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Abstract In this work we contribute to the formal analysis of the measurement of consensus in a society. Instead of approaching the topic from an absolute perspective we are concerned with a practical application: the proposal of a decision mechanism with respect to which consensus is measured. Surprisingly this produces a powerful unifying model, a restriction of which is deeply analysed. We also study the axiomatic properties of particular expressions for consensus with various salient social rules as a reference.

Keywords Consensus · measurement · Borda rule · Copeland rule

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1 Introduction

A classical group decision making problem is established in a context where a group of voters or experts have to make a decision on a set of alternatives or candidates. The experts' opinions about alternatives are usually characterized by their ideas, principles, knowledge, etc. and this fact causes difficulties when it comes to making a collective decision or select one alternative or candidate. In this paper we focus on measuring the degree of agreement between the voters and the final decision reached via voting systems.

Suppose the case of a (finite) committee that intends to offer vacancies to a (finite) list of candidates. Each member is assumed to produce a complete preorder on the candidates, that is, ties are allowed. For various plausible reasons the committee wants to agree on a complete preorder of the candidates, for example because the candidates may reject the offer, or because the number of candidates to be appointed is externally and independently decided. It is intuitively clear that some orderings convey "higher consensus" than others, whatever the formal meaning that we attach to that term.

We here propose a model that considers both aspects of the process, namely, the social preference on the alternatives and the consensus that arises from it. Generally speaking, the question we pose ourselves is: How should the design be for the committee to reach a consistent decision (in the form of a complete preorder on the candidates) with regard to favouring consensus? This issue is linked to different branches of the social choice literature. Firstly, voting rules come to mind as a well-established tool to reach a decision. Secondly, the measurement of consensus must be introduced in the analysis.

Regarding the latter problem we separate from the main trend in the literature, that consists of proposing and axiomatizing particular formulations for an absolute measure of consensus or coherence (v., e.g., Bosch [4] or Alcalde and Vorsatz [2]). We here provide an alternative methodology for approaching the measurement of consensus, which we call *referenced consensus measures*. In agreement with the discussion above, it can be specialized via two ways: the "voting rule" that is selected, and the measure of agreement between profiles and orderings. We prove that this model permits to unify all the approaches to measure consensus in the sense of the aforementioned literature. Then we perform a descriptive analysis of its formal properties with a special emphasis on a subclass of it –that we call *normal referenced consensus measures*– and particularly on two relevant cases whose explicit constructions are detailed. Adopting this latter position has several advantages. It gives a single and practical solution to a problem, which permits to compare proposals on the basis of the consensus that they yield and therefore favours descriptive analyses. Also, in view of the behavior of the general classes that we study one can conclude that the performance of this solution is sufficiently good to value it as a decision aiding tool.

The paper is organized as follows. Section 2 is devoted to introduce basic notation and definitions, as well as our proposal of measurement of consensus, the referenced consensus measure. We prove that it incorporates the usual model by consensus measures. Then we propose the particular subclass of normal referenced consensus measures as a suitable framework where a better normative behavior can be guaranteed. In Section 3 operational characterizations of some focal voting rules are provided, which helps us to deal with the two explicit proposals for measurement of consensus that we present. Also we perform a short analysis of the dichotomous case. In Section 4 we explore a list of appealing properties of normal referenced consensus measures, and particularly of our explicit proposals. Finally, in Section 5 we give some concluding remarks and pose questions for further research.

2 Notation and Definitions

We fix $X = \{x_1, ..., x_k\}$, a finite set of k options, alternatives or candidates. Abusing notation, on occasions we refer to option x_s as option s for convenience. A population of agents or voters is a finite subset $\mathbf{N} = \{1, 2, ..., N\}$ of natural numbers. We also denote $\mathbf{K} = \{\{i, j\} \subseteq \mathbb{N} : i, j \in \{1, 2, ..., k\},\ i \neq j\}$.

Let $W(X)$ be the set of weak orders or complete preorders on X, that is, the set of complete and transitive binary relations on X. If $R \in W(X)$ is a weak order on X that reflects the preferences of a voter, then by $x_k R x_j$ we mean "R-voter thinks that alternative x_k is at last as good as x_j". $L(X)$ denotes the set of linear orders on X.

A *profile* $\mathcal{R} = (R_1, \ldots, R_N) \in W(X) \times \overset{N}{\ldots} \times W(X)$ is a vector of weak orders, where $R_i \in W(X)$ represents the preferences of the individual i on the k alternatives or candidates for each $i = 1, \ldots, N$. The *reversal* of the profile $\mathcal{R}$, denoted by $\mathcal{R}^{-1}$, is the profile $(R_1^{-1}, ..., R_N^{-1})$ where $x_s\, R_i^{-1}\, x_t \Leftrightarrow x_t\, R_i\, x_s$ for each possible voter $i \in \{1, \ldots, N\}$ and candidates or alternatives $x_s,\, x_t \in \{1, \ldots, k\}$. We say that the profile $\mathcal{R}$ is constant to R if $\mathcal{R} = (R, \overset{N}{\ldots\ldots}, R)$.

Any permutation σ of the voters $\{1, 2, ..., N\}$ determines a permutation of $\mathcal{R}$ by $\mathcal{R}^\sigma = (R_{\sigma(1)}, ..., R_{\sigma(N)})$. Similarly, any permutation π of the candidates $\{1, 2, ..., k\}$ determines a permutation of every complete preorder $R \in W(X)$ via $x_s\, {}^{\pi}R_i\, x_t \Leftrightarrow x_{\pi^{-1}(s)}\, R_i\, x_{\pi^{-1}(t)}$ for all $s, t \in \{1, \ldots, k\}$ and $i \in \{1, \ldots, N\}$. Then with $\mathcal{R}$ and π we can associate ${}^{\pi}\mathcal{R} = ({}^{\pi}R_1, ..., {}^{\pi}R_N)$.

Finally, given any profile of weak orders $\mathcal{R} = (R_1, \ldots, R_N) \in W(X)^N$ and any weak order R' on X, we denote $\mathcal{R} \uplus R'$ the profile $(R_1, \ldots, R_N, R')$ of $N + 1$ weak orders. We denote by $\mathcal{P}(X)$ the set of all profiles, that is, $\mathcal{P}(X) = \bigcup_{N \geqslant 2} W(X)^N$.

2.1 Basic Definitions

A *Consensus measure with reference to a consensus function* (henceforth, *referenced consensus measure*, RCM for simplicity, when the consensus function is common knowledge) is a pair $\mathbf{M} = (\mathcal{C}, \partial)$ where:

1) $\mathcal{C}$ is a consensus function (cf., McMorris and Powers, 2009), that is, a mapping

$$\mathcal{C} : \mathcal{P}(X) \to W(X),$$

that associates a complete preorder $\mathcal{C}(\mathcal{R})$ with each profile of complete preorders $\mathcal{R}$. We speak of the *consensus preorder* $\mathcal{C}(\mathcal{R})$ associated with $\mathcal{R}$, and assume that

1.*a*) $\mathcal{C}(\mathcal{R}) = R$ for each profile $\mathcal{R}$ that is constant to the complete preorder R.
1.*b*) $\mathcal{C}(\mathcal{R}^\sigma) = \mathcal{C}(\mathcal{R})$ for each profile of complete preorders and σ permutation of the voters.
1.*c*) $\mathcal{C}({}^{\pi}\mathcal{R}) = {}^{\pi}\,\mathcal{C}(\mathcal{R})$ for each profile of complete preorders and π permutation of the candidates or alternatives.

Abusing notation, this can be replaced with a *voting rule* with suitable properties: for example, 1.*b*) and 1.*c*) just mean the usual anonymity and neutrality conditions, respectively.

