The Value for Actions-Set Games

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February 2009

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February 23, 2009

Abstract

Action-Set games are transferable utility games where the set of players is finite, every player has a finite set of actions, and the worth of the game is a function of the actions taken by the players. In this setting a rule has to determine individual payoffs at each combinations of actions. Following an axiomatic approach, we define the set of Consistent Bargaining Equilibria.

Keywords: Action-set games, Shapley value, Prekernel, Consistent Bargaining Equilibria.

1 Action-set games

One of the features common to most economic situations is that the interaction among agents through activities like production, exchange, etc. generates benefits shared among the participating agents. Moreover, the productivity gains of the specialization of cooperative labor in different task and roles (labor division) are important within any type of production process, ranging from pin manufacture to legal practice and medical care. Labor division refers to a cooperative situation where two agents have to choose tasks or actions to achieve the outcome. Efficient choices are those implying that agents master different tasks, i.e., an agent choosing "action a", and the other choosing "action b".

For instance, suppose that two friends, say A and B, go to deer hunting. Each of them can either line driving (action l_i, $i = 1,2$), i.e., flushing deers toward the hunter, or shooting deers (action s_i, $i = 1,2$). Obviously, the efficient outcomes entail that one friend line drives and the other shoots; if both line drive, then no deer will be shooted, and, finally, if both shoot deers, then less deers will be shooted, because nobody will flush them to the hunters. Payoffs could be represented by the following matrix.

v	s_2	l_2
s_1	4	10
l_1	10	0

Hunters' choice of actions will depend on the expected payoff redistributions of the outcome between them, under all the possible choices. Therefore, a redistribution rule has to specify a payoff vector for any pair of actions.

The three main features characterizing our setting are:

1.- *Unanimity.* The final agreements must be unanimous; hence partial cooperation is not allowed and strict subcoalitions of agents play no role in this setting.

2.- *No disagreement point.* The agents cannot take any particular action to guarantee themselves a minimum payoff. Whatever the actions taken by the players, they must agree with the redistribution of the outcome.

3.- *Transferable utility.* The outcome is a totally divisible good which can be redistributed among agents.

Formally, an *action-set game* is any $\Gamma = \left(N, \left(A^i\right)_{i \in N}, v\right)$, where N is a finite set of players, with $|N| = n$, and for any player i in N, A_i is a finite non empty set of actions available for player i. An *action profile* is a combination of actions that the players in N might choose. We let A denote the set of all possible action profiles, so that ψ

$$A = \underset{i \in N}{\times} A^i.$$

For any action profile $x = (x_i)_{i \in N}$ in A, the real number $v(x)$ represents the total worth that players would get if x where the combination of the actions taken by the players. Therefore, v is a mapping $v : A \to \mathbb{R}_+$.

Given an action-set game $\Gamma = (N, A, v)$, a value solution ψ is a mapping that specifies a payoff vector for every action profile, that is

$$\psi : A \to \mathbb{R}^N,$$

The number $\psi_i(x, v)$ represents the payoff that player $i \in N$ receives when action profile $x \in A$ is taken in Γ. When no confusion arises, we denote such a number by $\psi_i(x)$. As utility is totally transferable, a payoff vector $\psi(x)$ is *feasible* if $\sum_{i \in N} \psi_i(x) \leq v(x)$.

We wish to propose a rule to determine individual payoffs in this setting. For that purpose we will follow an axiomatic approach, that is, we will impose some appealing properties which the agents involved in the bargaining are ready to follow.

2 Axiomatics

The first minimum requirement that a rule ψ must satisfy is the budget restriction:

Definition 1 Efficiency: $\sum_{i\in N}\psi_i(x) = v(x)$*, for all* $x \in A$

To illustrate the next property suppose that players are involved in bargaining at some x, and some agent i disagree with the proposal at hand. She can reject that proposal by threatening to change her action to a different $y_i \in A^i$, where $v(x_{-i}, y_i) < v(x)$. In this way, she can impose a loss to the rest of players as far they do not take into account her claim. But, if players are in a location $\bar{x}$ where it is impossible to lower their payoffs by changing their action, then nobody will make this kind of threat. Assuming that players have equal bargaining skills, payoffs must be the same for all of them at the worst situation.

