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ABSTRACT

This paper studies the sequential sampling scheme as a solution to the problem of aliasing, where the sampling interval is restricted to a minimum allowable value ΔT. In sequential sampling, the signal is sampled at intervals of ΔT, $\Delta T+\Delta\tau$, $\Delta T+2\Delta\tau$, $\Delta T+3\Delta\tau$, ...; where $\Delta\tau < \Delta T$ and $\Delta\tau$ may be selected as desirable. Sequential sampling is, however, analyzed and it is proved that when the ratio $\Delta T/\Delta\tau$ is an integral number, the associated spectral estimates give a Nyquist frequency $\frac{1}{2\Delta\tau}$. This sampling scheme can, therefore, be employed to yield a required cut- off frequency.

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INTRODUCTION

Some data acquisition systems have a minimum allowable sampling interval and do not provide a desired sampling period less than a minimum allowable value. This may be due to some restrictions set by the measuring instrument that has to be used [1-7].

Let the minimum allowable sampling time be ΔT; if the uniform sampling scheme is employed then, the Nyquist or cut-off frequency is known [8] to be given as:

$$f_c = \frac{1}{2\Delta T} \qquad (1)$$

This would mean that if frequencies higher than f_c are present, aliasing will occur. Otherwise, the signal would have to be filtered so that only frequencies below f_c are passed and, therefore, the spectral analysis will be restricted [8].

The sequential sampling scheme can, however, be employed to obtain an autocorrelation function with estimates $\Delta\tau$ apart, where $\Delta\tau < \Delta T$, with the exception of coefficients lying inside the range $R(0) \to R(\Delta T)$. In this sampling scheme, the signal would be sampled at intervals of:

$\Delta T, \Delta T+\Delta\tau, \Delta T+2\Delta\tau, \Delta T+3\Delta\tau, \ldots$

From the sampled signal, an autocorrelation function can be obtained with the coefficients:

$R(0), R(\Delta T), R(\Delta T+\Delta\tau), R(\Delta T+2\Delta\tau), \ldots$

While ΔT is restricted, the value of $\Delta\tau$ may be chosen as desirable. It will be proved, in this paper, that the sequential sampling can give an increased cut-off frequency as:

$$f_{cs} = \frac{1}{2\Delta\tau} \qquad (2)$$

The sequential sampling can, therefore, be employed to overcome aliasing and the restrictions of spectral analysis, by selecting a sufficiently small value for $\Delta\tau$.

THE CUT OFF FREQUENCY IN THE SEQUENTIAL SAMPLING

The cut- off frequency provided by the sequential sampling scheme is considered in this section. The analysis employs the impulse representation of a continuous signal as an approach to discretization [9-14].

In the sequential sampling, the signal is sampled at intervals of:

$\Delta T, \Delta T+\Delta\tau, \Delta T+2\Delta\tau, \Delta T+3\Delta\tau, \ldots$

The sampling instants are, therefore, given by:

$$t_i = 0, \quad \Delta T, 2\Delta T+\Delta\tau, 3\Delta T+3\Delta\tau, 4\Delta T+6\Delta\tau, \ldots \qquad (3)$$

This can be written as:

$$t_i = i\Delta T + [\sum_{r=0}^{i} r - i]\Delta\tau\,, \quad i = 0,1,2,3,\ldots \qquad (4)$$

Since, $\sum_{r=0}^{i} r = \frac{i}{2}(i+1)$, then equation (4) gives:

$$t_i = i\Delta T + \frac{i}{2}(i-1)\Delta\tau \qquad (5)$$

When a continuous signal x(t) is sampled, the sample values $x(t_i)$ are acquired. A discrete autocorrelation function, with coefficients $R(\tau_j)$, may be obtained from the discrete signal, by contributions of the products $x(t_i)x(t_{i+j})$. Equation (5) can be used to give the time delay τ_j as:

$$\tau_j = t_{i+j} - t_i = \Delta T + [\frac{j}{2}(j-1) + ij + (j-1)\mu]\Delta\tau \qquad (6)$$

i= o, 1, 2, ...

j= o, 1, 2, ...

where μ is a constant given by:

$$\mu = \Delta T/\Delta\tau \qquad (7)$$

It is seen from equation (6) that for j=0, the time delay is zero and for j=1, the time delay is $\Delta T + i\Delta\tau$ (where i=0, 1, 2,3, ...). An autocorrelation function is, therefore, obtainable at discrete values of the time delay given as:

$$\tau_n = o,\ \Delta T + n\Delta\tau n = 0, 1, 2, 3, \ldots \qquad (8)$$

If the ratio μ is an integral number, then higher values of j would also provide more contributions to the autocorrelation estimates at the above time delays τ_n. This is because j(j-1)/2 is always even, and any value of j would hence add a multiple of $\Delta\tau$ to ΔT.

