The Provision of a Public Good with a direct Provision Technology and a Large Number of Agents

Behringer, Stefan

Goethe Universität Frankfurt

12. September 2008

The Provision of a Public Good with a direct Provision Technology and a Large Number of Agents

Stefan Behringer
Universität Frankfurt

September 12, 2008

Abstract

This paper provides a limit result for the provision of a public good in a mechanism design framework as the number of agents gets large. What distinguishes the public good investigated in this analysis is its direct provision technology which is commonplace in modern information technologies.

1 Motivation

Limit theorems for the provision of goods under asymmetric information about individual valuations with many agents yield strongly opposing results depending on whether the good to be provided is private or public. In the former case the impossibility result of ex-post efficient provision given the presence of both private information and voluntary participation follows from the Myerson and Satterthwaite (1983) Theorem.

In the case of a *public* good, i.e. a good that is non-rival in consumption, the limit result can be fundamentally different. Intuitively, with only few

agents, misrepresentations of valuations leading to lower contributions for public good provision are deterred by the fact that the chances that to good will be supplied at all would decrease significantly. With many agents, each agent will have a lower probability of being 'pivotal'. On the other hand a misrepresentation of valuations may still lead to lower contributions. Hence the inefficiency is expected to be larger in the limit. Mailath and Postlewaite (1990) give an example for such an inefficiency result where the probability of provision converges to zero as the number of agents grows out of bounds. Their negative limit result hinges on the assumption that the total cost of the public good is linear in the number of agents which prevents this magnitude from having any impact on the necessary level of per-capita contributions. This finding is further generalized in Lehrer and Neeman (2000).

More recently Hellwig (2003), in a more general analysis that allows for variable public good quantities, disentangles the two effects and shows that assuming independence of the cost function from the number of agents is sufficient for the negative limit result to be reversed. For a public good that is excludable, Norman (2004) shows that even with a linear total cost, a mechanism exits that charges a simple fixed fee from each non-excluded agent and allows for provision in the limit. The magnitude of the implied waste from exclusion relative to first best will be strictly reduced if there is more than one public good that can be bundled as shown in Fang & Norman (2008).

The discussion which cost assumption is more adequate when looking at public good provision in replica economies goes back to Roberts' (1976) analysis who, using national defense as a motivation, also opts against the constant cost specification. This paper avoids the difficulty of an explicit choice between these regimes as the corresponding cost function for the provision of the public good will be determined endogenously. We depart from existing approaches by specifying a benchmark innovation technology that allows for *direct provision* of the public good by the agents. The provision thus does not lie in the hand of some external entity but can be undertaken by each agent individually. Agents' dichotomous nature as beneficiaries and as direct providers of the public good allows us to set up a mechanism that consists of separate contracts.

Motivation for this particular technology can be found when looking at *information* as a commodity. Inventions, academic research, and the produc-

tion of knowledge in general share the feature of having a modular structure consisting of information. Progress is made at the individual level and for the potential benefit of the community at large. Processes that use information as an input as well as an output also benefit the individual providing it as it allows for further investigation. Similarly, specific innovation processes such as those made possible by modern information technology that allow for a segmentation of large and complex tasks (the creation of a multi-lingual online encyclopedia such as Wikipedia, the creation of complex network and community sites such as Myspace or Facebook, or the development of open source software) to a multitude of developers and researchers can be captured by it.

The core of all these processes consist of information as the essential and indivisible commodity that is non-rival in consumption and often non-excludable and if it is, e.g. if patenting is possible or spillovers are not inevitable, this may not be socially desirable. Thus treating information as public goods for the purpose of our analysis seems to be a very natural choice.

Following a standard Bayesian mechanism design approach we seek an incentive compatible, individual rational, and feasible mechanism allowing for side payments to generate the provision of the public good. We derive necessary and sufficient conditions for the existence of such a mechanism. We show that the conditions for existence of the mechanism will be relaxed as the number of agents gets large. Most importantly, given our technology and a large number of agents we can approximate the first best outcome arbitrarily closely.

The underlying intuition for our main result is as follows: as argued above with many agents, each agent will have a lower probability of being 'pivotal' increasing the incentives to misrepresent valuations. On the other hand the modular structure of independent development and research efforts given the direct provision technology imply that with many agents provision is almost certain at a cost that is increasing but bounded above. Hence the benefit of a misrepresentation on required individual contributions is decreasing but bounded below by zero. It will then be shown that the convergence behaviour of the two incentives is such that for any cost level there always exists a large enough number of agents such that the latter effect dominates.

2 The setup

There are n agents with private valuation parameter (type) $\theta_i \in \Theta, i = 1, 2...n$, $\Theta \equiv [\underline{\theta}, \bar{\theta}]$ which are realizations of independently and identically distributed (i.i.d.) random variables $\tilde{\theta}_i$. The random variables $\tilde{\theta}_i$ are drawn from identical distributions that have a continuous and strictly positive density $f(\theta_i)$, and a cumulative distribution function $F(\theta_i)$.

The provision of the public good is an all or nothing decision, i.e. the level of the public good provision is fixed and will take place with some probability $r(\boldsymbol{\theta}) : \Theta^n \to [0, 1]$ where the total vector of types is denoted by $\boldsymbol{\theta} \in [\underline{\theta}, \bar{\theta}]^n$. Employing the revelation principle we can restrict ourselves to incentive compatible direct mechanisms.

A mechanism (allocation) is a triple of functions $\langle r(\boldsymbol{\theta}), \mathbf{p}(\boldsymbol{\theta}), \mathbf{z}(\boldsymbol{\theta}) \rangle$ where $\mathbf{p}(\boldsymbol{\theta})$ is the vector of total effort or contributions to the public good with generic elements $(p_i(\boldsymbol{\theta})) : \Theta^n \to [0, 1]$ and $\mathbf{z}(\boldsymbol{\theta})$ is a vector of net side payments with generic elements $(z_i(\boldsymbol{\theta})) : \Theta^n \to \mathfrak{R}$.

Agent's net utilities will then consist of the real allocations $r(\boldsymbol{\theta})$ and $\mathbf{p}(\boldsymbol{\theta})$ and the allocation of net side payments $\mathbf{z}(\boldsymbol{\theta})$. Ex-post utilities are denoted as

$$u_i = \theta_i r(\boldsymbol{\theta}) - p_i(\boldsymbol{\theta}) - z_i(\boldsymbol{\theta}) \tag{1}$$

As the agent is a user as well as a direct provider of the public good the utility function can be decomposed into two parts as

$$u_i = \underbrace{\theta_i r(\boldsymbol{\theta}) - z_{Ui}(\boldsymbol{\theta})}_{user} \underbrace{-p_i(\boldsymbol{\theta}) + z_{Pi}(\boldsymbol{\theta})}_{provider} \tag{2}$$

respectively, so that net side payments are defined as

$$z_i(\boldsymbol{\theta}) \equiv z_{Ui}(\boldsymbol{\theta}) - z_{Pi}(\boldsymbol{\theta}) \tag{3}$$

The *direct provision technology* implies that each agent i can provide the good with some probability (effort) $p_i(\boldsymbol{\theta})$ directly. The total probability of the good being provided independently with n agents is thus

$$r(\boldsymbol{\theta}) = 1 - \Pi_{i=1}^n (1 - p_i(\boldsymbol{\theta})) \tag{4}$$

An allocation is *feasible* if side payments satisfy

$$\sum_{i=1}^n z_i(\boldsymbol{\theta}) \equiv \sum_{i=1}^n (z_{Ui}(\boldsymbol{\theta}) - z_{Pi}(\boldsymbol{\theta})) \geq 0 \ \forall \ \boldsymbol{\theta} \tag{5}$$

A *first best allocation* is given as a feasible allocation $\langle r(\boldsymbol{\theta}), \mathbf{p}(\boldsymbol{\theta}), \mathbf{z}(\boldsymbol{\theta}) \rangle$ for which no other feasible allocation achieves a higher aggregate expected surplus[1]

$$\sum_{i=1}^{n} \int \ldots \int \left[(r(\boldsymbol{\theta})\theta_i - p_i(\boldsymbol{\theta})) \right] f(\theta_1) f(\theta_2) \ldots f(\theta_n) d\theta_n \ldots d\theta_2 d\theta_1 = \quad (6)$$

$$\sum_{i=1}^{n} \int \left[(r(\boldsymbol{\theta})\theta_i - p_i(\boldsymbol{\theta})) \right] dF^n(\boldsymbol{\theta}).$$

The mechanism is divided into *two contracts* corresponding to the agents' functions. A first contract is written for the agents as providers and a second contract for the agents as users. Both contracts are signed at the interim stage, i.e. when agents know their valuation and are thus subject to the option of all agents with any valuations to not sign the contract.