Definition 2 Equal Minimum Rights*: Let* $\bar{x} \in A$ *such that* $v(\bar{x}) = Min_{x\in A}\{v(x)\}$*, then* $\psi_i(\bar{x}) = \psi_j(\bar{x})$*, for all* $i, j \in N$*.*

In this way, individual payoffs at the worst possible outcome of the game will act as a *reference point* for the remaining possible agreements.

Moreover, since unanimity of any agreement is a desiderable property, then it will be required that every player will obtain at least as much she will obtain at the reference point.

Definition 3 Individual Rationality*: Let* $\bar{x} \in A$ *such that* $v(\bar{x}) = Min_{x\in A}\{v(x)\}$*, then for all* $x \in A$*,* $\psi_i(x) \geq \psi_i(\bar{x})$*, for all* $i \in N$*.*

An example of a rule that satisfies the above properties and that, a priori, could be appealing is the *equal payoff division*, which in principle is a nice candidate for a fair rule. Let us consider the following matrix $[v] = [v(x)]_{x\in A}$, where $A_i = \{a_i, b_i\}$, $i = 1, 2$, are agent i' actions.

v	a_2	b_2
a_1	4	8
b_1	6	2

For the above example such a rule yields the payoff matrix $[\psi] = [(\psi_1(x), \psi_2(x))]_{x\in A}$:

ψ	a_2	b_2
a_1	$(2,2)$	$(4,4)$
b_1	$(3,3)$	$(1,1)$

A natural way to measure the bargaining position of agent i at each action profile is by comparing the variation of agent j's payoffs, under i's action changes. Suppose that agents choose the pair of actions (a_1,b_2), where the equal payoff division rule yields $(\psi_1(a_1,b_2) = 4, \psi_2(a_1,b_2) = 4)$. Here, agent 1 could argue that if she changed her action to b_1, then she could inflict a loss to agent 2 of :

$$\psi_2(a_1,b_2) - \psi_2(b_1,b_2) = 4 - 1 = 3$$

On the contrary, the loss on agent 1's payoffs that agent 2 could inflict by changing his action from b_2 to a_2 is:

$$\psi_1(a_1,b_2) - \psi_1(a_1,a_2) = 4 - 2 = 2$$

Clearly the bargaining position of agent 2 is weaker than that of agent 1, and therefore the equal payoff division does not take into account the agents' different bargaining power.

We wish to design a rule with the property that at any action profile, agent i's threat to agent j is balanced by agents j's counterthreat to agent i. This property will be called *Equal Punishments*.

Define the concept of "punishment" in the general setting as follows:

Definition 4 *Given a value ψ, for all $x \in A$, we define the* punishment *of player i to player j as*

$$P_{ij}\left[\psi(x)\right] := \psi_j(x) - Min_{y_{i\in A}}\ \psi_j(x_{-i},y_i).$$

The amount $P_{ij}\left[\psi(x)\right]$ measures the maximum payoff losses that player i can inflict to player j. Note that since $Min_{y_{i\in A}} \psi_j(x_{-i},y_i) \leq \psi_j(x)$, then $P_{ij}\left[\psi(x)\right] \geq 0$, for all $x \in A$, $i,j \in N$. The difference $P_{ij}\left[\psi(x)\right] - P_{ji}\left[\psi(x)\right]$ can also interpreted as the adjustment that player i could claim at $\psi(x)$ against player j based on a comparison of their punishments.

Because the two-person case does not present any ambiguity, we state firstly the equilibrium concept when $N = \{1,2\}$.

Definition 5 *Equal Punishments: $P_{12}\left[\psi(x)\right] = P_{21}\left[\psi(x)\right]$, for all $x \in A$.*

Definition 6 *Given a two-person game, a value $\psi(x)$ is a* Bargaining Equilibrium *if it satisfies Efficiency, Equal Minimum Rights, Individual Rationality and Equal Punishments.*

Given $\Gamma = (\{1,2\}, A, v)$, denote by $\mathscr{E}(\{1,2\}, A, v)$ the set of all Bargaining Equilibrium values. In the two-person case this set is always nonempty.

Theorem 1 *When $\Gamma = (\{1,2\}, A, v)$, $\mathscr{E}(\{1,2\}, A, v) \neq \varnothing$.*

The proof of this Theorem is a corollary of Theorem 2.

