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Optimal Capital Allocation Principles[*]

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Abstract

This paper develops a unifying framework for allocating the aggregate capital of a financial firm to its business units. The approach relies on an optimisation argument, requiring that the weighted sum of measures for the deviations of the business unit's losses from their respective allocated capitals be minimised. This enables the association of alternative allocation rules to specific decision criteria and thus provides the risk manager with flexibility to meet specific target objectives. The underlying general framework reproduces many capital allocation methods that have appeared in the literature and allows for several possible extensions. An application to an insurance market with policyholder protection is additionally provided as an illustration.

Keywords: Capital allocation; risk measure; comonotonicity; Euler allocation; default option; Lloyd's of London.

1   Introduction

The level of the capital held by a bank or an insurance company is a key issue for its stakeholders. The regulator, primarily sharing the interests of depositors and policyholders, establishes rules to determine the required capital to be held by the company. The level of this capital is determined such that the company will be able to meet its financial obligations with a high probability as they fall due, even in adverse situations. Rating agencies rely on the level of available capital to assess the financial strength of a company. Shareholders and investors alike are concerned with the risk of their capital investment and the return that it will generate.

The determination of a sufficient amount of capital to hold is only part of a larger risk management and solvency policy. The practice of Enterprise Risk Management (ERM) enhances identifying, measuring, pricing, and controlling risks. An important component of an ERM framework is the exercise of *capital allocation*, a term referring to the subdivision of the aggregate capital held by the firm across its various constituents, e.g. business lines, types of exposure, territories or even individual products in a portfolio of insurance policies.

Most financial firms write several lines of business and may want their total capital allocated across these lines for a number of reasons. First, there is a need to redistribute the total (frictional or opportunity) cost associated with holding capital across various business lines so that this cost is equitably transferred back to the depositors or policyholders in the form of charges.

Secondly, the allocation of expenses across lines of business is a necessary activity for financial reporting purposes. Lastly, capital allocation provides for a useful device of assessing and comparing the performance of the different lines of business by determining the return on allocated capital for each line. Comparing these returns allows one to distinguish the most profitable business lines and hence may assist in remunerating the business line managers or in making decisions concerning business expansions, reductions or even eliminations.

There is countless number of ways to allocate the aggregate capital of a company to its different business units. Mutual dependencies that may exist between the performances of the various business units make capital allocation a non-trivial exercise. Accordingly, there is an extensive amount of literature on this subject with a wide number of proposed capital allocation algorithms. Cummins (2000) provides an overview of several methods suggested for capital allocation in the insurance industry and relates capital allocation to management decision making tools such as RAROC (risk-adjusted return on capital) and EVA (economic value added). Myers and Read Jr. (2001) consider capital allocation principles based on the marginal contribution of each business unit to the company's default option. LeMaire (1984) and Denault (2001) discuss capital allocations based on game theory considerations, where a risk measure is used as a cost functional. In the case of coherent risk measures (see Artzner et al. (1999)), such capital allocations reduce to subdivisions according to marginal costs. In closely related works, marginal ('Euler') capital allocations are proposed within a portfolio optimisation context by Tasche (2004) and an axiomatic allocation system is proposed by Kalkbrener (2005). A commentary on the various approaches to allocating capital has appeared in Venter (2004). A recent work by Kim and Hardy (2008) proposed a method based on a solvency exchange option and which explicitly accounts for the notion of limited liability.

Panjer (2001) considers the particular case of multivariate normally distributed risks and provides an explicit expression of marginal cost based allocations, when the risk measure used is Tail Value-at-Risk (TVaR). Landsman and Valdez (2003) extends these explicit capital allocation formulas to the case where risks belong to the class of multivariate elliptical distributions, for which the class of multivariate normal is a special case. Dhaene et al. (2008) derive the results of Landsman and Valdez (2003) in a rather more straightforward manner and apply these to sums that involve normal as well as lognormal risks. In Valdez and Chernih (2003), expressions for covariance-based allocations are derived for multivariate elliptical risks. Tsanakas (2004) studies allocations where the relevant risk measure belongs to the class of distortion risk measures, while Tsanakas (2008) extends these allocation principles to the more general class of convex risk measures including the exponential risk measures. Furman and Zitikis (2008b) introduce the class of weighted risk capital allocations "which stems from the weighted premium calculation principle".

The multitude of allocation methods proposed in the literature is bewildering. Allocation methods are sometimes proposed in an ad-hoc fashion lacking usually with much economic justification and are thereby viewed as arbitrary. This motivated some authors to doubt the legitimate purpose of the exercise itself of allocating capital e.g. Gründl and Schmeiser (2007). Certain allocation methods may be best suited in order to address specific issues, but it is always unclear what these issues are. Indeed, Gründl and Schmeiser (2007) point out this importance because accordingly, certain allocation techniques can dangerously lead to wrong financial decisions.

This paper attempts to resolve several of these issues by constructing a unifying framework designed to address specific decision criteria in a comprehensive yet still highly stylised setting. We consider capital allocation as the outcome of a particular optimisation problem, in which the weighted sum of measures for the deviations of the business unit's losses from their respective allocated capitals is minimised. This idea has also been presented in Section 4.1 of Dhaene et al. (2003) in the context of stop-loss premiums. In this paper, we generalise the optimisation

approach of Dhaene et al. (2003) by considering more general deviation measures. We show that alternative specifications of the objective function, by applying different deviation measures, different weights on the possible outcomes of the random deviations and different exposure weights on the business units, reproduces several of the allocation techniques in the literature within the general framework we propose in this paper. Note that related results, in the context of premium allocation, have been presented in Zaks et al. (2006). A generalisation of the optimisation approach of Dhaene et al. (2003) in a different direction than the one we propose in this paper, may also be found in Laeven and Goovaerts (2004).

A quadratic deviation criterion gives rise to allocations that generally have the form of expectations. Allocations based on well-known risk measures such as the Conditional Tail Expectation (CTE) as well as allocations taking into account the insurer's default option are derived in this setting. An absolute deviation criterion will give rise to quantile-based allocations. In both cases, the allocations may or may not reflect the dependence structure of the portfolio, via dependence between the individual and the aggregate risks. Furthermore, the general optimal allocation framework proposed in this paper is also applied to the case of an insurance market with policyholder protection, found, for example, in Lloyd's of London. In that context, novel interpretations arise for allocation methods used in the past and present at Lloyd's.

The structure of this paper is as follows. In Section 2, risk measures and the capital allocation problem are discussed and an overview of some popular allocation methods is given. In Section 3, which forms the main contribution of the paper, a unifying optimal capital allocation approach is presented. From this general approach, a multitude of special cases is derived. In Section 4, we discuss the case of an insurance market with policyholder protection. Finally, brief conclusions are given in Section 5.

2 Capital allocation

2.1 Risk Measures

A risk measure is a mapping ρ from a set Γ of real-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to the real line $\mathbb{R}$:

$$\rho : \Gamma \to \mathbb{R} : X \in \Gamma \to \rho[X]. \tag{1}$$

The random variable X refers to the loss associated with conducting a business. Risk measures have traditionally been used for determining insurance premia (Goovaerts et al. (1984)). More recently, however, they have been applied in a risk management context, with $\rho[X]$ representing the amount of capital to be set aside in order to make the loss X an acceptable risk; see Artzner et al. (1999).

Some well known properties that risk measures may or may not satisfy are law invariance, monotonicity, positive homogeneity, translation invariance (or equivariance) and subadditivity. They are formally defined as:

- *Law invariance*: For any $X_1, X_2 \in \Gamma$ with $\mathbb{P}[X_1 \leq x] = \mathbb{P}[X_2 \leq x]$ for all $x \in \mathbb{R}$, $\rho[X_1] = \rho[X_2]$.

- *Monotonicity*: For any $X_1, X_2 \in \Gamma$, $X_1 \leq X_2$ implies $\rho[X_1] \leq \rho[X_2]$.

- *Positive homogeneity*: For any $X \in \Gamma$ and $a > 0$, $\rho[aX] = a\rho[X]$.

- *Translation invariance*: For any $X \in \Gamma$ and $b \in \mathbb{R}$, $\rho[X + b] = \rho[X] + b$.

- *Subadditivity*: For any $X_1, X_2 \in \Gamma$, $\rho[X_1 + X_2] \leq \rho[X_1] + \rho[X_2]$.

Artzner et al. (1999) call any risk measure that satisfies the last four properties a *coherent* risk measure. Föllmer and Schied (2002) and Dhaene et al. (2008) provide weaker sets of properties and discuss the desirability or otherwise of the properties of coherent risk measures.