Example 1 A tie-breaking Borda rule as given by Suzumura [9, pp. 107-108] attaches a complete preorder to each profile of complete preorders. It ranks the candidates according to their respective Borda score defined as follows:

$$\beta(x_s) = \sum_{i=1}^{N} (\#\{\, x_t \in X : x_s \, R_i \, x_t \,\} - \#\{\, x_t \in X : x_t \, R_i \, x_s\})$$

Because 1.a), 1.b) and 1.c) are immediate, we denote by $\mathcal{C}_{\mathcal{B}}$ such consensus function. If in fact we have a profile of linear orders the same ranking is obtained through the alternative Borda score given by:

$$\beta'(x_s) = \sum_{i=1}^{N} (\#\{\, x_t \in X : x_s \, R_i \, x_t \,\})$$

Example 2 The Copeland method is described in e.g., Saari and Merlin [8] or Suzumura [9, p. 108]. It ranks the candidates according to their respective Copeland score defined as follows:

$$\kappa(x_s) = \#\{\, x_t \in X : x_s \text{ beats } x_t \text{ by s.s.m.}\} - \#\{\, x_t \in X : x_t \text{ beats } x_s \text{ by s.s.m.}\}$$

where s.s.m. stands for "strict simple majority". This rule is widely used in tournament situations, and versions of it are adopted by sports leagues. Again, 1.a), 1.b) and 1.c) are immediate. We denote by $\mathcal{C}_{\mathcal{C}}$ its associated consensus function.

2) ∂ is a *referenced measure function* (RMF), that is, a mapping

$$\partial : \mathcal{P}(X) \times W(X) \to [0, 1],$$

that assigns a real number, $\partial(\mathcal{R}, R) \in [0, 1]$, to each pair of a profile of complete preorder $\mathcal{R}$, and a complete preorder R, with the following properties:

2.a) $\partial(\mathcal{R}, R) = 1$ if and only if $\mathcal{R}$ is constant to R.
2.b) $\partial(\mathcal{R}^{\sigma}, R) = \partial(\mathcal{R}, R)$ for each possible permutation σ of the voters.
2.c) $\partial({}^{\pi}\mathcal{R}, {}^{\pi} R) = \partial(\mathcal{R}, R)$ for each possible permutation π of the candidates.

With regard to $\mathbf{M} = (\mathcal{C}, \partial)$ each profile of complete preorders $\mathcal{R}$ on X has a consensus $\nabla_{\mathbf{M}}(\mathcal{R}) = \partial(\mathcal{R}, \mathcal{C}(\mathcal{R}))$.

It is important to observe that *each conventional consensus measure can be intrepreted as a referenced consensus measure*, in the following sense. Recall first that a consensus measure (cf., Bosch [4]) is a mapping:

$$\mathcal{M} : \mathcal{P} \to [0, 1]$$

that assigns a real number $\mathcal{M}(\mathcal{R})$ to each profile of complete preorder $\mathcal{R}$ with the following properties:

- $\mathcal{M}(\mathcal{R}) = 1$ if and only if $\mathcal{R}$ is a constant profile.
- $\mathcal{M}(\mathcal{R}^{\sigma}) = \mathcal{M}(\mathcal{R})$ for each permutation σ of the voters.
- $\mathcal{M}({}^{\pi}\mathcal{R}) = \mathcal{M}(\mathcal{R})$ for each permutation π of the candidates.

Then, given a consensus measure $\mathcal{M}$ we define its associated RMF as

$$\partial_{\mathcal{M}}(\mathcal{R}, R) = \begin{cases} \mathcal{M}(\mathcal{R} \uplus R) & \text{if } \mathcal{R} \text{ is constant,} \\ \mathcal{M}(\mathcal{R}) & \text{otherwise.} \end{cases}$$

Now it is straightforward to check:

- $\partial_{\mathcal{M}}$ satisfies 2.a), 2.b) and 2.c)
- For any $\mathcal{C}$ consensus function $\partial_{\mathcal{M}}(\mathcal{R}, \mathcal{C}(\mathcal{R})) = \mathcal{M}(\mathcal{R})$.

In conclusion, for every consensus measure $\mathcal{M}$, any RCM $(\mathcal{C}, \partial_{\mathcal{M}})$ associated with it is equivalent to $\mathcal{M}$ irrespective of $\mathcal{C}$, in the sense that both $\mathcal{M}$ and $(\mathcal{C}, \partial_{\mathcal{M}})$ produce the same *number* as a measure of the consensus in the society. Note that contrary to the spirit of our proposal, the role of $\mathcal{C}(\mathcal{R})$ is irrelevant in the previous construction. In order to enhance the influence of $\mathcal{C}(\mathcal{R})$ in the consensus measure we now demand an additional property to referenced measure functions and introduce the corresponding new subclass of consensus measures:

Definition 1 (Normal Referenced Consensus Measure) A referenced consensus measure $\mathbf{M} = (\mathcal{C}, \partial)$ is called *normal* if its referenced measure function ∂ verifies

2.d) $\partial(\mathcal{R}, R) > 0$ if $R \in \mathcal{R}$.

If we adopt the position that overall satisfaction is an aggregate of individual satisfaction then property 2.d) can be regarded as natural. We emphasize that *the subclass of normal referenced consensus measures does not include all the conventional ones.* For example, the trivial measure defined as

$$\mathcal{T}(\mathcal{R}) = \begin{cases} 1 & \text{if } \mathcal{R} \text{ is a constant profile,} \\ 0 & \text{otherwise,} \end{cases}$$

is not a normal RCM. Note that we can assume that there exists a non-constant profile $\mathcal{R}$ such that $\mathcal{C}(\mathcal{R}) \in \mathcal{R}$ (this forcefully holds e.g., when the number of voters is higher than the cardinality of $W(X)$). Since $\mathcal{T}(\mathcal{R}) = 0$ and $\partial(\mathcal{R}, \mathcal{C}(\mathcal{R})) > 0$ for any normal RCM we conclude the assertion.

We now propose a construction of RMFs based on conventional consensus measures, which verifies property 2.d). Let $\mathcal{M}$ be a consensus measure. Given a profile of complete preorders $\mathcal{R}$ and a complete preorder R we define the $\mu^p(\mathcal{M})$-RFM as the p-generalized mean of the $\mathbb{R}^N$ vector that has the i-th component equal to $\mathcal{M}(\mathcal{R}_i \uplus R)$, that is

$$\partial^p_{\mathcal{M}}(\mathcal{R}, R) = \left(\sum_{i=1}^{N} \frac{1}{N} \mathcal{M}(\mathcal{R}_i \uplus R)^p \right)^{1/p} \tag{1}$$

It is trivial to check that properties 2.a), 2.b), 2.c) and 2.d) hold true. We conclude this part with an example.