2.1 The Provider contract

In this contract, agents are given payments $z_{Pi}(\theta_i, \boldsymbol{\theta}_{-i})$ in exchange for the enforceable promise to develop and provide the public good with some probability $p_i(\theta_i, \boldsymbol{\theta}_{-i})$. Interim individual rationality which guarantees the voluntary acceptance of the contract for any given type requires that

$$U_{Pi}(\theta_i) = \int (-p(\theta_i, \boldsymbol{\theta}_{-i}) + z_{Pi}(\theta_i, \boldsymbol{\theta}_{-i})) dF^{n-1}(\boldsymbol{\theta}_{-i}) \geq 0 \; \forall \; \theta_i \quad (7)$$

where the outside options have been normalized to zero. This condition can be guaranteed to hold if we make the implementation assumption of ex-post individual rationality by equality between

$$p_1(\boldsymbol{\theta}) = \ldots = p_n(\boldsymbol{\theta}) = p(\boldsymbol{\theta}) = z_{Pi}(\boldsymbol{\theta}) \; \forall \; \boldsymbol{\theta} \quad (8)$$

hence no agent will ever regret to have taken part in the provision process, independently of the provision outcome. This implementation assumption trivially implies that ex-ante individual rationality also holds. Within the

---

[1] The second line follows from the assumed independence of the random variables $\tilde{\theta}_i$ so that the prior distribution is simply given by the product distribution F^n. Whenever we leave out the limits of the integrals we integrate over the full support of Θ.

context of or motivational examples this translates into some fundamental and equitable compensation mechanism (i.e. some base salary) that keeps the developer or researcher just indifferent to expanding the effort and ensure a minimum probability with which the next module will be provided. Clearly the implied duplication of effort implies a very conservative and thus robust starting point when investigating the total welfare consequences of the direct provision technology below. We also assume that effort levels $p(\boldsymbol{\theta})$ are observable which allows us to set aside potential moral hazard problems that may occur with development efforts that are chosen privately.[2]

The implementation assumption allows us to write the total provision probability (4) as

$$r(\boldsymbol{\theta}) = 1 - (1 - p(\boldsymbol{\theta}))^n \ \forall \ \boldsymbol{\theta} \tag{9}$$

2.2 The User contract

This contract is intended to generate the side payments $z_{Ui}(\theta_i, \boldsymbol{\theta}_{-i})$ necessary to finance the expenses incurred by the provider contract and thus to satisfy the overall feasibility constraint. The form of this contract is more involved as we have to guarantee that no agent has an incentive to misrepresent valuations that are unobservable to all others. With alternative analyses of direct mechanisms it shares the feature that it may have to be retranslated into more practical mechanisms that are feasible when looking at the public good nature of information absent any additional private contractual framework set up to limit dissipation. Again we guarantee that each agent will participate voluntarily after own valuations are become known. Interim incentive compatibility (IIC) then guarantees that no agent has an incentive to misrepresent valuations, i.e.

$$\int (r(\theta_i, \boldsymbol{\theta}_{-i})\theta_i - z_{Ui}(\theta_i, \boldsymbol{\theta}_{-i}))dF^{n-1}(\boldsymbol{\theta}_{-i}) \ \geq \tag{10}$$

$$\int (r(\hat{\theta}_i, \boldsymbol{\theta}_{-i})\theta_i - z_{Ui}(\hat{\theta}_i, \boldsymbol{\theta}_{-i}))dF^{n-1}(\boldsymbol{\theta}_{-i}) \ \forall \ \theta_i, \hat{\theta}_i \ \in \ \Theta, \ \forall \ \theta_i$$

---

[2]For an investigation of the optimal design of knowledge sharing in teams (with each member being important for the aggregate knowledge level) with adverse selection resulting from the public good characteristics of information and the additional complications of moral hazard from hidden research efforts in a finite economy setting see d'Aspremont, Bhattacharya, & Gérard-Varet, 1998.

Interim individual rationality (IIR) requires

$$\int \left(r(\theta_i, \boldsymbol{\theta}_{-i})\theta_i - z_{Ui}(\theta_i, \boldsymbol{\theta}_{-i}) \right) dF^{n-1}(\boldsymbol{\theta}_{-i}) \geq 0 \ \forall \ \theta_i \in \Theta, \ \forall \ \theta_i \qquad (11)$$

2.3 Bayesian-Nash Implementation

For ease of notation we define the interim probability of provision as perceived by agent i as

$$\rho_i(\theta_i) \equiv \int r(\theta_i, \boldsymbol{\theta}_{-i})dF^{n-1}(\boldsymbol{\theta}_{-i}) \qquad (12)$$

and the interim expected side payments from the user contract of agent i to the mechanism designer as

$$\xi_{Ui}(\theta_i) \equiv \int z_{Ui}(\theta_i, \boldsymbol{\theta}_{-i})dF^{n-1}(\boldsymbol{\theta}_{-i}) \qquad (13)$$

so that the agent's *expected* utility from the user contract can be written as

$$U_{Ui}(\theta_i) = \rho_i(\theta_i)\theta_i - \xi_{Ui}(\theta_i) \qquad (14)$$

We now use the following well known result:

Lemma 1 *A mechanism* $\langle r(\boldsymbol{\theta}), z_{U1}(\boldsymbol{\theta}), ... z_{Un}(\boldsymbol{\theta}) \rangle$ *is interim incentive compatible (IIC), iff*
  a) expected probability of provision $\rho_i(\theta_i)$ *is non-decreasing in* θ_i, *and*
  b) interim utility levels satisfy

$$U_{Ui}(\theta_i) = U_{Ui}(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} \rho_i(\eta)d\eta \qquad (15)$$

so that high valuation types will receive a higher expected utility level and each agents expected side payments satisfy

$$\xi_{Ui}(\theta_i) = \xi_{Ui}(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} \eta d\rho_i(\eta) \ \forall \theta_i. \qquad (16)$$

Proof: Standard.∎

If expectations of all side payment functions from the user contract $\xi_{Ui}(\theta_i)$ satisfy the above and the expectations of the probability of provision $\rho_i(\theta_i)$ are non-decreasing in type we can then chose a set of side payment functions $z_{Ui}(\boldsymbol{\theta})$ such that $\int z_{Ui}(\theta_i, \boldsymbol{\theta}_{-i})dF^{n-1}(\boldsymbol{\theta}_{-i}) = \xi_{Ui}(\theta_i) \ \forall \ \theta_i$. One obvious choice for such a function is $z_{Ui}(\boldsymbol{\theta}) = \xi_{Ui}(\theta_i)$. This procedure yields a Bayesian incentive compatible allocation. Note that the IIC constraint determines each agent's interim expected utility up to some constant of integration.