2.2 The allocation problem

Consider a portfolio of n individual losses $X_1, X_2, ..., X_n$ materialising at a fixed future date T. Assume that $(X_1, X_2, ..., X_n)$ is a random vector on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Throughout the paper, we will always assume that any loss X_i has a finite mean. The distribution function $\mathbb{P}[X_i \leq x]$ of X_i will be denoted by $F_{X_i}(x)$.

The aggregate loss is defined by the sum

$$S = \sum_{i=1}^{n} X_i, \tag{2}$$

where this aggregate loss S can be interpreted as:

- the total loss of a corporate, e.g. an insurance company, with the individual losses corresponding to the losses of the respective business units;

- the loss from an insurance portfolio, with the individual losses being those arising from the different policies;

- the loss suffered by a financial conglomerate, while the different individual losses correspond to the losses suffered by its subsidiaries; or

- the total loss across an insurance/financial market, with the individual losses being the ones from the different companies in this market.

It is the first of these interpretations we will use throughout this article, with the exception of Section 4. Hence S is the aggregate loss faced by an insurance company and X_i the loss of business unit i. We assume that the company has already determined the aggregate level of capital and denote this total risk capital by K. The company now wishes to allocate this exogenously given total risk capital K across its various business units, that is, to determine non-negative real numbers $K_1, \ldots, K_n$ satisfying the full allocation requirement:

$$\sum_{i=1}^{n} K_i = K. \tag{3}$$

This allocation is in some sense a notional exercise; it does not mean that capital is physically shifted across the various units, as the company's assets and liabilities continue to be pooled. The allocation exercise could be made in order to rank the business units according to levels of profitability. This task is performed by determining at time T, the respective returns on the respective allocated capital $[(K_i - X_i)/K_i] - 1$, $i = 1, \ldots, n$.

Given that a capital allocation can be carried out in a countless number of ways, additional criteria must be set up in order to determine the most suitable. A reasonable start is to require the allocated capital amounts K_i to be 'close' to their corresponding losses X_i in some appropriately defined sense. This underlies the approach proposed in the present paper. Prior to introducing the idea of 'closeness' between individual loss and allocated capital, we revisit some well-known capital allocation methods.

2.3   Some known allocation formulas

For a given probability level $p \in (0, 1)$, we denote the Value-at-Risk (VaR) or quantile of the loss random variable X by $F_X^{-1}(p)$. As usual, it is defined by

$$F_X^{-1}(p) = \inf \left\{ x \in \mathbb{R} \mid F_X(x) \geq p \right\}, \qquad p \in [0, 1]. \tag{4}$$

with $\inf\{\emptyset\} = +\infty$ by convention. Below we will also need so-called α–mixed inverse distribution functions, see Dhaene et al. (2002). Therefore, we first define the inverse distribution function $F_X^{-1+}(p)$ of the random variable X by

$$F_X^{-1+}(p) = \sup \left\{ x \in \mathbb{R} \mid F_X(x) \leq p \right\}, \qquad p \in [0, 1], \tag{5}$$

with $\sup\{\emptyset\} = -\infty$. The α–mixed inverse distribution function $F_X^{-1(\alpha)}$ of X is then defined as follows:

$$F_X^{-1(\alpha)}(p) = \alpha F_X^{-1}(p) + (1 - \alpha) F_X^{-1+}(p), \qquad p \in (0, 1) \; \alpha \in [0, 1]. \tag{6}$$

From this definition, one immediately finds that for any random variable X and for all x with $0 < F_X(x) < 1$, there exists an $\alpha_x \in [0, 1]$ such that $F_X^{-1(\alpha_x)}(F_X(x)) = x$.

2.3.1   The haircut allocation principle

A straightforward allocation method consists of allocating the capital $K_i = \gamma F_{X_i}^{-1}(p)$, $i = 1, \ldots, n$, to business unit i, where the factor γ is chosen such that the full allocation requirement (3) is satisfied. This gives rise to the *haircut allocation principle:*

$$K_i = \frac{K}{\sum_{j=1}^n F_{X_j}^{-1}(p)} F_{X_i}^{-1}(p), \qquad i = 1, \ldots, n. \tag{7}$$

For an exogenously given value for K, this principle leads to an allocation that is not influenced by the dependence structure between the losses X_i of the different business units. In this sense, one can say that the allocation method is independent of the portfolio context within which the individual losses X_i are embedded.

It is a common industry practice, driven by banking and insurance regulations to measure stand-alone losses by a VaR, for a given probability level p. Therefore, let us assume that $K = F_S^{-1}(p)$. In addition, we assume that in case business unit i was a stand-alone unit, then its loss would be measured by $F_{X_i}^{-1}(p)$. It is well-known that the quantile risk measure is not always subadditive. Consequently, using the p-quantile as stand-alone risk measure will not necessarily imply that the subportfolios will benefit from a pooling effect. This means that it may happen that the allocated capitals K_i exceed the respective stand-alone capitals $F_{X_i}^{-1}(p)$.

2.3.2   The quantile allocation principle

The haircut allocation rule (7) allocates to each business unit i a proportion γ of its p-quantile, with γ chosen such that the full allocation condition is fulfilled. This means that a constant proportional reduction (or increase) is applied on each of the quantiles $F_{X_i}^{-1}(p)$. Instead of applying a proportional cut on the monetary amounts $F_{X_i}^{-1}(p)$, one could adopt the probability level p equally among all business units and determine an α–mixed inverse with $\alpha \in [0, 1]$, such that the full allocation requirement is again satisfied. This approach gives rise to the *quantile allocation principle* with allocated capital amounts K_i given by

$$K_i = F_{X_i}^{-1(\alpha)}(\beta p), \text{ with } \alpha \text{ and } \beta \text{ such that } \sum_{i=1}^n K_i = K. \tag{8}$$

Similar to the haircut allocation principle, for a given aggregate capital K, the allocated capitals K_i are not influenced by the dependence structure between the different losses X_i, $i = 1, \ldots, n$.

The quantile allocation rule is in compliance with the principle of using equal quantiles to measure the risk associated with the different business units. If it is considered 'consistent' to measure each stand-alone loss X_i by the corresponding quantile $F_{X_i}^{-1}(p)$, then it makes sense to measure each 'pooled' loss by $F_{X_i}^{-1(\alpha)}(\beta p)$ where α and β are chosen such that the full allocation requirement is satisfied. This means that all losses X_i continue to be evaluated at the same probability level '$\beta \times p$' and the benefits from pooling are in some sense 'subdivided neutrally' across the different business units.

The haircut allocation principle (7) will in general not lead to a quantile-based allocation with the same probability level for all business units. Companies and regulators when debating that all risks should be evaluated using the same p-quantile measure may prefer the quantile allocation principle rather than the haircut principle.

The appropriate levels of α and β are to be determined as the solutions to

$$K = \sum_{i=1}^{n} F_{X_i}^{-1(\alpha)}(\beta p), \tag{9}$$

In order to solve this problem, we need to introduce the concept of a *comonotonic sum* S^c defined by

$$S^c = \sum_{i=1}^{n} F_{X_i}^{-1}(U), \tag{10}$$

where U is a uniform random variable on $(0, 1)$. It then holds that

$$K = F_{S^c}^{-1(\alpha)}(\beta p), \tag{11}$$

which leads us to

$$\beta p = F_{S^c}(K). \tag{12}$$

Furthermore, we have that

$$K = F_{S^c}^{-1(\alpha)}(F_{S^c}(K)). \tag{13}$$

Further additional details can be found in Dhaene et al. (2002). The quantile allocation rule in (8) can then be re-expressed as

$$K_i = F_{X_i}^{-1(\alpha)}\left(F_{S^c}(K)\right), \qquad i = 1, \ldots, n, \tag{14}$$

with α determined from (13).

In the special case that all distribution functions F_{X_i} are strictly increasing and continuous, this rule reduces to

$$K_i = F_{X_i}^{-1}\left(F_{S^c}(K)\right), \qquad i = 1, \ldots, n. \tag{15}$$

This allocation principle was proposed in Dhaene et al. (2003), where it was derived as the solution of an appropriate optimisation problem; See also Section 3.3. Notice that for strictly increasing and continuous distribution functions F_{X_i}, the quantile allocation principle can be considered as a special case of the haircut allocation principle (7) by choosing $p = F_{S^c}(K)$.

