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04. March 2009

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Abstract

We present new sufficient conditions for the existence of a continuous utility function for an arbitrary binary relation on a topological space. Such conditions are basically obtained by using both the concept of a weakly continuous binary relation on a topological space and the concept of a countable network weight. In particular, we are concerned with suitable topological notions which generalize the concept of compactness and do not imply second countability or local compactness.

Keywords: hereditarily Lindelöf space; weakly continuous binary relation; countable network weight; hemicompactness; submetrizability.

JEL classification: C60

MSC: primary 91B16; secondary 54F05

1 Introduction

The problem concerning the existence of a continuous utility function for a not necessarily *total* (*linear*) *preorder* (or for a *partial order*) on a topological space was extensively treated in the literature concerning the applications of mathematics to economics and social sciences.

Peleg [27] was the first who provided sufficient conditions for the existence of a continuous utility function for a partial order on a topological space. Peleg said that he was solving a problem raised by Aumann [1], who pointed out that it is realistic not assume that an individual may compare any two objects according with its own preferences, so that “incomparability” may take place in some cases (see also Ok ([25]).

Following the illuminating approach of Nachbin [24], who combined the classical results of mathematical utility theory with some of the most important achievements in elementary topology, Mehta was able to establish very general conditions for the existence of a continuous utility function for a not necessarily total preorder on a topological space (see e.g. Mehta [21] and the survey in Mehta [22]). The reader may also consult the book by Bridges and Mehta [6] for a miscellanea of theorems concerning the existence of continuous order isomorphisms.

Herden [15] found a characterization of the existence of a continuous utility function for a not necessarily total preorder on a topological space by using the concept of a separable system. Herden also showed that the classical utility representation theorems of Eilenberg-Debreu and Debreu (see Debreu [10, 11] and Eilenberg [12]) concerning the existence of a continuous utility function for a continuous total preorder on a connected and separable topological space and respectively on a second countable topological space are corollaries of his main result. By using similar arguments, Bosi and Mehta [5] presented a unified approach to the existence of a semicontinuous or continuous utility function on a preordered topological space, while the continuous utility representation problem in arbitrary concrete categories was discussed by Bosi and Herden [4].

In a slightly different context, Chateauneuf [9] characterized in a very elegant way the representability of a preference relation with pseudotransitive preference-indifference on a connected topological space by means of a pair of continuous real-valued functions.

In order to possibly generalize the theorems of Eilenberg-Debreu and Debreu to the case of a non-total preorder, Herden and Pallack [18] introduced the concept of a *weakly continuous* preorder $\precsim$ on a topological space (X, τ).

We recall that a preorder $\precsim$ on a topological space (X, τ) is said to be weakly continuous if for every $x, y \in X$ such that $x \prec y$ there exists a continuous increasing function $u_{xy} : (X, \tau, \precsim) \longrightarrow (\mathbb{R}, \tau_{nat}, \leq)$ such that $u_{xy}(x) < u_{xy}(y)$.

Herden and Pallack showed that Debreu theorem is generalizable to the case of a weakly continuous preorder while Eilenberg-Debreu theorem is not. Furthermore, looking at the proof of theorem 2.15 in Herden and Pallack [18], it is easily seen that there exists a continuous utility function for any weakly continuous preorder $\precsim$ on a topological space (X, τ) such that the product topology $\tau \times \tau$ on $X \times X$ is hereditarily Lindelöf (it is well known that this requirement generalizes the assumption of second countability of the topological space (X, τ)).

In this paper, we first generalize the aforementioned result presented by Herden and Pallack by showing that the existence of a continuous utility function for a binary relation R on a topological space (X, τ) is equivalent to the existence of a topology τ' coarser than τ such that R is weakly continuous on (X, τ') and (X, τ') has a *countable network weight* (or equivalently the product topology $\tau' \times \tau'$ on $X \times X$ is hereditarily Lindelöf). Then we use this result in order to derive some sufficient conditions for the existence of a continuous utility function. In this way, we generalize Debreu continuous utility representation theorem by showing that every weakly continuous binary relation on a topological space with a countable net weight has a continuous utility representation. This result may be viewed as a generalization of a proposition in Caterino, Ceppitelli and Mehta [8], where the authors consider the case of a continuous total preorder on a topological space with a countable net weight.

