \{\lambda \in D | (x, \lambda) \notin \overline{R}\}.$$

Then, $\{y\} \subseteq D^* \subseteq D$. We show that this last conclusion will lead to a contradiction. Indeed, suppose that $D^* = D$, then $D \setminus \{x\} \subset D$ is an undominated subset of X, which is impossible because of the minimal character of D. It remains to exclude the case $D^* \subset D$. In fact, if we were in this case, then for each $t \in D \setminus D^*$ and each $s \in D^*$ there must hold $(t, s) \notin \overline{R}$, for otherwise, from $(x, t) \in \overline{R}$ $(t \in D \setminus D^*)$ and $(t, s) \in \overline{R}$ we must have $(x, s) \in \overline{R}$ which is impossible $(s \in D^*)$. In such case we have $(t, s) \notin R$. Hence, $D^* \subset D$ is an undominated subset of X which is impossible. This contradiction implies that x is a maximal element of R. Conversely, if x is a maximal element of R, then $\{x\}$ is a minimal undominated element of X. □

Since acyclicity concludes that upper semicontinuity implies upper tc-semicontinuity, by using Theorem 3 and Proposition 4 we have the following result.

Corollary 5. Let (X, τ) be an R-upper compact topological space, and let R be an acyclic upper semicontinuous binary relation on X. Then the set of of maximal elements in X is non empty.

Next theorem characterizes the existence of the $\mathcal{GOCHA}(R)$ set for binary relations.

Theorem 6. Let R be a binary relation on X and let Y be a subset of X. The following conditions are equivalent:

(i) The $\mathcal{GOCHA}(R/Y)$ set is non empty,

(ii) there exists a topology τ on X such that R is upper tc-semicontinuous and Y is compact in relative topology,

(iii) there exists a topology τ on X such that R is upper tc-semicontinuous and Y is R-upper compact in relative topology.

Proof. It is then obvious that (ii) implies (iii). By Theorem 3, we have that (iii) implies (i). It remains to prove that (i) implies (ii). Indeed, let $Y \subseteq X$ and let $\mathcal{GOCHA}(R/Y) \neq \emptyset$. Then, $\mathcal{GOCHA}(R/Y) = \bigcup_{i \in I} D_i$ where D_i are minimum undominated subsets of Y. Let τ be the excluded set topology generated by $\mathcal{GOCHA}(R/Y)$ [12, page 48] (it has as open sets all those subsets of Y which are disjoint from $\mathcal{G}$, together with Y and X themselves). Then Y is compact under τ in relative topology since every open cover of Y includes $Y = Y \cap X$ itself. Hence, $\{Y\}$ is always a finite subcover. It remains to prove that R is upper tc-semicontinuous. We prove that for each $x \in X$ the sets $\{y \in X | xP(\overline{R})y\}$ are open in τ. We have two cases to consider: (a) $x \notin \mathcal{GOCHA}(R/Y)$; (b) $x \in \mathcal{GOCHA}(R/Y)$. In the first case, we show that for each $t \in \mathcal{GOCHA}(R/Y)$, there holds $(x,t) \notin \overline{R}$. Indeed, suppose to the contrary that $(x,t_0) \in \overline{R}$ for some $t_0 \in \mathcal{GOCHA}(R/Y)$. It then follows that, there exists a natural number n and alternatives $t_1, t_2, ..., t_{n-1}, t_n$ such that $xRt_1...t_{n-1}Rt_nRt_0$. Therefore, $t_n \in \mathcal{GOCHA}(R/Y)$, for suppose otherwise: since $t_0 \in \mathcal{GOCHA}(R/Y)$, we cannot have t_nRt_0. Similarly, $t_{n-1} \in \mathcal{GOCHA}(R/Y)$, and an induction argument based on this logic yields $x \in \mathcal{GOCHA}(R/Y)$, a contradiction. Hence, $\{y \in X|xP(\overline{R})y\} \cap \mathcal{GOCHA}(R/Y) = \emptyset$. To prove the second case above, let $x \in \mathcal{GOCHA}(R/Y) = \bigcup_{i \in I} D_i$ where D_i are minimum undominated subsets of Y. Then, there exists $i \in I$ such that $x \in D_i$. Suppose that $xP(\overline{R})y$ for some $y \in \mathcal{GOCHA}(R/Y)$. There are two subcases to consider: (b_1) $y \in D_i$; (b_2) $y \in D_j$ with $j \neq i$.

Subcase (b_1). Let $y \in D_i$. Put $A_y = \{t \in D_i \,|(y,t) \in \overline{R}\,\}$. We have that $A_y \neq \emptyset$, because otherwise, for each $t \in D_i$, $(y,t) \notin \overline{R} \supseteq R$, which implies that $D_i \setminus \{y\} \subset D_i$ is an undominated subset of X, a contradiction because of the minimal character of D_i. Let $D_i^* = D_i \setminus A_y$. We now show that $D_i^* = \emptyset$. We proceed by the way of contradiction. Suppose that $D_i^* \neq \emptyset$. Then, for each $t \in A_y$ and each $s \in D_i^*$ we have $(t,s) \notin R$ for suppose otherwise, $(t,s) \in R$ implies that $(y,s) \in \overline{R}$ contradicting $s \in D_i^*$. Therefore, $D_i^* \subset D_i$ is an undominated subset of X, again a contradiction. Hence, $A_y = D_i$. But then, since $x \in D_i$ we conclude that $(y,x) \in \overline{R}$ which contradicts $xP(\overline{R})y$.