The IIR condition

$$\int (r(\theta_i, \boldsymbol{\theta}_{-i})\theta_i - z_{Ui}(\theta_i, \boldsymbol{\theta}_{-i}))dF^{n-1}(\boldsymbol{\theta}_{-i}) \geq 0 \ \forall \ \theta_i \qquad (17)$$

can be rewritten using the IIC condition as

$$U_{Ui}(\theta_i) = \rho_i(\theta_i)\theta_i - \xi_{Ui}(\theta_i) = \int_{\underline{\theta}}^{\theta_i} \rho_i(\eta)d\eta - \xi_{Ui}(\underline{\theta}) + \rho_i(\underline{\theta})\underline{\theta} \geq 0 \ \forall \ \theta_i \quad (18)$$

As $\int_{\underline{\theta}}^{\theta_i} \rho_i(\eta)d\eta \geq 0 \ \forall \ \theta_i$ we have a necessary and sufficient condition for IIR to hold given by

$$U_{Ui}(\underline{\theta}) = \rho_i(\underline{\theta})\underline{\theta} - \xi_{Ui}(\underline{\theta}) \geq 0 \qquad (19)$$

Lemma 2 *Given the user contract is IIC, then ex-ante side payments satisfy*

$$\int z_{Ui}(\boldsymbol{\theta})dF^n(\boldsymbol{\theta}) = -U_{Ui}(\underline{\theta}) + \int \left(\theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)}\right) r(\boldsymbol{\theta})dF^n(\boldsymbol{\theta}) \qquad (20)$$

Proof: See online Appendix.∎

8

3 The Mechanisms

We are now in a position to derive a necessary and sufficient condition for the existence of an effort allocation function and an overall development probability that satisfy the key constraints of the analysis. We first weaken the requirement of strict (or ex-post) feasibility to weak (or ex-ante) feasibility. It is well known (e.g. Cramton, Gibbons, & Klemperer, 1987) that this allows for a more elegant analysis and once a mechanism is derived that satisfies the latter, a simple transformation of transfers allows for a satisfaction of the former. An allocation is then *weakly feasible* if side payments satisfy

$$\sum_{i=1}^{n} \int z_i(\boldsymbol{\theta})dF^n(\boldsymbol{\theta}) \equiv \sum_{i=1}^{n} \int (z_{Ui}(\boldsymbol{\theta}) - z_{Pi}(\boldsymbol{\theta}))dF^n(\boldsymbol{\theta}) \geq 0 \qquad (21)$$

We thus have the following Lemma:

Lemma 3 *For any probability of provision $r(\boldsymbol{\theta})$ such that $\rho_i(\theta_i)$ is non-decreasing in θ_i, there exist net side payments $\mathbf{z}(\boldsymbol{\theta})$ such that $< r(\boldsymbol{\theta}), \mathbf{z}(\boldsymbol{\theta}) >$ is interim incentive compatible, interim individual rational, and weakly feasible iff*

$$\int \left(\sum_{i=1}^{n} \left(\theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) r(\boldsymbol{\theta}) - np(\boldsymbol{\theta}) \right) dF^n(\boldsymbol{\theta}) \geq 0 \qquad (22)$$

Proof: See online Appendix.∎

9

3.1 The Program $\mathcal{P}$

The mechanism designer has to solve the following program $\mathcal{P}$ for the weakly feasible overall mechanism that implements the contracts under interim individual rationality and incentive compatibility and generates non-negative expected social benefit. Using Lemma 3 this program is

$$Max_{r(\boldsymbol{\theta})} \left\{ \int \left(\sum_{i=1}^{n} \theta_i r(\boldsymbol{\theta}) - k_n(r(\boldsymbol{\theta})) \right) dF^n(\boldsymbol{\theta}), 0 \right\} \tag{23}$$

$$s.t. \; \rho_i(\theta_i) \text{ non-decreasing in } \theta_i \text{ and} \tag{24}$$

$$s.t. \; \int \left(\sum_{i=1}^{n} \left(\theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) r(\boldsymbol{\theta}) - k_n(r(\boldsymbol{\theta})) \right) dF^n(\boldsymbol{\theta}) \geq 0 \tag{25}$$

The *ex-post efficient* (or **first best**) provision rule given by the first order condition for (23) equates

$$\frac{\partial k_n(r(\boldsymbol{\theta}))}{\partial r} = (1 - r)^{\frac{1}{n} - 1} = \sum_{i=1}^{n} \theta_i \tag{26}$$

The provision rule is thus

$$r^*(\boldsymbol{\theta}) = \begin{cases} 0 \text{ if } \sum_{i=1}^{n} \theta_i \leq 1 \\ 1 - \left(\sum_{i=1}^{n} \theta_i \right)^{-\frac{n}{n-1}} \text{ if } \sum_{i=1}^{n} \theta_i > 1 \end{cases} \tag{27}$$

Note that this firsts best development probability is *interior* for finite magnitudes. Using an indicator function we have

$$r^*(\boldsymbol{\theta}) = \left(1 - \left(\sum_{i=1}^{n} \theta_i \right)^{-\frac{n}{n-1}} \right) \mathbf{1}_{\left\{ \sum_{i=1}^{n} \theta_i > 1 \right\}} \tag{28}$$

at first best total costs

$$k_n(r^*(\boldsymbol{\theta})) = n \left(1 - \left(\sum_{i=1}^{n} \theta_i \right)^{\frac{1}{1-n}} \right) \mathbf{1}_{\left\{ \sum_{i=1}^{n} \theta_i > 1 \right\}} \tag{29}$$

10

A sufficient condition for constraint (24) is that the *hazard rate condition* holds, i.e. that the "virtual utility"

$$\theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \text{ is non-decreasing in } \theta_i \tag{30}$$

Looking at the objective function, note that the function is concave in $r(\boldsymbol{\theta})$ and hence first-order conditions are necessary and sufficient for the constraint optimization. Given (30), Program $\mathcal{P}$ can be solved by pointwise maximization and its solution satisfies the monotonicity constraint.

In the following Lemma we show that the constraints that the user and the provider contract put on the socially optimal outcome of the public good provision is non-trivial. Generically the implementable mechanism cannot achieve the first best outcome.

The **second best** maximization problem of program $\mathcal{P}$ can be rearranged using a Lagrange multiplier approach as

$$Max_{r(\boldsymbol{\theta}),\lambda} \left\{ \begin{array}{c} (1+\lambda)\left(\int \left(\sum_{i=1}^n \theta_i r(\boldsymbol{\theta}) - k_n(r(\boldsymbol{\theta}))\right) dF^n(\boldsymbol{\theta})\right) \\ -\lambda \int \left(\sum_{i=1}^n \left(\frac{1-F(\theta_i)}{f(\theta_i)}\right) r(\boldsymbol{\theta})\right) dF^n(\boldsymbol{\theta}), 0 \end{array} \right\} \tag{31}$$

We can show that the optimal Lagrange multiplier λ^* is bounded away from zero and hence the constraint is non-trivial. The proof parallels Hellwig's, 2003, proof of Proposition 1 and is relegated to the online appendix.

Lemma 4 *Define the probability of provision as an indicator function*

$$r_\lambda(\boldsymbol{\theta}) \equiv \left(1 - \left(\sum_{i=1}^n \theta_i\right)^{-\frac{n}{n-1}}\right) \mathbf{1}_{\left\{ \begin{array}{c} (1+\lambda)\times \\ \left[\begin{array}{c} \left(\sum_{i=1}^n \theta_i\right) \times \left(1 - \left(\sum_{i=1}^n \theta_i\right)^{-\frac{n}{n-1}}\right) - \\ k_n\left(r_\lambda(\boldsymbol{\theta}) = 1 - \left(\sum_{i=1}^n \theta_i\right)^{-\frac{n}{n-1}}\right) \end{array}\right] > \\ \lambda \sum_{i=1}^n \left[\frac{1-F(\theta_i)}{f(\theta_i)}\right] \times \left(1 - \left(\sum_{i=1}^n \theta_i\right)^{-\frac{n}{n-1}}\right) \end{array} \right\}} \tag{32}$$

Then there exists a unique Lagrange multiplier $\lambda^* > 0$ such that

$$G(\lambda^*) \equiv \int \left(\sum_{i=1}^n \left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right) r_{\lambda^*}(\boldsymbol{\theta}) - k_n(r_{\lambda^*}(\boldsymbol{\theta}))\right) dF^n(\boldsymbol{\theta}) = 0 \tag{33}$$

given $n > 1$. Also $r_\lambda(\boldsymbol{\theta})$ is non-increasing and $G(\lambda)$ is non-decreasing in λ.

11

Proof: See online Appendix.■

Looking at program $\mathcal{P}$ defined by (31) we can show that the 'shadow cost of the constraint', and therefore the value of λ^{*n} that satisfies the complementary slackness condition $\lambda^{*n}G(\lambda^{*n}) = 0$ increases in n. Given there are more agents, the probability that any particular agent's signal will be pivotal for the provision of the public good becomes negligible. It follows that extracting a given side payment under the user contract and still satisfying incentive compatibility and individual rationality becomes increasingly difficult and hence the constraint bears more heavily on the maximization program the more agents are involved. This intuition is formalized by showing that the critical Lagrange multiplier is not uniformly bounded.