2.3.3 The covariance allocation principle

The *covariance allocation principle* proposed by e.g. Overbeck (2000) is given by

$$K_i = \frac{K}{\text{Var}[S]} \text{Cov}[X_i, S], \qquad i = 1, \ldots, n, \tag{16}$$

where $\text{Cov}[X_i, S]$ is the covariance between the individual loss X_i and the aggregate loss S and $\text{Var}[S]$ is the variance of the aggregate loss S. Because clearly the sum of these individual covariances is equal to the variance of the aggregate loss, the full allocation requirement is automatically satisfied in this case.

The covariance allocation rule, unlike the haircut and the quantile allocation principles, explicitly takes into account the dependence structure of the random losses $(X_1, X_2, ..., X_n)$. Business units with a loss that is more correlated with the aggregate portfolio loss S are penalised by requiring them to hold a larger amount of capital than those which are less correlated.

2.3.4 The CTE allocation principle

For a given probability level $p \in (0, 1)$, the Conditional Tail Expectation (CTE) of the aggregate loss S is defined as

$$\text{CTE}_p[S] = \mathbb{E}\left[S \mid S > F_S^{-1}(p)\right]. \tag{17}$$

At a fixed level p, it gives the average of the top $(1 - p)\%$ losses. In general, the CTE as a risk measure does not necessarily satisfy the subadditivity property. However, it is known to be a coherent risk measure in case we restrict to random variables with continuous distribution function. See Acerbi and Tasche (2002) and Remark 4.2.3. in Dhaene et al. (2006).

The *CTE allocation principle*, for some fixed probability level $p \in (0, 1)$, has the form

$$K_i = \frac{K}{\text{CTE}_p[S]} \mathbb{E}\left[X_i \mid S > F_S^{-1}(p)\right], \qquad i = 1, \ldots, n \tag{18}$$

In the particular case that $K = \text{CTE}_p[S]$, formula (18) reduces to the allocation suggested by Overbeck (2000) and Denault (2001).

The CTE allocation rule explicitly takes into account the dependence structure of the random losses $(X_1, X_2, ..., X_n)$. Interpreting the event '$S > F_S^{-1}(p)$' as 'the aggregate portfolio loss S is large', we see from (18) that business units with larger conditional expected loss, given that the aggregate loss S is large, will be penalised with a larger amount of capital required than those with lesser conditional expected loss.

2.3.5 Proportional allocations

The capital allocation methods we have discussed so far are special cases of a more general class. Each member of this class is obtained by first choosing a risk measure ρ and then attributing the capital $K_i = \alpha \rho[X_i]$ to each business unit i, $i = 1, \ldots, n$. The factor α is chosen such that the full allocation requirement (3) is satisfied. This gives rise to the *proportional allocation principle*:

$$K_i = \frac{K}{\sum_{j=1}^n \rho[X_j]} \rho[X_i], \qquad i = 1, \ldots, n. \tag{19}$$

Clearly, all allocation principles discussed in the previous subsections follow from (19) by choosing the appropriate risk measure ρ:

$$\text{Haircut allocation:} \qquad \rho[X_i] = F_{X_i}^{-1}(p), \tag{20}$$

$$\text{Quantile allocation:} \qquad \rho[X_i] = F_{X_i}^{-1}\left(F_{S^c}(K)\right) \text{ with } \alpha \text{ from (13)}, \tag{21}$$

$$\text{Covariance allocation:} \qquad \rho[X_i] = \text{Cov}[X_i, S], \text{ and} \tag{22}$$

$$\text{CTE allocation:} \qquad \rho[X_i] = \mathbb{E}\left[X_i \mid S > F_S^{-1}(p)\right]. \tag{23}$$

Note that if ρ is law invariant, the proportional allocation derived from ρ is not influenced by the dependence structure between the losses X_i of the different business units.

Let us assume that stand-alone losses are measured by a risk measure ρ. This means that $K = \rho[S]$ and also that the risk of business unit i, considered as a stand-alone unit, is measured by $\rho[X_i]$. From (19) one finds that in case of a proportional allocation, each business unit benefits from a pooling effect in the sense that $K_i \leq \rho[X_i]$ if and only if

$$K = \rho[S] \leq \sum_{j=1}^{n} \rho[X_j]. \tag{24}$$

This condition is fulfilled for subadditive risk measures ρ. As we have observed before, the haircut allocation method, which chooses a VaR as a stand-alone risk measure, may lead to a positive or a negative pooling effect. On the other hand, choosing Tail Value-at-Risk (TVaR) as stand-alone risk measure such as in the CTE allocation method, will lead to $K_i \leq \rho[X_i]$.

2.3.6 Location-scale families of distributions

This section investigates the relationship that exists between the different allocation rules presented above, in case the losses X_i belong to the same location-scale family of distributions. Herewith, we assume that there exists a random variable Z with a zero mean and constants $a_i > 0$ and b_i such that

$$X_i \stackrel{d}{=} a_i Z + b_i, \qquad i = 1, \ldots, n, \tag{25}$$

where $\stackrel{d}{=}$ stands for 'equality in distribution'. We further assume that F_Z is strictly increasing and continuous on $\mathbb{R}$.

Let us first consider the general proportional allocation principle (19) where ρ is assumed to be law invariant, translation invariant and positive homogeneous. In this case, all $\rho[X_i]$ can be expressed as

$$\rho[X_i] = F_{X_i}^{-1}(p), \qquad i = 1, \ldots, n, \tag{26}$$

for some fixed probability level $p \in (0, 1)$, see e.g. Dhaene et al. (2009). This means that under the stated conditions, the proportional allocation principle (19) reduces to the haircut allocation principle (7).

Next we consider the CTE allocation principle (18) with $K = \text{CTE}_p[S]$ and where the vector of business losses $(X_1, X_2, ..., X_n)$ is multivariate elliptically distributed with $\mathbb{E}[X_i] = 0$, $i = 1, \ldots, n$. The assumption that all X_i have a zero mean may be relevant for practical situations where a provision equal to the expected value of the aggregate loss is set aside, and in addition capital is used as a buffer to protect against the 'uncertainty of the aggregate loss around its mean'. In this case, each X_i has to be interpreted as the loss of business unit i minus its expectation.

From Landsman and Valdez (2003), we find that in this case the allocated capitals K_i are given by

$$K_i = \mathbb{E}\left[X_i \,\middle|\, S > F_S^{-1}(p)\right] = \frac{\text{CTE}_p[S]}{\text{Var}[S]} \text{Cov}[X_i, S], \qquad i = 1, \ldots, n. \tag{27}$$

Hence, we can conclude that when $(X_1, X_2, ..., X_n)$ is multivariate elliptically distributed with zero means and in addition $K = \text{CTE}_p[S]$, the CTE allocation principle (18) coincides with the covariance allocation principle (16).

2.3.7 Allocation and the default option

A somewhat different class of approaches, based on the arguments of Myers and Read Jr. (2001), produces a capital allocation procedure by considering the value of the insurer's default option. Since the shareholders of the company have limited liability, in the event of default, i.e. when

$S > K$, they are not, in principle, obligated to pay the excess loss $S - K$. Therefore, the protection that the collective of policyholders purchases is $\min(S, K)$, which can be written as

$$S - (S - K)_+. \tag{28}$$

The quantity $(S - K)_+$ is called the *policyholder deficit* or alternatively the *insurer's default option*.

Myers and Read Jr. (2001) assume that markets are complete and that a proportional increase in the exposure to a particular line of business produces a proportional increase in its allocated capital. They subsequently allocate the value of the default option via the marginal contributions of each line of business to that value.

It can be shown that the Myers-Read allocation is given by the general formula (which does not appear in their paper in this form),

$$\mathbb{E}[(S - K)_+] = \sum_{j=1}^n \mathbb{E}[(X_j - K_j)\,\mathbb{I}(S > K)], \tag{29}$$

where $\mathbb{I}(A)$ is the indicator function of event A and expectations may be taken under a risk neutral measure.

It is worth noting that (29) is not a capital allocation formula in itself, as the $K_1 \ldots, K_n$ are considered as given and only the default option value is allocated. Capital allocation methods that consider the value of the default options have been derived by e.g. Sherris (2006).

3 Optimal capital allocations

3.1 General setting

The allocation of the exogenously given aggregate capital K to n parts $K_1, \ldots, K_n$, corresponding to the different subportfolios or business units, can be carried out in an infinite number of ways, some of which were illustrated in the previous section. At first glance, there seems to be a lack of a clear motivation for preferring to choose one method over another, although it appears obvious that different capital allocations must in some sense correspond to different questions that can be asked within the context of risk management. Hereafter we systematise capital allocation methods by viewing them as solutions to a particular decision problem. For that we need to formulate a decision criterion, such as:

Capital should be allocated such that for each business unit the allocated capital and the loss are sufficiently close to each other.