Finally, we show that suitable notions which generalize the concept of compactness such as *σ-compactness*, *hemicompactness* and the concept of *k-space* (see e.g. McCoy [20]) may be invoked in order to guarantee the continuous representability of a weakly continuous binary relation on a submetrizable topological space (i.e., on a space that admits a coarser metrizable topology). It is remarkable that these situations do not imply second countability or local compactness (see Levin [19] and Back [2]). On the other hand, assumptions of this kind are interesting in economics since they are very frequently applied to function spaces (for example, the *compact-open topology* on the space of all continuous functions is considered in Ok [26] in connection with the problem of representing continuous multifunctions).

2 Notation and preliminaries

Throughout this paper, we shall denote by R a binary relation on an arbitrary nonempty set X. The *strict part* R_S of R and the *symmetric part* I of R are defined as follows:

$$xR_Sy \Leftrightarrow (xRy) \wedge \neg(yRx) \quad (x, y \in X),$$

$$xIy \Leftrightarrow (xRy) \wedge (yRx) \quad (x, y \in X).$$

Further, we shall denote by $\mathcal{R}_{\mathcal{S}}$ and $\mathcal{R}'_{\mathcal{S}}$ the *graphs* of R_S and the *dual* of R_S, namely

$$\mathcal{R}_{\mathcal{S}} = \{(x,y) \in X \times X : xR_S y\},$$

$$\mathcal{R}'_S = \{(x,y) \in X \times X : (y,x) \in R_S\}.$$

A *preorder* R on X is a *reflexive* and *transitive* binary relation on X. A preorder is said to be *total* if for any two elements $x, y \in X$ either xRy or yRx. In the sequel, a preorder will be preferably denoted by the symbol $\precsim$. In this case, the strict part of a preorder $\precsim$ will be indicated by $\prec$.

The pair (X, R) will be referred to as a *related set* in the general case. If in addition τ is a topology on the set X, then the triplet (X, τ, R) will be referred to as a *topological related space*.

If (X, R) is a related set, then a subset A of X is said to be *decreasing* if, for every $x \in X$ and $y \in A$, xRy implies that $x \in A$.

Given a related set (X, R), a real-valued function u on X is said to be

(i) *increasing* if $u(x) \leq u(y)$ for all $x, y \in X$ such that xRy,

(ii) *order-preserving* if it is increasing and $u(x) < u(y)$ for all $x, y \in X$ such that $xR_S y$.

In the sequel, an order-preserving function will be referred to as a *utility function*.

If $\precsim$ is a total preorder on a set X, then the associated *order topology* will be denoted by $\tau^{\precsim}$. We recall that $\tau^{\precsim}$ is the topology generated by the sets $L(x) = \{z \in X : z \prec x\}$ and $U(x) = \{z \in X : x \prec z\}$ with $x \in X$.

From Herden and Pallack [18], a binary relation R on a topological space (X, τ) is said to be *weakly continuous* if for all $x, y \in X$ such that $xR_S y$ there exists a continuous increasing real-valued function u_{xy} on (X, τ, R) such that $u_{xy}(x) < u_{xy}(y)$.

Herden and Pallack [18, Lemma 2.2] proved that if R is a total preorder, then the above defined continuity of R on (X, τ) coincides with the classical requirement that $L(x) = \{z \in X : zR_S x\}$ and $U(x) = \{z \in X : xR_S z\}$ are open subsets of X for every $x \in X$. In this case, the total preorder R on (X, τ) is said to be *continuous*.

We recall that a preorder $\precsim$ on a topological space (X, τ) is said to be *closed* if $\precsim$ is a closed subset of $X \times X$ with respect to the product topology $\tau \times \tau$ on $X \times X$. Herden and Pallack [18, Proposition 2.11] proved that every weakly continuous binary relation R on a topological space (X, τ) has a *weakly continuous refinement* by a closed preorder (i.e., for every weakly continuous binary relation R on (X, τ) there exists a weakly continuous preorder $\precsim$ on (X, τ) such that $R \subset \precsim$ and $R_S \subset \prec$).

We recall that a topology τ on a set X is a *hereditarily Lindelöf topology* if for every subset A of X and every open covering $\mathcal{C}$ of A there exists some countable subcovering $\mathcal{C}^{'} \subset \mathcal{C}$ of A.