Example 3 Let us first recall the definition of *Kemeny's measure*. For every profile of complete preorders $\mathcal{R} = (R_1, \ldots, R_N)$, its Kemeny's measure $\mathcal{K}(\mathcal{R})$ is the probability that the binary ordering between a pair of randomly selected alternatives is the same for all voters. Given $p = 1$ and $\mathcal{M} = \mathcal{K}$ –the Kemeny's measure– the above construction produces the following RMF. Attending to (1), we have to compute the Kemeny's measure of a profile composed by two elements: R_i that represents the preferences of individual i and R the referenced complete preorder. Observe that there is a total of $\frac{k(k-1)}{2}$ possible random choices. Thus the proportion of pairwise comparisons where R_i and R coincide is

$$\mathcal{K}(R_i \uplus R) = \frac{2}{k(k-1)} \sum_{(s,t)\in \mathbf{K}} \mathcal{K}^{s,t}(R_i \uplus R)$$

with

$$\mathcal{K}^{s,t}(R_i \uplus R) = \begin{cases} 1 \text{ if } R_i \text{ and } R \text{ coincide on the binary comparison} \\ \quad \text{between } x_s \text{ and } x_t; \\ 0 \text{ otherwise.} \end{cases}$$

We then conclude that the $\mu^1(\mathcal{K})$-RMF is given by:

$$\partial^1_{\mathcal{K}}(\mathcal{R}, R) = \frac{\mathcal{K}(R_1 \uplus R) + \ldots + \mathcal{K}(R_N \uplus R)}{N}.$$

Convention 1 *In what follows we omit superscripts when they are* 1*, in particular we denote the* $\mu^1(\mathcal{K})$*-RMF as* $\partial_{\mathcal{K}}$*.*

Along the rest of the paper we restrict our attention to normal referenced consensus measure with referenced measure function based on generalized means.

3 Some proposals for normal referenced consensus measures

In this section we detail the construction of two relevant normal RCM proposals. These models reach the consensus decision with Borda and Copeland methods, respectively, and both of them measure the consensus with the $\mu^1(\mathcal{K})$-RMF. We first provide an operational characterizations of Borda and Copeland rules. We then analyse our proposals and finally, we shortly discuss the dichotomous case.

3.1 Some Operational Characterizations

Let us fix a profile $\mathcal{R} = (R_1, ..., R_N)$ of complete preorders on X. Its Borda and Copeland scores can be reinterpreted in terms of simple matrix operations. We denote by A^t the transpose of the matrix A. Besides, for any $m \times n$ real-valued matrix $A = (a_{i,j})_{m\times n}$ the notation $sig(A)$ refers to the $m\times n$ matrix whose (i, j) cell is 1 if $a_{i,j} > 0$, -1 if $a_{i,j} < 0$, and 0 otherwise. I_k denotes the identity matrix of size $k \times k$.

For each complete preorder R_s, the asymmetric part of which is denoted by P_s, its preference matrix $\mathbf{P}_s$ is defined as the $k \times k$ binary matrix whose (i, j) cell is 1 when $x_i \, P_s \, x_j$, and 0 otherwise. Observe that R_s is linear if and only if $\mathbf{P}_s + (\mathbf{P}_s)^t + I_k = (1)_{k\times k}$, the constant to 1 matrix of size $k \times k$. Besides, the sum of the cells in the i-th row of $\mathbf{P}_s - (\mathbf{P}_s)^t$ is $\#\{\, x_j \in X : \, x_i \, P_s \, x_j\} - \#\{\, x_j \in X : \, x_j \, P_s \, x_i\}$. We say that $\mathcal{R}$ has an aggregate preference matrix $\mathbf{A}(\mathcal{R}) = \mathbf{P}_1 + ... + \mathbf{P}_N$. Its (i, j) cell has the number of agents for which alternative x_i is strictly better than x_j. The sum of the cells in its i-th row is the usual Borda score $\beta'(x_i)$ when $\mathcal{R}$ is a profile of linear orders.

Define $\overline{\mathbf{A}}(\mathcal{R}) = \mathbf{A}(\mathcal{R}) - (\mathbf{A}(\mathcal{R}))^t$, then the sum of the cells in its i-th file is $\beta(x_i)$, the Borda score of alternative x_i (v. Example 1).

Observe that the fact that the (i,j) cell of $\overline{\mathbf{A}}(\mathcal{R})$ is greater than 0 is equivalent to the fact that alternative x_i beats x_j by strict simple majority under the profile $\mathcal{R}$. Thus if we define $\widetilde{\mathbf{A}}(\mathcal{R}) = sig(\overline{\mathbf{A}}(\mathcal{R}))$ then the sum of the cells in its i-th file is $\kappa(x_i)$, the Copeland score of alternative x_i (v. Example 2).

Example 4 Suppose $X = \{x, y, z, w\}$ thus $k = 4$. Let $\mathcal{R} = (R_1, R_2, R_3)$ be the profile of linear orders given by: $w\,P_1\,y\,P_1\,x\,P_1\,z$, $z\,P_2\,w\,P_2\,y\,P_2\,x$, $x\,P_3\,z\,P_3\,y\,P_3\,w$.

Then

$$\mathbf{P}_1 = \begin{pmatrix} 0\,0\,1\,0 \\ 1\,0\,1\,0 \\ 0\,0\,0\,0 \\ 1\,1\,1\,0 \end{pmatrix} \qquad \mathbf{P}_2 = \begin{pmatrix} 0\,0\,0\,0 \\ 1\,0\,0\,0 \\ 1\,1\,0\,1 \\ 1\,1\,0\,0 \end{pmatrix} \qquad \mathbf{P}_3 = \begin{pmatrix} 0\,1\,1\,1 \\ 0\,0\,0\,1 \\ 0\,1\,0\,1 \\ 0\,0\,0\,0 \end{pmatrix}$$

Some simple computations yield

$$\mathbf{A}(\mathcal{R}) = \begin{pmatrix} 0\,1\,2\,1 \\ 2\,0\,1\,1 \\ 1\,2\,0\,2 \\ 2\,2\,1\,0 \end{pmatrix} \qquad \overline{\mathbf{A}}(\mathcal{R}) = \widetilde{\mathbf{A}}(\mathcal{R}) = \begin{pmatrix} 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & -1 \\ -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$$

Thus for this setting the Borda and Copeland scores coincide throughout. Their values are -1 por options x and y, and 1 for options z and w. Therefore the social preference R that is derived from both choice rules is $w\,I\,z\,P\,x\,I\,y$ (cf., Suzumura [9, p. 108]).

Calculating $\partial_{\mathcal{K}}(\mathcal{R}, R)$ for $\mathcal{R} = (R_1, ..., R_N)$ profile of complete preorders and R complete preorder is trivial from the numbers $\mathcal{K}^{s,t}(R_i \uplus R)$. These amounts can be computed with the assistance of basic matrix manipulations too. Denote by $\mathbf{P}$ the preference matrix of R defined as above. Let us observe two facts.

1. Cell (s,t) of both $\mathbf{P}_i + (\mathbf{P}_i)^t$ and $\mathbf{P} + \mathbf{P}^t$ has a 0 if and only if both R_i and R are indifferent between x_s and x_t. This can not happen when $s \neq t$ if either R_i or R is linear.
2. Cell (s,t) of both $\mathbf{P}_i$ and $\mathbf{P}$ has a 1 if and only if $x_s\,P_i\,x_t$ and $x_s\,P\,x_t$.

This means that: the number of pairs of different options for which both R_i and R are indifferent is the number of cells strictly above the diagonal with a 0 for both $\mathbf{P}_i + (\mathbf{P}_i)^t$ and $\mathbf{P} + \mathbf{P}^t$ (and it is 0 if either R_i or R is linear); and the number of pairs of options for which R_i and R have equal strict preference is the number of cells (outside the diagonal) with a 1 for both $\mathbf{P}_i$ and $\mathbf{P}$.[1] The sum of these two amounts is $\frac{k(k-1)}{2}\mathcal{K}(R_i, R) = \sum_{\{s,t\}\in\mathbf{K}} \mathcal{K}^{s,t}(R_i \uplus, R)$.