For instance, in the above example, the following payoff matrix:

ψ	a_2	b_2
a_1	$(2,2)$	$(4.5,3.5)$
b_1	$(2.5,3.5)$	$(1,1)$

corresponds to an equilibrium. In particular:

$$P_{12}(a_1,b_2) = P_{21}(a_1,b_2) = 2.5;$$
$$P_{12}(b_1,a_2) = P_{21}(b_1,a_2) = 1.5;$$
$$P_{12}(a_1,a_2) = P_{21}(a_1,a_2) = P_{12}(b_1,b_2) = P_{21}(b_1,b_2) = 0.$$

In general, this equilibrium concept gives rise to a set of equilibria, for example the next payoff division is also an equilibrium:

ψ	a_2	b_2
a_1	$(1,3)$	$(4,4)$
b_1	$(2,4)$	$(1,1)$

Discussion It is important to notice that ex-ante symmetric environments, where players have the same skills to undertake different jobs, may give rise to ex-post individuals payoffs, with players being differently rewarded as a function of the different roles that they perform in labor division.

To illustrate this aspect, consider again the hunters' game of the introduction. The following surplus division is an equilibrium:

ψ	s_2	l_2
s_1	$(2,2)$	$(6,4)$
l_1	$(4,6)$	$(0,0)$

Individual payoffs are symmetric at action-pairs (l_1,l_2) and (s_1,s_2), but asymmetric at action-pairs (l_1,s_2) and (s_1,l_2). It is important to point out that to justify individual asymmetric payoffs there is no need to assume ex-ante different skills, distinct discount rate at their bargaining position, etc. In our ex-ante symmetric setting, the ex-post asymmetries are generated by the players' different locations inside the whole matrix. Thus, in the action-pair (l_1,s_2) the bargaining position of A is worse than that of B, while hunter A enjoys of a better position than that of B in the matrix cell (s_1,l_2). This is so because the hunter with the shooter's role can always threaten the line driver with doing the same activity and then getting no deer while the line driver hunter's thread of becoming a shooter is less dramatic, since it only means hunting less deers. Thus, once ex-ante symmetric players are at a particular cell in the outcome matrix, ex-ante symmetries need not be such and asymmetries in the players' bargaining power play a key role in the ex-post equilibrium.

3 More than two players

How to extend the equilibrium notion for three or more players? A straightforward option is by imposing that all pairs of players have balanced mutual punishments. Under this approach we have the following definitions:

Definition 7 *Bilateral Equal Punishments:* $P_{ij}[\psi(x)] = P_{ji}[\psi(x)]$, *for all* $i,j \in N$, *and* $x \in A$.

Definition 8 *A value* $\psi(x)$ *is a* Bilateral Bargaining Equilibrium *if it satisfies Efficiency, Equal Minimum Rights, Individual Rationality and Bilateral Equal Punishments.*

This concept' appeal is that of associating the very natural property of *Bilateral Consistency*: payoff division under the rule is the same when considering all the players together than when only taking two players into account while letting fixed both the actions and payoffs of the remaining players. More formally, consider the game (N,A,v) and a value ψ. For all $\bar{x} \in A$, define for each pair of players $i,j \in N$ and a fixed combination of actions of the remaining players $\bar{x}_{-ij} \in \times_{k \in N \setminus \{i,j\}} A^k$, the subgame $(\{1,2\}, A^i \times A^j, v^{\psi}_{\bar{x}_{-ij}})$, where

$$v^{\psi}_{\bar{x}_{-ij}}(x_i,x_j) := v\left(\bar{x}_{-ij},x_i,x_j\right) - \sum_{k \in N\{i,j\}} \psi_k\left(\bar{x}_{-ij},x_i,x_j\right), \text{ for all } (x_i,x_j) \in A^i \times A^j.$$

Then, the value ψ satisfies bilateral consistency whenever $\psi_r\left((x_i,x_j), v^{\psi}_{\bar{x}_{-ij}}\right) = \psi_r\left(\left(\bar{x}_{-ij},x_i,x_j\right), v\right)$, for $r = i, j$ an all $\left(\bar{x}_{-ij},x_i,x_j\right) \in A$. It is immediate to see that if a value is a bilateral bargaining equilibrium then it will satisfy bilateral consistency.

Unfortunately, the condition of bilateral equal punishments is too stringent: it yields an empty set most of the times. The next example illustrates this fact.