Lemma 5 *The Lagrange multiplier of program $\mathcal{P}$ grows out of bounds as n gets large for any $r(\boldsymbol{\theta}) \in [0, 1)$.*

Proof: See online Appendix.■

4   Limit Results

Given the characteristics of Program $\mathcal{P}$ we have analysed above it is now interesting to find out how the number of agents affects the possibility to obtain a desirable outcome.[3]

Looking at the first best provision rule we find the following Lemma:

Lemma 6 *The first best provision rule $r^*(\boldsymbol{\theta})$ converges to provision with certainty in economy n as n gets large.*

Proof: See online Appendix.■

Also

---

[3]In order to take limits we define an economy n as given by a vector of type distributions $\theta^n = (\theta_1, ..., \theta_n) \in \Theta^n$ and an economy $n + 1$ as given by a vector $\theta^{n+1} = (\theta_1, ..., \theta_{n+1}) \in \Theta^{n+1}$ where θ_{n+1} is a new i.i.d. draw from the distribution $F(\theta_i)$ accordingly.

Lemma 7 *The total cost function $k_n(r(\boldsymbol{\theta}))$ in economy n is increasing in n, continuous, and strictly concave. It approaches a limit as*

$$\lim_{n \to \infty} k_n(r(\boldsymbol{\theta})) = -\ln(1 - r(\boldsymbol{\theta})) \tag{34}$$

Proof: See Appendix.■

We now proceed with the construction of approximately efficient mechanisms. Assuming for the moment that the type function density $f(\cdot)$ is *degenerate* and has point mass half at the extremes of the support $\Theta \equiv [\underline{\theta}, \bar{\theta}]$ and that the random variables are i.i.d.. We may then use an argument by Al-Najjar & Smorodinsky, 2000 and show

Lemma 8 *The LHS of each agent's interim incentive compatibility constraint is bounded above by*

$$\frac{1}{\sqrt{2\pi}} \frac{2}{\sqrt{(n-1)}} e^{\frac{1}{12n}}.$$

Proof: See online Appendix.■

Intuitively, as n grows large and the discrete Binomial distribution approaches a continuous Normal distribution, the highest probability of having a precise number of occurrences of $\theta_i = \bar{\theta}$ (i.e. of knowing to be the pivotal agent) goes to zero as the Normal distribution has no atoms. For large n the exponential term above converges to one so that we see that the incentive to reveal one's type truthfully is approximately falling in $(\sqrt{n})^{-1}$.[4]

The perceived probability of being the pivotal agent can be interpreted as the underlying motivation for making contributions towards the provision of the public good in the user contract. We have shown that this motivation is decreasing in n for each individual agent at the rate of $(\sqrt{n})^{-1}$ so that the value of the virtual utility term by a theorem of Lindeberg-Levy satisfies

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) \xrightarrow{d} N(0, \sigma^2) \tag{35}$$

---

[4] An extension to non-degenerate type space is in their Section.3.3.

where we normalize $\underline{\theta} = 0$.[5] As the indicator constraint condition in Lemma 4 implies that for the *expected aggregate* side payments we integrate only over positive sums, the expected value for large n can be derived as $\int_0^\infty x d\Phi(x) \equiv \varphi$ where $\Phi(\cdot)$ is the Normal distribution function. There thus exists a constant $\varphi > 0$ such that the stabilized sum has a limit

$$\lim_{n \to \infty} \quad \frac{1}{\sqrt{n}} \int \left(1 - \left(\sum_{i=1}^n \theta_i \right)^{-\frac{n}{n-1}} \right) \times \tag{36}$$

$$\mathbf{1}_{\left\{ \sum_{i=1}^n \left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} \right) > 0 \right\}} \sum_{i=1}^n \left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} \right) dF^n(\boldsymbol{\theta}) = \varphi$$

and so the sum of expected virtual utilities that can be collected from the agents if strategic constraints are taken into account will grow out of bounds in $\sqrt{n}$.[6] The Theorem now shows that even under the constraints imposed, the probability of provision in a second best mechanism will converge in probability to the first best probability as n becomes large.

---

[5] As one referee has emphasized, such a normalization is typical but not completely innocuous. It implies that there remains the possibility that an agent does not benefit from the provision of the public good at all. This may be empirically plausible for public goods whose benefit can occur privately (without the possibility of some subset of agents being physically excluded) but for which agents are "taxed" (here with the opportunity costs of their effort) collectively. The notion of "taxation" in our context of public good provision may only be disputable for our assumption of participation constraints inasmuch as the designer can retain its coercive powers in a global and competitive environment. As we are concerned with large number results below the normalization is necessary in order to prevent the construction of simpler mechanisms that allow for concurrent welfare results with many agents. Normalizing the lower support to some strictly negative constant on the other hand will lead to asymptotic impossibility.

[6] Having exclusion as an additional instrument (as investigated in Hellwig, 2003 and Norman, 2004) allows for a simple, *ex-post* IR, fixed fee mechanism and a collection from participating agents that grows out of bounds linearly.

Theorem 9 *Let $\left\langle r_j^n(\boldsymbol{\theta}), \mathbf{p}_j^n(\boldsymbol{\theta}), \mathbf{z}_j^n(\boldsymbol{\theta}) \right\rangle \equiv \Psi_j^n(\boldsymbol{\theta})$ be a jth-best allocation given the direct provision technology and n agents. Then for a first best allocation $\Psi_1^n(\boldsymbol{\theta}): r_1^n(\boldsymbol{\theta}) \xrightarrow{a.s.} 1$ i.e. almost surely as $n \to \infty$. For a second best allocation $\Psi_2^n(\boldsymbol{\theta}): r_2^n(\boldsymbol{\theta}) \xrightarrow{p} 1$ i.e. in probability as $n \to \infty$. The expected loss from second best relative to first best thus converges to zero.*

Proof: See Appendix.∎

Despite convergence to the first best provision rule the direct provision technology implies that there is some waste of effort due to our conservative assumption that the designer requires direct provision from *all* agents. In the limit this total waste is $-\ln(1 - r^{n*}(\boldsymbol{\theta})) - 1$ by Lemma 7 and generically bounded whereas the gross benefit from the mechanism is unbounded above.

5 Conclusion

The analysis has shown that irrespective of an increased incentive to free ride on other agents' efforts, the conditions for the provision of a public good with a direct provision technology are relaxed as the number of agents gets large. Substituting any 'ad hoc' choice of the cost function for the provision of the public good with an endogenous motivation we depart from existing approaches by specifying a new provision technology benchmark. Despite our conservative choice of an inherent duplication of efforts in the corresponding Bayesian Mechanisms we are able to approximate the first best outcome arbitrarily closely in the limit. Our result thus sheds light on the conditions for the provision of certain public goods that are commonplace in modern information technologies which may be provided directly by its users when there is a large number of them.

Clearly the mapping from any abstract direct mechanisms that enables the analysis in the first place into more practical mechanisms constitutes a challenge. This is the case even more so when looking at the public good nature of information within the context of modern information technologies, where direct monetary payments do not seem constitute the rule.

An exception may be the case of academic research where dissipation usually cannot be fully prevented. Given that progress in such research can be fairly described by independent research efforts we have argued above that

signing a provider contract comes close to a base salary that is independent of ones stake in a particular venture. Academic publishers on the other hand will try to charge high valuation users differently for the timely receipt of the final articles. Hence this may generate something akin to the revenue collected in the user contract.

With many researchers then the next innovation module is almost certain and despite everyone putting in effort the costs imposed by the base salaries will be bounded above as more agents and higher individual efforts can be traded off. On the other hand the total benefit from those willing to learn about the development in print is unbounded above thus relaxing the overall feasibility constraint of the mechanism. The analysis and its positive limit result may therefore be interpreted as an argument for integrating academic publishing closely with the large research community in order to make sure that the proceeds of publishing activities can flow back to those who have provided the effort.

6 Appendix I

Proof of Lemma 7:

The first derivative of the total cost function w.r.t n is

$$\frac{\partial k_n(r(\boldsymbol{\theta}))}{\partial n} = 1 - (1 - r(\boldsymbol{\theta}))^{\frac{1}{n}}\left(1 - \frac{\ln\left(1 - r(\boldsymbol{\theta})\right)}{n}\right) \geq 0 \qquad (37)$$

i.e. non-negative for all $r(\boldsymbol{\theta}) \in [0, 1)$ which follows from noting that the LHS is increasing in n so that we can focus on the case $n = 1$. Now the first and second bracket are always strictly smaller than one for $r(\boldsymbol{\theta}) \in [0, 1)$ hence the result follows. The second derivative is

$$\frac{\partial^2 k_n(r(\boldsymbol{\theta}))}{\partial n^2} = -(1 - r(\boldsymbol{\theta}))^{\frac{1}{n}}\frac{\left(\ln\left(1 - r(\boldsymbol{\theta})\right)\right)^2}{n^3} < 0 \qquad (38)$$

hence the function is strictly concave in n.