In order to cast this statement in a more formal setting, consider the aggregate portfolio loss $S = X_1 + \cdots + X_n$ with aggregate capital K. Once the aggregate capital is allocated, the difference between aggregate loss and aggregate capital can be expressed as

$$S - K = \sum_{j=1}^n (X_j - K_j), \tag{30}$$

where the quantity $(X_j - K_j)$ expresses the loss minus the allocated capital for subportfolio j. Important to notice is that in this setting, the subportfolios are cross-subsidising each other, in the sense that the occurence of the event '$X_k > K_k$' does not necessarily lead to 'ruin'; such unfavorable performance of subportfolio k may be compensated by a favorable outcome for one or more values $(X_l - K_l)$ of the other subportfolios. A different setting will be considered in section 4.

We propose to determine the appropriate allocation by the following optimisation problem:

Optimal capital allocation problem: *Given the aggregate capital $K > 0$, determine the allocated capitals K_i, $i = 1, \ldots, n$, from the following optimisation problem:*

$$\min_{K_1,\ldots,K_n} \sum_{j=1}^n v_j \mathbb{E}\left[\zeta_j \, D\left(\frac{X_j - K_j}{v_j} \right) \right], \; such \; that \sum_{j=1}^n K_j = K, \tag{31}$$

where the v_j are non-negative real numbers such that $\sum_{j=1}^n v_j = 1$, the ζ_j are non-negative random variables such that $\mathbb{E}[\zeta_j] = 1$ and D is a non-negative function.

Before solving the general optimal capital allocation problem (31), we first elaborate on its various elements.

$D\left(\frac{X_j - K_j}{v_j} \right)$: For simplicity, we first assume that $v_j = 1$ and also that $\zeta_j \equiv 1$. The terms $D\left(X_j - K_j\right)$ quantify the deviations of the outcomes of the losses X_j from their allocated capital K_j. Minimising the sum of the expectations of these quantities essentially reflects the requirement that the allocated capitals should be 'as close as possible' to the losses they are allocated to.

ζ_j : The deviations of the losses X_j from their respective allocated capital levels K_j are measured by the terms $\mathbb{E}\left[\zeta_j \, D\left(X_j - K_j\right)\right]$. These expectations involve non-negative random variables ζ_j with $\mathbb{E}[\zeta_j] = 1$ that are used as weight factors to the different possible outcomes of $D\left(X_j - K_j\right)$.

One possible choice for the ζ_j could be $\zeta_j = h(X_j)$ for some non-negative and non-decreasing function h. In this case, the heaviest weights are attached to deviations that correspond to states-of-the-world leading to the largest outcomes of X_j. We will call allocations based on such a choice for the ζ_j *business unit driven allocations*.

Another choice is to let $\zeta_j = h(S)$ for some non-negative and non-decreasing function h, such that the outcomes of the deviations are weighted with respect to the aggregate portfolio performance. In this case, heavier weights are attached to deviations that correspond to states-of-the-world leading to larger outcomes of S. Allocations based on such a choice for the random variables ζ_j will be called *aggregate portfolio driven allocations*.

A yet different approach is to let $\zeta_j = \zeta_M$ for all j, where ζ_M can be interpreted as the loss on a reference (or market) portfolio. In this case, the weighting is market driven and the corresponding allocation is said to be a *market driven allocation*.

v_j : The non-negative real number v_j is a measure of exposure or business volume of the j-th unit, such as revenue, insurance premium, etc. These scalar quantities are chosen such that they sum to 1. Their inclusion in the expression $D\left(\frac{X_j - K_j}{v_j} \right)$ normalises the deviations of loss from allocated capital across business units to make them relatively more comparable. At the same time, the v_j's are used as weights attached to the different values of $\mathbb{E}\left[\zeta_j \, D\left(\frac{X_j - K_j}{v_j} \right) \right]$ in the minimisation problem (31), in order to reflect the relative importance of the different business units.

In summary, we propose in (31) to fully allocate the aggregate capital K to the different business units such that the exposure-weighted sum of the expectations of the weighted and normalised deviations of the losses X_j from their respective allocated capitals K_j, is minimised.

3.2 The quadratic optimisation criterion

3.2.1 General solution of the quadratic allocation problem

In this subsection we discuss optimal allocation under a quadratic criterion, that is, by letting

$$D(x) = x^2. \tag{32}$$

In this case, the optimal allocation problem (31) reduces to

$$\min_{K_1,\ldots,K_n} \sum_{j=1}^{n} \mathbb{E}\left[\zeta_j \frac{(X_j - K_j)^2}{v_j}\right], \text{ such that } \sum_{j=1}^{n} K_j = K. \tag{33}$$

The solution to this minimisation problem is given in the following theorem.

Theorem 1 *The optimal allocation problem (33) has the following unique solution:*

$$K_i = \mathbb{E}[\zeta_i X_i] + v_i \left(K - \sum_{j=1}^{n} \mathbb{E}[\zeta_j X_j]\right), \qquad i = 1, \ldots, n. \tag{34}$$

Proof. Problem (33) can be solved via a Lagrange optimisation. However, we will give a simple geometric proof, based on an argument given in Zaks et al. (2006) who considered the minimisation problem (33) in the special case that all $\zeta_j \equiv 1$.

Taking into account the relations

$$\mathbb{E}\left[\zeta_j (X_j - K_j)^2\right] = (\mathbb{E}[\zeta_j X_j] - K_j)^2 + \mathbb{E}\left[\zeta_j X_j^2\right] - (\mathbb{E}[\zeta_j X_j])^2, \qquad j = 1, \ldots, n,$$

we have that the solution of the minimisation problem (33) is identical to the solution of the following minimisation problem:

$$\min_{K_1,\ldots,K_n} \sum_{j=1}^{n} \frac{(\mathbb{E}[\zeta_j X_j] - K_j)^2}{v_j}, \text{ such that } \sum_{j=1}^{n} K_j = K. \tag{35}$$

Clearly, eliminating the term $\sum_{j=1}^{n} \left(\mathbb{E}\left[\zeta_j X_j^2\right] - (\mathbb{E}[\zeta_j X_j])^2\right)$ does not change the optimal allocation.

By introducing the notation

$$x_j = \frac{K_j - \mathbb{E}[\zeta_j X_j]}{\sqrt{v_j}} \qquad j = 1, \ldots, n, \tag{36}$$

we can transform (35) into

$$\min_{x_1,\ldots,x_n} \sum_{j=1}^{n} x_j^2, \text{ such that } \sum_{j=1}^{n} \sqrt{v_j} x_j = \left(K - \sum_{j=1}^{n} \mathbb{E}[\zeta_j X_j]\right). \tag{37}$$

Let us now interpret the set

$$\left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \ \Big| \ \sum_{j=1}^{n} \sqrt{v_j} x_j = \left(K - \sum_{j=1}^{n} \mathbb{E}[\zeta_j X_j]\right) \right\}$$

as a hyperplane in $\mathbb{R}^n$. The solution of (37) can then be interpreted as the point $(x_1, x_2, \ldots, x_n)$ on the hyperplane $\sum_{j=1}^{n} \sqrt{v_j} x_j = \left(K - \sum_{j=1}^{n} \mathbb{E}[\zeta_j X_j] \right)$ that is closest to the origin $(0, 0, \ldots, 0)$. Hence,

$$x_i = \sqrt{v_i} \left(K - \sum_{j=1}^{n} \mathbb{E}[\zeta_j X_j] \right), \qquad i = 1, \ldots, n.$$

Translating this result in terms of the K_i via (36) immediately leads to (34). ∎

The capital K_i given by (34) equals the weighted expected loss of X_i, in addition to a term proportional to the volume of the unit. This second term is a redistribution of the difference between the amount of aggregate capital K held and $\sum_{j=1}^{n} \mathbb{E}[\zeta_j X]$. This redistribution is assigned using weights v_i based on the 'volume' or on some other measure of the 'riskiness' of the corresponding business units.