Let us consider a three player game, where $N = \{i, j, k\}$. The set of actions are: Player i chooses between two matrices t_i and b_i, $A_i = \{t_i, b_i\}$. Player j, chooses between rows u_j and d_j, $A_j = \{u_j, d_j\}$. And Player k, chooses between columns l_k and r_k, $A_k = \{l_k, r_k\}$. The matrix $[v]$ is as displayed in Figure 1.

v	l_k	r_k	
u_j	0	3	t_i
d_j	4	8	
	l_k	r_k	
u_j	0	0	b_i
d_j	3	8	

One is tempted to extend the Equal Punishments property to any pair of players, i.e. $P_{ij}[\psi(x)] = P_{ji}[\psi(x)]$, for any i, j in N. Doing so, the closest payoffs satisfying this property are given in Figure 2.

	l_k			r_k			
u_j	0,000	0,000	**0,000**	1,222	0,278	**1,500**	t_i
	0,000	**0,000**	0,000	0,556	**0,278**	0,000	
	0,000	0,000	0,000	**1,222**	0,278	1,500	
d_j	0,000	0,000	**0,000**	0,167	1,167	**2,667**	
	0,500	**3,500**	0,000	0,000	**4,667**	1,167	
	0,500	0,500	0,000	**0,667**	0,000	0,167	

	l_k			r_k			
u_j	0,000	0,000	**0,000**	0,000	0,000	**0,000**	b_i
	0,000	**0,000**	0,000	0,000	**0,000**	0,000	
	0,000	0,000	0,000	**0,000**	0,000	0,000	
d_j	0,000	0,000	**0,000**	0,278	2,222	**2,500**	
	0,000	**3,000**	0,000	0,278	**5,222**	2,500	
	0,000	0,000	0,000	**0,278**	0,556	0,000	

Each cell in this matrix displays both the individual payoffs and punishments of players. For example, at (t_i, d_j, r_k), the information from the table of Figure 2 is:

$P_{ki}(t_i, d_j, r_k) = 0.16\hat{6}$	$P_{kj}(t_i, d_j, r_k) = 1.6\hat{6}$	$\psi_k(t_i, d_j, r_k) = 2.6\hat{6}$
$P_{ji}(t_i, d_j, r_k) = 0$	$\psi_j(t_i, d_j, r_k) = 4.6\hat{6}$	$P_{jk}(t_i, d_j, r_k) = 1.16\hat{6}$
$\psi_i(t_i, d_j, r_k) = 0.6\hat{6}$	$P_{ij}(t_i, d_j, r_k) = 0$	$P_{ik}(t_i, d_j, r_k) = 0.16\hat{6}$

and hence: $P_{ki} = P_{ik} = 0.16\hat{6}$, $P_{kj} = P_{jk} = 1.6\hat{6}$, and $P_{ji} = P_{ii} = 0$. Therefore all bilateral threats are balanced. However, at (b_i, d_j, r_k), we have that:

$P_{ki}(b_i, d_j, r_k) = 0.27\hat{7}$	$P_{kj}(b_i, d_j, r_k) = 2.2\hat{2}$	$\psi_k(b_i, d_j, r_k) = 2.5$
$P_{ji}(b_i, d_j, r_k) = 0.27\hat{7}$	$\psi_j(b_i, d_j, r_k) = 5.2\hat{2}$	$P_{jk}(b_i, d_j, r_k) = 2.5$
$\psi_i(b_i, d_j, r_k) = 0.27\hat{7}$	$P_{ij}(b_i, d_j, r_k) = 0.5\hat{5}$	$P_{ik}(b_i, d_j, r_k) = 0$

and then players' punishments follow the inequalities $P_{ki} = 0.27\hat{7} > 0 = P_{ik}$, $P_{ij} = 0.5\hat{5} > 0.27\hat{7} = P_{ji}$, and $P_{jk} = 2.5 > 2.2\hat{2} = P_{kj}$. Adjusting the corresponding payoffs in order to reduce these inequalities, means rising ψ_k with respect ψ_i according to $P_{ki} > P_{ik}$, also rising ψ_i with respect ψ_j, following the inequality $P_{ij} > P_{ji}$, and finally rising ψ_j with respect ψ_k by $P_{jk} > P_{kj}$. Obviously, it is impossible to make these three changes simultaneously.