In the limit, total costs will converge which follows from rewriting

$$(1 - r(\boldsymbol{\theta}))^{\frac{1}{n}} = \exp(-\frac{1}{n}\ln(\frac{1}{1 - r(\boldsymbol{\theta})})) \qquad (39)$$

where the exponent will go to zero as n becomes large. We then use a Taylor approximation around zero, which implies that from the expansion of the exponential function into $\exp(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ we use the first two terms and find

$$(1 - r(\boldsymbol{\theta}))^{\frac{1}{n}} \approx \left(1 - \frac{1}{n}\ln(\frac{1}{1 - r(\boldsymbol{\theta})})\right) \qquad (40)$$

and this will be a strict equality for $n \to \infty$ as all remaining terms can be neglected. We thus have that the limit of $k_n(r(\boldsymbol{\theta}))$ becomes

$$\lim_{n \to \infty}\left(n(1 - (1 - r(\boldsymbol{\theta}))^{\frac{1}{n}})\right) = \ln(\frac{1}{1 - r(\boldsymbol{\theta})}) = -\ln(1 - r(\boldsymbol{\theta})). \qquad (41)$$

Note that total costs will only be uniformly bounded if $r(\boldsymbol{\theta}) < 1$.∎

Proof of Theorem 9:

For large n the sum of valuations will be unbounded above for almost all type realizations whereas Lemma 7 reveals an upper bound on total costs for all n so that from a first best perspective the probability of provision approaches certainty almost surely which can be seen from Lemma 6. In order to show uniform convergence in provision probability for a second best allocation we need to show that for any $\varepsilon > 0$, there exists a number of agents $\check{n}(\varepsilon)$ so that for $n \geq \check{n}(\varepsilon)$ we have

$$\Pr\left\{|r_n(\boldsymbol{\theta}) - 1| > \varepsilon\right\} \leq \varepsilon \tag{42}$$

so that as n grows out of bounds $r_n(\boldsymbol{\theta})$ converges in probability to the first best provision probability.

The second best probability of provision as given by Program P is

$$r_n(\boldsymbol{\theta}) \equiv \left(1 - \left(\sum_{i=1}^{n}\theta_i\right)^{-\frac{n}{n-1}}\right)\mathbf{1}_{\left\{\begin{array}{c}(1+\lambda)\times \\ \left[\begin{array}{c}(\sum_{i=1}^{n}\theta_i)\times\left(1-(\sum_{i=1}^{n}\theta_i)^{-\frac{n}{n-1}}\right)- \\ k_n\left(r_n(\boldsymbol{\theta})=1-(\sum_{i=1}^{n}\theta_i)^{-\frac{n}{n-1}}\right)\end{array}\right]> \\ \lambda\sum_{i=1}^{n}\left[\frac{1-F(\theta_i)}{f(\theta_i)}\right]\times\left(1-(\sum_{i=1}^{n}\theta_i)^{-\frac{n}{n-1}}\right)\end{array}\right\}} \tag{43}$$

As the Lagrange multiplier is unbounded above by Lemma 5 for large n we can focus on

$$r_n(\boldsymbol{\theta}) = \left(1 - \left(\sum_{i=1}^{n}\theta_i\right)^{-\frac{n}{n-1}}\right)\mathbf{1}_{\left\{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\theta_i-\frac{1-F(\theta_i)}{f(\theta_i)}\right)>\frac{\frac{1}{\sqrt{n}}k_n\left(r_n(\boldsymbol{\theta})=1-\left(\sum_{i=1}^{n}\theta_i\right)^{-\frac{n}{n-1}}\right)}{1-\left(\sum_{i=1}^{n}\theta_i\right)^{-\frac{n}{n-1}}}\right\}} \tag{44}$$

Using Lemma 8 in its modification for a continuous type space we know that individual incentive constraints will be bounded above by some positive constant that is decreasing in $(\sqrt{n})^{-1}$. Whence the term on the LHS of the indicator condition will converge to some strictly positive constant. Using Lemma 7 we know that

$$k_n\left(r_n(\boldsymbol{\theta}) = 1 - \left(\sum_{i=1}^{n}\theta_i\right)^{-\frac{n}{n-1}}\right) \tag{45}$$

18

will be bounded above and the RHS of the indicator condition converges to zero. Thus we find that for any $\varepsilon > 0$, there exists a number of agents $\check{n}(\varepsilon)$ so that for $n \geq \check{n}(\varepsilon)$ we have convergence in probability, i.e.

$$\Pr\left\{|r_n(\boldsymbol{\theta}) - 1| > \varepsilon\right\} = \tag{46}$$

$$\Pr\left\{\mathbf{1}\left\{\begin{array}{c}\left(1 - \left(\sum_{i=1}^{n}\theta_i\right)^{-\frac{n}{n-1}}\right) \times \\ \frac{1}{\sqrt{n}}k_n\left(r_n(\boldsymbol{\theta}) = 1 - \left(\sum_{i=1}^{n}\theta_i\right)^{-\frac{n}{n-1}}\right) < \\ \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right)\left(1 - \left(\sum_{i=1}^{n}\theta_i\right)^{-\frac{n}{n-1}}\right)\end{array}\right\}^{< 1 - \varepsilon}\right\} \leq \varepsilon. \blacksquare$$

7 References

D'Aspremont, C., Bhattacharya, S., & Gérard-Varet, L.-A. (1998): "Knowledge as a public good: efficient sharing and incentives for development effort", *Journal of Mathematical Economics*, 30, p.389-404.

Al-Najjar, N.I. & Smorodinsky, R. (2000): "Pivotal Players and the Characterization of Influence", *Journal of Economic Theory*, 92, p.318-342.

Cramton, P., Gibbons, R., & Klemperer, P. (1987): "Dissolving a Partnership Efficiently", *Econometica*, 55, p.615-632.

Fang, H. & Norman, P. (2008): "Optimal Provision of Multiple Excludable Public Goods", mimeo.

Hellwig, M.F. (2003): "Public-Good Provision with Many Participants", *Review of Economic Studies*, 70, p.589-614.

Lehrer, E. & Neeman, Z. (2000): The Scope of Anonymous Voluntary Bargaining Under Asymmetric Information", *Review of Economic Studies*, 67, p.309-326.

Mailath, G. & Postlewaite, A. (1990): "Asymmetric Information Bargaining Problems with Many Agents", *Review of Economic Studies*, 57, p.351-367.

Myerson, R. & Satterthwaite, M. (1983): "Efficient Mechanisms for Bilateral Trading", *Journal of Economic Theory*, 28, p.265-281.

Norman, P. (2004): "Efficient Mechanisms for Public Goods with Use Exclusions", *Review of Economic Studies*, 71, p.1163-1180.

Roberts, J. (1976): "The incentives for correct revelation of preferences and the number of consumers", *Journal of Public Economics*, 6, p.359-374.