Example 5 In the situation of Example 4 one has

$$\mathbf{P}_1 + \mathbf{P}_1^t = \mathbf{P}_2 + \mathbf{P}_2^t = \mathbf{P}_3 + \mathbf{P}_3^t = \begin{pmatrix} 0\,1\,1\,1 \\ 1\,0\,1\,1 \\ 1\,1\,0\,1 \\ 1\,1\,1\,0 \end{pmatrix}$$

[1] This number is obtained with a computer assistant very easily: do the cell-by-cell multiplication and sum up all the cells in the result.

because all $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$ are linear orders. The preference matrix of the complete preorder R that is prescribed by both the Borda and Copeland rule is

$$\mathbf{P} = \begin{pmatrix} 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ 1\,1\,0\,0 \\ 1\,1\,0\,0 \end{pmatrix}$$

Some simple computations yield $\partial_{\mathcal{K}}(\mathcal{R}, R) = \frac{1}{3}(\frac{2+4+1}{6}) = \frac{7}{18}$ since

- No pair of different options is indifferent under any of the P_i's.
- Only cells $(4,1)$ and $(4,2)$ have a 1 in both $\mathbf{P}_1$ and $\mathbf{P}$.
- Only cells $(3,1)$, $(3,2)$, $(4,1)$ and $(4,2)$ have a 1 in both $\mathbf{P}_2$ and $\mathbf{P}$.
- Only cell $(3,2)$ has a 1 in both $\mathbf{P}_3$ and $\mathbf{P}$.

3.2 The RCM-B proposal

In this Subsection we analyse the referenced consensus measure given by the tie-breaking Borda rule that was detailed in Example 1 and $\partial_{\mathcal{K}}$ introduced in Example 3. We refer to this model as RCM-B, that is, $\mathbf{M}_B = (\mathcal{C}_{\mathcal{B}}, \partial_{\mathcal{K}})$.

Let us first show how this proposal produces its output with a simple Example.

Example 6 In the situation of Example 4 we checked that $\mathcal{C}_{\mathcal{B}}(\mathcal{R})$ is the complete preorder R_B determined by $w\, I_B\, z\, P_B\, x\, I_B\, y$. According to Example 5

$$\nabla_{\mathbf{M}_B}(\mathcal{R}) = \partial_{\mathcal{K}}(\mathcal{R}, \mathcal{C}_{\mathcal{B}}(\mathcal{R})) = \frac{7}{18}$$

This means that 7 out of $18 = \frac{3\cdot 4\cdot(4-1)}{2}$ possible pairwise comparisons made by a member of the society $\{1, 2, 3\}$ coincide with the binary ordering given by the consensus function in the model.

3.3 The RCM-C proposal

In this Subsection we analyse the referenced consensus measure given by the Copeland method (cf. Example 2) and $\partial_{\mathcal{K}}$. We refer to this model as RCM-C, that is, $\mathbf{M}_C = (\mathcal{C}_{\mathcal{C}}, \partial_{\mathcal{K}})$.

Let us first show how this proposal produces its output with a simple Example.

Example 7 In the situation of Example 4 (v. Example 6) we found $\mathcal{C}_{\mathcal{C}}(\mathcal{R}) = R_B$ thus

$$\nabla_{\mathbf{M}_C}(\mathcal{R}) = \partial_{\mathcal{K}}(\mathcal{R}, \mathcal{C}_{\mathcal{C}}(\mathcal{R})) = \frac{7}{18}$$

Again, 7 out of 18 possible pairwise comparisons made by a member of the society $\{1, 2, 3\}$ coincide with the binary ordering given by the consensus function in the model.

3.4 The case of a dichotomous choice

Suppose $k = 2$, i.e., the dichotomous case. To simplify notation let $X = \{x, y\}$. We also denote $n_1 = |\{i \in \mathbf{N} : x\, P_i\, y\}|$ and $n_2 = |\{i \in \mathbf{N} : y\, P_i\, x\}|$, thus $N - n_1 - n_2 = |\{i \in \mathbf{N} : x\, I_i\, y\}|$. Due to properties $1.b$) and $2.b$) we can reorder the voters as convenient, and we assume that voters $1, ..., n_1$ prefer x strictly over y, that voters $n_1+1, ..., n_1+n_2$ prefer y strictly over x, and that the last $N - n_1 - n_2$ voters are indifferent between x and y. Let $n_{x,y}(\mathcal{R})$ denote the majority margin of x over y under $\mathcal{R}$, that is, the number of voters that prefer x strictly over y minus the number of voters that prefer y strictly over x, or $n_{x,y}(\mathcal{R}) = n_1 - n_2$. Now the Borda and Copeland voting rule coincide with strict simple majority: the preference matrix of the complete preorder R_0 that is prescribed by them is

$$\mathbf{P_0} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ if } n_{x,y}(\mathcal{R}) > 0 \;\; (\text{or } n_1 > n_2)$$

$$\mathbf{P_0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ if } n_{x,y}(\mathcal{R}) = 0 \;\; (\text{or } n_1 = n_2)$$

$$\mathbf{P_0} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ if } n_{x,y}(\mathcal{R}) < 0 \;\; (\text{or } n_1 < n_2)$$

Because $k = 2$ we obtain:

$$\begin{aligned}
&\text{if } n_1 > n_2, \text{ then } \;\mathcal{K}(R_i \uplus R_0) = \begin{cases} 1 & \text{for } i = 1, ..., n_1 \\ 0 & \text{otherwise} \end{cases} \\
&\text{if } n_2 > n_1, \text{ then } \;\mathcal{K}(R_i \uplus R_0) = \begin{cases} 1 & \text{for } i = n_1+1, ..., n_1+n_2 \\ 0 & \text{otherwise} \end{cases} \\
&\text{if } n_1 = n_2, \text{ then } \;\mathcal{K}(R_i \uplus R_0) = \begin{cases} 1 & \text{for } i = n_1+n_2+1, ..., N \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Therefore

$$\partial_{\mathcal{K}}(\mathcal{R}, R_0) = \begin{cases} \frac{n_1}{N} & \text{if } n_1 > n_2 \\ \frac{n_2}{N} & \text{if } n_2 > n_1 \\ \frac{N-n_1-n_2}{N} = 1 - 2\frac{n_1}{N} & \text{if } n_1 = n_2 \end{cases}$$

This means that under either RCM-B or RCM-C, total lack of consensus only happens under a precise fifty-fifty division among all the voters (half prefer x strictly over y, half the other way around), which is commonly agreed upon (see e.g., Alcalde and Vorsatz [1, pp. 2-3]). Obviously when $k = 2$ and $n_1 + n_2 = N$ (i.e., the dichotomous and *binary* case) one has

$$\partial_{\mathcal{K}}(\mathcal{R}, R_0) = \begin{cases} \frac{n_1}{N} & \text{if } n_1 > n_2 \\ \frac{n_2}{N} & \text{if } n_2 > n_1 \\ 0 & \text{if } n_1 = n_2 \end{cases}$$

and an odd number of voters can not produce zero consensus under these models.

4 Normal referenced consensus measures: a critical analysis of their properties

Along this Section, $\mathbf{M} = (\mathcal{C}, \partial^p_{\mathcal{M}})$ denotes a normal referenced consensus measure with a $\mu^p(\mathcal{M})$-RMF. In other words, $\partial^p_{\mathcal{M}}$ is based on a conventional consensus measure $\mathcal{M}$ and computed as a p-generalized mean according to (1). We proceed to check that such model agrees with certain axioms that are common use in the literature. At this point we remark that Axioms 1 to 3 below hold true in the larger class of referenced consensus measures. Finally, a critical analysis of other *ad-hoc* properties is performed along this study.

4.1 Some properties of referenced consensus measures

The following axiom is trivial from the definition of a referenced consensus measure. It means that maximum consensus is reached under commonly held preferences across agents.

Axiom 1 $\mathbf{M}$ *is* unanimous *if for each constant profile* $\mathcal{R}$ *it is true that* $\nabla_{\mathbf{M}}(\mathcal{R}) = 1$.