Nevertheless, the same example shows that it is possible to modify the equal punishment property by balancing instead the *total sum* of he punishments that a player can inflict to the rest of the players with respect the total punishments that the others can inflict to her. In particular, in our example it holds that

$$P_{ij}+P_{ik}=0.5\hat{5}+0=P_{ji}(b_i,d_j,r_k)+P_{ki}(b_i,d_j,r_k)=0.27\hat{7}+0.27\hat{7},$$

$$P_{ji}(b_i,d_j,r_k)+P_{jk}(b_i,d_j,r_k)=0.27\hat{7}+2.5=P_{ij}(b_i,d_j,r_k)+P_{kj}(b_i,d_j,r_k)=0.5\hat{5}+2.2\hat{2},$$

$$P_{ki}(b_i,d_j,r_k)+P_{kj}(b_i,d_j,r_k)=0.27\hat{7}+2.2\hat{2}=P_{ik}(b_i,d_j,r_k)+P_{jk}(b_i,d_j,r_k)=0+2.5.$$

We consider then, the much less restrictive punishment property:

Definition 9 *Total Equal Punishments:* $\sum_{j\in N\setminus i} P_{ij}\left[\psi(x)\right]=\sum_{j\in N\setminus i} P_{ji}\left[\psi(x)\right]$*, for all* $i,j\in N$ *and* $x\in A$.

Notice that when $n=2$, the above property translates to Equal Punishments, i.e. $P_{12}=P_{21}$. The corresponding Equilibrium concept is then:

Definition 10 *A value* ψ *is a* Consistent Bargaining Equilibrium *if it satisfies Efficiency, Equal Minimum Rights, Individual Rationality and Total Equal Punishments.*

Given $\Gamma=(N,A,v)$, denote by $\mathscr{E}(N,A,v)$ the set of all Consistent Bargaining Equilibrium values. It is clear that if a bilateral bargaining equilibrium exists it must be also a consistent bargaining equilibrium.

Remark: Let $\Gamma=(N,A,v)$ and $\Gamma'=(N,A,w)$, where for all $x\in A, w(x)=v(x)+a$, $a\in R$.

Definition 11 *A rule* ψ *verifies* strategic equivalence *if* $\psi_i(x,w)=\psi_i(x,v)+\frac{a}{n}$

To illustrate this property, suppose that the worth of the game is raised in the same amount, i.e., $w(x)=v(x)+a$ for any $x\in A$. In this case, the increase in the "productivity" cannot be attributed to any player in particular, so that it should be equally redistributed among all the players. It is not difficult to show that any $\psi\in\mathscr{E}(N,A,v)$ satisfies strategic equivalence. This property will be useful in the proof of Theorem 2 below.

Theorem 2 $\mathscr{E}(N,A,v)\neq\varnothing$

Proof. Consider a game $\Gamma=(N,A,v)$. By strategic equivalence, we can assume without loss of generality that $Min_{x\in A}\{v(x)\}=0$.

Let $\Delta[v]$ be the set:

$$\Delta[v] := \left\{ g \equiv [g(x)]_{x\in A} : \sum_{i\in N} g_i(x) = v(x) \text{ and } g_i(x) \geq 0,\, i \in N \right\}.$$

By construction $\Delta[v]$ is a compact and convex set.

Define the following adjusting map f :

$$f_i(g(x)) = g_i(x) + \frac{1}{n-1}\left[\sum_{j\in N\setminus i} P_{ij}[g(x)] - \sum_{j\in N\setminus i} P_{ji}[g(x)]\right],$$

for all $x \in A$ and for all $i, j \in N$.

By the definition of punishments P_{ij} it follows that f is a continuous map.

Now, note that by construction:

$$0 \leq P_{ij}[g(x)] \leq g_j(x), \text{for all } g \in \Delta[v], \text{ and } x \in A.$$

Therefore,

$$\begin{aligned} 0 &\leq \sum_{j\in N\setminus i} P_{ij}[g(x)] \leq \sum_{j\in N\setminus i} g_j(x), \text{ and} \\ 0 &\leq \sum_{j\in N\setminus i} P_{ji}[g(x)] \leq (n-1)g_i(x). \end{aligned}$$

and then $f_i(g(x))$ can be at most:

$$f_i(g(x)) \leq g_i(x) + \frac{1}{n-1}\sum_{j\in N\setminus i} g_j(x) \leq g_i(x) + \sum_{j\in N\setminus i} g_j(x) = v(x),$$

and at least:

$$f_i(g(x)) \geq g_i(x) - \frac{1}{n-1}[(n-1)g_i(x)] = 0$$

Moreover, $\sum_{i\in N} f_i(g(x)) = v(x)$, since:

$$\sum_{i\in N}\left(\sum_{j\in N\setminus i} P_{ij}[g(x)] - \sum_{j\in N\setminus i} P_{ji}[g(x)]\right) = 0$$

Hence, the mapping f goes from $\Delta[v]$ into $\Delta[v]$. Then, by Brower's fixed point Theorem, there exists a g^*, such that $f(g^*) = g^*$.