Subcase (b_2). In this subcase we have $xP(\overline{R})y$ with $x \in D_i$, $y \in D_j$ and $i \neq j$. But then, by using $x\overline{R}y$ as in the case (i) above we conclude that $x \in D_j$, an absurd. Hence, in any case we have $\{y \in X | xP(\overline{R})y\} \cap \mathcal{GOCHA}(R/Y) = \emptyset$. Hence the proof is complete. □

The following corollary is an immediate consequence of Proposition 4 and Theorem 6.

Corollary 7. Let R be an acyclic binary relation on X and let Y be a subset of X. The following conditions are equivalent:

(i) The set of maximal elements in Y is non empty,

(ii) there exists a topology τ on X such that R is upper semicontinuous and Y is compact in relative topology,

(iii) there exists a topology τ on X such that R is upper semicontinuous and Y is R-upper compact in relative topology.

If we apply Corollary 7 for $Y = X$, we have the main result of Alcantud in [1].

3.2. **Transfer continuities.** It is trivial that if a binary relation R is acyclic, then R is upper tc-semicontinuous if and only if for every $x, y \in X$ such that xRy there is a neighborhood $N(y)$ of y such that $x\overline{R}z$ for each $z \in N(y)$. Some authors use this alternative expression for upper tc-semicontinuity; cf., e.g., Peris and Subiza [18], where that property is called *lower quasi-continuity.* Tian and Zhou [27] prove a similar result by using a condition that they call *transfer continuity* which states that whenever xRy there exists a point x' and a neighborhood $N(y)$ of y such that $x'Rz$ for each $z \in N(y)$. This condition was used by Sonnenschein [17] to prove the existence of maximal elements for relations (not necessarily acyclic) satisfying a convexity condition. Subiza and Peris give an example [18, Example 2] which show that transfer continuity and lower quasi-continuity are independent properties. They also give a kind of continuity that generalizes both lower quasi-continuity and transfer continuity. More precisely: A binary relation R defined on a topological space (X, τ) is *transfer lower quasi-continuous* ([18]) if for all $x \in X$ there is x' and a neighborhood $N(y)$ of y such that $x'\overline{R}z$ for all $z \in N(y)$. If we suppose that R satisfies the continuity condition $\mathcal{P}$, it can be seen that R is *transfer* $\mathcal{P}$ on X if it is $\mathcal{P}$ by choosing $x' = x$. The basic idea behind the transfer continuities is the following: For a preference R to have, maximal elements, given $y \in R(x)$, the conventional continuity conditions describe relations between x and a neighborhood of y. However, to characterize the existence of maximal elements for a preference, the topological structure of R below the level of y is irrelevant and only the topological structure of R above the level of y is important. Therefore, we do not have to know the topological relations between x and a neighborhood of y. We only need to know the relation between a neighborhood of y and an element x' in its "upper"part of this neighborhood.

Results on the existence of maximal elements by using transfer continuities, have been given by Subiza and Peris [18], Mechta [13], Tian and Zhou [27] and Alcantud [1]. Although these results ensure the existence of maximal elements on compact topological spaces for weaker continuity conditions from that described in §3.1, they cannot do that in compact subspaces. This is due to the fact that transfer lower quasi-continuity condition is not a "local"property, in the sense that we cannot ensure the existence of maximal elements in closed subsets of the underlying set. We now give a general theorem which ensures the existence of maximal elements in compact sets. To do that, we extend the notion of transfer lower (quasi-) continuous defined by Subiza and Peris.

Definition 8. Let (X,τ) be a topological space and let R be a consistent binary relation defined on X. Then, R is said to be *generalized transfer lower (quasi-) continuous* if, whenever $xP(R)y$, there exists a point x' and a neighborhood $N(y)$ of y such that $x'P(R)z$ ($x'P(\overline{R})z$) for all $z \in N(y)$.

Clearly, in acyclic binary relations the notions of generalized transfer lower (quasi-)continuity and transfer lower (quasi-)continuity coincide.

Proposition 9. Let X be a compact topological space and let R be a generalized transfer lower quasi-continuous binary relation defined on X. Then the set of maximal elements of R on X is non-empty.

Proof. Suppose that there is not a maximal element. Then for every element $x \in X$ there exists $y \in X$ such that $yP(R)x$. Since the space is generalized transfer lower quasi-continuous, there exists $y(x) \in X$ and an open neighborhood $N(x)$ of x which satisfies $y(x)P(\overline{R})z$ for every $z \in N(x)$.