Let us recall some classical definitions in the theory of cardinal functions. As usual, the symbol $\aleph_0$ will stand for the smallest infinite cardinal.
A family $\mathcal{N}$ of subsets of a topological space (X, τ) is called a *network* for X if every non empty open subset of X is a union of elements of $\mathcal{N}$.
The *network weight* (or *net weight*) of (X, τ) is defined by

$$nw(X, \tau) = \min\{|\mathcal{N}| : \mathcal{N} \text{ is a network for } (X, \tau)\} + \aleph_0.$$

As usual, define by

$$w(X, \tau) = \min\{| \mathcal{B} |: \mathcal{B} \text{ is a base for } (X, \tau)\} + \aleph_0$$

the *weight* of (X, τ). We recall that if (X, τ) is either metrizable or locally compact or else linearly ordered then $nw(X, \tau) = w(X, \tau)$ (see Engelking [13]). For what concerns subspaces and topological products we have that if (Y, τ') is a subspace of (X, τ) then

$$nw(Y, \tau') \leq nw(X, \tau)$$

and

$$nw(\prod_{s \in S} X_s, \prod_{s \in S} \tau_s) = \max\{|S|, \sup_{s \in S} nw(X_s, \prod_{s \in S} \tau_s)\}.$$

So every subspace of a countable product of spaces having countable net weight has countable net weight. Since $nw(X, \tau) = \aleph_0$ implies that (X, τ) is Lindelöf, a countable product of spaces with countable net weight is hereditarily Lindelöf.

3 Existence of continuous utilities

Herden and Pallack [18, Theorem 2.15] proved that every weakly continuous binary relation on a second countable space has a continuous utility representation. This result generalizes the famous Debreu utility representation theorem which states that every continuous total preorder on a second countable topological space admits a continuous utility representation (see Debreu [10, 11]).

The following theorem characterizes the existence of a continuous utility function for an arbitrary binary relation on a topological space and therefore generalizes the aforementioned result proved by Herden and Pallack.

Theorem 3.1 *Let R be a binary relation on a topological space (X, τ). Then the following conditions are equivalent:*

(i) *There exists a continuous utility function u on (X, τ, R);*

(ii) *There exists a topology τ' on X coarser than τ such that R is weakly continuous on (X, τ') and (X, τ') is second countable;*

(iii) *There exists a topology τ' on X coarser than τ such that R is weakly continuous on (X, τ') and (X, τ') has a countable net weight;*

(iv) *There exists a topology τ' on X coarser than τ such that R is weakly continuous on (X, τ') and the product topology $\tau' \times \tau'$ on $X \times X$ is hereditarily Lindelöf;*

(v) *There exists a topology τ' on X coarser than τ such that R is weakly continuous on (X, τ') and the topology $(\tau' \times \tau')_{\mathcal{R_S}}$ induced by the product topology $\tau' \times \tau'$ on the graph $\mathcal{R_S}$ of R_S is Lindelöf.*

Proof. (i) $\Rightarrow$ (ii). Let u be a continuous utility function on (X, τ, R). Consider the total preorder $\precsim$ on X defined by

$$x \precsim y \Leftrightarrow u(x) \leq u(y) \quad (x, y \in X),$$

and let $\tau' = \tau^{\precsim}$ be the order topology associated to $\precsim$. Observe that from the Debreu Open Gap Lemma (see e.g. Bridges and Mehta [6, Lemma 3.13]), since there exists a utility function u on the totally preordered set $(X, \precsim)$ there also exists a continuous utility function u' on the totally preordered topological space $(X, \tau', \precsim)$. Since $\precsim$ is (continuously) representable we have that τ' is second countable (see Bridges and Mehta [6, Proposition 1.6.11]). It is clear that τ' is coarser than τ from the definition of the total preorder $\precsim$ and the continuity of the function u on the topological space (X, τ). Further, we have that R is weakly continuous on (X, τ') since u' is continuous on (X, τ') and we have that, for all $x, y \in X$,

$$xRy \Rightarrow u(x) \leq u(y) \Rightarrow x \precsim y \Rightarrow u'(x) \leq u'(y),$$

$$xR_Sy \Rightarrow u(x) < u(y) \Rightarrow x < y \Rightarrow u'(x) < u'(y).$$

(ii) $\Rightarrow$ (iii). Trivial.
(iii) $\Rightarrow$ (iv). See the considerations at the end of section 2.
(iv) $\Rightarrow$ (v). Immediate.
(v) $\Rightarrow$ (i). Since the binary relation R on the topological space (X, τ') is weakly continuous, we have that for every pair $(x, y) \in X \times X$ such that xR_Sy there exists a continuous increasing function u_{xy} on (X, τ', R) such that $u_{xy}(x) < u_{xy}(y)$. It is not restrictive to assume that u_{xy} takes values in $[0, 1]$. Define for every pair $(x, y) \in X \times X$ such that xR_Sy

$$A_{u_{xy}}(x) := u_{xy}^{-1}([0, \frac{u_{xy}(x) + u_{xy}(y)}{2}[), \; B_{u_{xy}}(y) := u_{xy}^{-1}(]\frac{u_{xy}(x) + u_{xy}(y)}{2}, 1]).$$