We show now that $g^* \in \mathscr{E}(N, A, v)$.

Firstly, efficiency follows by construction of $f(g)$. Secondly, for all $x \in A$ such that $v(x) = 0$, it holds that $g_i^*(x) = 0$, for all i and then Equal Minimum Rights is verified.

Moreover, by construction $g_i^*(x) \geq 0$, for all $x \in A$ and then Individual Rationality holds. Finally, $f(g^*) = g^*$ if and only if:

$$\sum_{j \in N \setminus i} P_{ij}[g(x)] - \sum_{j \in N \setminus i} P_{ji}[g(x)] = 0,$$

and then the Equal Punishments property is satisfied. ■

4 Related Literature

The Prekernel.

The *prekernel* was introduced for the class of transferable utility (TU) games in Davis and Maschler (1965), and extended to the class of non-transferable utility (NTU) games in Moldovanu (1990) and Serrano (1997). The prekernel consists of those efficient payoffs x in which each player is in a "bilateral equilibrium" with any other player. This balanced condition is expressed in terms of the *individual excess* of player i against player j, $e_{ij}(x)$[1], that is $e_{ij}(x) = e_{ji}(x)$ for all $i, j \in N$. When considering NTU-games, these excesses must be weighted by the normal vector components at x, $\lambda(x)$, so that the balanced condition translates to $\lambda_i(x)e_{ij}(x) = \lambda_j(x)e_{ji}(x)$, for all $i, j \in N$. For two-person problems, this solution coincides with the Nash bargaining solution. For three or more players, as pointed out in Moldovanu (1990) and Serrano (1997), the prekernel is often an empty set. It should be stressed both the similarity of the bilateral equilibrium condition for the prekernel with respect to the bilateral punishments condition of the bilateral bargaining equilibrium, and the same non existence problem of both concepts in their corresponding settings.

Moreover, the total equal punishments property that we have imposed to overcome the existence problem is similar to the property used by Orshan and Zarzuelo (2000). They define the *average prekernel,* by imposing the equilibrium condition that the average (aggregate) excesses of a player against all the others must be equal to that of all of the others against her:

$$\sum_{j \in N \setminus i} \lambda_i(x)e_{ij}(x) = \sum_{j \in N \setminus i} \lambda_j(x)e_{ji}(x), \text{ for all } i \in N.$$

[1]The reader is referred to the cited bibliography for the formal definitions.

Orshan and Zarzuelo prove that the average prekernel is non-empty over a large significant class of NTU-games. They also show that if an allocation x of the average prekernel verifies that $\lambda_i(x)e_{ij}(x) > \lambda_j(x)e_{ji}(x)$ for some i, j, then it will exist a set of players $\{h_1, ..., h_k\} \subset N \backslash \{i, j\}$ for which the following chain of inequalities holds:

$$\lambda_i(x)e_{ij}(x) > \lambda_j(x)e_{ji}(x);\ \lambda_j(x)e_{jh_1}(x) > \lambda_{h_1}(x)e_{h_1h_2}(x); ...$$
$$...\lambda_{h_k}(x)e_{h_ki}(x) > \lambda_i(x)e_{ih_k}(x).$$

That is, if a player has a claim against any other player, then this will cause a sequence of claims that will end in a claim toward herself. We reproduce a parallel result with respect to the total equal punishment condition of our consistent bargaining equilibrium.

Denote by $i \succ_{\psi(x)} j$ iff $P_{ij}[\psi(x)] > P_{ji}[\psi(x)]$, and $i \preceq_{\psi(x)} j$ iff $P_{ij}[\psi(x)] \leq P_{ji}[\psi(x)]$, and say that m players form a *chain* at $\psi(x)$ if $i_1 \succ_{\psi(x)} i_2 \succ_{\psi(x)} \cdots \succ_{\psi(x)} i_m$.