8 Appendix II

Proof of Lemma 2:

Interim side payments have been found above as

$$\xi_{Ui}(\theta_i) = -U_{Ui}(\underline{\theta}) + \rho_i(\theta_i)\theta_i - \int_{\underline{\theta}}^{\theta_i} \rho_i(\eta)d\eta \tag{47}$$

Now integrating over the full support we find that from the independence assumption

$$\int z_{Ui}(\boldsymbol{\theta})dF^n(\boldsymbol{\theta}) = \int \xi_{Ui}(\theta_i)dF(\theta_i) \tag{48}$$

so that

$$\int z_{Ui}(\boldsymbol{\theta})dF^n(\boldsymbol{\theta}) = \tag{49}$$

$$\int \left(-U_{Ui}(\underline{\theta}) + \rho_i(\theta_i)\theta_i - \int_{\underline{\theta}}^{\theta_i} \rho_i(\eta)d\eta \right) dF(\theta_i) =$$

$$-U_{Ui}(\underline{\theta}) + \rho_i(\theta_i)\theta_i - \int \int_{\underline{\theta}}^{\theta_i} \rho_i(\eta)d\eta dF(\theta_i)$$

Integration by parts of the last term and some rearranging yields

$$\int z_{Ui}(\boldsymbol{\theta})dF^n(\boldsymbol{\theta}) = -U_{Ui}(\underline{\theta}) + \int \left(\theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) \rho_i(\theta_i)dF(\theta_i) \tag{50}$$

Thus

$$\int z_{Ui}(\boldsymbol{\theta})dF^n(\boldsymbol{\theta}) = -U_{Ui}(\underline{\theta}) + \int \left(\theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) r(\boldsymbol{\theta})dF^n(\boldsymbol{\theta}).\blacksquare \tag{51}$$

Proof of Lemma 3:

Only-if-part: Suppose that $\langle r(\boldsymbol{\theta}), z_{U1}(\boldsymbol{\theta}), ...z_{Un}(\boldsymbol{\theta}) \rangle$ is interim incentive compatible and interim individually rational. Then summing (20) over all n we have

$$\sum_{i=1}^{n} \int z_{Ui}(\boldsymbol{\theta}) dF^n(\boldsymbol{\theta}) = -\sum_{i=1}^{n} U_{Ui}(\underline{\theta}) + \int \sum_{i=1}^{n} \left(\theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) r(\boldsymbol{\theta}) dF^n(\boldsymbol{\theta}) \tag{52}$$

Using the definition of weak feasibility this becomes

$$\sum_{i=1}^{n} \int z_{Pi}(\boldsymbol{\theta}) dF^n(\boldsymbol{\theta}) + \sum_{i=1}^{n} U_{Ui}(\underline{\theta}) - \int \sum_{i=1}^{n} \left(\theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) r(\boldsymbol{\theta}) dF^n(\boldsymbol{\theta}) \leq 0 \tag{53}$$

Using the degree of freedom in the analysis which follows from the constant of integration in Lemma 1 we normalize

$$U_{Ui}(\underline{\theta}) = \rho_i(\underline{\theta})\underline{\theta} - \xi_{Ui}(\underline{\theta}) = 0 \tag{54}$$

so that the interim individual rationality constraint for the lowest type is strictly binding. Finally using the implementation assumption of the provision contract (8) we find the constraint as

$$\int \left(\sum_{i=1}^{n} \left(\theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) r(\boldsymbol{\theta}) - np(\boldsymbol{\theta}) \right) dF^n(\boldsymbol{\theta}) \geq 0 \tag{55}$$

If-part: is again well known. We now show that, as claimed in Section 3 above, if (22) is satisfied, then side payments can be chosen so that even *feasibility* holds. The proof is by construction and follows the "If-part" of the Proof of Lemma 4 in Cramton, et. al. (1987) (see also Börgers and Norman (2008))[7]. Let the new side payments be

$$\check{z}_{Ui}(\boldsymbol{\theta}) = \xi_{Ui}(\theta_i) + \ z_{Pi}(\cdot) - \frac{1}{n-1} \sum_{j \neq i} \xi_{Uj}(\hat{\theta}_j) + \tag{56}$$

$$\frac{1}{n-1} \sum_{j \neq i} \int \xi_{Uj}(\hat{\theta}_j) dF(\hat{\theta}_{-j}) - \frac{1}{n-1} \sum_{j \neq i} \int z_{Pj}(\cdot) dF^{n-1}(\theta_i, \boldsymbol{\theta}_{-i})$$

---

[7]Börgers, T., & Norman, P. (2008): "A note on budget balance under interim participation constraints: the case of independent types", *Economic Theory*.

22

then summing over all n we find

$$\sum_{i=1}^{n} \check{z}_{Ui}(\boldsymbol{\theta}) = \sum_{i=1}^{n} z_{Pi}(\cdot) + \sum \int \xi_{Uj}(\hat{\theta}_j) dF(\hat{\theta}_{-j}) - \sum \int z_{Pj}(\cdot) dF^{n-1}(\theta_i, \boldsymbol{\theta}_{-i})$$

(57)

and by *weak feasibility* the sum of the last two termis non-negative we thus find that also *feasibility* holds:

$$\sum_{i=1}^{n} \check{z}_i(\boldsymbol{\theta}) \geq 0$$

(58)

Furthermore, taking expectations over all other types $-i$ we find that

$$\int \check{z}_{Ui}(\boldsymbol{\theta}) dF^{n-1}(\theta_i, \boldsymbol{\theta}_{-i}) = \xi_{Ui}(\theta_i)$$

(59)

and so the new transfers also interim incentive compatible. ∎

Lemma 4: *Define the probability of provision as an indicator function*

$$r_\lambda(\boldsymbol{\theta}) \equiv \left(1 - \left(\sum_{i=1}^n \theta_i\right)^{-\frac{n}{n-1}}\right) \mathbf{1}_{\left\{\begin{array}{c} (1+\lambda)\times \\ \left[\begin{array}{c} \left(\sum_{i=1}^n \theta_i\right) \times \left(1 - \left(\sum_{i=1}^n \theta_i\right)^{-\frac{n}{n-1}}\right) - \\ k_n\left(r_\lambda(\boldsymbol{\theta}) = 1 - \left(\sum_{i=1}^n \theta_i\right)^{-\frac{n}{n-1}}\right) \end{array}\right] \\ \lambda \sum_{i=1}^n \left[\frac{1-F(\theta_i)}{f(\theta_i)}\right] \times \left(1 - \left(\sum_{i=1}^n \theta_i\right)^{-\frac{n}{n-1}}\right) \end{array} > \right\}}$$

$$(60)$$

Then there exists a unique Lagrange multiplier $\lambda^* > 0$ such that

$$G(\lambda^*) \equiv \int \left(\sum_{i=1}^n \left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right) r_{\lambda^*}(\boldsymbol{\theta}) - k_n(r_{\lambda^*}(\boldsymbol{\theta}))\right) dF^n(\boldsymbol{\theta}) = 0 \quad (61)$$

given $n > 1$. Furthermore $r_\lambda(\theta)$ is non-increasing and $G(\lambda)$ is non-decreasing in λ.

Proof:

Following arguments similar to Myerson and Satterthwaite (1983) and Hellwig, (2003), we show that $\lambda \to G(\lambda)$ is increasing and negative for $\lambda = 0$ as show below that $G(0) < 0$ for $n > 1$ and $G(\lambda)$ is continuous.

As λ gets large the indicator function above approaches one of the form

$$r_\lambda(\boldsymbol{\theta}) \equiv 1 - \left(\sum_{i=1}^n \theta_i\right)^{-\frac{n}{n-1}} \mathbf{1}_{\substack{\sum_{i=1}^n \left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right)\times\left(1-\left(\sum_{i=1}^n \theta_i\right)^{-\frac{n}{n-1}}\right) \\ > k_n\left(r_\lambda(\boldsymbol{\theta})=1-\left(\sum_{i=1}^n \theta_i\right)^{-\frac{n}{n-1}}\right)}} \qquad (62)$$

so that there exists some $\hat{\lambda} > 0$ for which we have $G(\hat{\lambda}) > 0$ given $k_n(r_\lambda(\boldsymbol{\theta}) = 1 - (\sum_{i=1}^n \theta_i)^{-\frac{n}{n-1}}) = n\left(1 - (\sum_{i=1}^n \theta_i)^{\frac{1}{1-n}}\right) < n$ which holds for all $n < \infty$. Note that this indicator function $r_\lambda(\boldsymbol{\theta})$ is chosen such that for $\lambda = 0$ we get the first best result as in that case $(\sum_{i=1}^n \theta_i) \times \left(1 - (\sum_{i=1}^n \theta_i)^{-\frac{n}{n-1}}\right) - k_n\left(r_\lambda(\boldsymbol{\theta}) = 1 - (\sum_{i=1}^n \theta_i)^{-\frac{n}{n-1}}\right) > 0$ is indeed the unconstrained maximum welfare.