In the particular case that the volume weights are given by

$$v_i = \frac{\mathbb{E}[\zeta_i X_i]}{\sum_{j=1}^{n} \mathbb{E}[\zeta_j X_j]}, \tag{38}$$

it immediately follows that (34) reduces to

$$K_i = \frac{K}{\sum_{j=1}^{n} \mathbb{E}[\zeta_j X_j]} \mathbb{E}[\zeta_i X_i], \qquad i = 1, \ldots, n. \tag{39}$$

This allocation rule can be seen as a special case of the proportional allocation rule (19), by choosing

$$\rho[X_i] = \mathbb{E}[\zeta_i X_i], \qquad i = 1, \ldots, n. \tag{40}$$

Notice that (39) can be rearranged as

$$\frac{K_i - \mathbb{E}[\zeta_i X_i]}{\mathbb{E}[\zeta_i X_i]} = \frac{K - \sum_{j=1}^{n} \mathbb{E}[\zeta_j X_j]}{\sum_{j=1}^{n} \mathbb{E}[\zeta_j X_j]}, \qquad i = 1, \ldots, n. \tag{41}$$

In the special case that the aggregate capital K is given by $K = \sum_{j=1}^{n} \mathbb{E}[\zeta_j X_j]$, the allocation rule (34) reduces to

$$K_i = \mathbb{E}[\zeta_i X_i], \qquad i = 1, \ldots, n. \tag{42}$$

This class of allocations is investigated in Furman and Zitikis (2008b) who call the members of this class *weighted risk capital allocations*.

3.2.2 Business unit driven allocations

In this subsection we consider the case where the weighting random variables ζ_i's in the quadratic allocation problem (33) are given by

$$\zeta_i = h_i(X_i), \tag{43}$$

with h_i being a non-negative and non-decreasing function such that $\mathbb{E}[h_i(X_i)] = 1$, for $i = 1, \ldots, n$. Hence, for each business unit i, the states-of-the-world to which we want to assign the heaviest weights are those under which the business unit performs the worst. As earlier pointed out, we call allocations based on (43) *business unit driven allocations*. In this case, the allocation rule (34) can be rewritten as

$$K_i = \mathbb{E}[X_i h_i(X_i)] + v_i \left(K - \sum_{j=1}^{n} \mathbb{E}[X_j h_i(X_j)] \right), \qquad i = 1, \ldots, n. \tag{44}$$

For an exogeneously given value of K, the allocations K_i are not influenced by the mutual dependence structure between the losses X_i of the different business units. In this sense, one can say that the allocation principle (44) is independent of the portfolio context within which the X_i's are embedded, and hence, is indeed business unit driven. Such allocations might be a useful instrument for determining the performance bonuses of the business unit managers, in case one assumes that each manager should be rewarded for the performance of his own business unit, but not extra rewarded (or penalised) for the interrelationship that exists between the performance of his business unit and that of the other units of the company.

The law invariant risk measure $\mathrm{E}[X_i h_i(X_i)]$ assigns to any loss X_i the expected value of the weighted outcomes of this loss, where higher weights correspond to larger outcomes of the loss, that is, to more adverse scenarios. Risk measures and premium principles of this general type have been proposed and investigated in Heilmann (1989), Tsanakas (2007) and Furman and Zitikis (2008a).

A particular choice of the random variables $h_i(X_i)$ considered in (44) is given by

$$h_i(X_i) = \frac{\mathbb{I}\left(X_i > F_{X_i}^{-1}(p)\right)}{1 - F_{X_i}\left(F_{X_i}^{-1}(p)\right)}, \qquad i = 1, \dots, n, \tag{45}$$

for some $p \in (0, 1)$. In this case, we find that $\mathbb{E}[X_i h_i(X_i)]$ transforms into

$$\mathbb{E}[X_i h_i(X_i)] = \mathrm{CTE}_p[X_i], \qquad i = 1, \dots, n. \tag{46}$$

More generally, consider the random variables $h_i(X_i)$ defined by

$$h_i(X_i) = g'\left(\overline{F}_{X_i}(X_i)\right), \qquad i = 1, \dots, n, \tag{47}$$

with $g : [0, 1] \to [0, 1]$ an increasing and concave function with derivative g' if it exists, and $\overline{F}_X$ the decumulative function of X. We then find that

$$\mathbb{E}[X_i h_i(X_i)] = \mathbb{E}[X_i \, g'\left(\overline{F}_{X_i}(X_i)\right)], \qquad i = 1, \dots, n, \tag{48}$$

and $\mathbb{E}[X_i h_i(X_i)]$ is a concave distortion risk measure, also called spectral risk measure. See Wang (1996), Acerbi (2002), or Dhaene et al. (2006).

Other examples of risk measures of the form $\mathbb{E}[X_i h_i(X_i)]$ are the standard deviation principle, the Esscher principle and the exponential principle. These are summarised in Table 1.

Defining the volumes v_i by

$$v_i = \frac{\mathbb{E}[X_i h_i(X_i)]}{\sum_{j=1}^n \mathbb{E}[X_j h_j(X_j)]}, \qquad i = 1, \dots, n, \tag{49}$$

we find that the allocation principle (44) reduces to

$$K_i = \frac{K}{\sum_{j=1}^n \mathbb{E}[X_j h_j(X_j)]} \mathbb{E}[X_i h_i(X_i)], \qquad i = 1, \dots, n, \tag{50}$$

which is a special case of the proportional allocation principles discussed in Section 2.3.5.

3.2.3 Aggregate portfolio driven allocations

Let us now consider the case where

$$\zeta_i = h(S), \qquad i = 1, \dots, n, \tag{51}$$

Table 1: *Business unit driven risk measures*

Risk measure	$h_i(X_i)$	$\mathbb{E}[X_i h_i(X_i)]$
Standard deviation principle Bühlmann (1970)	$1 + a\dfrac{X_i - \mathbb{E}[X_i]}{\sigma_{X_i}},\ a \geq 0$	$\mathbb{E}[X_i] + a\sigma_{X_i}$
Conditional tail expectation Overbeck (2000)	$\dfrac{1}{1-p}\mathbb{I}\left(X_i > F_{X_i}^{-1}(p)\right),\ p \in (0,1)$	$\mathrm{CTE}_p[X_i]$
Distortion risk measure Wang (1996), Acerbi (2002)	$g'\left(\overline{F}_{X_i}(X_i)\right), g : [0,1] \mapsto [0,1],$ $g' > 0, g'' < 0$	$\mathbb{E}\left[X_i g'\left(\overline{F}_{X_i}(X_i)\right)\right]$
Exponential principle Gerber (1974)	$\displaystyle\int_0^1 \dfrac{e^{\gamma a X_i}}{\mathbb{E}[e^{\gamma a X_i}]}d\gamma,\ a > 0$	$\dfrac{1}{a}\ln\mathbb{E}\left[e^{a X_i}\right]$
Esscher principle Gerber (1981)	$\dfrac{e^{a X_i}}{\mathbb{E}[e^{a X_i}]},\ a > 0$	$\dfrac{\mathbb{E}[X_i e^{a X_i}]}{\mathbb{E}[e^{a X_i}]}$

with h being a non-negative and non-decreasing function such that $\mathbb{E}[h(S)] = 1$. In this case, the states-of-the-world to which we assign the heaviest weights are those under which the aggregate portfolio performs worst. Therefore, we call such allocations *aggregate portfolio driven allocations*. The allocation rule (34) can now be rewritten as

$$K_i = \mathbb{E}[X_i h(S)] + v_i\left(K - \mathbb{E}[S h(S)]\right), \qquad i = 1,\ldots,n. \tag{52}$$

Hence, the capital K_i allocated to unit i is determined using a weighted expectation of the loss X_i, with higher weights attached to states-of-the-world that involve a large aggregate loss S. Notice that the allocation principle (52) can be reformulated as

$$K_i = \mathbb{E}[X_i] + \mathrm{Cov}[X_i, h(S)] + v_i\left(K - \mathbb{E}[S h(S)]\right), \qquad i = 1,\ldots,n. \tag{53}$$

This means that the capital allocated to the i-th business unit is given by the sum of the expected loss $\mathbb{E}[X_i]$, a loading which depends on the covariance between the individual and aggregate losses X_i and $h(S)$, plus a term proportional to the volume of the business unit. A strong positive correlation between X_i and $h(S)$, which reflects that X_i could be a substantial driver of the aggregate loss S, produces a higher allocated capital K_i. Allocation principles of the form (52) are closely related to the 'Euler' allocations proposed in Tasche (2004).

Using aggregate portfolio driven allocations might be appropriate when one wants to investigate each individual portfolio's contribution to the aggregate loss of the entire company. In other words, the company wishes to evaluate the subportfolio performances, e.g. the returns on the allocated capitals, in the presence of the other subportfolios. This can provide relevant information to the company within which it can further be used to evaluate either business expansions or reductions.