Theorem 3 *Let $\psi \in \mathscr{E}(N, A, v)$ and suppose that $k \succ_{\psi(x)} l$ for some $k, l \in N$, $|N| \geq 3$ and $x \in A$. Then there exists a chain from l to k.*

Proof. Let $\psi(x)$ be a payoff equilibrium and suppose that $k \succ_{\psi(x)} l$. Assume on the contrary that there does not exist a chain from l back to k and let $S = \{l\} \cup \{i \in N \backslash k : \exists$ a chain from l to $i\}$. By the total equal punishment condition

$$\sum_{i \in S} \sum_{j \in N \backslash i} \left(P_{ij}[\psi(x)] - P_{ji}[\psi(x)]\right) = \sum_{i \in S} \sum_{j \notin S} \left(P_{ij}[\psi(x)] - P_{ji}[\psi(x)]\right) = 0$$

For every $i \in S$ and $j \notin S$, $P_{ij}[\psi(x)] \leq P_{ji}[\psi(x)]$, otherwise j will be in S. Moreover by assumption, $P_{lk}[\psi(x)] < P_{kl}[\psi(x)]$. Therefore:

$$\sum_{i \in S} \sum_{j \notin S} \left(P_{ij}[\psi(x)] - P_{ji}[\psi(x)]\right) < 0$$

which is a contradiction. ■

The Shapley value.

The *Shapley value* was introduced for the class of TU-games in Shapley (1953). Given a TU-game v with player set N, the value $\phi_i(N, v)$ is the expectation of what player i will obtain in v if, for any possible order in which players arrive to the game, all equally likely, she is paid according to her marginal contribution to her predecessors. Myerson (1980) gives a characterization of the Shapley value by imposing the property of *balanced contributions,* which is very close in spirit to our equal punishment property. Suppose that player i leaves the game and then compute the value in the subgame $(N\backslash i,v)$ for the remaining players. The difference $\phi_j(N,v)-\phi_j(N\backslash i,v)$ is just the variation in j's payoffs due to player i' decision of leaving the game. The balanced contribution axiom says that a value ψ satisfies this property if these differences are balanced for every pair of players, i.e.

$$\psi_i(N,v)-\psi_i(N\backslash j,v)=\psi_j(N,v)-\psi_j(N\backslash i,v), \text{ for all } i,j\in N.$$

Myerson proves that the Shapley value is the unique value satisfying efficiency and balanced contributions. Trying to impose the balanced contributions property to NTU-games, means running into the same existence problem than that of the prekernel. For two-person NTU-games $(\{i,j\},V)$, it suffices to look at both a payoff vector $\psi(\{i,j\},V)=a$ and vector of weights (λ_i,λ_j) satisfying

$$\text{Efficiency: } \lambda_i a_i+\lambda_j a_j\geq\lambda_i b_i+\lambda_j b_j, \text{ for all feasible } (b_i,b_j),$$

and

$$\text{Balanced contributions: } \lambda_i\left(a_i-d_i\right)=\lambda_j\left(a_j-d_j\right),$$

where $d_i=\psi_i(\{i\},V)\equiv\psi_i(\{i,j\}\backslash j,V)$. As it is well known, the Nash bargaining solution (Nash, 1950) is the only one satisfying these two properties (Harsanyi, 1963). Unfortunately again, in coalitional games with three or more players the condition

$$\lambda_i\left(\psi_i(N,V)-\psi_i(N\backslash j,V)\right)=\lambda_j\left(\psi_j(N,V)-\psi_j(N\backslash i,V)\right), \text{ for all } i,j\in N,$$

cannot be imposed, because the existence of such payoff vectors is not guaranteed. Remarkably, replacing the above property by the *average (aggregate) balanced contributions*:

$$\sum_{j\in N\backslash i}\lambda_j\left(\psi_j(N,V)-\psi_j(N\backslash i,V)\right)=\sum_{j\in N\backslash i}\lambda_i\left(\psi_i(N,V)-\psi_i(N\backslash j,V)\right), \text{ for all } i\in N,$$

as in Hart and Mas-Colell (1996) makes it possible to show the existence of payoffs allocations satisfying this property jointly with efficiency; this is the set of *Consistent allocations.* This concept was previously (and independently) introduced by Maschler and Owen (1989) for hyperplane games, and later defined in Maschler and Owen (1992) for the more general setting of NTU-games.

Acknowledgments: The authors thank both the Spanish Ministry of Science and Technology and the European Feder Founds for financial support under project SEJ2007-66581. The first author also thanks partial support from "Subvención a grupos consolidados del Gobierno Vasco".

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