Then the family $C := \{A_{u_{xy}(x)} \times B_{u_{xy}(y)}\}_{(x,y)\in\mathcal{R}_S}$ is an open cover of the graph $\mathcal{R}_S$ of R_S. Since the topology $(\tau' \times \tau')_{\mathcal{R}_S}$ induced by the product topology $\tau' \times \tau'$ on $\mathcal{R}_S$ is Lindelöf, there exists a countable subfamily C' of C which also covers $\mathcal{R}_S$, and therefore there exists a countable family $\{u_n\}_{n\in\mathbb{N}}$ of continuous increasing functions on (X, τ', R) such that for every $(x, y) \in X \times X$ with xR_Sy there exists some $n \in \mathbb{N}$ such that $u_n(x) < u_n(y)$. Hence, $u := \sum_{n=0}^{\infty} 2^{-n}u_n$ is a continuous utility function on the topological related space (X, τ', R). Since τ' is coarser than τ, we have that u is also a continuous utility function on the topological related space (X, τ, R) and the proof is complete. □

We recall that from Herden [16] a topology τ on a set X is said to be *useful* if every continuous total preorder $\precsim$ on the topological space (X, τ) is representable by a continuous utility function $u : (X, \tau, \precsim) \longrightarrow (\mathbb{R}, \tau_{nat}, \leq)$ (see also Herden and Pallack [17]). From Theorem 3.1 we immediately obtain the following corollary which provides a condition under which a topology is useful.

Corollary 3.2 *A topology τ on a set X is useful provided that the product topology $\tau \times \tau$ on $X \times X$ is hereditarily Lindelöf (in particular, in case that τ has a countable net weight).*

Remark 3.3 It is clear that Theorem 3.1 implies that whenever the product topology $\tau \times \tau$ on $X \times X$ is hereditarily Lindelöf then there exists a continuous utility function u for every weakly continuous binary relation R on (X, τ) (see the considerations in the introduction). On the other hand, the condition according to which the product topology $\tau \times \tau$ on $X \times X$ is hereditarily Lindelöf is not necessary for the topology τ to be useful. An example can be constructed in the following way. Consider a Tychonoff space Y (that is a completely regular Hausdorff space) such that $Y \times Y$ is not Lindelöf, for instance the Sorgenfrey line. It is know that Y can be embedded in a Tychonoff cube $X = [0, 1]^J$ and so $Y \times Y$ is homeomorphic to a subspace of $X \times X$. Hence $X \times X$ is not hereditarily Lindelöf. But, since X is compact, every continuous total preorder on X has a maximum and minimum. Therefore applying Theorem 3 in Monteiro [23] to X, which is pathwise connected, we get that every continuous total preorder on X is representable by a continuous utility function.

We say that a topology τ on a set X is *strongly useful* (see Bosi and Herden [3]) if every weakly continuous preorder $\precsim$ on the topological space (X, τ) is representable by a continuous utility function $u : (X, \tau, \precsim) \longrightarrow (\mathbb{R}, \tau_{nat}, \leq)$. It is clear that a strongly useful topology on a set X is also useful. Further, we say that a topology τ on a set X is *R-strongly useful* if every weakly continuous binary relation R on the topological space (X, τ) admits a continuous utility function $u : (X, \tau, R) \longrightarrow (\mathbb{R}, \tau_{nat}, \leq)$. Indeed the two definitions are equivalent. In fact we can state the following proposition whose proof is based on Proposition 2.11 in Herden and Pallack [18] (see the introduction).

Proposition 3.4 *Let (X, τ) be a topological space. The following conditions are equivalent:*

(i) *τ is R-strongly useful;*

(ii) *τ is strongly useful;*

(iii) *every closed and weakly continuous preorder $\precsim$ on (X, τ) admits a continuous utility function.*

The following corollary of Theorem 3.1 provides a characterization of R-strongly useful topologies in the metrizable case. The proof is based on the theorem in Estévez and Hervés [14].