Similarly, Proposition 1 below proves that the following property obtains:

Axiom 2 $\mathbf{M}$ *is* anonymous *if for each permutation of the voters* σ *and each profile* $\mathcal{R}$*, it is true that* $\nabla_{\mathbf{M}}(\mathcal{R}) = \nabla_{\mathbf{M}}(\mathcal{R}^{\sigma})$.

As is apparent, anonymity of a normal referenced consensus measure means that the consensus measure does not change if we rename the voters.

Proposition 1 *Any* $\mathbf{M}$ *is anonymous.*

Proof This holds because $\mathbf{M}$ has properties 1.b) and 2.b). Specifically,

$$\nabla_{\mathbf{M}}(\mathcal{R}) = \partial(\mathcal{R}, \mathcal{C}(\mathcal{R})) \stackrel{2.b)}{=} \partial(\mathcal{R}^{\sigma}, \mathcal{C}(\mathcal{R})) \stackrel{1.b)}{=} \partial(\mathcal{R}^{\sigma}, \mathcal{C}(\mathcal{R}^{\sigma})) = \nabla_{\mathbf{M}}(\mathcal{R}^{\sigma})$$

□

In particular, both RCM-B and RCM-C satisfy Axiom 2. We now argue that normal referenced consensus measures verify the following property too:

Axiom 3 $\mathbf{M}$ *is* neutral *if the consensus measure does not change when we rename the candidates.*

Proposition 2 *Any* $\mathbf{M}$ *is neutral.*

Proof From the fact that the consensus function associated with $\mathbf{M}$ satisfies 1.c) it is tedious but straightforward to check that $\mathbf{M}$ verifies Axiom 3. □

In order to introduce a further property of normal referenced consensus measures, we first give some notation. For each profile $\mathcal{R} = (R_1, ..., R_N)$ and $m \in \mathbb{N}$ we denote

$${}^{m}\mathcal{R} = (R_1, \overset{m}{\ldots}, R_1, R_2, \overset{m}{\ldots}, R_2, ..., R_N, \overset{m}{\ldots}, R_N)$$

that we call an m-replication of the profile $\mathcal{R}$. Then we say that the consensus function $\mathcal{C}$ verifies *replication* if $\mathcal{C}(\mathcal{R}) = \mathcal{C}({}^{m}\mathcal{R})$ throughout. This means that for each fixed society, the same consensus ordering is proposed if we repeatedly clone it. Likewise we define:

Axiom 4 **M** *verifies the* replication *axiom if for each profile* $\mathcal{R}$ *and* $m \in \mathbb{N}$ *it is true that* $\nabla_{\mathbf{M}}(\mathcal{R}) = \nabla_{\mathbf{M}}({}^m\mathcal{R})$.

Coupled with Axiom 2, this property is the analogous of the replication axiom in Alcalde and Vorsatz [1].[2] They interpret it as an invariance property asking that exact replications of a society are attached the same level of coherence as the original. The following result checks the model under inspection against Axiom 4:

Proposition 3 *Given* $\mathbf{M} = (\mathcal{C}, \partial^p_{\mathcal{M}})$, *if* $\mathcal{C}$ *verifies replication then* **M** *satisfies the replication axiom.*

Proof Let us fix a profile $\mathcal{R} = (R_1, ..., R_N)$ and $m \in \mathbb{N}$. By definition of ${}^m\mathcal{R}$, it is clear that for any complete peorder R one has

$$\begin{aligned}\partial^p_{\mathcal{M}}({}^m\mathcal{R}, R) &= \left(\frac{\sum_{i=1}^N \mathcal{M}(R_i \uplus R)^p + \overset{m}{\cdots} + \sum_{i=1}^N \mathcal{M}(R_i \uplus R)^p}{m \cdot N}\right)^p \\ &= \left(\frac{\sum_{i=1}^N \mathcal{M}(R_i \uplus R)^p}{N}\right)^p = \partial^p_{\mathcal{M}}(\mathcal{R}, R).\end{aligned}$$

Thus because $\mathcal{C}$ satisfies replication we finally obtain

$$\nabla_{\mathbf{M}}({}^m\mathcal{R}) = \partial^p_{\mathcal{M}}(\mathcal{R}, \mathcal{C}({}^m\mathcal{R})) = \partial^p_{\mathcal{M}}(\mathcal{R}, \mathcal{C}(\mathcal{R})) = \nabla_{\mathbf{M}}(\mathcal{R}).$$

□

Corollary 1 *Both RCM-B and RCM-C verify the replication axiom.*

Proof By Proposition 3 it suffices to prove that both Borda and Copeland rankings satisfy replication. Let us fix a profile $\mathcal{R} = (R_1, ..., R_N)$ and $m \in \mathbb{N}$. We observe that because $\mathbf{A}({}^m\mathcal{R}) = m \cdot \mathbf{A}(\mathcal{R})$ and $\overline{\mathbf{A}}({}^m\mathcal{R}) = m \cdot \overline{\mathbf{A}}(\mathcal{R})$ the Borda ranking is preserved by m-replication of the profile. Further, the fact that $\widetilde{\mathbf{A}}(\mathcal{R}) = sig(\overline{\mathbf{A}}(\mathcal{R})) = sig(\overline{\mathbf{A}}({}^m\mathcal{R})) = \widetilde{\mathbf{A}}({}^m\mathcal{R})$ implies that the Copeland ranking is preserved by m-replication of the profile. □

The next Axiom captures the intuitively appealing property that the consensus measure should not change if all the agents simultaneously reverse their orderings of the alternatives:

Axiom 5 **M** *verifies* reversal invariance *if the reversal of any profile* $\mathcal{R}$, *namely* $\mathcal{R}^{-1}$, *produces the same consensus, i.e.,*

$$\nabla_{\mathbf{M}}(\mathcal{R}) = \nabla_{\mathbf{M}}(\mathcal{R}^{-1}) \text{ for each possible profile } \mathcal{R}$$

To discuss this property, we have to introduce some additional notations. A consensus function $\mathcal{C}$ satifies the *reversal* property if $\mathcal{C}(R^{-1}) = \mathcal{C}(R)^{-1}$ for any complete peorder R. This means that when all voters in a profile reverse their rankings of the candidates then the outcome is reversed. A consensus measure $\mathcal{M}$ verifies the *reversal* property if $\mathcal{M}(\mathcal{R}^{-1}) = \mathcal{M}(\mathcal{R})$ for any profile of complete preorders. That is, the consensus measure is unchanged when the profile is reversed. Let us analyse this property in detail.

[2] These authors acknowledge inspiration by the scale invariance axiom in Allison's [3] characterization of the Gini index.

Proposition 4 *Given* $\mathbf{M} = (\mathcal{C}, \partial^p_{\mathcal{M}})$, *if* $\mathcal{C}$ *and* $\mathcal{M}$ *verify the reversal property, then* $\mathbf{M}$ *satisfies the reversal invariance axiom.*

Proof Since $(R_i \uplus \mathcal{C}(\mathcal{R}))^{-1} = R_i^{-1} \uplus \mathcal{C}(\mathcal{R})^{-1}$, by hypotesis we infer for all $i = 1 \dots N$ that

$$\mathcal{M}(R_i \uplus \mathcal{C}(\mathcal{R})) = \mathcal{M}((R_i \uplus \mathcal{C}(\mathcal{R}))^{-1}) = \mathcal{M}(R_i^{-1} \uplus \mathcal{C}(\mathcal{R})^{-1}) = \mathcal{M}(R_i^{-1} \uplus \mathcal{C}(\mathcal{R}^{-1})),$$

and thus from definition of $\mu^p(\mathcal{M})$-RMF we conclude as follow

$$\nabla_{\mathbf{M}}(\mathcal{R}) = \partial^p_{\mathcal{M}}(\mathcal{R}, \mathcal{C}(\mathcal{R})) = \partial^p_{\mathcal{M}}(\mathcal{R}^{-1}, \mathcal{C}(\mathcal{R}^{-1})) = \nabla_{\mathbf{M}}(\mathcal{R}^{-1}).$$

□

Our particular proposals in Section 3 verify this property too.