Defining the volumes v_i by

$$v_i = \frac{\mathbb{E}[X_i h(S)]}{\mathbb{E}[S h(S)]}, \qquad i = 1,\ldots,n, \tag{54}$$

we find that the allocation principle (52) reduces to the proportional allocation rule

$$K_i = \frac{K}{\mathbb{E}[S h(S)]}\mathbb{E}[X_i h(S)], \qquad i = 1,\ldots,n. \tag{55}$$

A particular choice of the random variable $h(S)$ considered in (52) is given by

$$h(S) = \frac{\mathbb{I}\left(S > F_S^{-1}(p)\right)}{1 - F_S\left(F_S^{-1}(p)\right)}, \qquad i = 1, \ldots, n, \tag{56}$$

for some $p \in (0,1)$. In this case, we find that $\mathbb{E}[X_i h(S)]$ and $\mathbb{E}[Sh(S)]$ transform into

$$\mathbb{E}[X_i h(S)] = \mathbb{E}\left[X_i \,\middle|\, S > F_S^{-1}(p)\right], \qquad i = 1, \ldots, n \tag{57}$$

and

$$\mathbb{E}[Sh(S)] = \mathrm{CTE}_p\left[S\right], \tag{58}$$

respectively. The CTE allocation principle (18) follows as a special case of the allocation principle (55) by choosing $h(S)$ as in (56).

Taking

$$h(S) = S - \mathbb{E}[S], \tag{59}$$

we find

$$\mathbb{E}[X_i h(S)] = \mathrm{Cov}\left[X_i, S\right], \qquad i = 1, \ldots, n \tag{60}$$

and

$$\mathbb{E}[Sh(S)] = \mathrm{Var}[S], \tag{61}$$

which implies that the allocation principle (55) reduces to the covariance allocation principle (16) discussed in Section 2.3.3.

Other choices for the random variable $h(S)$ and the related expressions for $\mathrm{E}[X_i h(S)]$ can be found in Table 2. They correspond to capital allocation principles that have been considered in Overbeck (2000), Tsanakas (2004), and Tsanakas (2008).

Table 2: *Aggregate portfolio driven allocations*

Reference	$h(S)$	$\mathbb{E}[X_i h(S)]$
Overbeck (2000)	$1 + a\dfrac{S - \mathbb{E}[S]}{\sigma_S}, \ a \geq 0$	$\mathbb{E}[X_i] + a\dfrac{\mathrm{Cov}[X_i, S]}{\sigma_S}$
Overbeck (2000)	$\dfrac{1}{1 - F_S\left(F_S^{-1}(p)\right)}\mathbb{I}\left(S > F_S^{-1}(p)\right), \ p \in (0,1)$	$\mathbb{E}[X_i | S > F_S^{-1}(p)]$
Tsanakas (2004)	$g'(\overline{F}_S(S)), g : [0,1] \mapsto [0,1], g' > 0, \ g'' < 0$	$\mathbb{E}\left[X_i g'(\overline{F}_S(S))\right]$
Tsanakas (2008)	$\displaystyle\int_0^1 \frac{e^{\gamma a S}}{\mathbb{E}[e^{\gamma a S}]}d\gamma, a > 0$	$\mathbb{E}\left[X_i \displaystyle\int_0^1 \frac{e^{\gamma a S}}{E[e^{\gamma a S}]}d\gamma\right]$
Wang (2007)	$\dfrac{e^{aS}}{\mathbb{E}[e^{aS}]}, \ a > 0$	$\dfrac{\mathbb{E}[X_i e^{aS}]}{\mathbb{E}[e^{aS}]}$

3.2.4 Market driven allocations

Let ζ_M be a random variable such that market-consistent values of the aggregate portfolio loss S and the business unit losses X_i are given by

$$\pi[S] = \mathbb{E}[\zeta_M S] \tag{62}$$

and

$$\pi[X_i] = \mathbb{E}[\zeta_M X_i], \qquad i = 1, \ldots, n, \tag{63}$$

respectively. Further suppose that at the aggregate portfolio level, a provision $\pi[S]$ is set aside to cover future liabilities S. Apart from the aggregate provision $\pi[S]$, the aggregate portfolio has an available solvency capital equal to $(K - \pi[S])$. The *solvency ratio* of the aggregate portolio is then given by

$$\frac{K - \pi[S]}{\pi[S]}. \tag{64}$$

In order to determine an optimal capital allocation over the different business units, we let in (31) $\zeta_i = \zeta_M$, $i = 1, \ldots, n$, thus allowing the market to determine which states-of-the-world are to be regarded adverse. This yields:

$$K_i = \pi[X_i] + v_i \left(K - \pi[S] \right). \tag{65}$$

If we now use the market-consistent prices as volume measures, after substituting

$$v_i = \frac{\pi[X_i]}{\pi[S]}, \qquad i = 1, \ldots, n, \tag{66}$$

in (65), we find

$$K_i = \frac{K}{\pi[S]} \pi[X_i], \qquad i = 1, \ldots, n. \tag{67}$$

Rearranging these expressions leads to

$$\frac{K_i - \pi[X_i]}{\pi[X_i]} = \frac{K - \pi[S]}{\pi[S]}, \qquad i = 1, \ldots, n. \tag{68}$$

The quantities $\pi[X_i]$ and $(K_i - \pi[X_i])$ can be interpreted as the market-consistent provision and the solvency capital attached to business unit i, while $\frac{K_i - \pi[X_i]}{\pi[X_i]}$ is its corresponding solvency ratio. From (68), we can conclude that the optimisation criterion (33) with $\zeta_i = \zeta_M$, $i = 1, \ldots, n$, and volume measures given by (66), leads to a capital allocation whereby the solvency ratio for each business unit is the same and equal to that of the aggregate portfolio. A similar allocation principle has been proposed by Sherris (2006) within the context of allocating the company's total equity to the different business units "to determine an expected return on equity by line of business".

3.2.5 Allocation with respect to the default option

An alternative choice for the weighting random variable ζ_i is given by

$$\zeta_i = h(S) = \frac{\mathbb{I}(S > K)}{\mathbb{P}[S > K]}, \qquad i = 1, \ldots, n, \tag{69}$$

such that only those states-of-the-world that correspond to insolvency are considered when determining the expectations $\mathbb{E}[X_i h(S)]$. The allocation rule (34) then becomes

$$K_i = \mathbb{E}\left[X_i \mid S > K\right] + v_i \left(K - \mathbb{E}\left[S \mid S > K\right] \right). \tag{70}$$

Notice the similarity between (69) and the choice of $h(S)$ made in (56) which led to the CTE allocation rule.

Expression (70) can be rearranged as follows:

$$\mathbb{E}[(X_i - K_i)\, \mathbb{I}(S > K)] = v_i \mathbb{E}[(S - K)_+], \qquad i = 1, \ldots, n. \tag{71}$$

Summing the left and right hand sides of this expression over $i = 1, \ldots, n$, leads to expression (29). The quantity $\mathbb{E}[(S - K)_+]$ represents the *expected policyholder deficit* or alternatively, the expected value of the *default option* that shareholders of an insurance company hold, given their limited liability. The allocation principle in (71) is such that the marginal contribution of each business unit to the expected value of the policyholder deficit is the same per unit of business volume, and hence is consistent with the arguments of Myers and Read Jr. (2001).

3.3 The absolute deviation optimisation criterion

In this section we discuss the optimal allocation problem under an absolute deviation criterion, that is, by letting

$$D(x) = |x| \,. \tag{72}$$

In this case the optimisation problem (31) reduces to

$$\min_{K_1, \ldots, K_n} \sum_{j=1}^{n} \mathbb{E}\left[\zeta_j \left|X_j - K_j\right|\right], \text{such that} \sum_{j=1}^{n} K_j = K. \tag{73}$$

From the relation

$$|x| = 2\left(x\right)_+ - x, \tag{74}$$

we immediately find that the optimal solution of (73) is identical to the solution of the following problem:

$$\min_{K_1, \ldots, K_n} \sum_{j=1}^{n} \mathbb{E}\left[\zeta_j \left(X_j - K_j\right)_+\right], \text{ such that} \sum_{j=1}^{n} K_j = K. \tag{75}$$

This means that the absolute deviation optimisation problem (73) only takes into account the outcomes of the business unit losses X_j that lead to technical insolvency $X_j > K_j$ in that unit.

In order to solve the optimisation problem (75), we first consider the special case where all ζ_j's are identical to 1.