The discrete autocorrelation function may be represented as:

$$R^*(\tau) = \Delta\tau . R(\tau)\delta_b(\tau) \qquad (9)$$

where $R(\tau)$ is the continuous autocorrelation function and $\delta_b(\tau)$ is the following form of the delta comb:

$$\delta_b(\tau) = \delta(\tau) + \delta(\tau - \Delta T) + \delta(\tau - \Delta T - \Delta\tau) + \delta(\tau - \Delta T - 2\Delta\tau) + L \qquad (10)$$

It is established [9-10] that the Fourier transform of equation (10) can be written as:

$$\Delta_b(\omega) = 1 + e^{-j\omega\Delta T}\sum_{0}^{\infty} e^{-j\omega n\Delta\tau} \qquad (11)$$

which by manipulation [9-10] can be re-written as:

$$\Delta_b(\omega) = \frac{1 - e^{-j\omega\Delta\tau} + e^{-j\omega\mu\Delta\tau}}{1 - e^{-j\omega\Delta\tau}} \qquad (12)$$

where substitution has also been made for ΔT from equation (7).

The Fourier transformation of R*(τ) gives the spectral density S*(ω) corresponding to the sampled signal and that of R(τ) would yield the spectral density S(ω) of the original continuous signal. The approach adopted for the Fourier transformation of equation (9) is based on the convolution and residue theorems [9]. By evaluating the residue terms [9] and using the convolution property [9], for substitution into equation (9), the Fourier transform of this equation ca be obtained as:

$$S^*(\omega) = \sum_{-\infty}^{\infty} e^{j2\pi n\mu} S(\omega + 2n\omega_{cs}), \quad \omega_{cs} = \pi/_{\Delta\tau} \qquad (13)$$

However, if the ratio $\mu=(\Delta T/\Delta\tau)$ is an integral number then,

$$e^{j2\pi n\mu} = 1$$

noting that n is also an integer. Substituting this into equation (13) gives:

$$S^*(\omega) = \sum_{-\infty}^{\infty} S(\omega + 2n\omega_{cs}), \quad \omega_{cs} = \pi/_{\Delta\tau} \qquad (14)$$

Now, consider the periodicity of S*(ω); this can also be examined by applying the corresponding methods [9-14]. Using equations (9), (10), (11) and the rules established for discrete Fourier transformation [9-10], it can be written:

$$\Delta\tau[R(0) + e^{-j\omega\Delta T}\sum_{0}^{\infty} R(\Delta T + n\Delta\tau)e^{-j\omega n\Delta\tau}] = \Delta\tau[R(0) + e^{-j\omega\mu\Delta\tau}\sum_{0}^{\infty} R(\Delta T + n\Delta\tau)e^{-j\omega n\Delta\tau}] \qquad (15)$$

and then for an integer m:

$$\Delta\tau[R(0) + e^{-j(\omega+2m\omega_{cs})\mu\Delta\tau}\sum_{0}^{\infty} R(\Delta T + n\Delta\tau)e^{-j(\omega+2m\omega_{cs})n\Delta\tau}] =$$

$$\Delta\tau[R(0) + e^{-j\omega\mu\Delta\tau}e^{-j2m\omega_{cs}\mu\Delta\tau}\sum_{0}^{\infty} R(\Delta T + n\Delta\tau)e^{-j\omega n\Delta\tau}.e^{-j2m\omega_{cs}n\Delta\tau}] =$$

$$\Delta\tau[R(0) + e^{-j\omega\mu\Delta\tau}.e^{-j2m\mu\pi}\sum_{0}^{\infty} R(\Delta T + n\Delta\tau)e^{-j\omega n\Delta\tau}.e^{-j2mn\pi}] =$$

$$\Delta\tau[R(0) + e^{-j\omega\mu\Delta\tau}.e^{-j2m\mu\pi}\sum_{0}^{\infty} R(\Delta T + n\Delta\tau)e^{-j\omega n\Delta\tau}] \qquad (16)$$

since m and n are integers. If μ is also an integral number, this would reduce to:

$$\Delta\tau[R(0) + e^{-j\omega\mu\Delta\tau}\sum_{0}^{\infty} R(\Delta T + n\Delta\tau)e^{-j\omega n\Delta\tau}]$$

from which it follows that:

$$S^{*}(\omega + 2m\omega_{cs}) = S^{*}(\omega) \qquad (17)$$

This is the mathematical statement for $S^{*}(\omega)$ to be periodic with period $2\omega_{cs}$. Otherwise, if μ is not an integral number,

$$S^{*}(\omega + 2m\omega_{cs}) \neq S^{*}(\omega) \qquad (18)$$

and the requirement for periodicity is not met.