Thus,

$$X = \bigcup_{x\in X} N(x).$$

Since the space is compact, there exist $\{x_1, ..., x_n\}$ such that

$$X = \bigcup_{i\in\{1,...,n\}} N(x_i).$$

Consider the finite set $\{y(x_1), ..., y(x_n)\}$. Then by following a similar argument as in Proposition 1 we obtain the existence of a $P(\overline{R})$-cycle in $\{y(x_1), ..., y(x_n)\}$, and therefore a contradiction. □

The previous Proposition generalizes the corresponding theorems of Mehta [13], Subiza and Peris [18] and Alcantud [1].

The following Theorem can be derived from Proposition 9, in the same way as Theorem 3 is derived by Proposition 1.

Theorem 10. Let (X, τ) be an R-upper compact topological space, and let R be a generalized transfer lower quasi-continuous binary relation on X. Then, the $\mathcal{GOCHA}(R)$ set is non empty.

In a similar way, as in Theorem 6, we can prove the following theorem.

Theorem 11. Let R be a binary relation on X. The following conditions are equivalent:
(i) The $\mathcal{GOCHA}$ set is non empty,
(ii) there exists a compact topology τ on X such that R is generalized transfer lower continuous,
(iii) there exists a compact topology τ on X such that R is generalized transfer lower quasi-continuous.

The following corollary is an immediate consequence of Proposition 4 and Theorem 11.

Corollary 12. [1, Theorem 5]. Let R be an acyclic binary relation on X. The following conditions are equivalent:
(i) The set of maximal elements is non empty,
(ii) there exists a compact topology τ on X such that R is transfer lower continuous,
(iii) there exists a compact topology τ on X such that R is transfer lower quasi-continuous.

4. The structure of the $\mathcal{GOCHA}(R)$ set

In the cases considered by Sloss, Brown, Bergstrom and Walker, if the binary relation is upper semicontinuous and the underlying set is compact then the set of maximal elements obtained is compact too (see 7.12 in Border [4]). Alcantud considers a condition weaker than upper semicontinuity (in fact upper tc-semicontinuity) for the binary relation and a weaker condition than compactness (R-upper compactness) in the underlying set and he proves that the set of maximal elements is R-upper compact.

Similarly, we shall show that in the case where an arbitrary binary relation is upper tc-semicontinuous and the underlying set is R-upper compact then the $\mathcal{GOCHA}(R)$ set is R-upper compact.

Proposition 13. Let R be a binary relation on (X,τ). Suppose that X is R-upper compact and the relation R is upper tc-semicontinuous. Then, $\mathcal{GOCHA}(R)$ is R-upper compact.

Proof. We firstly prove that the $\mathcal{GOCHA}(R)$ set equals to $\bigcap_{x\in X}(X-\{y\in X|xP(\overline{R})y\})$. Indeed, let $y^*\in\mathcal{GOCHA}(R)$. Then, as in the proof of Theorem 7, we conclude that $\mathcal{GOCHA}(R)\cap\{y\in X|xP(\overline{R})y\})=\emptyset$ and hence, for each $x\in X$ we have $(x,y^*)\notin P(\overline{R})$. Therefore, for each $x\in X$, $y^*\notin\{y\in X|xP(\overline{R})y\}$. Thus, $y^*\in\bigcap_{x\in X}(X-\{y\in X|xP(\overline{R})y\})$. Conversely, if $y^*\in\bigcap_{x\in X}(X-\{y\in X|xP(\overline{R})y\})$, then for each $x\in X$ we have $(x,y^*)\notin P(\overline{R})$. We prove that y^* belongs to a minimum undominated subset of X. If for each $x\in X$ we have $(x,y^*)\notin R$, then $\{y^*\}$ is an undominated subset of X. It is a minimum one because it is a unit set. Otherwise, there exists $x\in X$ such that $(x,y^*)\in R\subseteq\overline{R}$. Since $(x,y^*)\notin P(\overline{R})$, we conclude that $(y^*,x)\in\overline{R}$. Let $\mathcal{C}(y^*)$ be the cycle containing y^* that is maximal in the sense that it is not a proper subset of any other cycle. We prove that $\mathcal{C}(y^*)$ is an undominated subset of X. Suppose on the contrary, that $(t,z)\in R$ for some $t\in X\setminus\mathcal{C}(y^*)$ and $z\in\mathcal{C}(y^*)$; to deduce a contradiction. It follows that $(t,y^*)\in\overline{R}$ which implies that $(y^*,t)\in\overline{R}$. Hence, $t\in\mathcal{C}(y^*)$, a contradiction. Clearly, $\mathcal{C}(y^*)$ is a minimal undominated subset of X. Hence, $y^*\in\mathcal{GOCHA}(R)$. Because the intersection of closed upper sets is again a closed upper set, the $\mathcal{GOCHA}$ set is R-upper compact, by [1, Lemma 2] . □

Since acyclicity of R implies $P(\overline{R})=\overline{R}$, the following result is an immediate consequence of the previous proposition and Proposition 4.

Corollary 14. ([1, Proposition 2]). Let R be an acyclic binary relation on (X,τ). Suppose that X is R-upper compact and R is upper tc-semicontinuous. Then, the set of maximal elements is non-empty and R-upper compact.

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