Theorem 2 *Assuming that $F_{S^c}^{-1+}(0) < K < F_{S^c}^{-1}(1)$, the optimal allocation problem*

$$\min_{K_1, \ldots, K_n} \sum_{j=1}^{n} \mathbb{E}\left[(X_j - K_j)_+\right] \text{ such that} \sum_{j=1}^{n} K_j = K \tag{76}$$

has the following solution:

$$K_i = F_{X_i}^{-1(\alpha)}(F_{S^c}(K)), \qquad i = 1, \ldots, n, \tag{77}$$

where S^c is defined in (10) and $\alpha \in [0, 1]$ follows from

$$F_{S^c}^{-1(\alpha)}(F_{S^c}(K)) = K. \tag{78}$$

Proof. Let α be determined from (78). Then we immediately find from Dhaene et al. (2002) that

$$
\begin{aligned}
\mathbb{E}\left[(S^c - K)_+\right] &= \sum_{j=1}^{n} \mathbb{E}\left[\left(X_j - F_{X_i}^{-1(\alpha)}(F_{S^c}(K))\right)_+\right] \\
&\leq \sum_{j=1}^{n} \mathbb{E}\left[(X_j - K_j)_+\right], \qquad K \in \left(F_{S^c}^{-1+}(0), F_{S^c}^{-1}(1)\right),
\end{aligned}
$$

holds for all $(K_1, K_2, ..., K_n)$ such that $\sum_{j=1}^{n} K_j = K$. This proves the stated result. ■

We can conclude that the quantile allocation principle (14) considered in Section 2.3.2 is a solution of the minimisation problem (76). This optimisation problem and its solution were previously considered in Dhaene et al. (2003) in the particular case that the F_{X_i}'s are all strictly increasing. A proof of Theorem 2 using Lagrange techniques can be found in Laeven and Goovaerts (2004).

The solution to the general optimisation problem (73) will be expressed in terms of functions $F_{X_i}^{(\zeta_i)}$ defined as follows:

$$F_{X_i}^{(\zeta_i)}(x) = \mathbb{E}[\zeta_i \, \mathbb{I}(X_i \leq x)] = \mathbb{E}[\zeta_i \mid X_i \leq x] \, F_{X_i}(x), \qquad i = 1, \dots, n. \tag{79}$$

One can prove that each function $F_{X_i}^{(\zeta_i)}$ defines a proper distribution function, which we will call the ζ_i–weighted distribution of X_i; See Rao (1997), Furman and Zitikis (2008a) and the references therein. The decumulative distribution function $\overline{F}_{X_i}^{(\zeta_i)}(x) = 1 - F_{X_i}^{(\zeta_i)}(x)$ is given by

$$\overline{F}_{X_i}^{(\zeta_i)}(x) = \mathbb{E}\left[\zeta_i \, \mathbb{I}(X_i > x)\right] = \mathbb{E}[\zeta_i \mid X_i > x] \, \overline{F}_{X_i}(x), \qquad i = 1, \dots, n. \tag{80}$$

A sufficient condition for $F_{X_i}^{(\zeta_i)}$ to be continuous is that F_{X_i} be continuous. A sufficient condition for $F_{X_i}^{(\zeta_i)}$ to be strictly increasing is that F_{X_i} be strictly increasing and that $\mathbb{P}[\zeta_i > 0] = 1$. For any $p \in (0, 1)$ and any $\alpha \in [0, 1]$, we denote the α–mixed inverse of $F_{X_i}^{(\zeta_i)}$ at level p by $\left(F_{X_i}^{(\zeta_i)}\right)^{-1(\alpha)}(p)$.

In the following lemma, we prove that the deviation measure $\mathbb{E}\left[\zeta_i \, (X_i - K_i)_+\right]$ can be transformed to a stop-loss premium of X_i with retention K_i, where the expectation is taken with respect to the ζ_i–weighted distribution of X_i.

Lemma 3 *Let U be a uniform random variable on the unit interval $(0, 1)$. Then it holds that*

$$\mathbb{E}\left[\zeta_i \, (X_i - K_i)_+\right] = \mathbb{E}\left[\left(\left(F_{X_i}^{(\zeta_i)}\right)^{-1}(U) - K_i\right)_+\right], \qquad i = 1, \dots, n. \tag{81}$$

Proof. From the tower property of the expectation operator, we find

$$\mathbb{E}\left[\zeta_i \, (X_i - K_i)_+\right] = \mathbb{E}\left[\zeta_i \mathbb{E}\left[(X_i - K_i)_+ \mid \zeta_i\right]\right].$$

Substituting $\mathbb{E}\left[(X_i - K_i)_+ \mid \zeta_i\right]$ by $\int_{K_i}^{\infty} \mathbb{P}[X_i > x \mid \zeta_i] \, dx$ and changing the order of the integrations, we find

$$
\begin{aligned}
\mathbb{E}\left[\zeta_i \, (X_i - K_i)_+\right] &= \int_{K_i}^{\infty} \mathbb{E}\left[\zeta_i \, \mathbb{P}[X_i > x \mid \zeta_i]\right] \, dx \\
&= \int_{K_i}^{\infty} \mathbb{E}\left[\zeta_i \, \mathbb{E}[\mathbb{I}(X_i > x) \mid \zeta_i]\right] \, dx.
\end{aligned}
$$

Taking into account the power property once more leads to

$$\mathbb{E}\left[\zeta_i \, (X_i - K_i)_+\right] = \int_{K_i}^{\infty} \mathbb{E}\left[\zeta_i \, \mathbb{I}(X_i > x)\right] \, dx = \int_{K_i}^{\infty} \overline{F}_{X_i}^{(\zeta_i)}(x) \, dx.$$

The stated result follows then from observing that the distribution function of $\left(F_{X_i}^{(\zeta_i)}\right)^{-1}(U)$ is given by $F_{X_i}^{(\zeta_i)}$ so that $\int_{K_i}^{\infty} \overline{F}_{X_i}^{(\zeta_i)}(x) \, dx$ is an expression for the stop-loss premium of $\left(F_{X_i}^{(\zeta_i)}\right)^{-1}(U)$ with retention K_i. ■

Now we are able to prove our main result concerning the absolute deviation optimisation problem.

Theorem 4 *Let $\overline{S}^c$ be the comonotonic sum defined by*

$$\overline{S}^c = \sum_{i=1}^{n} \left(F_{X_i}^{(\zeta_i)} \right)^{-1} (U), \tag{82}$$

where the random variable U is uniformly distributed on the unit interval $(0,1)$. In case $F_{\overline{S}^c}^{-1+}(0) < K < F_{\overline{S}^c}^{-1}(1)$, the optimal allocation problem (73) has the following solution:

$$K_i = \left(F_{X_i}^{(\zeta_i)} \right)^{-1(\alpha)} (F_{\overline{S}^c} K), \qquad i = 1, \ldots, n, \tag{83}$$

where $\alpha \in [0,1]$ follows from

$$F_{\overline{S}^c}^{-1(\alpha)}(F_{\overline{S}^c}(K)) = K. \tag{84}$$

Proof. From Lemma 3, we find that the optimisation problem (73) can be rewritten as

$$\min_{K_1,\ldots,K_n} \sum_{j=1}^{n} \mathbb{E}\left[\left(\left(F_{X_j}^{(\zeta_j)} \right)^{-1} (U) - K_j \right)_{+} \right], \text{ such that } \sum_{j=1}^{n} K_j = K.$$

The stated result follows then by applying Theorem 2. ∎

From the theorem above, we can conclude that the mean absolute deviation optimality criterion (73) gives rise to a quantile-based allocation principle: Each allocated capital K_i is given by the α–mixed inverse of the ζ_i–weighted distribution function of X_i at a fixed probability level, which is chosen such that the full allocation requirement is satisfied. From (83), we find that the optimal allocations K_i satisfy the following conditions:

$$F_{X_i}^{(\zeta_i)}(K_i) = F_{\overline{S}^c}(K), \qquad i = 1, \ldots, n. \tag{85}$$

In the case where

$$\mathbb{P}[\zeta_i > 0] = 1, \qquad i = 1, \ldots, n, \tag{86}$$

and the distributions $F_{X_j}^{(\zeta_j)}$ are strictly increasing, then the optimal allocations in (83) reduce to

$$K_i = \left(F_{X_i}^{(\zeta_i)} \right)^{-1} (F_{\overline{S}^c} K), \qquad i = 1, \ldots, n. \tag{87}$$