Corollary 2 *Both RCM-B and RCM-C verify the reversal invariance axiom.*

Proof Because the Borda and the Copeland rules satisfy the reversal property (cf., Saari and Merlin [8, Section 1]) we only have to prove that the Kemeny's measure verifies the reversal property. This is straightfoward since $\mathcal{K}^{s,t}(R_i \uplus \mathcal{C}(\mathcal{R})) = \mathcal{K}^{s,t}(R_i^{-1} \uplus \mathcal{C}(\mathcal{R}^{-1}))$ for each possible voter i and candidates s and t. □

We now investigate if normal referenced consensus measures verify the following *reinforcement* property:

Axiom 6 $\mathbf{M}$ *verifies* reinforcement *if adding* $\mathcal{C}(R)$ *to the profile* R *does not reduce the consensus, i.e.,*

$$\nabla_{\mathbf{M}}(\mathcal{R} \uplus \mathcal{C}(R)) \geqslant \nabla_{\mathbf{M}}(\mathcal{R}) \text{ for each possible profile } \mathcal{R}$$

We proceed to state a criterion for satisfaction of this property that depends upon the behavior of $\mathcal{C}$, and then we check that both RCM-B and RCM-C meet such criterion. We say that a consensus function $\mathcal{C}$ verifies *decision invariance* if $\mathcal{C}(\mathcal{R} \uplus \mathcal{C}(\mathcal{R})) = \mathcal{C}(\mathcal{R})$ for each profile $\mathcal{R}$. This means that the consensus ordering does not change if we add to the society a new agent whose preferences coincide with the previous consensus preorder. Under this restriction we obtain:

Proposition 5 *Given* $\mathbf{M} = (\mathcal{C}, \partial^p_{\mathcal{M}})$, *if* $\mathcal{C}$ *verifies* decision invariance *then* $\mathbf{M}$ *verifies reinforcement.*

Proof Since $\mathcal{C}(\mathcal{R} \uplus \mathcal{C}(\mathcal{R})) = \mathcal{C}(\mathcal{R})$, and using $\mathcal{M}(\mathcal{C}(R) \uplus \mathcal{C}(R)) = 1$, one has

$$\begin{aligned}
\nabla_{\mathbf{M}}(\mathcal{R} \uplus \mathcal{C}(R)) &= \partial^p_{\mathcal{M}}(\mathcal{R} \uplus \mathcal{C}(\mathcal{R}), \mathcal{C}(\mathcal{R})) \\
&= \left(\frac{\sum_{i=1}^N \mathcal{M}(R_i, \mathcal{C}(\mathcal{R}))^p + \mathcal{M}(\mathcal{C}(\mathcal{R}), \mathcal{C}(\mathcal{R}))^p}{N+1} \right)^{1/p} \\
&= \left(\frac{N}{N+1} \left[\partial^p_{\mathcal{M}}(\mathcal{R}, \mathcal{C}(\mathcal{R})) \right]^p + \frac{1}{N+1} \right)^{1/p} \geqslant \partial^p_{\mathcal{M}}(\mathcal{R}, \mathcal{C}(\mathcal{R})) = \nabla_{\mathbf{M}}(\mathcal{R}),
\end{aligned}$$

where the last inequality derives from the fact $\nabla_{\mathbf{M}}(\mathcal{R}) \leqslant 1$. Such inequality becomes strict provided $\nabla_{\mathbf{M}}(\mathcal{R}) < 1$. □

An appeal to Proposition 5 permits us to prove that both RCM-B and RCM-C verify reinforcement:

Proposition 6 *RCM-B and RCM-C verify reinforcement.*

Proof We just need to prove that their respective consensus functions verify the decision invariance property which in conjunction with Proposition 5, proves the assertion. Firstly we analyse RCM-B. Let us take the profile $\mathcal{R}$ and denote $R_B = \mathcal{C}_{\mathcal{B}}(\mathcal{R})$ with preference matrix $\mathbf{P}_B$. Recall that for $\overline{\mathbf{A}}(\mathcal{R}) = \mathbf{A}(\mathcal{R}) - (\mathbf{A}(\mathcal{R}))^t$, the sum of the cells in its i-th file is $\beta(x_i)$, the Borda score of alternative x_i. We claim $\mathcal{C}_{\mathcal{B}}(\mathcal{R} \uplus \mathcal{C}_{\mathcal{B}}(\mathcal{R})) = \mathcal{C}_{\mathcal{B}}(\mathcal{R})$. These orders arise from the respective Borda scores, namely β_B and β, obtained from $\overline{\mathbf{A}}(\mathcal{R} \uplus \mathcal{C}_{\mathcal{B}}(\mathcal{R}))$ and $\overline{\mathbf{A}}(\mathcal{R})$ by summing up the cells in their rows. Observe $\overline{\mathbf{A}}(\mathcal{R} \uplus \mathcal{C}_{\mathcal{B}}(\mathcal{R})) = \overline{\mathbf{A}}(\mathcal{R}) + \mathbf{P}_B - (\mathbf{P}_B)^t$. By construction $\beta(x_i) \geqslant \beta(x_j)$ if and only if $x_i \, R_B \, x_j$. Because the sum of the cells in the i-th row of $\mathbf{P}_B - (\mathbf{P}_B)^t$ is $\#\{ x_l \in X : x_i \, P_B \, x_l\} - \#\{ x_l \in X : x_l \, P_B \, x_i\}$, one has that $x_i \, R_B \, x_j$ if and only if the sum of the cells in the i-th row of $\mathbf{P}_B - (\mathbf{P}_B)^t$ is greater or equal than the sum of the cells in the j-th row of $\mathbf{P}_B - (\mathbf{P}_B)^t$. This proves our claim $\beta_B(x_i) \geqslant \beta_B(x_j)$ if and only if $\beta(x_i) \geqslant \beta(x_j)$ throughout.

We now analyse RCM-C. Let us take the profile $\mathcal{R}$ and denote $R_C = \mathcal{C}_{\mathcal{C}}(\mathcal{R})$ with preference matrix $\mathbf{P}_C$. Recall that for $\widetilde{\mathbf{A}}(\mathcal{R}) = sig(\overline{\mathbf{A}}(\mathcal{R}))$, the sum of the cells in its i-th file is $\kappa(x_i)$, the Copeland score of alternative x_i. We claim $\mathcal{C}_{\mathcal{C}}(\mathcal{R} \uplus \mathcal{C}_{\mathcal{C}}(\mathcal{R})) = \mathcal{C}_{\mathcal{C}}(\mathcal{R})$. These orders arise from the respective Copeland scores, namely κ_c and κ, obtained from $\widetilde{\mathbf{A}}(\mathcal{R} \uplus \mathcal{C}_{\mathcal{C}}(\mathcal{R}))$ and $\widetilde{\mathbf{A}}(\mathcal{R})$ by summing up the cells in their rows. Thus our claim boils down to $\kappa_c(x_i) \geqslant \kappa_c(x_j)$ if and only if $\kappa(x_i) \geqslant \kappa(x_j)$ throughout. This holds if we prove $sig(\overline{\mathbf{A}}(\mathcal{R} \uplus \mathcal{C}_{\mathcal{C}}(\mathcal{R}))) = sig(\overline{\mathbf{A}}(\mathcal{R}))$. Observe $\overline{\mathbf{A}}(\mathcal{R} \uplus \mathcal{C}_{\mathcal{C}}(\mathcal{R})) = \overline{\mathbf{A}}(\mathcal{R}) + \mathbf{P}_C - (\mathbf{P}_C)^t$. By construction $\kappa(x_i) \geqslant \kappa(x_j)$ if and only if $x_i \, R_C \, x_j$. Because cell (i,j) in $\overline{\mathbf{A}}(\mathcal{R})$ is positive (resp., negative) if and only if x_i beats x_j by s.s.m. (resp., x_j beats x_i by s.s.m.) if and only if cell (i,j) in $\mathbf{P}_C - (\mathbf{P}_C)^t$ is positive (resp., negative), the claim $sig(\overline{\mathbf{A}}(\mathcal{R} \uplus \mathcal{C}_{\mathcal{C}}(\mathcal{R}))) = sig(\overline{\mathbf{A}}(\mathcal{R}))$ easily follows from a cell-by-cell inspection. □