Corollary 3.5 *Let τ be a metrizable topology on a set X. Then the following conditions are equivalent:*

(i) *τ is R-strongly useful;*

(ii) *τ is useful;*

(iii) *τ is separable.*

Denote by $\Delta(X)$ the *diagonal* of a set X (i.e., $\Delta(X) = \{(x, x) : x \in X\}$). We recall that if (X, τ) is a topological space, then a subset of X is said to be a *G_δ-set* if it is a countable intersection of open subsets of X.

Corollary 3.6 *Let (X, τ, R) be a linearly ordered topological space and assume that R is continuous. If the product topology $\tau \times \tau$ on $X \times X$ is Lindelöf and X has a G_δ- diagonal, then R has a continuous utility representation.*

Proof. Since R is a linear order then $\{\Delta(X), \mathcal{R}_S, \mathcal{R}'_S\}$ is a partition of $X \times X$. Hence $\mathcal{R}_S \cup \mathcal{R}'_S = (X \times X) \setminus \Delta(X)$ is Lindelöf because it is a countable union of closed subsets of $X \times X$. Further, since R is a continuous linear order on (X, τ), we have that $\mathcal{R}_S$ ($\mathcal{R}'_S$) is open in $X \times X$ since for every $(x, y) \in \mathcal{R}_S$ ($(x, y) \in \mathcal{R}'_S$) there exists a continuous increasing function u_{xy} on (X, τ, R) such that $u_{xy}(x) < u_{xy}(y)$ ($u_{xy}(x) > u_{xy}(y)$) and therefore if we adopt the notation in the proof of Theorem 3.1 we have that $A_{u_{xy}}(x) \times B_{u_{xy}}(y)$

is contained in $\mathcal{R}_S$ ($B_{u_{xy}(x)} \times A_{u_{xy}(y)}$ is contained in $\mathcal{R}'_S$). In particular, we have that $\mathcal{R}_S$ is a Lindelöf spaces when endowed with the induced topology $\tau \times \tau_{\mathcal{R}_S}$, and therefore Theorem 3.1 applies (see in particular the equivalence of the statements (i) and (v)). $\square$

Corollary 3.7 *Let $\precsim$ be a total preorder on a set X. Then $\precsim$ has a utility representation if and only if the product topology $\tau^{\precsim} \times \tau^{\precsim}$ on $X \times X$ is hereditarily Lindelöf.*

Proof. If $\precsim$ is a total preorder on a set X and there exists a utility representation for $\precsim$, then the order topology $\tau^{\precsim}$ on X is second countable (see e.g. Proposition 1.6.11 and Corollary 1.6.14 in Bridges and Mehta [6]) and therefore it is clear that the product topology $\tau^{\precsim} \times \tau^{\precsim}$ on $X \times X$ is hereditarily Lindelöf. The converse is an immediate consequence of Theorem 3.1 (see in particular the equivalence of the statements (i) and (iv)) since it is clear that $\precsim$ is (weakly) continuous on $(X, \tau^{\precsim})$. $\square$

Remark 3.8 Using the proof of Corollary 3.7 we may immediately conclude that if $(X, \precsim)$ is a linearly preordered set, then the following equivalence holds:

$$\tau^{\precsim} \times \tau^{\precsim} \text{ is hereditarily Lindelöf} \Leftrightarrow \tau^{\precsim} \text{ is second countable.}$$

From Theorem 3.1 we can also immediately deduce the following proposition (see in particular the equivalence of the statements (i) and (iii)) which generalizes the classical Debreu continuous utility representation theorem.

Proposition 3.9 *Let (X, τ) be a topological space with $nw(X, \tau) = \aleph_0$ and let R be a weakly continuous binary relation defined on (X, τ). Then R has a continuous utility representation.*

The following corollary is an immediate consequence of Proposition 3.9.