Strict monotonicity of $G(\lambda)$ is shown by looking at $r_{\lambda_1}(\boldsymbol{\theta}) \neq r_{\lambda_2}(\boldsymbol{\theta})$ for any $\lambda_1 \neq \lambda_2$. W.l.o.g. let $r_{\lambda_1}(\boldsymbol{\theta}) = 1 - (\sum_{i=1}^n \theta_i)^{-\frac{n}{n-1}}$, then from the indicator

function this implies that for small λ

$$\left(\sum_{i=1}^n \theta_i\right) \times \left(1 - \left(\sum_{i=1}^n \theta_i\right)^{-\frac{n}{n-1}}\right) - k_n \left(r_\lambda(\boldsymbol{\theta}) = 1 - \left(\sum_{i=1}^n \theta_i\right)^{-\frac{n}{n-1}}\right) > 0 \tag{63}$$

and for $r_{\lambda_2}(\boldsymbol{\theta}) = 0 < r_{\lambda_1}(\boldsymbol{\theta})$ and large λ the condition

$$\sum_{i=1}^n \left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right) \times \left(1 - \left(\sum_{i=1}^n \theta_i\right)^{-\frac{n}{n-1}}\right) - k_n \left(r_\lambda(\boldsymbol{\theta}) = 1 - \left(\sum_{i=1}^n \theta_i\right)^{-\frac{n}{n-1}}\right) \leq 0 \tag{64}$$

holds and thus $r_{\lambda_1}(\boldsymbol{\theta}) \lesseqgtr r_{\lambda_2}(\boldsymbol{\theta})$ as $\lambda_1 \gtreqless \lambda_2$.

Furthermore by definition we have

$$G(\lambda_1) - G(\lambda_2) =$$
$$\int \left(\begin{array}{c} (r_{\lambda_1}(\boldsymbol{\theta}) - r_{\lambda_2}(\boldsymbol{\theta})) \sum_{i=1}^n \left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right) - \\ (k_n(r_{\lambda_1}(\boldsymbol{\theta})) - k_n(r_{\lambda_2}(\boldsymbol{\theta}))) \end{array} \right) dF^n(\boldsymbol{\theta}) \tag{65}$$

Using the monotonicity of $k_n(r(\boldsymbol{\theta}))$ in $r(\boldsymbol{\theta})$ for $n > 1$ and (64) we find the final result that $G(\lambda_1) \lesseqgtr G(\lambda_2)$ implies that $\lambda_2 \gtreqless \lambda_1$ has to hold.∎

Intermediate result for Lemma 4:
See that $G(0) < 0$ for $n > 1$.

Proof: Note that for $n = 1$ the problem of providing a public good or a private good are isomorphic. Following Güth and Hellwig, (1986), Proposition 5.4.[8] we assume $n > 1$ agents rewriting condition (33) using the ex-post efficient provision probability $r(\boldsymbol{\theta})^*$ as given by the indicator function (28) as

$$G(0) = \int \left(\sum_{i=1}^n \left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right) r(\boldsymbol{\theta})^* - k_n(r(\boldsymbol{\theta})^*) \right) dF^n(\boldsymbol{\theta}) \geq 0 \tag{66}$$

Our aim is to show that this condition will not hold. We rewrite the condition by integrating the ex-post efficiency condition into the limits of the integral as

---

[8] Güth, W. & Hellwig, M.F. (1986): "The Private Supply of a Public Good", *Journal of Economics*, Supplement, 5, p.121-159.

$$\int_{\Delta} \left(\sum_{i=1}^{n} \left(\left(\theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) - k_n(r(\boldsymbol{\theta})) \right) \right) dF^n(\boldsymbol{\theta}) \geq 0 \qquad (67)$$

where

$$\Delta \equiv \left\{ \boldsymbol{\theta} \in \Theta^n \left| \sum_{i=1}^{n} \theta_i > \frac{k_n(r(\boldsymbol{\theta}))}{r(\boldsymbol{\theta})} \right. \right\} \qquad (68)$$

We aim to show that this leads to a contradiction. In order to separate out integrals, let $k = 1...n$ and define two critical type vectors as

$$\Delta^k \equiv \left\{ \boldsymbol{\theta}^k \in \Theta^k \left| \sum_{i=1}^{k} \theta_i > \frac{k_n(r(\boldsymbol{\theta}))}{r(\boldsymbol{\theta})} \geq \sum_{i=1}^{k-1} \theta_i \right. \right\} \qquad (69)$$

and

$$\Delta^{k-1} \equiv \left\{ \boldsymbol{\theta}^{k-1} \in \Theta^{k-1} \left| \sum_{i=1}^{k-1} \theta_i + 1 > \frac{k_n(r(\boldsymbol{\theta}))}{r(\boldsymbol{\theta})} \geq \sum_{i=1}^{k-1} \theta_i \right. \right\} \qquad (70)$$

so that $\Delta \equiv \cup_{k=1}^{n} \left[\Delta^k \times [0,1]^{n-k} \right]$ as we only need to integrate over the types for which the constraint holds. We can rewrite the constraint (25) using the independence assumption as

$$\sum_{k=1}^{n} \int_{\Delta^k \times [0,1]^{n-k}} \left(\sum_{i=1}^{n} \left(\theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) - k_n(r(\boldsymbol{\theta})) \right) dF^n(\boldsymbol{\theta}) \geq 0 \qquad (71)$$

For $k > 1$ we decompose the integral and write

$$\int_{\Delta^{k-1}} \int_{k_n(r(\boldsymbol{\theta})) - \sum_{i=1}^{k-1} \theta_i}^{1} \left(\sum_{i=1}^{k} \left(\left(\theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) - k_n(r(\boldsymbol{\theta})) \right) \right) \times \quad (72)$$

$$dF_k(\theta_k) dF^{k-1}(\boldsymbol{\theta}^{k-1}) + F^k(\Delta^k) \sum_{j=k+1}^{n} \int \left(\theta_j - \frac{1 - F(\theta_j)}{f(\theta_j)} \right) dF_j(\theta_j)$$

For $j = k + 1...n$ the expected value of the virtual utility is $\underline{\theta}(= 0)$ and hence the second term falls out.

Integrating by parts we can show that the first part becomes

$$\int_{k_n(r(\boldsymbol{\theta}))-\sum_{i=1}^{k-1}\theta_i}^{1} \left(\theta_u - \frac{1-F(\theta_u)}{f(\theta_u)}\right) dF_u(\theta_u) = \qquad (73)$$

$$\left(k_n(r(\boldsymbol{\theta})) - \sum_{i=1}^{k-1}\theta_i\right)\left(1 - F_k(k_n(r(\boldsymbol{\theta})) - \sum_{i=1}^{k-1}\theta_i)\right)$$

As we have the first-best choice in the integral condition we find

$$
\begin{aligned}
G(0) &= \int_{\Delta^k \times [0,1]^{n-k}} \left(\sum_{i=1}^{n}\left(\left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right) - k_n(r(\boldsymbol{\theta}))\right)\right) dF^n(\boldsymbol{\theta}) = \\
&\quad \int_{\Delta^{k-1}} \left(\sum_{i=1}^{k-1}\left(\left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right) - k_n(r(\boldsymbol{\theta})) + k_n(r(\boldsymbol{\theta})) - \sum_{i=1}^{k-1}\theta_i\right)\right) \times \\
&\quad \left(1 - F_k\left(k_n(r(\boldsymbol{\theta})) - \sum_{i=1}^{k-1}\theta_i\right)\right) dF^{k-1}(\boldsymbol{\theta}^{k-1}) = \\
&\quad -\int_{\Delta^{k-1}} \sum_{i=1}^{k-1}\left(\frac{1-F(\theta_i)}{f(\theta_i)}\right)\left(1 - F_k\left(k_n(r(\boldsymbol{\theta})) - \sum_{i=1}^{k-1}\theta_i\right)\right) dF^{k-1}(\boldsymbol{\theta}^{k-1}) \leq 0
\end{aligned}
$$

$$(74)$$

thus the first best outcome violates the constraint. If $n > 1$ and (at least) $k > 1$, Δ^{k-1} has positive measure and thus the inequality is strict so that we have $G(0) < 0$. One way of interpreting this result is that for a non-trivial number of agents a mechanism that imposes the first best outcome would imply a loss to the designer and hence the public good can only be provided with a subsidy.∎

27

Proof of Lemma 5:

By contradiction: The program $\mathcal{P}$ can be written as

$$Max_{r(\boldsymbol{\theta}),\lambda}\left\{\int\left(\sum_{i=1}^{n}\left(\theta_i - \frac{\lambda}{1+\lambda}\left(\frac{1-F(\theta_i)}{f(\theta_i)}\right)\right)r(\boldsymbol{\theta}) - k_n(r(\boldsymbol{\theta}))\right)dF^n(\boldsymbol{\theta}), 0\right\}$$

Denote $r_{\lambda^{n*}}(\boldsymbol{\theta})$ as the provision rule that maximizes the probability of provision and we assume that $\lambda = \bar{\lambda}$ is *constant*, so that $r_{\lambda^{n*}}(\boldsymbol{\theta}) = r_{\bar{\lambda}}(\boldsymbol{\theta})$. For a constant multiplier $\bar{\lambda}$, $r_{\bar{\lambda}}(\boldsymbol{\theta}) \overset{a.s.}{\rightarrow} 1$ if

$$\lim_{n\to\infty}\left(\frac{k_n(r(\boldsymbol{\theta}))}{\sum_{i=1}^{n}\left(\theta_i - \frac{\bar{\lambda}}{1+\bar{\lambda}}\left(\frac{1-F(\theta_i)}{f(\theta_i)}\right)\right)}\right) \overset{a.s.}{\rightarrow} 0 \qquad (75)$$

for all $r(\boldsymbol{\theta}) \in [0,1)$. Looking at

$$(1-(1-r_{\bar{\lambda}}(\boldsymbol{\theta}))^{\frac{1}{n}}) = \frac{1}{n}\sum_{i=1}^{n}\left(\theta_i - \frac{\bar{\lambda}}{1+\bar{\lambda}}\left(\frac{1-F(\theta_i)}{f(\theta_i)}\right)\right) \qquad (76)$$

we solve into

$$1 - r_{\bar{\lambda}}(\boldsymbol{\theta}) = \left(1 - \frac{1}{n}\sum_{i=1}^{n}\left(\theta_i - \frac{\bar{\lambda}}{1+\bar{\lambda}}\left(\frac{1-F(\theta_i)}{f(\theta_i)}\right)\right)\right)^n \qquad (77)$$

Using the LLN in the numerator we know that the realization of the sum of random variables converges to its expectation almost surely, i.e.

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\theta_i - \frac{\bar{\lambda}}{1+\bar{\lambda}}\left(\frac{1-F(\theta_i)}{f(\theta_i)}\right)\right)\right) = \frac{E\{\theta\}}{1+\bar{\lambda}} < 1 \qquad (78)$$

i.e. a constant, which follows from an integration by parts. Hence for large n the equality can only be satisfied if we have that $r_{\bar{\lambda}}(\boldsymbol{\theta}) \overset{a.s.}{\rightarrow} 1$, given a constant multiplier. This contradicts the assumption, as almost sure convergence implies that for large n, the provision probability $r(\tilde{\theta}_i, \boldsymbol{\theta}_{-i}^{n-1})$ cannot vary very much with the signal of the ith agent and hence the side payment extracted from him under the user contract will be close to zero. As this holds for any agent the feasibility constraint will be violated. Thus the Lagrange multiplier cannot be constant and therefore λ^{n*} has to grow out of bounds as

the number of agents gets large. We conclude that $\lim_{n\to\infty} \lambda^{n*} = \infty$ has to hold so that both numerator and denominator go to zero which prevents $r_{\bar{\lambda}}(\boldsymbol{\theta}) \overset{a.s.}{\to} 1$.∎

Proof of Lemma 6:
The first best provision rule satisfies

$$\lim_{n\to\infty} r^*(\boldsymbol{\theta}) = \lim_{n\to\infty} \left(1 - \left(\sum_{i=1}^{n} \theta_i \right)^{-\frac{n}{n-1}} \right) \mathbf{1}_{\left\{\sum_{i=1}^{n} \theta_i > 1\right\}} = 1. \blacksquare \qquad (79)$$

Proof of Lemma 8:
The interim incentive compatibility of a mechanism (satisfying the two contracts) guarantees that no agent has an incentive to misrepresent valuations, i.e.

$$\int (r(\theta_i, \boldsymbol{\theta}_{-i})\theta_i - p(\theta_i, \boldsymbol{\theta}_{-i}))dF^{n-1}(\boldsymbol{\theta}_{-i}) \geq$$

$$\int (r(\hat{\theta}_i, \boldsymbol{\theta}_{-i})\theta_i - p(\hat{\theta}_i, \boldsymbol{\theta}_{-i}))dF^{n-1}(\boldsymbol{\theta}_{-i}) \ \forall \ \theta_i, \hat{\theta}_i \ \in \ \Theta, \ \forall \ \theta_i \qquad (80)$$

Consider only mechanisms which treat all agents symmetrically and let $r(\boldsymbol{\theta}) \ \in \ \left\{ 0, \left(1 - (\sum_{i=1}^{n} \theta_i)^{-\frac{n}{n-1}} \right) \mathbf{1}_{\left\{\sum_{i=1}^{n} \theta_i > 1\right\}} \right\}$ be non-decreasing in $k \equiv \# \left\{ i \, | \, \theta_i = \bar{\theta} \right\}$. The strictest interim incentive compatibility condition is given as

$$\int (r(\theta_i \ = \ \bar{\theta}, \boldsymbol{\theta}_{-i}) - r(\theta_i = \underline{\theta}, \boldsymbol{\theta}_{-i}))dF^{n-1}(\boldsymbol{\theta}_{-i}) \geq$$

$$\frac{1}{n\underline{\theta}} \int (k_n(r(\theta_i \ = \ \bar{\theta}, \boldsymbol{\theta}_{-i})) - k_n(r(\theta_i = \underline{\theta}, \boldsymbol{\theta}_{-i})))dF^{n-1}(\boldsymbol{\theta}_{-i}) \qquad (81)$$

Due to symmetry we can write r as a function of k and there is *at most one* value of k (defining the 'pivotal' agent k^*) for which

$$(r(\theta_{k^*} = \bar{\theta}, \boldsymbol{\theta}_{-k^*}) - r(\theta_{k^*} = \underline{\theta}, \boldsymbol{\theta}_{-k^*})) = \left(1 - \left(\sum_{i=1}^{n} \theta_i \right)^{-\frac{n}{n-1}} \right) \mathbf{1}_{\left\{\sum_{i=1}^{n} \theta_i > 1\right\}}$$

$$(82)$$

Let h be the number of agents besides i to have $\theta_i = \bar{\theta}$ we can rewrite the incentive compatibility using the Binomial distribution for equal probability as

$$\max_h \left[\left(\frac{1}{2}\right)^{n-1} \frac{(n-1)!}{(n-1-h)!h!} \right] \geq$$
$$\frac{1}{n\underline{\theta}} \int \left(k_n(r(\theta_i = 1, \boldsymbol{\theta}_{-i})) - k_n(r(\theta_i = 0, \boldsymbol{\theta}_{-i})) \right) dF^{n-1}(\boldsymbol{\theta}_{-i}) \qquad (83)$$

where the LHS is now the probability of being the pivotal agent k^*. An upper bound on the LHS is found by choosing $h = (n-1)/2$ (as in the simple majority rule case) so that using Stirling's formula given as

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{\frac{\varrho(n)}{12n}} \text{ for some } \varrho(n) \in (0,1) \qquad (84)$$

we have

$$\left(\frac{1}{2}\right)^{n-1} \frac{\sqrt{2\pi(n-1)} \left(\frac{n-1}{e}\right)^{n-1} e^{\frac{\varrho(n)}{12n}}}{\left(\sqrt{2\pi(\frac{n-1}{2})} \left(\frac{n-1}{2e}\right)^{\frac{n-1}{2}} e^{\frac{\varrho(n)}{12n}}\right)^2} \leq \qquad (85)$$

$$\left(\frac{1}{2}\right)^{n-1} \frac{\sqrt{(n-1)} \left(\frac{n-1}{e}\right)^{n-1} e^{\frac{1}{12n}}}{\sqrt{2\pi} \left(\sqrt{(\frac{n-1}{2})} \left(\frac{n-1}{2e}\right)^{\frac{n-1}{2}}\right)^2} =$$

$$\frac{1}{\sqrt{2\pi}} \frac{2}{\sqrt{(n-1)}} e^{\frac{1}{12n}}. \blacksquare$$