In order to prove another interesting property of a suitable subclass of normal referenced consensus measures we need some previous elaboration. The consensus function $\mathcal{C}$ verifies *responsiveness* if for every $R' \in W(X)$ and $\mathcal{R} \in W(X)^N$, the following equality holds eventually (i.e., for all sufficiently large m):

$$\mathcal{C}(\mathcal{R} \uplus R' \uplus \overset{m}{\cdots} \uplus R') = R' \tag{2}$$

We proceed to prove that the Borda rule and the Copeland rule verify a restricted version of this property, namely restricted responsiveness: for every $R' \in L(X)$ and $\mathcal{R} \in W(X)^N$, Equation (2) holds eventually.

Lemma 1 *The Borda rule and the Copeland rule verify restricted responsiveness.*

Proof We fix $X = \{x_1, ..., x_k\}$, $R' \in L(X)$, and $\mathcal{R} \in W(X)^N$.

Firstly we analyse the Borda rule. Given $x_s \neq x_t$ we can assume $x_s P' x_t$ without loss of generality. Now irrespective of the Borda score that $\mathcal{R}$ attaches to them –namely, $\beta_{\mathcal{R}}(x_s)$ and $\beta_{\mathcal{R}}(x_t)$– it must be the case that for sufficiently large m the Borda score with respect to $\mathcal{R}^m = \mathcal{R} \uplus R' \uplus \overset{m}{\cdots} \uplus R'$ –which we denote by $\beta_{\mathcal{R}^m}$– is strictly higher for x_s, since

$$\beta_{\mathcal{R}^m}(x_s) - \beta_{\mathcal{R}^m}(x_t) \geqslant m + \beta_{\mathcal{R}}(x_s) - \beta_{\mathcal{R}}(x_t)$$

If m_0 is such that $m > m_0$ implies $m + \beta_{\mathcal{R}}(x_s) - \beta_{\mathcal{R}}(x_t) > 0$ then $m > m_0$ implies that the ordering between x_s and x_t according to $\mathcal{R}^m = \mathcal{R} \uplus R' \uplus \overset{m}{\cdots} \uplus R'$ coincides with its ordering according to R'. Because there are finitely many pairs in $\mathbf{K}$, this conclusion can be simultaneously reached for every pair $x_s \neq x_t$ of elements in X.

We now analyse the Copeland rule. Given $x_s \neq x_t$ we can assume $x_s P' x_t$ without loss of generality. It is clear that for sufficiently large m the alternative x_s beats x_t by strict simple majority according to $\mathcal{R}^m = \mathcal{R} \uplus R' \uplus \overset{m}{\cdots} \uplus R'$. Formally: denote by κ' the Copeland score of the profile with the linear order R' only, and by κ_m the Copeland score of the profile $\mathcal{R}^m$, then $\kappa'(x_s) > \kappa'(x_t)$ and it is eventually true that $\kappa_m(x_s) = \kappa'(x_s) > \kappa'(x_t) = \kappa_m(x_t)$. Now the argument goes through as above. □

Responsiveness can not be guaranteed in Lemma 1: even in the simplest non-trivial instance where there are two candidates both the Borda rule and the Copeland rule fail to be responsive as the next Example shows.

Example 8 Suppose $X = \{x, y\}$ thus $k = 2$. Let $\mathcal{R} = (R_1)$ be the profile of one linear order given by $x\, P_1\, y$. We also let R' be the complete preorder with $x\, I'\, y$, which is not a linear order. Then $\mathcal{C}_{\mathcal{B}}(\mathcal{R} \uplus R' \uplus \overset{m}{\cdots} \uplus R') = \mathcal{C}_{\mathcal{C}}(\mathcal{R} \uplus R' \uplus \overset{m}{\cdots} \uplus R') = R_1$ for each m, that is, both the Borda and Copeland methods suggest the consensus ordering $R_1 \neq R'$. The reason is that irrespective of m, the Borda score of x is a unit higher than the Borda score of y, and the Copeland score of x is 1 but the Copeland score of y is 0.

We are now ready to define:

Axiom 7 **M** *verifies* convergence to unanimity *if for every* $R' \in W(X)$*,*

$$\lim_{m \longrightarrow \infty} \nabla_{\mathbf{M}}(\mathcal{R} \uplus R' \uplus \overset{m}{\cdots} \uplus R') = 1$$

Axiom 8 **M** *verifies* restricted convergence to unanimity *if for every* $R' \in L(X)$ *and* $\mathcal{R} \in W(X)^N$*,* $\lim_{m \longrightarrow \infty} \nabla_{\mathbf{M}}(\mathcal{R} \uplus R' \uplus \overset{m}{\cdots} \uplus R') = 1$.

We proceed to ellucidate to which extent normal referenced consensus measures verify convergence to unanimity, with an especial attention to the RCM-B and RCM-C cases.

Proposition 7 *Given* $\mathbf{M} = (\mathcal{C}, \partial^p_{\mathcal{M}})$*, if* $\mathcal{C}$ *is responsive (resp., restrictedly responsive) then* **M** *verifies Axiom 7 (resp., Axiom 8). In particular, RCM-B and RCM-C verify restricted convergence to unanimity.*

Proof Suppose $\mathcal{C}$ is responsive, that is, for every $R' \in W(X)$ and $\mathcal{R} \in W(X)^N$ the equality $\mathcal{C}(\mathcal{R} \uplus R' \uplus \overset{m}{\cdots} \uplus R') = R'$ is eventually true. Then one has

$$\begin{aligned}
\lim_{m \longrightarrow \infty} \nabla_{\mathbf{M}}(\mathcal{R} \uplus R' \uplus \overset{m}{\cdots} \uplus R') &= \lim_{m \longrightarrow \infty} \partial^p_{\mathcal{M}}(\mathcal{R} \uplus R' \uplus \overset{m}{\cdots} \uplus R', \mathcal{C}(\mathcal{R} \uplus R' \uplus \overset{m}{\cdots} \uplus R')) \\
&= \lim_{m \longrightarrow \infty} \partial^p_{\mathcal{M}}(\mathcal{R} \uplus R' \uplus \overset{m}{\cdots} \uplus R', R') \\
&= \lim_{m \longrightarrow \infty} \left(\frac{\sum_{i=1}^N \mathcal{M}(R_i \uplus R')^p + m}{N + m} \right)^{1/p} \\
&= \lim_{m \longrightarrow \infty} \left(\frac{N}{N+m} [\partial^p_{\mathcal{M}}(\mathcal{R}, R')]^p + \frac{m}{N+m} \right)^{1/p} = 1
\end{aligned}$$

where we are using that $\mathcal{M}(R' \uplus R') = 1$ and $\partial^p_{\mathcal{M}}(\mathcal{R}, R') \leqslant 1$.