We end this subsection with an example of an absolute deviation allocation principle. Consider the following choice for the weighting random variables:

$$\zeta_i = \frac{\mathbb{I}(S > K)}{\mathbb{P}[S > K]}, \qquad i = 1, \ldots, n. \tag{88}$$

This means that the optimisation procedure only considers those outcomes that lead to insolvency, i.e. the case where $S > K$, on the aggregate portfolio level. In this instance, the optimisation problem (75) reduces to

$$\min_{K_1,\ldots,K_n} \sum_{j=1}^{n} \mathbb{E}\left[(X_j - K_j)_{+} \mid S > K \right], \text{ such that } \sum_{j=1}^{n} K_j = K. \tag{89}$$

From (79), we find that the ζ_i–weighted distribution function of X_i is given by

$$F_{X_i}^{(\zeta_i)}(x) = \mathbb{P}\left[X_i \leq x | S > K\right], \qquad i = 1, \ldots, n. \tag{90}$$

In the particular case that $x = K_i$, we find from (85) that:

$$\overline{F}_{X_i}^{(\zeta_i)}(K_i) = \mathbb{P}\left[X_i > K_i | S > K\right] = \overline{F}_{\overline{S}^c}(K), \qquad i = 1, \ldots, n. \tag{91}$$

Hence, capital is allocated such that the conditional probability of a business unit's loss exceeding its allocated capital, given that the whole company defaults, is identical for all business units.

4 Capital allocation from the viewpoint of the policyholders

In this section we apply our general optimal allocation framework (31) to the case of an insurance market consisting of n member companies, each exposed to a loss X_i, $i = 1, \ldots, n$. Suppose that the current insurance regulation requires each company to hold a capital $\overline{K}_i$, $i = 1, \ldots, n$. The aggregate required capital across the market is then given by

$$K = \sum_{j=1}^{n} \overline{K}_j. \tag{92}$$

Companies do not cross-subsidise such that they retain their own realised profits, even in the case where other market participants default. In an extreme scenario where $X_i > \overline{K}_i$, the i-th company will default and the shortfall to policyholders from that company's default is given by $(X_i - \overline{K}_i)_+$. Obviously, the aggregate market shortfall is given by $\sum_{j=1}^{n}(X_j - \overline{K}_j)_+$.

The set up in this section is also applicable to an insurance market that guarantees payments to policyholders via a dedicated fund, in the case where one or more insurers default. In this case, the shortfall is paid from a pool to which all companies contribute. It is clear that the aggregate liability of this pool is equal to $\sum_{j=1}^{n}(X_j - \overline{K}_j)_+$. This is an arrangement encountered most prominently in the Lloyd's of London insurance market, with Lloyd's members in the role of the individual portfolios and the Lloyd's New Central Fund guaranteeing policyholder liabilities[1]. Cross-subsidy only takes place at the level of the pool guaranteeing the policyholder liabilities.

We now suppose that the regulator (or the pool manager) wants to investigate whether the aggregate capital K in the market could be redistributed over the different companies leading to a more beneficial financial situation for the policyholders (or the pool), according to a pre-defined criterion. More generally, starting with the market capital K given by (92), assume that the regulator wants to determine the optimal re-allocation $(K_1, K_2, \ldots, K_n)$ of the aggregate capital K.

In order to solve the regulator's (or pool manager's) concern, we first consider the quadratic criterion where the optimal capital requirements K_i of the different companies are found by solving problem (31), with the function $D(x)$ given by

$$D(x) = ((x)_+)^2. \tag{93}$$

This leads us to the optimisation problem expressed as follows:

$$\min_{K_1, \ldots, K_n} \sum_{j=1}^{n} v_j \mathbb{E}\left[\zeta_j \left(\frac{(X_j - K_j)_+}{v_j}\right)^2\right], \text{ such that } \sum_{j=1}^{n} K_j = K. \tag{94}$$

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[1]The views expressed in this section by the authors are their own and do not necessarily represent the views of Lloyd's.

The above general problem can be solved by a straightforward Lagrange optimisation, assuming that its objective function is differentiable, that is, each function $\mathbb{E}\left[\zeta_i(X_i - K_i)_+\right]$ is differentiable with respect to K_i.

Lemma 5 *Assume that the objective function of optimisation problem (94) is differentiable. Then the solution $(K_1, K_2, \ldots, K_n)$ satisfies*

$$\mathbb{E}\left[\zeta_i(X_i - K_i)_+\right] = v_i \mathbb{E}\left[\sum_{j=1}^n \zeta_j(X_j - K_j)_+\right], \qquad i = 1, \ldots, n. \tag{95}$$

Proof. The proof follows immediately from

$$\frac{\partial}{\partial K_i}\mathbb{E}\left[\zeta_i\left((X_j - K_j)_+\right)^2\right] = -2\mathbb{E}\left[\zeta_i(X_i - K_i)_+\right]),$$

together with the relevant first order conditions. ∎

From equations (95) we can conclude that under the optimisation problem (94), companies are supposed to hold capital such that the expected weighted shortfalls per unit of exposure

$$\frac{\mathbb{E}\left[\zeta_i(X_i - K_i)_+\right]}{v_i}, \qquad i = 1, \ldots, n,$$

are equal across the market.

Let us now consider the situation where the regulator (pool manager) seeks to find the optimal capital requirements K_i for the different companies by solving problem (31), with the function $D(x)$ given by

$$D(x) = (x)_+. \tag{96}$$

This is equivalent to the optimisation problem given by (76) which seeks to minimise the aggregate expected shortfall in the market. From Theorem 4, we know the solution of this problem is the quantile-based allocation:

$$K_i = \left(F_{X_i}^{(\zeta_i)}\right)^{-1(\alpha)}(F_{\overline{S}^c}K), \qquad i = 1, \ldots, n. \tag{97}$$

In case all $\zeta_i \equiv 1$ and all F_{X_i} are strictly increasing, this optimal allocation simplifies to

$$K_i = F_{X_i}^{-1}(F_{S^c}(K)), \qquad i = 1, \ldots, n. \tag{98}$$

When additionally the initial capital requirements $\overline{K}_i$ are given by

$$\overline{K}_i = F_{X_i}^{-1}(p), \qquad i = 1, \ldots, n \tag{99}$$

for some $p \in (0, 1)$, one immediately finds that $F_{S^c}(K) = p$, so that the optimal allocation of the market capital coincides with the existing capital requirement regulation. This observation provides additional justification for quantile-based regulatory regimes.

We end this section by a short discussion on allocation methods used in the past and present at Lloyd's. Taking $\zeta_i \equiv 1$, $i = 1, \ldots, n$, (95) reduces in essence to the principle behind the Risk-Based Capital system operated by Lloyd's between the period 1996 and 2005, with the fraction on the left-hand-side of equation (95) termed the *Expected Loss Cost*, the quantity $\mathbb{E}\left[\sum_{j=1}^n (X_j - K_j)_+\right]$ called the *Aggregate Expected Loss* and provisions used as measures of business volume v_i. This allocation implies that each company has a proportional claim on the Lloyd's New Central Fund.

In 2006, capital allocation at Lloyd's was harmonised to be in sync with the new British regulatory regime (see e.g. Lloyd's (2008)), under which capital is calculated based on a Value-at-Risk (VaR) risk measure:

$$K_i = F_{X_i}^{-1}(p), \qquad i = 1, \ldots, n,$$

with p determined such that the full allocation principle is satisfied. From Theorem 2, we know that this allocation is optimal, in the sense that the expected aggregate liability of the pool is minimised, taking into account the full allocation requirement.

5 Conclusion

In this article, we developed a general and unifying optimisation framework that produces several of the capital allocation approaches that are encountered both in the literature and in practice. This general framework is based on the idea of minimising the sum of the divergences between the losses and the allocated capital of the different subportfolios. Depending on how this divergence is defined, several alternative allocation methods arise. It turns out that many allocation methods postulated in an ad-hoc fashion, are attributable to particular decision problems linked to specific purposes of allocating risk capital. If each capital allocation method is an answer to a particular question, in this paper we delineate what this question might be.

The ideas in the paper were first applied to a company with different business units. They were further applied to the case of a market with policyholder protection via a financial pool, as is the case, for example, at Lloyd's of London. Our framework helps to explain the optimality properties of the various allocation methods that have historically been used by Lloyd's.

Finally, the framework presented in this paper provides the flexibility to produce new capital allocation methods, by varying, for example, the choices of the weights ζ_i and v_i. This is not an avenue we pursued here at great lengths but remains a subject of importance for future work.

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