Corollary 3.10 *Let (X, τ) be a countable topological space and let R be a weakly continuous binary relation defined on (X, τ). Then R has a continuous utility representation.* $\square$

Remark 3.11 We may observe that since there exist countable spaces which are not second countable, Corollary 3.10 is not a consequence of Theorem 2.15 in Herden and Pallack [18]. $\square$

In order to present further implications of Theorem 3.1, let us now recall some definitions. A topological space (X, τ) is said to be *submetrizable* if there is a metric topology τ' on X which is coarser than τ. Moreover, (X, τ) is *hemicompact* if there is a countable family $\{K_n\}$ of compact subsets of X such that every compact subset of X is contained in some K_n. Of course, every hemicompact space is a countable union of compact sets, that is every hemicompact space is σ -compact. Finally, X is a *k-space* if a subset $A \subset X$ is open if and only if $A \cap K$ is open in K for every compact subset K of X. Theorem 2.15 in Herden and Pallack [18] generalizes the well known Levin's Theorem (see Levin [19]) which states that every closed preorder defined on a second countable locally compact topological space is representable by a continuous utility function. Caterino, Ceppitelli, Maccarino [7] extended Levin's Theorem to submetrizable hemicompact k-spaces. These spaces are, in general, neither locally compact nor second countable. Therefore, Theorem 2.15 in Herden and Pallack [18] cannot be applied in this case.

Proposition 3.12 *Let (X, τ) be a submetrizable, σ -compact space and let R be a weakly continuous binary relation defined on (X, τ). Then R has a continuous utility representation.*

Proof. Since compact topologies are minimal among T_2 topologies, every compact submetrizable space is metrizable, hence second countable. By σ-compactness, we have that $X = \cup_n K_n$ with K_n compact, for every $n \in \mathbb{N}$. Let $\mathcal{B}_n$ be a countable base for K_n. Then it is easily seen that $\mathcal{N} = \cup_n \mathcal{B}_n$ is a countable network for (X, τ). Hence, the thesis follows from Proposition 3.9. $\square$

Remark 3.13 We recall that every submetrizable σ -compact space is separable. The weaker assumptions of submetrizability and separability are not sufficient to guarantee the existence of a continuous utility representation for every continuous linear preorder. As an example of this fact, consider the Sorgenfrey line $(\mathbb{R}, \tau)$ (see Remark 3.3).
Let $\precsim$ be the preorder on $\mathbb{R}$ defined by:

$$x \precsim y \Leftrightarrow \begin{cases} \mid x \mid > \mid y \mid \text{ or } (\mid x \mid = \mid y \mid \text{ and } x < 0) \text{ or } x = y \quad \forall x, y \in]-1, 1] \\ \textit{or} \\ x \in]-\infty, -1] \cup]1, +\infty[\text{ and } y \in \mathbb{R} \end{cases}.$$

Then it is not hard to show that $\precsim$ is a continuous linear preorder on $\mathbb{R}$. Further $\precsim$ has uncountably many *jumps* (i.e., uncountably many pairs $(x, y) \in \mathbb{R} \times \mathbb{R}$ such that $x \prec y$ and for no $z \in \mathbb{R}$ it happens that $x \prec z \prec y$). Indeed, $(-a, a)$ is a jump for every $0 < a < 1$. Hence we may conclude that $\precsim$ cannot be representable by a (continuous) utility function. This preorder could be also constructed by means of a chain of open and closed subsets of $\mathbb{R}$ (see Bosi and Herden [3]).

The following proposition generalizes Proposition 2.12 in Herden and Pallack [18], who showed that every closed preorder $\precsim$ on a topological space (X, τ) is weakly continuous provided that (X, τ) is either a compact (Hausdorff-)space or a locally compact second countable (Hausdorff-)space. Indeed, if (X, τ) is either a compact (Hausdorff-)space or a locally compact second countable (Hausdorff-)space then (X, τ) is a hemicompact k-space.

Proposition 3.14 *Let (X, τ) be a hemicompact k-space and let $\precsim$ be a closed preorder on (X, τ). Then $\precsim$ is weakly continuous.*

Proof. Assume that $X = \cup_n K_n$ with K_n compact and $K_n \subset K_{n+1}$ for every $n \in \mathbb{N}$. Consider any two elements $x, y \in X$ with $x \prec y$. Then the set $F = \{x, y\}$ is contained in K_n for some integer n. The function $f : F \to I\!R$ defined by $f(x) = 0,\ f(y) = 1$ can be extended to a continuous increasing function on all of K_n (see from Levin [19, Lemma 2]). Since X is a k-space, by using a recursive process, we obtain an extension of the function f on all of X which is increasing and continuous. $\square$

Finally, from Proposition 3.12 and from the above Proposition 3.14 we immediately obtain the following result which was already proved by Caterino, Ceppitelli and Maccarino [7, Theorem 3] by using a different technique.

Proposition 3.15 *Let (X, τ) be a submetrizable, hemicompact k-space and let $\precsim$ be a closed preorder on (X, τ). Then $\precsim$ has a continuous utility representation.*

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