It is, therefore, seen that when the ratio $\mu(=\Delta T/\Delta\tau)$ is an integral number, the periodic pattern, conforming with the Nyquist theorem,[9-14] is obtained. That is, the sequential sampling gives a cut-off frequency $\omega_{cs} = \pi_{/\Delta\tau}$ or $f_{cs} = \frac{1}{2\Delta\tau}$. On the contrary, when μ is not a whole number, $S^{*}(\omega)$ is related to the true spectral density by equation (13); it includes a complex term and is not periodic.

CONCLUSIONS

This paper has considered the sequential sampling scheme, as a solution to the problem of aliasing, where the sampling interval is restricted to a minimum allowable value ΔT. In the sequential sampling, the signal is sampled at intervals of $\Delta T, \Delta T + \Delta\tau, \Delta T + 2\Delta\tau, \Delta T + 3\Delta\tau, \ldots;$ where $\Delta\tau$ p ΔT and may be selected as desirable.

The sequential sampling was considered analytically and it was proved that, when the ratio $\Delta T / \Delta\tau$ is an integral number, the corresponding spectral estimates give a cut- off frequency of $\frac{1}{2\Delta\tau}$. On the contrary, when the ratio is not a whole number, the associated spectrum of the sequentially sampled data was found to comprise a complex term in its relation to the true spectrum and would not be periodic in terms of the cut-off frequency.

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REFERENCES

1 - Wirth, W.D. (1995), “Energy Saving by Coherent Sequential Detection of Radar Signals with Unknown Doppler Shift”, IEE Proceedings on Radar, Sonar and Navigation, 142, 145-52.

2 - Willis, N.P. and Bresler, Y.(1992), “A New Approach to the Time-Sequential Sampling Problem”, Proceedings of the IEEE International Conference on Acoustics, Speech and Signal Processing, 3, 277-80.

3 - Allebach, J.P. (1984), “Design of Antialiasing Patterns for Time-Sequential Sampling of Spatiotemporal Signals”, IEEE Transactions on Acoustics, Speech and Signal Processing, 32, 1, 137- 44.

4 - Allebach, J.P. (1981), “Design of Sampling Patterns for Time Sequential Sampling of Spatio- Temporal Signals”, Proceedings of Micro-Delcon Delaware Bay Computer Conference, 9-13.

5 - Aizawa, A.N. and Wah, B.W. (1994), “A Sequential Sampling Procedure for Genetic Algorithms”, Computer and Mathematics with Applications, 27, 9-10, 77-82.

6 - Gaster, M. and Bradbury, L.J.S. (1976), “The Measurement of the Spectra of Highly Turbulent Flows by a Randomly Triggered Pulsed-Wire Anemometer”, J. Fluid Mech, 77, 499-509.

7 - Bradbury, L.J.S. (1978), “Examples of the Use of the Pulsed Wire Anemometer in Highly Turbulent Flow”, Proceedings of the Dynamic Flow Conference, Marseille, 489-509.

8 - Bendat, J.S. and Piersol, A.G. (1986), “Random Data: Analysis and Measurement Procedures”, Wiley-Interscience.

9 - Saucedo, R. and Schiring, E. (1968), “Introduction to Continuous and Digital Control Systems”, Macmillan.

10 - Papoulis, A. (1962), “The Fourier Integral and its Applications”, McGraw- Hill.

11 - Oppenheim, A.V. et al (1983), “Signals and Systems”, Prentice- Hall.

12 - Carlson, A.B. (1986), “Communication Systems”, McGraw-Hill.

13 - Jones, R.H. and Steele, N.C. (1989), “Mathematics in Communication Theory”, Ellis Horwood Publishers.

14 - Steward, E.G. (1989), “Fourier Optics: an Introduction”, Ellis Horwood Publishers.