The case of a restrictedly responsive consensus function is proved analogously. In particular, from Lemma 1 RCM-B and RCM-C verify restricted convergence to unanimity. □

4.2 Other properties of referenced consensus measures

The literature on measurement of consensus has dealt with other desirable properties that we briefly analyse in this Subsection. Axiom 9 below requests that null and full consensus are possible.

Axiom 9 **M** *verifies* full range *if there are two profiles* $\mathcal{R}$ *and* $\mathcal{R}'$ *such that* $\nabla_{\mathbf{M}}(\mathcal{R}) = 0$, $\nabla_{\mathbf{M}}(\mathcal{R}') = 1$.

Neither RCM-B nor RCM-C verify this property in the sense that zero consensus is impossible for particular values of N as seen in Subsection 3.4.

Similarly, we proceed to analyze the property of Monotonicity, whose formal definition is given in Alcalde and Vorsatz [1]. Intuitively it say as follows. Suppose that you measure the consensus in a society. Now one agent reverses her/his opinion about the ordering of one particular pair of alternatives only. [3] If the alternative that the agent favours after the change beats the other alternative in a pairwise comparison *for the rest of the society* then the consensus should increase. And if both alternatives tie in a pairwise comparison *for the rest of the society* then the consensus should not vary after the change.

Examples 9 and 10 below show that RCM-B does not verify any of the two statements that jointly define Monotonicity. The same goes for Examples 11 and 12 regarding RCM-C.

Example 9 Suppose $X = \{x, y, z\}$ thus $k = 3$. Let $\mathcal{R} = (R_1, R_2, R_3)$ be the profile of linear orders given by: $y\, P_1\, x\, P_1\, z$, $x\, P_2\, y\, P_2\, z$, $y\, P_3\, z\, P_3\, x$. Then $\mathcal{C}_{\mathcal{B}}(\mathcal{R})$ is P_1, that is, the Borda method produces P_1. Some simple computations yield $\nabla_{\mathbf{M}_B}(\mathcal{R}) = \frac{7}{9}$.

Consider the profile $\mathcal{R}' = (R_1', R_2, R_3)$ where R_1' is the linear order $y\, P_1'\, z\, P_1'\, x$. Under monotonicity this profile would have consensus $\frac{7}{9}$. However $\nabla_{\mathbf{M}_B}(\mathcal{R}') = \frac{5}{9}$ because $\mathcal{C}_{\mathcal{B}}(\mathcal{R}')$ is the complete preorder R' for which $y\, P'\, x\, I'\, z$.

Example 10 Suppose $X = \{x, y, z\}$ thus $k = 3$. Let $\mathcal{R} = (R_1, R_2, R_3)$ be the profile of linear orders given by: $y\, P_1\, x\, P_1\, z$, $z\, P_2\, x\, P_2\, y$, $y\, P_3\, z\, P_3\, x$. Then $\mathcal{C}_{\mathcal{B}}(\mathcal{R})$ is P_3, that is, the Borda method produces P_3. Some simple computations yield $\nabla_{\mathbf{M}_B}(\mathcal{R}) = \frac{2}{3}$.

Consider the profile $\mathcal{R}' = (R_1', R_2, R_3)$ where R_1' is the linear order $y\, P_1'\, z\, P_1'\, x$. Under monotonicity this profile would yield a higher consensus. However $\nabla_{\mathbf{M}_B}(\mathcal{R}') = \frac{5}{9}$ because $\mathcal{C}_{\mathcal{B}}(\mathcal{R}')$ is the complete preorder R' for which $y\, I'\, z\, P'\, x$.

Example 11 Suppose $X = \{x, y, z\}$ thus $k = 3$. Let $\mathcal{R} = (R_1, R_2, R_3)$ be the profile of linear orders given by: $x\, P_1\, y\, P_1\, z$, $y\, P_2\, x\, P_2\, z$, $z\, P_3\, y\, P_3\, x$. Then $\mathcal{C}_{\mathcal{C}}(\mathcal{R})$ is P_2, that is, the Copeland method produces P_2. Some simple computations yield $\nabla_{\mathbf{M}_C}(\mathcal{R}) = \frac{2}{3}$.

Consider the profile $\mathcal{R}' = (R_1', R_2, R_3)$ where R_1' is the linear order $x\, P_1'\, z\, P_1'\, y$. Under monotonicity this profile would have consensus $\frac{2}{3}$. However $\nabla_{\mathbf{M}_C}(\mathcal{R}') = 0$ because $\mathcal{C}_{\mathcal{C}}(\mathcal{R}')$ is the complete preorder R' for which $x\, I'\, y\, I'\, z$.

[3] Observe that this excludes from the analysis the case of a reversal of the order between x and y e.g., in $y\, I_1\, z\, P_1\, x$ or in $y\, P_1\, z\, P_1\, x$. These reversals modify the ordering between other pairs of alternatives too.

Example 12 Suppose $X = \{x, y, z\}$ thus $k = 3$. Let $\mathcal{R} = (R_1, R_2, R_3, R_4)$ be the profile of linear orders given by: $x\,P_1\,y\,P_1\,z,\ x\,P_2\,y\,P_2\,z,\ z\,P_3\,x\,P_3\,y,\ z\,P_4\,x\,P_4\,y$. Then $\mathcal{C}_{\mathcal{C}}(\mathcal{R})$ is the linear order $x\,P\,z\,P\,y$ (that is, the Copeland method produces P_1). Some simple computations yield $\nabla_{\mathbf{M}_C}(\mathcal{R}) = \frac{2}{3}$.

Consider the profile $\mathcal{R}' = (R_1', R_2, R_3, R_4)$ where R_1' is the linear order whose asymmetric part is P above (that is, $x\,P_1'\,z\,P_1'\,y$). Under monotonicity this profile would yield a higher consensus. However $\nabla_{\mathbf{M}_C}(\mathcal{R}') = \frac{7}{12}$ because $\mathcal{C}_{\mathcal{C}}(\mathcal{R}')$ is the complete preorder R' for which $x\,I'\,z\,P'\,y$.

5 Concluding remarks and future research

Alongside with normative approaches like the foundational Bosch [4] or Alcalde and Vorsatz [2], in this paper we analyse the measurement of consensus from a descriptive point of view. We have presented a general framework whose performance has been explored. Also we have given two particular specifications that link this proposal to voting theory. Some particular properties of theirs were presented too.

Our formulation permits to compare a finite list of proposals on a common ground so that the society can decide which one conveys a higher consensus. Nonetheless its primary objective is to assess the coherence within a society with reference to a given voting rule. As is apparent, this may serve to discriminate among the voting rule that should be selected if we aim at producing indisputable results.

Several questions remain open. Clearly, the performance of other measures with reference to alternative voting rules is a direct variation of our analysis. Also, different subclasses besides normal referenced consensus measures can yield a good normative performance. The computational aspects of the notion can be explored too. Analogously to the inspiring Saari [7] or Pritchard and Wilson [6], a more thorough inspection of small sets of candidates (e.g., the three-alternative case) can shed light on the study. This is a natural continuation of our Subsection 3.4. An ambitious project is the identification of the consensus function that yields the highest consensus as a function of the consensus distance (or at least, for focal examples like $\partial_{\mathcal{K}}$) [4]. Obviously when such procedure is used to make social decisions, the researcher can elaborate on manipulability issues too.

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[4] This has slight resemblances to the approach by Meskanen and Nurmi [5].

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